TRIANGULATED CORES OF PUNCTURED-TORUS GROUPS

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Abstract

We show that the interior of the convex core of a quasifuchsian punctured-torus group admits an ideal decomposition (usually an infinite triangulation) which is canonical in two very different senses: in a combinatorial sense *via* the pleating invariants, and in a geometric sense *via* an Epstein-Penner convex hull construction in Minkowski space. This result re-proves the Pleating Lamination Theorem for quasifuchsian punctured-torus groups, and extends to all punctured-torus groups if a strong version of the Pleating Lamination Conjecture is true.

0. Introduction

Among Kleinian groups with infinite covolume, quasifuchsian groups, which are deformations of Fuchsian surface groups, are fundamental examples. It has been noted [**Jø**, **MSW**] that such groups can be analyzed much more explicitly when the base surface is just a once-punctured torus — the curve complex is then dual to a planar tree, which simplifies many issues. Punctured-torus groups (which still retain many features of general quasifuchsian groups) have thus become a favorite "training ground": Minsky's work [**Min**] on end invariants is an example. Similarly, Caroline Series [**Se**] was able to prove the Pleating Lamination Conjecture for quasifuchsian punctured-torus groups (general pleating laminations, in contrast, seem to pose tremendous technical challenges: see [**BO**] for a partial treatment). In this paper, we prove that the interior of the convex core of any quasifuchsian punctured-torus group admits an ideal triangulation (or slightly more general decomposition) which relates "as nicely as one could possibly hope" both to the intrinsic geometry of the convex core, and to the combinatorics of its boundary pleatings. This answers several conjectures made in [**ASWY2**]. As a byproduct, we get enough control to re-prove the aforementioned result of [**Se**].

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic manifold with one cusp. Akiyoshi and Sakuma [**AS**] generalized the Epstein-Penner convex hull construction in Minkowski space $\mathbb{R}^{3+1}$ (see [**EP**]) to show that the interior V of the convex core of M admits a (unique) decomposition $\mathcal{D}^{Geom}$, canonical in a purely geometric sense. Roughly speaking, $\mathcal{D}^{Geom}$ is defined by considering the convex hull $C \subset \mathbb{R}^{3+1}$ of the discrete Γ-orbit of any light-like (i.e. isotropic) vector representing the cusp, and by projecting the cell decomposition of ∂C back to M (see Section 9 for more detail).

This decomposition $\mathcal{D}^{Geom}$ of V is quite mysterious in general. It seems to be unknown, for instance, whether $\mathcal{D}^{Geom}$ is always locally finite (see [**AS**]). However, those top-dimensional cells of $\mathcal{D}^{Geom}$ which come from space-like faces of C are contractible ideal polyhedra and are dual to the vertices of the Ford-Voronoi domain of M. Other top-dimensional cells of $\mathcal{D}^{Geom}$ may be non-contractible, are not seen in the Ford-Voronoi domain, and will typically accumulate near ∂V.

For quasifuchsian punctured-torus groups (i.e. geometrically finite discrete subgroups of $\mathrm{Isom}^+(\mathbb{H}^3)$ freely generated by two elements with parabolic commutator), a nice guess can be made at the combinatorics of $\mathcal{D}^{Geom}$. Indeed, the complex of ideal triangulations of the punctured torus is a tree T, and the projective classes of the two boundary pleatings of V can be seen as two points at infinity in T: the unique geodesic in T connecting these two points will define an explicit combinatorial ideal triangulation $\mathcal{D}^{Comb}$ of V (see Section 1 for details). The triangulation $\mathcal{D}^{Comb}$ is canonical in a combinatorial sense, with respect to the pleating data of V. It is natural to ask if $\mathcal{D}^{Comb}$ can be straightened out to be made a totally geodesic triangulation of V (without turning any tetrahedra "inside out"). Our main theorem answers this question in the affirmative:

Theorem 0.1. *If V is the interior of the convex core of a quasifuchsian punctured-torus group, the ideal triangulation $\mathcal{D}^{Geom}$ of V is the totally geodesic straightening of $\mathcal{D}^{Comb}$.*

In fact, aside from $\mathcal{D}^{Comb}$ and its coarsenings, the author does not know of any ideal cell decomposition that is invariant under the hyperelliptic involution (a property which $\mathcal{D}^{Geom}$ must enjoy by symmetry).

Thurston and Bonahon [**B1**] showed how to associate a projective class of measured laminations, or *ending lamination*, to a geometrically infinite end of a hyperbolic manifold. The Ending Lamination Theorem [**BCM**] roughly states that the asymptotic geometry of the end is completely described by its ending lamination. In view of extending Theorem 0.1 to all punctured-torus groups, it is reasonable to understand ending laminations as *infinitely strong pleatings* (see Section 10-D), and to conjecture that the group is determined by its ending and/or pleating laminations (this is a strong version of the Pleating Lamination Conjecture). Indeed, our method allows to construct punctured-torus groups with arbitrary admissible ending and/or pleating laminations, and these groups satisfy an analogue of Theorem 0.1: the precise statement, with a full description of $\mathcal{D}^{Geom}$ (especially in the case of rational laminations), is Theorem 10.1 in Section 10. When exactly one of the two ends is geometrically infinite, the uniqueness of these groups remains an open problem.

The identity $\mathcal{D}^{Geom} = \mathcal{D}^{Comb} =: \mathcal{D}$ of Theorem 0.1 (and the existence of $\mathcal{D}^{Comb}$, as realized by positively oriented cells) was Conjecture 8.2 in [**ASWY2**], also called the *Elliptic-Parabolic-Hyperbolic (EPH) Conjecture*. Akiyoshi subsequently established the identity in the case of two infinite ends, in [**Ak**]. Near a finite end however, the ideal tetrahedra of $\mathcal{D}$ flatten at a very quick rate: the smallest angle of the i-th tetrahedron typically goes to 0 faster than any geometric sequence $(\alpha^i)_{i \geq 0}$. In a sense, the difficulty is to show that these angles nevertheless stay positive for the hyperbolic metric.

The geometric realization of $\mathcal{D}^{Comb}$ (i.e. the construction of angles for the tetrahedra) will be fairly explicit: the solution will arise as the maximum of an explicit concave "volume" functional $\mathcal{V}$ over an explicit convex domain. The domain has infinite dimension, but there are explicit bounds on the contributions to $\mathcal{V}$ of the "tail" coordinates (suggesting efficient numerical implementations). This feature also allows us to show that the Kleinian group is a continuous function of its pleatings, which is the key to the Pleating Lamination Theorem (Section 10-E).

Acknowledgements. My thanks go to Makoto Sakuma for having drawn my attention to the EPH conjecture. I am also deeply indebted to Francis Bonahon, Frédéric Paulin, Makoto Sakuma and David Futer for many discussions, insights, and suggestions.

1. Strategy

The triangulation $\mathcal{D}^{Geom}$ can be seen (dually) as a refinement of the Ford-Voronoi domain of the quasifuchsian punctured-torus group. The latter domain was described in great detail in [**Jø**, **ASWY1**, **ASWY2**] and [**ASWY3**]: this produced a combinatorial description of *finitely many* tetrahedra of $\mathcal{D}^{Geom}$, based on a "geometric continuity" argument in the (connected) space of all quasifuchsian groups. To study *all* tetrahedra at once, including those not seen in the Ford-Voronoi domain, the present paper takes a somewhat opposite approach: first describe geometric shapes for the tetrahedra of the candidate triangulation $\mathcal{D}^{Comb}$, then establish that the gluing of these tetrahedra defines (the interior of the convex core of) a quasifuchsian group. A computation in Minkowski space then yields Theorem 0.1 ($\mathcal{D}^{Comb} = \mathcal{D}^{Geom}$), as a bonus, so to speak.

Identities of the form $\mathcal{D}^{Comb} = \mathcal{D}^{Geom}$ have been established by specialized techniques in several related contexts: punctured-torus bundles (by Lackenby [**La**]); complements of two-bridge links (see the announcement in [**ASWY3**]) or of certain arborescent links (see [**G2**]); totally degenerate punctured-torus groups [**Ak**]. The methods of the present paper essentially cover all these cases, but its true novelty is in dealing with pleating laminations, i.e. with groups whose limit set is smaller than $\mathbb{P}^1\mathbb{C}$. We will build on results (and notation) from [**GF**].

1-A. Setup. Let $S := (\mathbb{R}^2 \smallsetminus \mathbb{Z}^2)/\mathbb{Z}^2$ be the once-punctured torus endowed with its differential structure. Let $\mathcal{C}$ be the set of isotopy classes of simple closed curves in S that are not the loop around the puncture. Let also $\mathcal{C}'$ be the set of isotopy classes of properly embedded lines (running from the puncture to itself) in S that do not bound a disk.

Consider the maximal abelian cover $\pi : \mathbb{R}^2 \smallsetminus \mathbb{Z}^2 \to S$, and two coprime integers ξ, η. If ℓ is a line of slope η/ξ in $\mathbb{R}^2$ missing (resp. intersecting) $\mathbb{Z}^2$, then $\pi(\ell \smallsetminus \mathbb{Z}^2)$ defines a class $\gamma \in \mathcal{C}$ (resp. $\gamma' \in \mathcal{C}'$). We call $\eta/\xi \in \mathbb{P}^1\mathbb{Q}$ the *slope* of γ (resp. γ'). It is well-known that both $\mathcal{C}$ and $\mathcal{C}'$ are parametrized by $\mathbb{P}^1\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$, *via* the slope: we have bijections

$$s : \mathcal{C} \xrightarrow{\sim} \mathbb{P}^1\mathbb{Q} \qquad ; \qquad s' : \mathcal{C}' \xrightarrow{\sim} \mathbb{P}^1\mathbb{Q} \ .$$

When there is no ambiguity, we will sometimes omit s and identify $\mathcal{C}$ with $\mathbb{P}^1\mathbb{Q}$.

We will focus primarily on *irrational* pleating laminations (for rational ones, see Section 10). Let (α^+, β^+) and (α^-, β^-) be elements of $\mathbb{R}^{*2}$ such that β^+/α^+ and β^-/α^- are distinct irrationals. Define the *pleatings* $\lambda^{\pm} : \mathbb{P}^1\mathbb{Q} \to \mathbb{R}^{\pm}$ by

$$(1) \qquad \lambda^+(\eta/\xi) = \left\| \begin{array}{cc} \beta^+ & \eta \\ \alpha^+ & \xi \end{array} \right\| \;\; \text{and} \;\; \lambda^-(\eta/\xi) = - \left\| \begin{array}{cc} \beta^- & \eta \\ \alpha^- & \xi \end{array} \right\|,$$

where the double bars denote the absolute value of the determinant (and ξ, η are coprime integers). Replacing a pair $(\alpha^{\pm}, \beta^{\pm})$ by its negative does not change $\lambda^{\pm}$. Notice that $\lambda^- < 0 < \lambda^+$ (as functions on $\mathbb{P}^1\mathbb{Q}$).

In the hyperbolic plane $\mathbb{H}^2$ with boundary $\partial\mathbb{H}^2 = \mathbb{P}^1\mathbb{R}$, consider the Farey triangulation (the ideal triangle 01∞ iteratedly reflected in its sides, see e.g. Section 2 of [**Min**]). The irrationals β^+/α^+ and β^-/α^- belong to $\partial\mathbb{H}^2$, and the oriented (geodesic) line Λ from β^-/α^- to β^+/α^+ crosses infinitely many Farey edges $(e_i)_{i\in\mathbb{Z}}$ (the choice of e_0 is arbitrary). To every pair of consecutive integers $(i-1, i)$ is associated a letter, R or L, according to whether e_{i-1} and e_i share their Right or Left end, with respect to the orientation of Λ (we say that Λ *makes a Right* or *makes a Left* across the Farey triangle between e_{i-1} and e_i). We thus get a bi-infinite word $...RLLLRR...$ with infinitely many R's (resp. L's) near either end.

