Veering triangulations and the Cannon-Thurston map

François Guéritaud

Hyperbolic mapping tori. Let S be an oriented surface with at least one puncture, and $\varphi : S \to S$ an orientation-preserving homeomorphism. Define the *mapping torus* $M := S \times [0,1]/ \sim_\varphi$, where $\sim_\varphi$ identifies $(x,0)$ with $(\varphi(x),1)$. The topological type of the 3-manifold M depends only on the isotopy type of φ.

In what follows, we shall assume that φ is *pseudo-Anosov*, a technical condition meaning that the isotopy class $[\varphi]$ preserves no finite system of curves on S. A landmark result of Thurston's [6] is that M then admits a (unique) complete *hyperbolic metric*: $M \simeq \Gamma\backslash\mathbb{H}^3$ for some discrete group of isometries Γ. An important step towards this is the existence of a transverse pair of $[\varphi]$-invariant *singular foliations* λ^+, λ^- of S into lines (called *leaves*).

In [1], Agol described a canonical way of triangulating a mapping torus M, provided all singularities of the foliations λ^+, λ^- occur at punctures of the fiber S. These (ideal) triangulations enjoy a combinatorial property called *veeringness.* In [5] and [4], veering triangulations are shown to admit *positive angle structures*: this is a linearized version of the problem of finding the complete hyperbolic metric on M (endowed with a geodesic triangulation).

Combinatorics of the veering triangulation. I first presented an alternative construction of Agol's triangulation, which can be summarized as follows. Endow S with a flat (incomplete) metric for which the lines of the measured foliations λ^+ and λ^- are vertical and horizontal, respectively. Look for all possible maximal rectangles $R \subset S$ with edges along leaf segments. By maximality, such a rectangle R contains one singularity in each of its four sides. Connecting these four points and thickening, we get a tetrahedron $\Delta_R \subset S \times [0,1]$. It only remains to check that the tetrahedra Δ_R glue up to yield a triangulation of $S \times [0,1]$ (naturally compatible with the equivalence relation $\sim_\varphi$ since the foliations λ^+, λ^- are $[\varphi]$-invariant).

Unlike Agol's original definition, this does not rely on any auxiliary choices (*e.g.* of train tracks). One upshot is that it allows a detailed description of the induced 2-dimensional triangulations $\mathcal{T}$ of the vertex links (which are tori). The details do not matter, but each torus turns out to be decomposed into an even number of parallel annuli, with each triangle of $\mathcal{T}$ having its basis on a boundary component of some annulus, and its tip on the other boundary component.

The Cannon-Thurston map. Next, I showed that the combinatorics of a veering triangulation are also related to the hyperbolic geometry of M via the *Cannon-Thurston map*, which we now define. Let D (a disk) be the universal cover of the fiber S. The inclusion $S \to M$ lifts to a map $\iota : D \to \mathbb{H}^3$ between the universal covers, which turns out to extend continuously to a boundary map $\overline{\iota} : \mathbb{S}^1 \to \mathbb{S}^2$.

Cannon and Thurston [3] proved the surprising fact that $\bar{\iota}$ is a (continuous) *surjection* from the circle to the sphere. The endpoints of any leaf of $\lambda^\pm$ have the same image under $\bar{\iota}$, and this in fact generates all the identifications occurring under $\bar{\iota}$.

The connection with the veering triangulation and $\mathcal{T}$ is as follows. Choose an ideal vertex of the ideal triangulation of M; call it ∞. The hyperbolic metric gives a natural identification between $\mathbb{S}^2 - \{\infty\}$ and the universal cover of the toroidal link of ∞ in M. This universal cover (a plane Π) receives a topological triangulation $\widetilde{\mathcal{T}}$, lifting $\mathcal{T}$, in which the annuli of $\mathcal{T}$ become infinite vertical strips. (The vertices of $\widetilde{\mathcal{T}}$ are well-defined points with algebraic coordinates in $\mathbb{R}^2$, although higher skeleta of $\widetilde{\mathcal{T}}$ are only defined up to isotopy.) It turns out that the surjection $\bar{\iota} : \mathbb{S}^1 \to \Pi \cup \{\infty\}$ fills out Π by filling out in ordered succession a $\mathbb{Z}^2$-collection of topological disks, column by column, with columns being travelled alternately up and down. Each column corresponds to the interface A between adjacent infinite strips of $\widetilde{\mathcal{T}}$, and each topological disk δ corresponds to a basis $\beta \subset A$ of a triangle of $\widetilde{\mathcal{T}}$, with $\partial\beta \subset \partial\delta$. Two consecutive disks intersect at exactly one point, a vertex of $\widetilde{\mathcal{T}}$. Arbitrary disks intersect only (if at all) along their Jordan-curve boundaries, and the disks meet four at each vertex of $\widetilde{\mathcal{T}}$.

Although describing the full combinatorics requires a more elaborate dictionary between the foliations $\lambda^\pm$, the triangulation $\mathcal{T}$, and the Cannon-Thurston map $\bar{\iota}$, we can state the first entry of this dictionary as follows.

Theorem. *Suppose the hyperbolic 3-manifold M is a pseudo-Anosov mapping torus such that all singularities of the invariant foliations $\lambda^\pm$ occur at punctures of the fiber S. Let $\widetilde{\mathcal{T}}$ be the topological (doubly periodic) triangulation of the plane arising from the veering triangulation of M, and $\mathcal{D} = \{\delta_i\}_{i\in I}$ be the decomposition of the plane into topological disks arising from the Cannon-Thurston map. Then $\widetilde{\mathcal{T}}$ and $\mathcal{D}$ have the same vertex set.*

This connection was previously known for the punctured torus by work of Cannon and Dicks [2].

References

[1] I. Agol, *Ideal Triangulations of Pseudo-Anosov Mapping Tori*, arXiv:1008.1606, in Contemp. Math. 560, Amer. Math. Soc., Providence, RI, (2011), 1–17.

[2] J.W. Cannon, W. Dicks, *On hyperbolic once-punctured-torus bundles. II. Fractal tessellations of the plane*, Geom. Dedicata 123 (2006), 11–63.

[3] J.W. Cannon and W.P. Thurston, *Group Invariant Peano Curves*, Geometry and Topology 11 (2007), 1315–1356.

[4] D. Futer, F. Guéritaud, *Explicit angle structures for veering triangulations*, arxiv:1012.5134

[5] C.D. Hodgson, J. H. Rubinstein, H. Segerman, S. Tillmann, *Veering triangulations admit strict angle structures*, arXiv:1011.3695, G&T 15 (2011), 2073–2089.

[6] J.-P. Otal, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3*, SMF – Astérisque 235 (1996), 159 pages.