Two distinct rationals of $\mathbb{P}^1\mathbb{Q}$ are Farey neighbors exactly when the corresponding elements (properly embedded lines) of $\mathcal{C}'$ are disjoint up to homotopy. Therefore, each Farey triangle τ defines an *ideal triangulation* of the punctured torus S: indeed, the vertices of τ are the slopes of three disjoint (properly embedded) lines of $\mathcal{C}'$, whose union separates S into two *ideal triangles*. Moreover, two triangulations of S corresponding to Farey triangles which share an edge always differ by a *diagonal move* (see Figure 1). Such a diagonal move must be seen as a (topological) ideal tetrahedron in $S \times \mathbb{R}$ filling the space between two (topological) pleated surfaces, pleated along the two ideal triangulations.
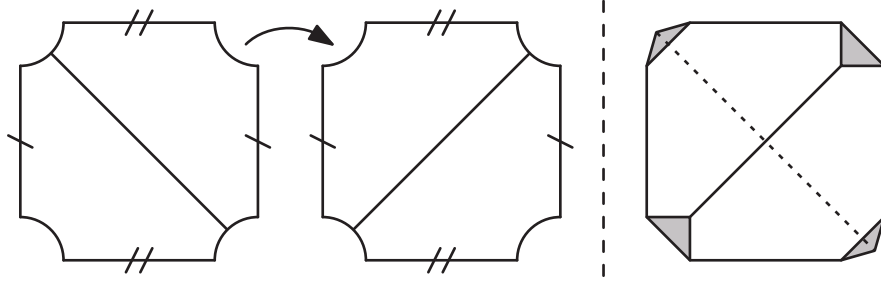

Figure 1.   Left: a diagonal move in S (the puncture is in the corners), associated to adjacent Farey triangles $-1\infty 0$ and $1\infty 0$. Right: how to see the diagonal move as an ideal tetrahedron, here with truncated vertices (grey).

Our strategy will be to consider triangulated surfaces S_i (pleated punctured tori) corresponding to the Farey triangles living between e_{i-1} and e_i for i ranging over $\mathbb{Z}$, interpolate these surfaces with tetrahedra Δ_i corresponding to the diagonal moves (i.e. to the Farey edges e_i themselves), and provide geometric

parameters (dihedral angles) for all these objects, using Rivin's Maximum Volume Principle (see [**Ri**]). Note that the $\lambda^\pm$ encode the combinatorial gluing rule between Δ_{i-1} and Δ_i. Also, each $S_i \subset S \times \mathbb{R}$, being homotopic to a fiber $S \times \{*\}$, comes with a natural *marking* $S_i \simeq S$ (defined up to isotopy).

Theorem 1.1. *There exists a cusped, non-complete hyperbolic 3-manifold V homeomorphic to $S \times \mathbb{R}$ such that*

  i – *The metric completion $\overline{V}$ of V is homeomorphic to $S \times [0,1] = V \sqcup S_{-\infty} \sqcup S_{+\infty}$, where $S_{-\infty}, S_{+\infty}$ are pleated surfaces whose pleating measures restrict to $\lambda^- \circ s$ and $\lambda^+ \circ s$ (respectively) on the set $\mathcal{C}$ of simple closed curves, where s is the slope function;*

  ii – *The geometric realizations of the $\{\Delta_i\}_{i\in\mathbb{Z}}$ in V define a totally geodesic ideal triangulation $\mathcal{D}^{Comb}$ of V.*

Sections 2 to 8 are devoted to producing such a V. The puncture of S is required to correspond to a cusp of V, so $\overline{V}$ will be the convex core of a hyperbolic manifold $\mathbb{H}^3/\Gamma$, where Γ is a quasifuchsian punctured-torus group (and we say that $\overline{V}$ has *pleating data* $\lambda^\pm$).

By [**Se**], or Corollary 10.8 below, the group Γ is determined uniquely up to conjugacy in $\mathrm{Isom}(\mathbb{H}^3)$ by the $\lambda^\pm$. Our construction therefore provides a decomposition $\mathcal{D}^{Comb}$ of the interior V of the convex core of an *arbitrary* quasifuchsian punctured-torus group Γ with irrational pleating laminations. The universal cover of V is drawn in Figure 3 of [**Th**].

Throughout the paper, we will deal with an infinite family of tetrahedra $(\Delta_i)_{i\in\mathbb{Z}}$, separated by pleated once-punctured tori $(S_i)_{i\in\mathbb{Z}}$. By an arbitrary choice, we have resolved that S_i is the surface between Δ_i and Δ_{i-1} — or equivalently, that Δ_i is bounded by the surfaces S_i and S_{i+1}. However, the numbering of the tetrahedra should be seen as the more essential one (see especially Definition 3.1 below).

1-B. Plan of the paper. In Section 2 (borrowed from [**GF**]), we describe the space of possible dihedral angle assignments x_i, y_i, z_i for the $\{\Delta_i\}_{i\in\mathbb{Z}}$. In Section 3, we encode $\lambda^\pm$ into constraints on the x_i, y_i, z_i. In Section 4, we carry out (constrained) volume maximization. Important asymptotic features of the solution are analyzed in Section 5. In Section 6, we describe the Euclidean triangulation of the cusp. In Sections 7 and 8, we show that the pleated surfaces S_i converge in a strong enough sense, so that their limit as i goes to $\pm\infty$ describes the (pleated) boundary of the metric completion of $V = \bigcup_{i\in\mathbb{Z}} \Delta_i$. At that point, we have constructed (the convex core of) a quasifuchsian group. The corresponding instance of Theorem 0.1 then follows from a computation, carried out in Section 9. In Section 10, we provide a similar construction of punctured-torus groups with *rational* pleating slopes $\beta^\pm/\alpha^\pm$ and/or with infinite ends, and re-prove that (λ^+, λ^-) are continuous coordinates for the space of quasifuchsian groups (see [**Se**]).

2. Dihedral angles

In this section we find positive dihedral angles for the ideal tetrahedra Δ_i, following Section 5 of [**GF**]. More precisely, we describe the convex space Σ of positive angle configurations for the Δ_i such that:

- the three dihedral angles near each ideal vertex of Δ_i add up to π (this is true in any ideal tetrahedron of $\mathbb{H}^3$);
- the dihedral angles around any edge of $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$ add up to 2π (this is necessary, though not sufficient, for a hyperbolic structure at the edge);
- the three pleating angles of each pleated punctured torus S_i add up to 0 (this is necessary, though not sufficient, to make the puncture p of S a cusp of V, i.e. make the loop around p lift to a parabolic isometry of $\mathbb{H}^3$).

(The first condition implies that opposite edges in Δ_i have the same dihedral angle.) Later on we shall apply Rivin's Maximum Volume Principle [**Ri**] on a certain convex subset of (the closure of) Σ.

If the tetrahedron Δ_i realizes a diagonal move (Figure 1) that kills an edge ε' of S_i and replaces it with ε, denote by $\pi - w_i$ the interior dihedral angle of Δ_i at ε and ε'. Observe that the slope of ε (resp. ε') is the rational located opposite the Farey edge e_i in the Farey diagram, on the side of β^+/α^+ (resp β^-/α^-).

Thus, the pleating angles of the surface S_i living between Δ_{i-1} and Δ_i are

$$(2) \qquad\qquad w_{i-1} \ , \ -w_i \ \text{ and } \ w_i - w_{i-1} \ .$$

Observe the sign convention: the pleated punctured torus embedded in $S \times \mathbb{R}$ receives an *upward* transverse orientation from $\mathbb{R}$, and the angles we consider are the dihedral angles *above* the surface, minus π. Thus, the "new" edge of Δ_{i-1}, pointing upward, accounts for a positive pleating w_{i-1}, while the "old" edge of Δ_i, pointing downward, accounts for a negative pleating $-w_i$. This is in accordance with the convention $\lambda^- < 0 < \lambda^+$ of (1), as we expect $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$ to be convex. We may write the three numbers (2) in the associated corners of the Farey triangle corresponding to S_i (Figure 2, top), and repeat the operation for S_{i+1} (bottom). For notational convenience, we write $(w_{i-1}, w_i, w_{i+1}) = (a, b, c)$.

In the tetrahedron Δ_i, let x_i (resp. y_i) be the interior dihedral angle at the edge whose slope is given by the right (resp. left) end of the Farey edge e_i. Let $z_i = \pi - w_i$ be the third dihedral angle of Δ_i. For instance, $2x_i$ (resp. $2y_i$) is the difference between the numbers written just below and just above the right (resp. left) end of e_i in Figure 2 (the factor 2 comes from the fact that the two edges of Δ_i with angle x_i [resp. y_i] are identified). By computing differences between the pleating angles given in Figure 2 (bottom), we find the following formulae for x_i, y_i, z_i (depending on the letters, R or L, living just before and just after the Farey edge e_i):

	$L\ L$	$R\ R$	$L\ R$	$R\ L$
x_i	$\frac{1}{2}(a+c)$	$\frac{1}{2}(-a+2b-c)$	$\frac{1}{2}(a+b-c)$	$\frac{1}{2}(-a+b+c)$
y_i	$\frac{1}{2}(-a+2b-c)$	$\frac{1}{2}(a+c)$	$\frac{1}{2}(-a+b+c)$	$\frac{1}{2}(a+b-c)$
z_i	$\pi-b$	$\pi-b$	$\pi-b$	$\pi-b$

(3)

The first of the three conditions defining Σ can be checked immediately; the other two are true by construction. From (3), the condition for all angles to be positive is that:
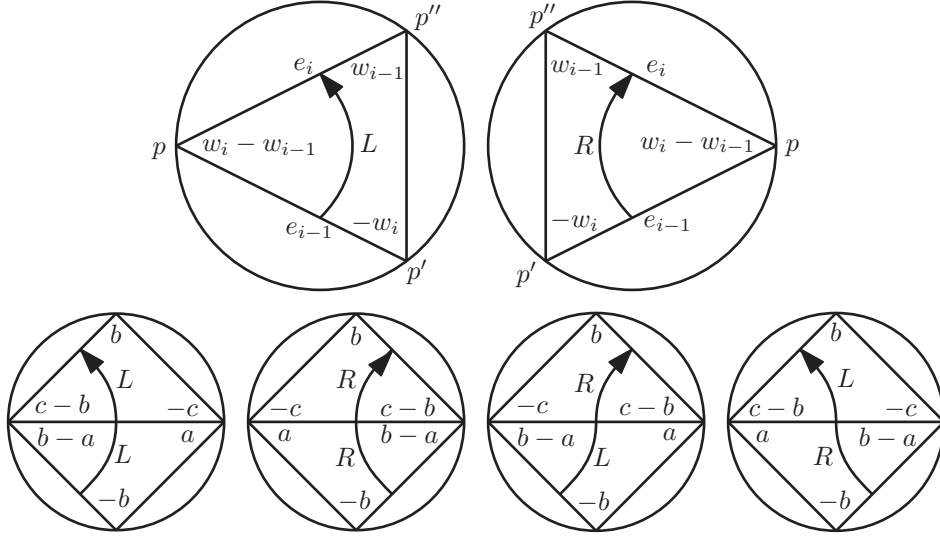
Figure 2. Bottom: e_i is the horizontal edge and $(a, b, c) = (w_{i-1}, w_i, w_{i+1})$.

(4) $\begin{cases} \text{For all } i \in \mathbb{Z} \text{ one has } 0 < w_i < \pi. \\ \text{If } i \text{ separates identical letters (first two cases), } 2w_i > w_{i+1} + w_{i-1}. \\ \text{If } i \text{ separates different letters (last two cases), } |w_{i+1} - w_{i-1}| < w_i. \end{cases}$

Denote by Σ the non-empty, convex solution set of (4).

3. Bounding the Bending

3-A. A natural constraint on the pleating of S_i. Next, we describe a certain convex subset of Σ. It will be defined by decay conditions on the $\{w_i\}_{i \in \mathbb{Z}}$ near $\pm\infty$. These conditions will encode $\lambda^\pm$, and are obtained in the following way. Consider the pleated punctured torus, S_i, lying between the tetrahedra Δ_i and Δ_{i-1}. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the edges of S_i and $\delta_1, \delta_2, \delta_3$ the corresponding pleating angles (exterior dihedral angles, counted positively for salient edges as in (2) above). Recall the set $\mathcal{C}$ of (isotopy classes of) simple closed curves in the punctured torus S. Define the pleating measure $\lambda_i : \mathcal{C} \to \mathbb{R}^+$ of S_i by

(5) $$\lambda_i(\gamma) = \delta_1 \nu_1(\gamma) + \delta_2 \nu_2(\gamma) + \delta_3 \nu_3(\gamma),$$

where $\nu_\sigma(\gamma) \in \mathbb{N}$ is the intersection number of γ with the edge ϵ_σ. Denote by $\varepsilon_\sigma \in \mathcal{C}$ the simple closed curve that has the same slope as ϵ_σ. We shall require that

$$\lambda^-(\varepsilon_\sigma) < \lambda_i(\varepsilon_\sigma) < \lambda^+(\varepsilon_\sigma)$$

for each $\sigma \in \{1, 2, 3\}$, and the same thing for every pleated punctured torus S_i in the $\mathbb{Z}$-family. (Note: in fact, a similar inequality will follow for all simple closed curves γ of $\mathcal{C}$ [see Lemma 3.5 below], namely, $\lambda^- < \lambda_i < \lambda^+$ as functions on $\mathcal{C}$. Forcing this "natural" inequality is the whole point of our constraint.)

In other words, denote by η/ξ the (rational) slope of ε_σ. Observe that $\lambda_i(\varepsilon_\sigma) = \delta_{\sigma'} + \delta_{\sigma''} = -\delta_\sigma$ where $\{\sigma, \sigma', \sigma''\} = \{1, 2, 3\}$, because the slopes of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are Farey neighbors. Therefore, by (1), the requirement is that for every ε_σ as above,

$$(6) \qquad -\left\|\begin{matrix} \beta^+ & \eta \\ \alpha^+ & \xi \end{matrix}\right\| < \delta_\sigma < \left\|\begin{matrix} \beta^- & \eta \\ \alpha^- & \xi \end{matrix}\right\|.$$

To express (6) in terms of the w_i, we need some notation (the δ_σ are determined by the w_i via (2) above). For each Farey edge e_i, let q_i^+ (resp. q_i^-) be the rational located opposite e_i, on the same side of e_i as β^+/α^+ (resp. β^-/α^-). For an arbitrary rational $p = \eta/\xi$ (reduced form), introduce the slightly abusive notation (abusive in that it depends on the ordered pair (α, β) rather than just on the real $\frac{\beta}{\alpha}$):

$$(7) \qquad \frac{\beta}{\alpha} \wedge p := \left\|\begin{matrix} \beta & \eta \\ \alpha & \xi \end{matrix}\right\| \text{ (absolute value of the determinant).}$$

Then, if l, r are the rationals living at the left and right ends of the Farey edge e_i, one has

$$(8) \quad (\tfrac{\beta^+}{\alpha^+} \wedge l) + (\tfrac{\beta^+}{\alpha^+} \wedge r) = \tfrac{\beta^+}{\alpha^+} \wedge q_i^- \ ; \ (\tfrac{\beta^-}{\alpha^-} \wedge l) + (\tfrac{\beta^-}{\alpha^-} \wedge r) = \tfrac{\beta^-}{\alpha^-} \wedge q_i^+ \ .$$

Indeed, the $\|\cdot\|$-notation is invariant under $PSL_2(\mathbb{Z})$, acting on $(\mathbb{H}^2, \partial\mathbb{H}^2)$ by isometries and on ordered pairs $\pm(\alpha, \beta)$ as a matrix group. Since $PSL_2(\mathbb{Z})$ acts transitively on oriented Farey edges, we are reduced to the case $(l, r, q_i^+, q_i^-) = (\infty, 0, 1, -1)$ where $\beta^-/\alpha^- < 0 < \beta^+/\alpha^+$, which is straightforward.

Let us now translate Equation (6) in terms of the w_i. Let e_{i-1}, e_i be two consecutive Farey edges; p and p' are the ends of e_{i-1}; p and p'' are the ends of e_i. One has $q_{i-1}^+ = p''$ and $q_i^- = p'$. In view of (2) and Figure 2 (top), Equation (6) translates to

$$\begin{array}{ccccc}
-(\tfrac{\beta^+}{\alpha^+} \wedge p'') & < & w_{i-1} & < & \tfrac{\beta^-}{\alpha^-} \wedge p'' \\
-(\tfrac{\beta^+}{\alpha^+} \wedge p') & < & -w_i & < & \tfrac{\beta^-}{\alpha^-} \wedge p' \\
-(\tfrac{\beta^+}{\alpha^+} \wedge p) & < & w_i - w_{i-1} & < & \tfrac{\beta^-}{\alpha^-} \wedge p
\end{array}$$

Using the fact that the w_i are positive (4), this simplifies to (respectively)

$$\begin{array}{ccccc}
 & & w_{i-1} & < & \tfrac{\beta^-}{\alpha^-} \wedge q_{i-1}^+ \\
-(\tfrac{\beta^+}{\alpha^+} \wedge q_i^-) & < & -w_i & & \\
-(\tfrac{\beta^+}{\alpha^+} \wedge p) & < & w_i - w_{i-1} & < & \tfrac{\beta^-}{\alpha^-} \wedge p
\end{array}$$

Finally, observe that $\frac{\beta^+}{\alpha^+} \wedge p = (\frac{\beta^+}{\alpha^+} \wedge q_{i-1}^-) - (\frac{\beta^+}{\alpha^+} \wedge q_i^-)$ while $\frac{\beta^-}{\alpha^-} \wedge p = (\frac{\beta^-}{\alpha^-} \wedge q_i^+) - (\frac{\beta^-}{\alpha^-} \wedge q_{i-1}^+)$, by Equation (8). Therefore, if we introduce

$$(9) \qquad \phi_i^+ := \frac{\beta^+}{\alpha^+} \wedge q_i^- \text{ and } \phi_i^- := \frac{\beta^-}{\alpha^-} \wedge q_i^+$$

then Equation (6) reduces to

$$(10) \qquad \left.\begin{array}{ccc} w_i & < & \min\{\phi_i^+, \phi_i^-\} \\ \phi_{i-1}^- - \phi_i^- & < & w_{i-1} - w_i & < & \phi_{i-1}^+ - \phi_i^+ \end{array}\right\} \forall i \in \mathbb{Z}.$$

3-B. Study of ϕ^+ and ϕ^-.

DEFINITION 3.1. In 1-A we associated to each $i \in \mathbb{Z}$ a Farey edge e_i living between two letters of $\{R, L\}$. We call i a *hinge index*, and Δ_i a *hinge tetrahedron*, if the two letters are distinct (RL or LR). *Non-hinges* correspond to RR or LL.

Lemma 3.2. *The following holds concerning the sequences* $\phi^+, \phi^- : \mathbb{Z} \to \mathbb{R}_+^*$.
  i − *First, ϕ^- is strictly increasing and ϕ^+ is strictly decreasing.*
  ii − *For all i one has $1 < \phi_{i+1}^-/\phi_i^- < 2$ and $1 < \phi_{i-1}^+/\phi_i^+ < 2$.*
  iii − *If $i \in \mathbb{Z}$ is non-hinge then $\phi_{i-1}^+ + \phi_{i+1}^+ = 2\phi_i^+$ and $\phi_{i-1}^- + \phi_{i+1}^- = 2\phi_i^-$.*
  iv − *If $i \in \mathbb{Z}$ is hinge then $\phi_{i+1}^- = \phi_i^- + \phi_{i-1}^-$ and $\phi_{i-1}^+ = \phi_i^+ + \phi_{i+1}^+$.*
  v − *The sequences $\phi^\pm$ are (weakly) convex, i.e. $2\phi_i^\pm \le \phi_{i-1}^\pm + \phi_{i+1}^\pm$.*
  vi − *If $i < j$ are consecutive hinge indices, then $1 + \frac{j-i}{2} \le \{ \frac{\phi_i^+}{\phi_j^+}, \frac{\phi_j^-}{\phi_i^-} \} \le 1 + j - i$.*
  vii − *One has $\lim_{+\infty} \phi^+ = \lim_{-\infty} \phi^- = 0$ and $\lim_{-\infty} \phi^+ = \lim_{+\infty} \phi^- = +\infty$.*
  viii − *One has $\lim_{-\infty} (\phi_{i-1}^+ - \phi_i^+) = \lim_{+\infty} (\phi_{i+1}^- - \phi_i^-) = +\infty$.*

REMARK 3.3. Points iii and iv are just equality cases of the inequalities (4), which encode positivity of the angles x_i, y_i, z_i.

Proof. We will deal only with ϕ^+: the arguments for ϕ^- are analogous. Let $e_{i-1} = pr_{i-1}$ and $e_i = pr_i$ be consecutive Farey edges. We have $\phi_i^+ = \frac{\beta^+}{\alpha^+} \wedge r_{i-1}$ while $\phi_{i-1}^+ = \frac{\beta^+}{\alpha^+} \wedge r_{i-1} + \frac{\beta^+}{\alpha^+} \wedge p$ by Equation (8), hence i.

For ii, we need care only about the upper bound. Just observe that $\phi_{i-1}^+ = (\frac{\beta^+}{\alpha^+} \wedge r_i + \frac{\beta^+}{\alpha^+} \wedge p) + \frac{\beta^+}{\alpha^+} \wedge p$ while $\phi_i^+ = \frac{\beta^+}{\alpha^+} \wedge r_i + \frac{\beta^+}{\alpha^+} \wedge p$.

For iii, assume $e_{i+1} = pr_{i+1}$ so that $\phi_k^+ = \frac{\beta^+}{\alpha^+} \wedge p + \frac{\beta^+}{\alpha^+} \wedge r_k$ for $|i - k| \le 1$. For $k \in \{i, i+1\}$ the right hand side is $\frac{\beta^+}{\alpha^+} \wedge r_{k-1}$, so $(\phi_{k-1}^+ - \phi_k^+) = \frac{\beta^+}{\alpha^+} \wedge p$, which is sufficient.

For iv, assume $e_{i+1} = p'r_i$. In the notations of Formula (9), we have $q_{i+1}^- = p$ and $q_i^- = r_{i-1}$, the ends of e_{i-1}. This together with Equation (8) yields the result. Point v follows from iv and ii at hinges indices, and from iii at other indices. For Point vi, observe that $\phi_i^+ = \phi_j^+ + (j-i)\phi_{j+1}^+$ by iii-iv, and conclude using ii. Point vii follows from vi, and the presence of infinitely many hinge indices near either end. Point viii follows from vii, v, and iv.   q.e.d.

3-C. Behavior of the pleatings $\lambda_i : \mathcal{C} \to \mathbb{R}$.

For any real sequence $u \in \mathbb{R}^{\mathbb{Z}}$, define the real sequence ∇u by $(\nabla u)_i = u_{i-1} - u_i$. Let us summarize the conditions imposed on $w : \mathbb{Z} \to \mathbb{R}_+^*$ (Eq. 4 and 10 above):

$$(11) \quad \begin{cases} 0 & < & w_i & < & \min\{\phi_i^+, \phi_i^-, \pi\} & \\ \nabla\phi_i^- & < & \nabla w_i & < & \nabla\phi_i^+ & \text{for all } i \in \mathbb{Z}; \\ & & |w_{i+1} - w_{i-1}| & < & w_i & \text{if } i \text{ is a hinge;} \\ & & w_{i+1} + w_{i-1} & < & 2w_i & \text{otherwise.} \end{cases}$$

It is a simple exercise to check that $w_i = \tanh \phi_i^+ \tanh \phi_i^-$, for instance, satisfies this system (tanh may be replaced by any strictly concave monotonous function from $\mathbb{R}_+$ to $[0, 1)$ with the same 1-jet at 0).

DEFINITION 3.4. If (11') denotes the system (11) in which all strong inequalities have been turned into weak ones, let $W \subset \mathbb{R}^{\mathbb{Z}}$ be the solution space of (11').

Suppose $(w_i)_{i \in \mathbb{Z}} \in W$ and recall (5) the corresponding pleating measure $\lambda_i : \mathcal{C} \to \mathbb{R}$ of S_i, the pleated punctured torus between the tetrahedra Δ_i and Δ_{i-1}.

Lemma 3.5. *For any curve* $\gamma \in \mathcal{C}$, *the sequence* $(\lambda_i(\gamma))_{i \in \mathbb{Z}}$ *is nondecreasing, with*

$$\lambda^-(\gamma) \;\leq\; \lim_{i \to -\infty} \lambda_i(\gamma) \;\leq\; \lim_{i \to +\infty} \lambda_i(\gamma) \;\leq\; \lambda^+(\gamma).$$

Proof. First, observe that the triples of pleating angles of S_i, S_{i+1} are always of the form $(A, B, -x_i - y_i)$; $(A - 2x_i, B - 2y_i, x_i + y_i)$ given in Figure 3, where x_i (resp. y_i) is positive, equal to the interior dihedral angle of the tetrahedron Δ_i at the horizontal (resp. vertical) edges of the (square) fundamental domain of the surface S. The closed curve γ in S traverses that fundamental domain a number of times, either vertically, or horizontally, or diagonally (cutting off one of the four corners). The pleating along γ increases by $2y_i$ per vertical passage, $2x_i$ per horizontal passage, and 0 per diagonal passage, hence the monotonicity statement. (This argument, or a variant of it, is also valid for higher-genus surfaces and non-simple closed curves, as long as the tetrahedron has positive angles.)
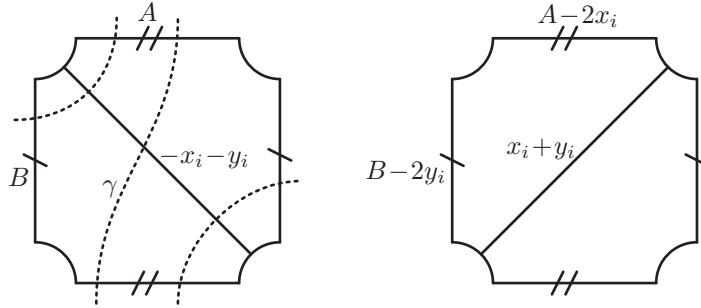

Figure 3. The pleatings of the surfaces S_i and S_{i+1} bounding Δ_i.

For the bounding, we will focus only on the λ^+-side. Consider the slope $\sigma = \frac{\eta}{\xi} \in \mathbb{P}^1 \mathbb{Q}$ of γ (a reduced fraction); recall the definition $\lambda^+(\sigma) = ||\begin{smallmatrix} \beta^+ & \eta \\ \alpha^+ & \xi \end{smallmatrix}||$ from (1). Consider a large enough index i, such that the Farey edge e_{i-1} separates σ from β^+/α^+. Consider the points $p, p', p'' \in \mathbb{P}^1 \mathbb{Q}$ such that $(e_{i-1}, e_i) = (pp', pp'')$: the points $(\sigma, p, \frac{\beta^+}{\alpha^+}, p'', p', \sigma)$ are cyclically arranged in $\mathbb{P}^1 \mathbb{R}$. Observe that the $\wedge$-notation (7) applied at rationals is just the (geometric) intersection number of the corresponding curves. Therefore, using the

angle information given in Figure 2 (top), one has
$$
\begin{aligned}
\lambda_i(\gamma) &= (p \wedge \sigma)(w_i - w_{i-1}) - (p' \wedge \sigma)w_i + (p'' \wedge \sigma)w_{i-1} \\
&= w_i(p \wedge \sigma) + \nabla w_i(p' \wedge \sigma) \qquad \text{since } p'' \wedge \sigma = p \wedge \sigma + p' \wedge \sigma \\
&\leq \phi_i^+(p \wedge \sigma) + \nabla \phi_i^+(p' \wedge \sigma) \\
&= \left(\tfrac{\beta^+}{\alpha^+} \wedge p'\right)(p \wedge \sigma) + \left(\tfrac{\beta^+}{\alpha^+} \wedge p\right)(p' \wedge \sigma) \qquad (*)
\end{aligned}
$$

(12)

by definition (9) of ϕ^+. The last quantity is $\frac{\beta^+}{\alpha^+} \wedge \sigma$ (hence the upper bound): by SL_2-invariance of the $\wedge$-notation, it is enough to check this when $p = \infty$ and $p' = 0$ — in that case, $\frac{\beta^+}{\alpha^+}$ and $\sigma = \frac{\eta}{\xi}$ have opposite signs, and $(*)$ does indeed become $|\beta^+ \xi| + |\alpha^+ \eta| = \left\|\begin{smallmatrix} \beta^+ & \eta \\ \alpha^+ & \xi \end{smallmatrix}\right\|$. q.e.d.

4. Hyperbolic volume

The product topology on $\mathbb{R}^{\mathbb{Z}}$ induces a natural topology on the space W of Definition 3.4: clearly, W is nonempty, convex, and compact.

If (x, y, z) is a nonnegative triple such that $x + y + z = \pi$, let $\mathcal{V}(x, y, z)$ be the hyperbolic volume of an ideal tetrahedron whose interior dihedral angles are x, y, z. We wish to compute the total hyperbolic volume of all tetrahedra when $w \in W$, i.e.
$$
\mathcal{V}(w) := \sum_{i \in \mathbb{Z}} \mathcal{V}(x_i, y_i, z_i)
$$
where x_i, y_i, z_i are defined from the w_i via Table (3). This poses the problem of well-definedness — the sum of the volumes might diverge. Let us estimate $\mathcal{V}$: a well-known explicit formula [**Mil**] gives

(13)
$$
\begin{aligned}
\mathcal{V}(x, y, z) &= -\int_0^x \log 2\sin - \int_0^y \log 2\sin - \int_0^z \log 2\sin \\
&= \int_0^x \log \frac{\sin(\tau + y)}{\sin \tau} d\tau \qquad \left(\text{as } \int_0^\pi \log 2\sin = 0\right) \\
&\leq \int_0^x \log \frac{\tau + y}{\tau} d\tau \qquad \text{(by concavity of sin)} \\
&= x \log \frac{x + y}{x} + y \log \frac{x + y}{y} \\
&\leq (x + y) \log 2 \qquad \text{(concavity of log)}.
\end{aligned}
$$

Lemma 4.1. *There exists a universal constant $K > 0$ such that the sum of the volumes of the tetrahedra Δ_j for $j \geq i$ (resp. $j \leq i$) is at most $K\phi_i^+$ (resp. $K\phi_i^-$).*

Proof. We will focus only on the ϕ^+-statement. First, by the computation above, the volume of the tetrahedron Δ_i is at most $w_i \log 2 \leq \phi_i^+ \log 2$ (see Table 3). In view of Lemma 3.2-vi, this implies that the total volume of all *hinge* tetrahedra beyond the index i is at most $3\phi_i^+ \log 2$. For the same reason, it is sufficient to prove

Sublemma 4.2. *There exists a universal constant $L > 0$ such that if 0 and $N \in \mathbb{N}$ are two consecutive hinge indices, then the sum of the volumes of the tetrahedra $\Delta_1, \Delta_2, \ldots, \Delta_{N-1}$ is at most $L\phi_0^+$.*

Proof. In view of homogeneity in the estimation (13), it is sufficient to assume $\phi_0^+ = 1$ and replace the volume with its estimate. Therefore, let $(w_i)_{0 \le i \le N}$ be a concave sequence in $[0,1]$: following Table (3), we want to find a universal upper bound L (not depending on N) for

$$\sum_{i=1}^{N-1} \frac{w_{i+1} + w_{i-1}}{2} \log \frac{2w_i}{w_{i+1} + w_{i-1}} + \frac{2w_i - w_{i+1} - w_{i-1}}{2} \log \frac{2w_i}{2w_i - w_{i+1} - w_{i-1}}.$$

If $A < B$ are positive integers, denote by Σ_A^B the restriction of the above sum to indices $A \le i < B$. Observe that the general term of Σ_A^B is bounded by $2e^{-1}$, because $\frac{\tau}{2} \log \frac{2}{\tau} \le e^{-1}$ for all positive τ. For yet better control, we bound the first half of the general term of Σ_A^B by $(\frac{w_{i+1}+w_{i-1}}{2})(\frac{2w_i}{w_{i+1}+w_{i-1}} - 1)$, and the second half, by concavity of log. This produces

$$
\begin{aligned}
\Sigma_A^B \;\le\;& \sum_{i=A}^{B-1} \left(\frac{2w_i - w_{i+1} - w_{i-1}}{2} \times \frac{w_{i+1} + w_{i-1}}{w_{i+1} + w_{i-1}} \right) \\
& + \left(\sum_{i=A}^{B-1} \frac{2w_i - w_{i+1} - w_{i-1}}{2} \right) \log \frac{\sum_{i=A}^{B-1} w_i}{\sum_{i=A}^{B-1} \frac{2w_i - w_{i+1} - w_{i-1}}{2}} \\
=\;& \sigma \log \frac{e \sum_{i=A}^{B-1} w_i}{\sigma} \;, \quad \text{where } \sigma = \frac{w_A - w_{A-1} + w_{B-1} - w_B}{2}.
\end{aligned}
$$

Denote by $M \in [\![0, N]\!]$ a value of the index i for which w_i is maximal. If $A < B \le M$, we have

$$0 \le \sigma \le \frac{w_A - w_{A-1}}{2} \le \frac{1}{2A} \le 1$$

(the third inequality follows from concavity of w between 0 and A). Also notice that $f : \tau \mapsto \tau \log \frac{e}{\tau}$ is nondecreasing on $[0,1]$. We shall apply these facts for $A = 2^{k-1}$ and $B = \min\{2^k, M\}$: the previous bound on Σ_A^B can be rewritten

$$
\begin{aligned}
\Sigma_A^B \;\le\;& f(\sigma) + \sigma \log \left(\sum_{i=A}^{B-1} w_i \right) \\
\le\;& f(2^{-k}) + 2^{-k} \log 2^{k-1} = 2^{-k}[1 + (2k-1)\log 2].
\end{aligned}
$$

The latter numbers (for k ranging over $\mathbb{N}^*$) add up to some universal $L' < +\infty$. After a similar argument for the indices $M < i < N$, we can take $L = 2e^{-1} + 2L'$.   q.e.d.

Finally, we can take $K = 3L + 3\log 2$. Lemma 4.1 is proved.   q.e.d.

Corollary 4.3. *The volume functional* $\mathcal{V} : W \to \mathbb{R}^+$ *is well-defined, continuous, and concave.*

Proof. Well-definedness is the point of Lemma 4.1. Given $\varepsilon > 0$, only finitely many indices i satisfy $\min\{\phi_i^+, \phi_i^-\} > \varepsilon/K$, and the others contribute at most 2ε to the volume: hence continuity in the product topology. Concavity follows from the concavity of the volume of *one* tetrahedron (parametrized by its angles): see e.g. Proposition 8 of [**GF**].   q.e.d.

Therefore, by compactness, there exists a sequence $(w_i)_{i \in \mathbb{Z}} \in W$ which maximizes the hyperbolic volume $\mathcal{V}$. From this point on, w will denote that maximizer.

Proposition 4.4. *For each $j \in \mathbb{Z}$, if $x_j y_j z_j = 0$ then $\max\{x_j, y_j, z_j\} = \pi$.*

Proof. Assume the tetrahedron Δ_j has exactly one vanishing angle, and aim for a contradiction. If $\mathcal{V}(\Delta^t)$ is the volume of a tetrahedron Δ^t having angles $x^t, y^t, (\pi - x^t - y^t)$ with $(x^t, y^t)_{t \geq 0}$ smooth, $x^0 = 0 < y^0 < \pi$ and $dx^t/dt_{|t=0} > 0$, then $d\mathcal{V}(\Delta^t)/dt_{|t=0} = +\infty$ (by Formula (13) above).

Let $(w_i')_{i \in \mathbb{Z}}$ be a sequence satisfying all (strict) inequalities of (11) and define $w^t := w + t(w' - w)$ for $0 \leq t \leq 1$. Denote by Δ_i^t the i-th tetrahedron determined via (3) by $(w_i^t)_{i \in \mathbb{Z}}$: then the angles of Δ_j^t satisfy the hypotheses above, so

$$\mathcal{V}(w^t) = \mathcal{V}(\Delta_j^t) + \left(\sum_{i \neq j} \mathcal{V}(\Delta_i^t)\right)$$

has right derivative $+\infty$ at $t = 0$ (the second summand is concave and continuous, so it has a well-defined right derivative in $\mathbb{R} \cup \{+\infty\}$ at 0). Therefore, $\mathcal{V}$ was not maximal at w. q.e.d.

In Section 10 we will need the following consequence of Proposition 4.4:

Proposition 4.5. *If $j \in \mathbb{Z}$ and $x_j y_j z_j = 0$, then j is a hinge index and $w_j = 0$.*

This is Proposition 13 in [**GF**]. But in fact much more is true:

Proposition 4.6. *All the (strict) inequalities of (11) are true at w.*

Proof. If some inequality of (11) involving ϕ^+ fails to be strict, it is easy to see by induction that $w = \phi^+$ near $+\infty$, so all tetrahedra Δ_i (for i large enough) have exactly one vanishing angle (see Remark 3.3): w was not maximal, by Proposition 4.4. Therefore all inequalities of (11) involving $\phi^{\pm}$ are strict. The arguments in [**GF**] (especially Lemma 16 and the argument of Section 9 there) can then be used to show that *all* inequalities (11) are strict at w, so w is a *critical* point of the volume functional $\mathcal{V}$. q.e.d.

Proposition 4.6 implies that the holonomy representation is trivial, i.e. the gluing of any finite number of consecutive tetrahedra Δ_i defines a complete hyperbolic metric with (polyhedral) boundary (the shapes of the Δ_i "fit together correctly" around the edges): see [**Ri**], [**CH**] or [**GF**]. Therefore, the links of the vertices of the ideal tetrahedra (Euclidean triangles) form a triangulation of the link of the puncture: the latter is naturally endowed with a Euclidean structure and its universal cover can be drawn in the plane (Figure 4 — more on the combinatorics of this triangulation in Section 6; see also [**GF**]). Denote by Γ the image of the induced holonomy representation $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$.

5. Behavior of (w_i)

We want to prove that the pleatings λ_i (see Section 3) of the pleated punctured tori S_i converge (in the weak-* topology, i.e. on any test curve $\gamma \in \mathcal{C}$) to the pleatings $\lambda^{\pm}$ near infinity. For this we must study (w_i), and especially show that the bounds specified by $\phi^{\pm}$ in Equations (11) are almost (but not quite) achieved. We use the ∇-notation as in Section 3-C.
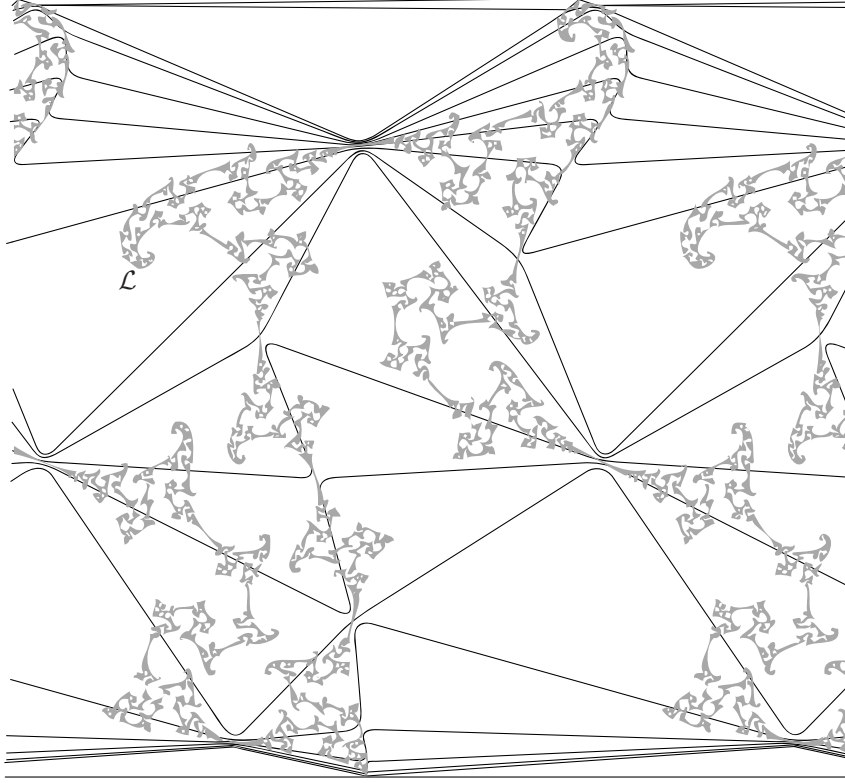
Figure 4. The cusp triangulation is shown, in anticipation, against the limit set $\mathcal{L} \subset \mathbb{P}^1\mathbb{C}$ of the quasifuchsian group (a complicated Jordan curve). The picture extends periodically to the right and left. Each broken line is the puncture link of a pleated surface S_i; the vertices (whose design artificially sets the S_i apart from each other) are all parabolic fixed points at which the limit set $\mathcal{L}$ becomes pinched. Infinitely many very flat triangles accumulate along the top and bottom horizontal lines. This picture was generated with Masaaki Wada's computer program Opti [**Wa**].

Lemma 5.1. *Recall that $\nabla\phi^- < 0 < \nabla\phi^+$ (Lemma 3.2). One has*

$$\max\left\{\overline{\lim_{\mathbb{Z}}}\frac{w}{\phi^+} \ , \ \overline{\lim_{\mathbb{Z}}}\frac{\nabla w}{\nabla\phi^+}\right\} = \max\left\{\overline{\lim_{\mathbb{Z}}}\frac{w}{\phi^-} \ , \ \overline{\lim_{\mathbb{Z}}}\frac{-\nabla w}{-\nabla\phi^-}\right\} = 1.$$

Proof. We focus on the ϕ^+-statement; the ϕ^--part is analogous. Since $\frac{w}{\phi^+} < 1$, assuming $\overline{\lim}\frac{w}{\phi^+} < 1$ implies $\sup\frac{w}{\phi^+} < 1$, and the same holds true for $\frac{\nabla w}{\nabla\phi^+}$ (see Equation 11). Therefore, suppose $\sup\frac{w}{\phi^+} \leq 1-\varepsilon$ and $\sup\frac{\nabla w}{\nabla\phi^+} \leq 1-\varepsilon$ for some $\varepsilon > 0$, and aim at a contradiction.

Recall the ordered pair (α^+, β^+) that helped define ϕ^+. For each $\mu > 0$, define $(\alpha^\mu, \beta^\mu) := (\mu\alpha^+, \mu\beta^+)$. This defines a new pleating function $\lambda^\mu = \mu\lambda^+ : \mathcal{C} \to \mathbb{R}^+$, a new $\phi^\mu = \mu\phi^+$ and a new domain W^μ by (11) (the numbers

α^- and β^- are left unchanged). By definition, $W = W^1$ and $W^\mu \subset W^{\mu'}$ if and only if $\mu \leq \mu'$. Let $(w_i^\mu)_{i \in \mathbb{Z}}$ be the maximizer of the volume functional $\mathcal{V}$ on W^μ. By assumption, we have $w^1 \in W^{1-\varepsilon} \subset W^1$, so $w^1 = w^{1-\varepsilon}$.

Write $V(\mu) = \mathcal{V}(w^\mu)$, so that $V(1) = V(1-\varepsilon)$. It is straightforward to check that for any $t \in [0,1]$, one has $(t \cdot w^\mu + (1-t) \cdot w^{\mu'}) \in W^{t\mu + (1-t)\mu'}$. Since the volume of any tetrahedron is a concave function of its angles, this is enough to imply that $V : \mathbb{R}^+ \to \mathbb{R}^+$ is (weakly) concave. By inclusion, V is also nondecreasing. In fact, V is strictly increasing (which will finish the proof by contradiction): to prove this, since V is concave, we just need to produce arbitrarily large values of the volume $\mathcal{V}$ at points $v^\mu \in W^\mu$, for large enough μ. We may assume (up to a translation of indices, see Lemma 3.2, vii – viii) that $\phi_0^- > \pi$ and $-\nabla\phi_0^- > \pi$. Then, start by defining $v_i^\mu = \min\{\phi_i^\mu, \phi_i^-, \pi\}$, so that $v^\mu = \pi$ on $[\![0, N+1]\!]$ for arbitrarily large N. Without loss of generality, by just taking μ large enough, we may further assume $\nabla\phi^\mu > \pi$ on $[\![0, N+1]\!]$. Then, each time $i \in [\![1, N]\!]$ is a *hinge* index, replace v_i^μ by $2\pi/3$: this is allowed in W^μ, by our assumptions on $\nabla\phi^-, \nabla\phi^\mu$. The angles of the tetrahedron Δ_i are then all in $\{\pi/2, \pi/3, \pi/6\}$ (see Table 3). Since there are infinitely many hinge indices near $+\infty$, the volume can become arbitrarily large: Lemma 5.1 is proved. q.e.d.

Corollary 5.2. *In fact,*

$$\varlimsup_{i \to +\infty} \min\left\{\frac{w_i}{\phi_i^+}, \frac{\nabla w_i}{\nabla\phi_i^+}\right\} = \varlimsup_{i \to -\infty} \min\left\{\frac{w_{i-1}}{\phi_{i-1}^-}, \frac{-\nabla w_i}{-\nabla\phi_i^-}\right\} = 1.$$

Proof. Again, we focus only on ϕ^+. By Lemma 5.1, there exists a subsequence $(w_{\nu(i)})_{i \in \mathbb{N}}$ such that $w_{\nu(i)} \sim \phi_{\nu(i)}^+$ or $\nabla w_{\nu(i)} \sim \nabla\phi_{\nu(i)}^+$. Suppose the latter is the case. For an arbitrary integer i, let n be the smallest hinge index larger than or equal to $\nu(i)$: observe that $\phi_{n+1}^+ = \nabla\phi_n^+ = \nabla\phi_{\nu(i)}^+$ by Lemma 3.2-iii-iv, while $w_{n+1} \geq \nabla w_n \geq \nabla w_{\nu(i)}$ by the positivity conditions (11). Therefore, $\frac{w_{n+1}}{\phi_{n+1}^+} \geq \frac{\nabla w_{\nu(i)}}{\nabla\phi_{\nu(i)}^+}$ so up to redefining ν we may assume simply $w_{\nu(i)} \sim \phi_{\nu(i)}^+$.

Pick $\varepsilon > 0$. Take i such that

$$(14) \qquad\qquad w_{\nu(i)} \geq (1-\varepsilon)\phi_{\nu(i)}^+.$$

Let n be the smallest hinge index strictly larger than $\nu(i)$. If $n = \nu(i)+1$ then

$$w_n \geq w_{n-1} - \nabla\phi_n^+ \geq (1-\varepsilon)\phi_{n-1}^+ - \phi_{n+1}^+ = \phi_n^+ - \varepsilon\phi_{n-1}^+ \geq (1-2\varepsilon)\phi_n^+;$$

$$\nabla w_n = w_{n-1} - w_n \geq (1-\varepsilon)\phi_{n-1}^+ - \phi_n^+ = \phi_{n+1}^+ - \varepsilon\phi_{n-1}^+ \geq (1-3\varepsilon)\phi_{n+1}^+ = (1-3\varepsilon)\nabla\phi_n^+$$

where Lemma 3.2 has been used several times. Therefore, $\min\left\{\frac{w_n}{\phi_n^+}, \frac{\nabla w_n}{\nabla\phi_n^+}\right\} \geq 1 - 3\varepsilon$.

If $n \geq \nu(i)+2$, we can find an index k such that $\frac{k-\nu(i)}{n-\nu(i)} \in [\frac{1}{2}, \frac{2}{3}]$. We will show that $\min\left\{\frac{w_k}{\phi_k^+}, \frac{\nabla w_k}{\nabla\phi_k^+}\right\} \geq 1 - 8\varepsilon$, which will finish the proof.

• By positivity of w and concavity of w between the points $(\nu(i), k, n)$ one has $w_k \geq \frac{1}{3}w_{\nu(i)}$. Therefore,

$$\phi_k^+ - w_k \leq \phi_{\nu(i)}^+ - w_{\nu(i)} \leq \left(\frac{1}{1-\varepsilon} - 1\right)w_{\nu(i)} \leq \frac{3\varepsilon}{1-\varepsilon}w_k$$

(here the first inequality holds because $(\phi^+ - w)$ is decreasing by Condition (11), and the second follows from the assumption (14) above). Hence,

$$\frac{w_k}{\phi_k^+} \geq \frac{1 - \varepsilon}{1 + 2\varepsilon} \geq 1 - 3\varepsilon.$$

• Observe that

$$(1 - \varepsilon)\phi_{\nu(i)}^+ \leq w_{\nu(i)} \leq w_k + (k - \nu(i))\nabla w_k \leq \phi_k^+ + (k - \nu(i))\nabla w_k$$

where the second inequality follows from concavity of w between the points $(\nu(i), k-1, k)$. It follows that

$$(k - \nu(i))\nabla w_k \geq \phi_{\nu(i)}^+ - \phi_k^+ - \varepsilon\phi_{\nu(i)}^+ = (k - \nu(i))\nabla\phi_k^+ - \varepsilon\phi_{\nu(i)}^+$$

hence

$$\nabla w_k \geq \nabla\phi_k^+ - \frac{\varepsilon}{k - \nu(i)}\phi_{\nu(i)}^+ \geq \nabla\phi_k^+ - \frac{2\varepsilon}{n - \nu(i)}\phi_{\nu(i)}^+.$$

Since Lemma 3.2-vi gives us $\phi_{\nu(i)}^+ \leq 2(n - \nu(i))\phi_n^+$, this yields

$$\nabla w_k \geq \nabla\phi_k^+ - 4\varepsilon\phi_n^+ \geq \nabla\phi_k^+ - 8\varepsilon\phi_{n+1}^+ = (1 - 8\varepsilon)\nabla\phi_k^+.$$

<div align="right">q.e.d.</div>

Corollary 5.3. *Recall the pleating λ_i of the pleated surface S_i. For any simple closed curve $\gamma \in \mathcal{C}$ we have $\lim_{i \to +\infty} \lambda_i(\gamma) = \lambda^+(\gamma)$ and $\lim_{i \to -\infty} \lambda_i(\gamma) = \lambda^-(\gamma)$.*

Proof. By Corollary 5.2, the member ratio in the inequality (12) can be made arbitrarily close to 1.   q.e.d.

6. The cusp link

We now aim to investigate the behavior of the pleated surfaces S_i as i goes to $\pm\infty$ — or, more precisely, to find two limiting pleated surfaces $S_{\pm\infty}$ with pleatings $\lambda^\pm$ such that $\overline{V} = \bigcup_{i \in \mathbb{Z}} \Delta_i \sqcup S_{+\infty} \sqcup S_{-\infty}$ is metrically complete with locally convex boundary. The difficult part is to prove that the intrinsic moduli of the S_i converge in Teichmüller space. This question will be addressed in the next section. In this section, we just describe the cusp link, introduce notation and prove a few inequalities.

As always, we will mainly work near $+\infty$. Define $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$. We begin by orienting all the edges of V in a way that will be consistent with the pleating data $(\alpha^\pm, \beta^\pm)$ from (1). Namely, denote by $\mathcal{E} \subset \mathbb{P}^1\mathbb{Q}$ the collection of all the endpoints of all the Farey edges $(e_i)_{i \in \mathbb{Z}}$. For $\sigma \in \mathcal{E}$, let Q_σ be the edge of V of slope σ. Recall that the punctured torus is defined as $(\mathbb{R}^2 \smallsetminus \mathbb{Z}^2)/\mathbb{Z}^2$. Orienting Q_σ is therefore equivalent to orienting the line F_σ of slope σ in $\mathbb{R}^2$. We decide that the positive half of F_σ should be on the same side of the line $\mathbb{R}(\alpha^-, \beta^-)$ as (α^+, β^+), and orient Q_σ accordingly.

Let us now describe the link of the puncture, or cusp triangulation (whose underlying space is a Euclidean annulus $\mathbb{S}^1 \times \mathbb{R}$: we refer to $\mathbb{R}$ as the "vertical" direction). Each tetrahedron Δ_i contributes four similar Euclidean triangles at infinity to the link of the puncture, corresponding to the four ideal vertices of Δ_i. The bases of these four triangles form a closed $\mathbb{S}^1$-homotopic curve which is a broken line of four segments, and the triangles point alternatively up and

down from this broken line (see Figure 5). The two upward (resp. downward) pointing triangles have the same Euclidean size, an effect of the hyperelliptic involution (rotation of 180° around the puncture) which acts isometrically on V (reversing all edge orientations) and as a horizontal translation on the cusp link.
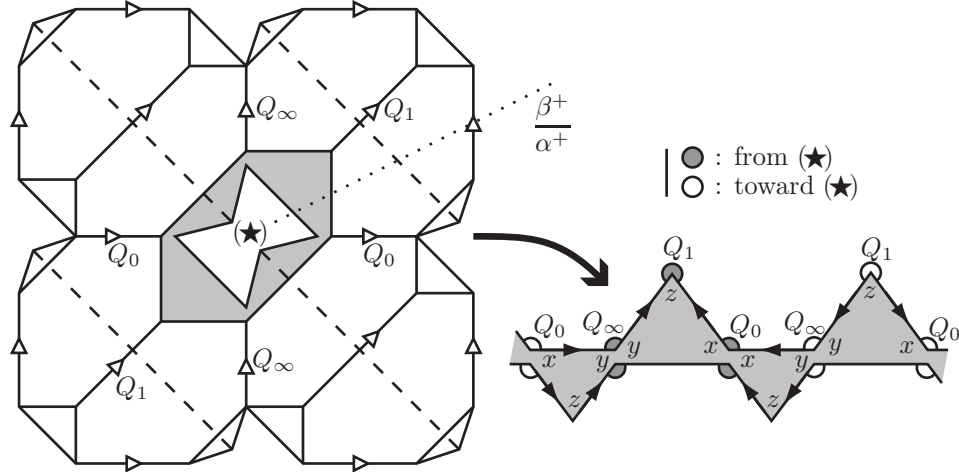

Figure 5. In the left panel, (★) marks the cusp. In the right panel, the cusp is at infinity.

DEFINITION 6.1. If the loop around the cusp has Euclidean length 4, let b_i (resp. b_i') be the length of the base of a downward-pointing (resp. upward-pointing) triangle contributed by the tetrahedron Δ_i, and define $\chi_i = b_i'/b_i$ (see e.g. Figure 7).

Figure 5 also shows some additional information, assuming that $e_i = 0\infty$ and $\alpha^+, \beta^+ > 0$ (hence $\beta^-/\alpha^- < 0$). Namely, the pleated surfaces S_{i+1} and S_i (above and below Δ_i) are pleated along Q_0, Q_∞, Q_1 and Q_0, Q_∞, Q_{-1} respectively, and the orientations of the Q_σ are as shown in the left panel of Figure 5 (the orientation of the edge Q_{-1} is not determined, as it depends on the position of β^-/α^- relative to -1). The orientations of the lines to/from the puncture are also shown in the right panel, at the vertices, with the help of a color code. Moreover, each segment ϵ of the upper broken line in the right panel corresponds to an arc about a vertex v of a face f of S_{i+1} in the left panel, so ϵ receives the orientation of the edge of f opposite v. If τ is one of the upward-pointing triangles drawn in the plane $\mathbb{C}$ (right panel), consider the tetrahedron Δ whose vertices are ∞ and those of τ: all edges of Δ, except one, receive orientations from the construction above, and Δ is isometric (respecting these orientations) to Δ_i. Finally, notice the labels in the 3 corners of each triangle in the right panel: the corner of the free vertex is labeled z (the angle there being z_i); the other two corners are labeled x and y accordingly. The labels x-y-z appear clockwise in each triangle.

The contribution of the tetrahedron Δ_{i-1} to the cusp triangulation is also a union of four triangles bounded by two broken lines. Moreover, the upper

broken line from Δ_{i-1} is the lower broken line from Δ_i, and the orientations of the lines to/from infinity must agree. Inspection shows that there are only two possibilities, corresponding to whether the letter living on the surface S_i is R or L: for R, the x-corners of the two 4-triangle families live near a pair of common vertices C, C'; for L, the same is true of the y-corners (Figure 6).
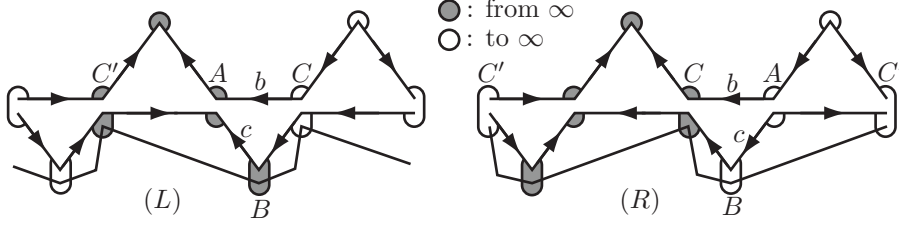

Figure 6. Left and Right transitions. Compare with the right panel of Figure 5.

DEFINITION 6.2. In a downward-pointing triangle defined by Δ_i, the edge lengths are b_i (the basis from Def. 6.1), b_{i-1}, and a third number which we call c_i. In either panel of Figure 6, we thus have $b = b_i$; $c = c_i$ and $BC = b_{i-1}$, where the tetrahedron Δ_i defines the top four triangles.

Property 6.3. *For i large enough, (b_i) is increasing and (b_i') is decreasing.*

Proof. Let τ_i (resp. τ_i') be a downward-pointing (resp. upward-pointing) triangle defined by the tetrahedron Δ_i. For large enough i we have $z_i = \pi - w_i > \pi - \phi_i^+ > \pi/2$, so b_i is the longest edge of τ_i, and b_i' the longest edge of τ_i'. Since b_{i-1} is an edge of τ_i and b_{i+1}' is an edge of τ_i' (Figure 6 or 7), the conclusion follows.   q.e.d.

Property 6.4. *We have $\lim_{i \to +\infty} \chi_i = 0$ (see Definition 6.1).*

Proof. We already know that $(\chi_i) = (b_i'/b_i)$ is ultimately decreasing. It is therefore enough to show that $\chi_{i+1}/\chi_{i-1} \le 1/2$ for large enough hinge indices i. Consider Figure 7, where angles labeled z are obtuse (as a rule, we shade a cusp triangle contributed by Δ_i whenever i is a hinge index). Check that

$$\frac{\chi_{i+1}}{\chi_{i-1}} = \frac{b_{i+1}'}{b_{i-1}'} \frac{b_{i-1}}{b_{i+1}} < \frac{b_{i+1}'}{b_i'} \frac{b_{i-1}}{b_i} = \frac{\sin x_i \sin y_i}{\sin^2 z_i} \le \frac{\sin^2(w_i/2)}{\sin^2 w_i} < \frac{1}{2}$$

(the last two inequalities follow from an easy study of sin, using $x_i + y_i = \pi - z_i = w_i < \pi/2$). As an immediate consequence, we find $\lim_{+\infty} b_i' = 0$ and $\lim_{+\infty} b_i = 2$. Similarly, $\lim_{-\infty} b_i' = 2$ and $\lim_{-\infty} b_i = 0$. (Compare with Figure 4).   q.e.d.

DEFINITION 6.5. Let $J \subset \mathbb{Z}$ be the set of all integers j such that $j - 1$ is a hinge index.

Proposition 6.6. *If $j < l$ are two large enough consecutive elements of J, and k is not in J, then*

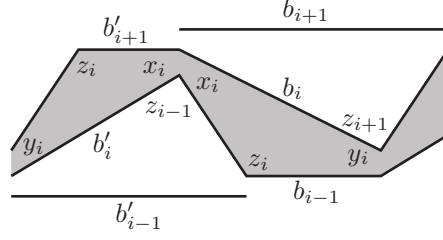$$\frac{c_k}{c_{k-1}} = \chi_{k-1} \ \text{ and } \ \frac{c_l}{c_j} \le \chi_{j-2}.$$

Figure 7. Here the index i is a hinge; the nature of $i \pm 1$ is undetermined.

Proof. Since $k - 1$ is not a hinge index, b_{k-1} shares the same end with b_k and with b_{k-2} (see Figure 8, left) so $c_k/c_{k-1} = b'_{k-1}/b_{k-1} = \chi_{k-1}$.

Since c_l is always an edge of τ'_{l-1}, we have $c_l \leq b'_{l-1}$, hence

$$(15) \qquad \frac{c_l}{c_j} \leq \frac{b'_{l-1}}{c_j} \leq \frac{b'_j}{c_j} = \frac{c_{j-1}}{b_{j-2}} \leq \frac{b'_{j-2}}{b_{j-2}} = \chi_{j-2}$$

where the equality in the middle just translates the similarity property of the "hinge" triangles τ_{j-1}, τ'_{j-1} (shaded in Figure 8, right). q.e.d.
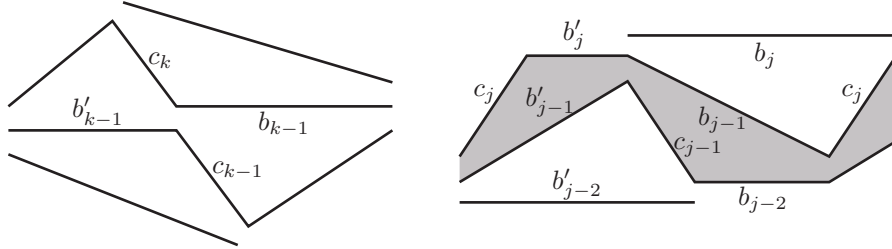

Figure 8. Left: $k - 1$ is not a hinge. Right: $j - 1$ is a hinge.

7. Intrinsic convergence of the surfaces S_i

7-A. Thickness of the tetrahedra. Consider a tetrahedron Δ_i bounded by the pleated surfaces S_i and S_{i+1}. Let Q (resp. Q') be the pleating edge of S_i (resp. S_{i+1}) not lying in $S_i \cap S_{i+1}$. Let s_i be the shortest segment between Q and Q', across Δ_i.

DEFINITION 7.1. Recall the orientations on the edges of $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$. Let ℓ_i be the complex length of the hyperbolic loxodromy along s_i sending Q to Q', respecting the orientations of Q, Q' (with $-\pi < \operatorname{Im} \ell_i \leq \pi$).

Proposition 7.2. *The series* $(\ell_i)_{i \in \mathbb{Z}}$ *is absolutely convergent.*

Proof. Consider a downward-pointing triangle τ contributed by Δ_i. Label the vertices of τ by A, B, C in such a way that $AC = b_i$, $BC = b_{i-1}$ and $AB = c_i$ (see Figure 6). Let γ_i be the hyperbolic loxodromy of complex length ℓ_i along the common perpendicular to $B\infty$ and AC, sending B, ∞ to A, C (in that order).

Sublemma 7.3. *Let $\ell = \ell_i = \rho + \theta\sqrt{-1}$ be the complex length of γ_i, with $\theta \in]-\pi, \pi]$. Then $\max\{\rho, |\theta|\} \leq \pi\sqrt{c_i/b_i}$.*

Proof. Up to a plane similarity, we may assume $A = 1$ and $C = -1$. Also, for convenience, relabel the edges of ABC by a, b, c. Let L be the fixed line of γ_i. The hyperbolic isometry defined by $z \mapsto f(z) = \frac{2B+1-z}{z+1}$ exchanges the oriented lines AC and $B\infty$, so it reverses the orientation of L around the center of the tetrahedron $ABC\infty$. Therefore, γ_i is given by $\gamma_i(z) = 2B - f(z)$. If M is a matrix of γ_i, one has $\frac{\mathrm{tr}^2 M}{4\det M} = \frac{\cosh\ell + 1}{2}$. Use $M = \begin{pmatrix} 2B+1 & -1 \\ 1 & 1 \end{pmatrix}$ to find $B = \cosh\ell$. Compute

$$
\frac{a \pm c}{b} = \frac{|B+1| \pm |B-1|}{2} = \left|\cosh^2\frac{\ell}{2}\right| \pm \left|\sinh^2\frac{\ell}{2}\right|
$$

$$
= \cosh\tfrac{\ell}{2}\cosh\tfrac{\overline{\ell}}{2} \pm \sinh\tfrac{\ell}{2}\sinh\tfrac{\overline{\ell}}{2} = \cosh\tfrac{\ell\pm\overline{\ell}}{2}.
$$

Use $\cosh(i\theta) = \cos\theta$ to get $\cosh(\rho) = \frac{a+c}{b}$ and $\cos(\theta) = \frac{a-c}{b}$. The estimates $\mathrm{Argcosh}(y) \leq 2\sqrt{\frac{y-1}{2}}$ and $\mathrm{Arccos}(y) \leq \pi\sqrt{\frac{1-y}{2}}$, since $\frac{1}{2}\left|\frac{a\pm c}{b} - 1\right| \leq \frac{c}{b}$, finally yield $\rho \leq 2\sqrt{c/b}$ and $|\theta| \leq \pi\sqrt{c/b}$, hence Sublemma 7.3.   q.e.d.

Since (b_i) goes to 2, Proposition 7.2 will follow if the c_i go to 0 fast enough near $+\infty$ (with a similar argument near $-\infty$). Such fast decay is given by Proposition 6.6: using the fact that (χ_i) goes to 0, we can bound the c_i by decreasing geometric sequences on intervals of the form $[\![j, l-1]\!]$ where $j < l$ are consecutive elements of the set J.   q.e.d.

7-B. Teichmüller charts. Let $\mathcal{T}_S$ be the Teichmüller space of the punctured torus: for each $i \in \mathbb{Z}$ we denote by $\mu(S_i) \in \mathcal{T}_S$ the intrinsic modulus of the marked surface S_i (for the marking $S_i \simeq S$ defined before Theorem 1.1). In order to prove that the $\mu(S_i)$ converge in $\mathcal{T}_S$, let us first introduce appropriate charts for $\mathcal{T}_S$.

Consider a topological ideal triangulation E of the punctured torus S, with labeled edges $\epsilon_1, \epsilon_2, \epsilon_3$. Then E defines an isomorphism

$$
h_E : \mathbb{P}^2\mathbb{R}_+^* = (\mathbb{R}_+^*)^{\{\epsilon_1,\epsilon_2,\epsilon_3\}}/\mathbb{R}_+^* \xrightarrow{\sim} \mathcal{T}_S.
$$

Namely, given a hyperbolic metric g on S, in order to compute $h_E^{-1}(g)$, straighten E to an ideal triangulation for g and return the positive projective triple of Euclidean lengths defined (in the link of the puncture) by the sectors opposite $\epsilon_1, \epsilon_2, \epsilon_3$. We consider the h_E as charts of $\mathcal{T}_S$. We endow $\mathbb{P}^2\mathbb{R}_+^*$ with the distance d given by

$$
d([a:b:c], [a':b':c']) := \min_{\lambda > 0} \max\left\{\left|\log\frac{\lambda a}{a'}\right|, \left|\log\frac{\lambda b}{b'}\right|, \left|\log\frac{\lambda c}{c'}\right|\right\}.
$$

Also define $\overline{h}_E : \mathbb{P}^2\mathbb{C}^* \to \mathcal{T}_S$ by $\overline{h}_E([a:b:c]) := h_E([\,|a| : |b| : |c|\,])$.

In particular, if the pleated punctured torus S_{i+1}, pleated along the ideal triangulation E_{i+1}, gives rise in the cusp link to a broken (oriented) line whose segments have complex coordinates (a, b, c), then

$$
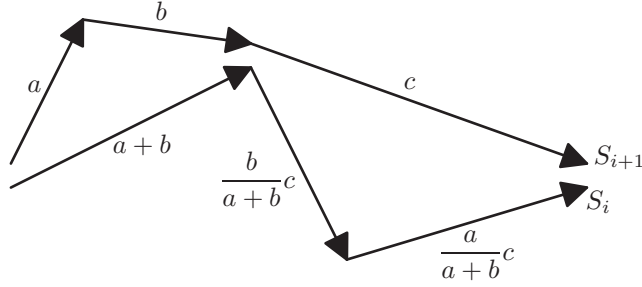\mu(S_{i+1}) = \overline{h}_{E_{i+1}}([a:b:c]).
$$

Figure 9. The complex projective triples associated to S_{i+1} and S_i. These edge orientations *do not* coincide with those of Figures 5 and 6.

Moreover, Figure 9 shows *two* broken lines corresponding respectively to S_{i+1} and the previous pleated surface S_i. Since the triangles (links of vertices of Δ_i) are similar, the complex coordinates of the segments forming the lower broken line are functions of (a, b, c), as shown. Therefore, if we define the substitution formula

$$\Psi([a : b : c]) := \left(\left[a + b \ : \ \frac{b}{a+b} c \ : \ \frac{a}{a+b} c \right] \right)$$

(which is a birational isomorphism from $\mathbb{P}^2\mathbb{C}$ to itself), then

$$\mu(S_i) = \overline{h}_{E_i}(\Psi([a : b : c])).$$

Finally, by viewing the diagonal move between triangulations E_i and E_{i+1} as a *flat* tetrahedron (as in Figure 1), we see that

$$\mu(S_{i+1}) = h_{E_i}(\Psi([\,|a| : |b| : |c|\,])).$$

In other words, the restriction ψ of Ψ to $\mathbb{P}^2\mathbb{R}_+^*$ is the chart map $h_{E_{i+1}} \to h_{E_i}$, i.e.

$$\begin{array}{ccc} \mathbb{P}^2\mathbb{R}_+^* & \xleftarrow{\ \psi\ } & \mathbb{P}^2\mathbb{R}_+^* \\ h_{E_i} \searrow & & \swarrow h_{E_{i+1}} \\ & \mathcal{T}_S & \end{array}$$

commutes. If there are no hinge indices between i and $i+k$ (bounds included), it is easy to check that the chart map $h_{E_{i+k}} \to h_{E_i}$ is, up to a permutation of the coordinates, the map ψ^k.

Property 7.4. *We have* $d\left(h_{E_i}^{-1}\mu(S_i), h_{E_i}^{-1}\mu(S_{i+1}) \right) = \log \dfrac{|a| + |b|}{|a + b|}.$

Proof. The right member is

$$d\left(\left[|a+b| : \left| \frac{b}{a+b} c \right| : \left| \frac{a}{a+b} c \right| \right] , \left[|a| + |b| : \frac{|b|}{|a| + |b|} |c| : \frac{|a|}{|a| + |b|} |c| \right] \right).$$

<div align="right">q.e.d.</div>

It is easy to see that $\psi : \mathbb{P}^2\mathbb{R}_+^* \to \mathbb{P}^2\mathbb{R}_+^*$ is 3-Lipschitz for d. In fact,

Proposition 7.5. *There exists* $K > 0$ *such that the n-th iterate* ψ^n *is* Kn*-bilipschitz for all* $n > 0$.

Proof. Since

$$\psi([a:b:c]) = \left[\frac{(a+b)^2}{c} : b : a\right] \quad \text{and} \quad \psi^{-1}([a:b:c]) = \left[c : b : \frac{(c+b)^2}{a}\right]$$

for all $a, b, c > 0$, it is enough to check the Lipschitz statement. Set $A := \sqrt{a}$; $C := \sqrt{c}$;

$$P_n := \frac{A^{n+1}}{C^n} + \sum_{i,j \in \mathbb{Z}} \binom{j}{i-1}\binom{n-i}{j-i} A^{2i-n-1} C^{n-2j} \quad \text{for all } n \geq -1$$

(the sum is really on $0 < i \leq j \leq n$), so that $P_0 = A$, $P_{-1} = C$, $P_1 = \frac{A^2+1}{C}$. We claim that $\psi^n[a:1:c] = [P_n^2 : 1 : P_{n-1}^2]$ for all $n \geq 0$. The claim is seen by induction on n: the only difficult thing is the induction step $P_{n+1}^2 = \frac{(P_n^2+1)^2}{P_{n-1}^2}$. First, it is straightforward to check that $P_{n+1} + P_{n-1} = (\frac{A}{C} + \frac{C}{A} + \frac{1}{AC})P_n$ (using the Pascal relation twice). Hence, for all $n \geq 1$, one has

$$
\begin{aligned}
&(P_{n+1}P_{n-1} - P_n^2) - (P_n P_{n-2} - P_{n-1}^2) \\
=\ & P_{n-1}(P_{n+1} + P_{n-1}) - P_n(P_n + P_{n-2}) \\
=\ & (P_{n-1}P_n - P_n P_{n-1})\left(\tfrac{A}{C} + \tfrac{C}{A} + \tfrac{1}{AC}\right) = 0.
\end{aligned}
$$

Therefore $P_{n+1}P_{n-1} - P_n^2 = P_1 P_{-1} - P_0^2 = 1$, which proves the induction step. Since P_n is a Laurent polynomial in A, C with partial degrees of order n and positive coefficients, we see that $\log P_n$ is Ln-bilipschitz in $\log a, \log c$ for some universal L. The Proposition follows. q.e.d.

The proof of Proposition 7.5 may seem extremely *ad hoc* and unsatisfactory. However, Proposition 7.5 is a special case of a more general phenomenon for *Markoff maps* (in the sense of [**Bo**]), which we describe in [**G1**].

7-C. Convergence of the moduli.

Proposition 7.6. *The moduli* $(\mu(S_i))_{i \to \pm\infty}$ *converge in Teichmüller space* $\mathcal{T}_S$.

Proof. To prove the $+\infty$-statement, we will fix a large enough index i, and prove that the series

$$\eta_j := d\left(h_{E_i}^{-1}\mu(S_j), h_{E_i}^{-1}\mu(S_{j+1})\right),$$

defined for $j > i$, has finite sum.

For $j > i$, consider a downward-pointing triangle τ contributed by Δ_j, with its edge lengths b_j, b_{j-1} and c_j. The angles of τ at the ends of b_j are x_j and y_j. By Property 7.4, we have $d\left(h_{E_j}^{-1}\mu(S_j), h_{E_j}^{-1}\mu(S_{j+1})\right) = \log\frac{c_j+b_{j-1}}{b_j}$. Compute

$$
\begin{aligned}
\log\frac{c_j+b_{j-1}}{b_j} &\leq \frac{c_j+b_{j-1}-b_j}{b_j} = \frac{\sin x_j + \sin y_j - \sin z_j}{\sin z_j} = \frac{2\sin\frac{x_j}{2}\sin\frac{y_j}{2}}{\cos\frac{x_j+y_j}{2}} \\
&\leq \frac{\sin x_j \sin y_j}{\sin^2 z_j}\sin^2 z_j = \frac{c_j b_{j-1}}{b_j^2}\sin^2 w_j \leq \frac{c_j}{b_j}\phi_j^{+2}
\end{aligned}
$$

(the inequality at the start of the second line holds for large enough j because $x_j + y_j = 2w_j \to 0$). Define $\delta_j = \frac{c_j}{b_j}\phi_j^{+2}$, and let M_j denote the best bilipschitz constant for the chart map $h_{E_j} \to h_{E_i}$. Observe that $\eta_j \leq M_j \delta_j$.

Let $j < l$ be two consecutive elements of J (see Definition 6.5), and $k \notin J$ an integer. We shall bound the $M_n \delta_n$ by geometric sequences on intervals of the form $[\![j, l-1]\!]$, using Proposition 6.6 as in the proof of Proposition 7.2. Since ψ is 3-bilipschitz, one clearly has $M_k \leq 3 M_{k-1}$. By Proposition 6.6,

$$\frac{M_k \delta_k}{M_{k-1} \delta_{k-1}} \leq 3 \frac{\delta_k}{\delta_{k-1}} \leq 3 \frac{c_k}{c_{k-1}} = 3 \chi_{k-1}$$

because (ϕ_k^+) and $(1/b_k)$ are decreasing. The right member goes to 0 for large k.

By Sublemma 7.5, $M_l \leq (l-j) L \cdot M_j$ for some universal L, and by Lemma 3.2 (i-iii-iv), $\phi_j^+ = \phi_{l-1}^+ + (l - j - 1)\phi_l^+ \geq (l-j)\phi_l^+$. Using Proposition 6.6, it follows that

$$\frac{M_l \delta_l}{M_j \delta_j} \leq (l-j) L \cdot \frac{c_l}{c_j} \left(\frac{\phi_l^+}{\phi_j^+} \right)^2 \leq \frac{L \chi_{j-2}}{l - j} \leq L \chi_{j-2}.$$

The right member goes to 0 as j goes to infinity, hence Proposition 7.6. q.e.d.

8. Extrinsic convergence of the surfaces S_i

8-A. Pleated surfaces. Propositions 7.2 and 7.6, together with Corollary 5.3, are the key ingredients to prove that the metric completion of $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$ has two boundary components which are pleated punctured tori with pleating measure $\lambda^{\pm}$. A pleated surface in a hyperbolic 3-manifold M is by definition a map $\varphi : \Sigma \to M$ (where Σ is a hyperbolic surface) which sends rectifiable arcs to rectifiable arcs of the same length, and such that through each point p of Σ runs an open segment s_p on which φ is totally geodesic. It is known (see [**CEG**], 5.1.4) that the direction of s_p is unique if and only if p belongs to a certain *geodesic lamination* Λ (closed union of disjoint geodesics) in Σ, and that φ is totally geodesic away from Λ.

To wrestle with pleated surfaces, we will use the fact that if φ is a locally convex immersion, and Λ has zero Lebesgue measure, then Λ comes with a transverse (pleating) measure ν_Λ. More precisely, ν_Λ can be defined on any sufficiently short oriented segment s transverse to Λ in the following way (see Sections 7 to 9 of [**B2**]). First, assume (taking universal covers if necessary) that $\Sigma = \mathbb{H}^2$ and $M = \mathbb{H}^3$. Immerse $\varphi(\mathbb{H}^2)$ into the Poincaré upper half-space model. Each component of $\mathbb{H}^2 \setminus \Lambda$ crossed by s can be extended to a subset A of $\mathbb{H}^2$ bounded by only (one or) two lines of Λ crossed by s. Endow s with a transverse orientation. The boundary component of A in $\partial \mathbb{H}^3 = \mathbb{P}^1 \mathbb{C}$ on (say) the positive side of the transverse orientation defines an oriented circle arc or line segment $c_A \subset \partial \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$, of angle $\theta_A \in (-2\pi, 2\pi)$ (the orientation of c_A is induced from s, and we may assume $\infty \notin c_A$). If c_A is a line segment or is reduced to a point, we fix $\theta_A = 0$. The closure of the union of all the c_A forms a rectifiable curve $c = c(s) \subset \mathbb{C}$, of length $\sum_A \mathsf{length}(c_A)$. Then c has a well-defined *regular curvature* $RC(c)$, defined as the absolutely convergent sum $\sum_A \theta_A$. But c also has a *total curvature* $TC(c)$, defined (up to an appropriate multiple of 2π) as the difference between the arguments of the initial and final tangent vectors to c. (The appropriate multiple of 2π can be determined by closing off the *embedded* arc c with a broken line, and requiring that the

resulting Jordan curve have total curvature 2π). Then, $\nu_\Lambda(s)$ is defined as the *singular curvature* $SC(c) = TC(c) - RC(c)$. If s is an arbitrarily long segment of $\mathbb{H}^2$ transverse to Λ, we can define $\nu_\Lambda(s)$ by subdividing s into sufficiently short segments $(s_1, \ldots, s_n)$ and define $SC(c(s)) = \nu_\Lambda(s) = \sum_{i=1}^n \nu_\Lambda(s_i)$.

Conversely, if $\varphi : \mathbb{H}^2 \to \mathbb{H}^3$ is a pleated immersion and $SC(c(s))$ is well-defined and non-negative for all transverse segments s, then φ is locally convex with pleating measure ν_Λ as above. We refer to [**B2**] for greater detail.

8-B. Setup. Define the marked hyperbolic once-punctured torus $S_{+\infty}$, endowed with the hyperbolic metric $\lim_{i \to +\infty} \mu(S_i)$. There exists a unique compact geodesic lamination Λ_c^+ of slope β^+/α^+ on $S_{+\infty}$ (namely, Λ_c^+ is the topological limit of any sequence of simple closed curves whose slopes converge to β^+/α^+). Exactly two lines ℓ, ℓ' issued from the puncture of $S_{+\infty}$ fail to meet Λ_c^+: define $\Lambda^+ = \Lambda_c^+ \sqcup \ell \sqcup \ell'$. Then $S_{+\infty}$ is the disjoint union of Λ^+ and the interiors of two ideal triangles A, A'. The union $A \cup A' \cup \ell \cup \ell'$ is a punctured ideal bigon.

Recall the ideal tetrahedra Δ_i, the space $V = \bigcup_{i \in \mathbb{Z}} \Delta_i$ and the hyperelliptic involution $\iota : V \to V$ which reverses all edge orientations. The slopes of the (oriented) pleating lines of the surface S_i (between Δ_{i-1} and Δ_i) are elements of $\mathbb{P}^1\mathbb{Q}$ projecting to $0, 1, \infty$ in $\mathbb{P}^1(\mathbb{Z}/2\mathbb{Z})$: accordingly, we call these pleating lines l_i^0, l_i^1, l_i^∞. For $* \in \{0, 1, \infty\}$, denote by ω_i^* the unique point of l_i^* fixed under the hyperelliptic involution (the ω_i^* are called *Weierstrass points*). Let s_i^* be the segment from ω_i^* to ω_{i+1}^* (across the tetrahedraon Δ_i if $\omega_i^* \neq \omega_{i+1}^*$): each s_i^* is contained in a (pointwise) fixed line Ω^* of the hyperelliptic involution.

Fix the value of the superscript $*$ (soon we shall omit it). Fix a point ω in $\mathbb{H}^2$ and an oriented line l through ω. For each (oriented) surface S_i, consider the oriented, marked universal covering $\pi_i : (\mathbb{H}^2, \omega, l) \to (S_i, \omega_i^*, l_i^*) = (S_i, \omega_i, l_i)$. Endow a universal cover $\widetilde{V}$ of V with lifts $\widetilde{\omega}_i$ of the ω_i connected by lifts of the segments $s_i^* = s_i$, and fix a developing map $\Phi : \widetilde{V} \to \mathbb{H}^3$. There is a unique map h_i such that

$$(16) \qquad \begin{array}{ccccc} (\mathbb{H}^2, \omega, l) & \xrightarrow{h_i} & (\widetilde{V}, \widetilde{\omega}_i) & \xrightarrow{\Phi} & \mathbb{H}^3 \\ \pi_i \downarrow & & \downarrow & & \\ S_i & \longrightarrow & V & & \end{array}$$

commutes. We will prove that the (developing) pleated immersions

$$(17) \qquad \qquad \varphi_i = \Phi \circ h_i : \mathbb{H}^2 \to \mathbb{H}^3$$

converge as pleated maps to a pleated immersion $\varphi_{+\infty}$.

8-C. Convergence of the φ_i. By Proposition 7.2, since the segments $s_i^* = s_i$ all belong to the same fixed line of the hyperelliptic involution, it is already clear that the restriction of φ_i to l converges to a totally geodesic embedding of the line l into $\mathbb{H}^3$ (the convergence is uniform on all compacts of l). By Ascoli's theorem, since the φ_i are 1-Lipschitz, there exists an increasing sequence ν such that the $\varphi_{\nu(i)}$ converge to a certain map $\varphi_{+\infty}$, uniformly on all compact sets of $\mathbb{H}^2$. In Section 8-E below, $\varphi_{+\infty}$ is shown to be independent of the subsequence ν: in anticipation, we now abusively write φ_i instead of $\varphi_{\nu(i)}$.

We shall work in the projective tangent bundle $\mathcal{E} = \mathbb{P}T\mathbb{H}^2$, a circle bundle over $\mathbb{H}^2$ in which geodesic laminations naturally live as closed sets. Length

and angle measurements define a (canonical) complete Riemannian metric on $\mathcal{E}$. For $K \subset \mathbb{H}^2$ compact and $A, B \subset \mathcal{E}$ closed, let $K^{\mathcal{E}} \subset \mathcal{E}$ be the preimage of K under the natural projection $\mathcal{E} \to \mathbb{H}^2$, and define

$$d_K(A, B) := \inf \left\{ \delta > 0 \mid A \cap K^{\mathcal{E}} \subset \mathcal{N}_\delta(B) \ , \ B \cap K^{\mathcal{E}} \subset \mathcal{N}_\delta(B) \right\}$$

where $\mathcal{N}_\delta(X)$ denotes the set of points within δ of X. Then $\inf(1, d_K)$ is a pseudometric, and the set of closed subsets of $\mathcal{E}$ is compact for the Hausdorff metric

$$d_H = \sum_{n>0} 2^{-n} \inf(1, d_{K_n})$$

where the K_n are concentric balls of radius n.

Observe that $S_{+\infty}$ also has Weierstrass points $\omega_{+\infty}^*$, belonging to leaves $l_{+\infty}^*$ of the geodesic lamination Λ^+. Fixing the value of $*$ as before, denote by $\pi_{+\infty} : (\mathbb{H}^2, \omega, l) \to (S_{+\infty}, \omega_{+\infty}^*, l_{+\infty}^*)$ an oriented universal cover.

A consequence of Proposition 7.6 is that the lifts to $\mathcal{E}$ of the $\pi_i^{-1}(l_i^0 \cup l_i^1 \cup l_i^\infty)$ converge for d_H to the lift of $\pi_{+\infty}^{-1}(\Lambda^+)$. We know that $U = \pi_{+\infty}^{-1}(S_{+\infty} \smallsetminus \Lambda^+) \subset \mathbb{H}^2$ is a disjoint union of (open) ideal triangles, of full Lebesgue measure in $\mathbb{H}^2$. For any connected compact set $K \subset U$, we have $K \cap \pi_i^{-1}(l_i^0 \cup l_i^1 \cup l_i^\infty) = \emptyset$ for i large enough, so $\varphi_{+\infty}$ is totally geodesic on K. Therefore, $\varphi_{+\infty}$ is totally geodesic on each component of U. Since $\varphi_{+\infty}$ is clearly 1-Lipschitz, we can approximate any segment in $\mathbb{H}^2 \smallsetminus U$ by segments in U to show that $\varphi_{+\infty}$ is totally geodesic on each leaf of $\mathbb{H}^2 \smallsetminus U$. By Lemma 5.2.8 in [**CEG**], $\varphi_{+\infty}$ sends rectifiable segments to rectifiable segments of the same length, and is a pleated map.

8-D. The map $\varphi_{+\infty}$ is a topological immersion. Define $\mathcal{P} := \pi_{+\infty}^{-1}(\Lambda^+)$, which contains the pleating locus of $\varphi_{+\infty}$. To prove $\varphi_{+\infty}$ is an immersion, it is enough to find a short geodesic segment m of $\mathbb{H}^2$, through the base point ω, transverse to $\mathcal{P}$, and prove that $\varphi_{+\infty}$ is an immersion on the union Υ of all strata (lines and complementary ideal triangles) of $\mathcal{P}$ crossed by m (indeed, $\pi_{+\infty}(\Upsilon) = S_{+\infty}$). Clearly, $\varphi_{+\infty}$ is already an immersion near any point of $\mathbb{H}^2 \smallsetminus \mathcal{P}$. At other points, the key fact will be an "equidistribution" property of the 3 pleating lines of the surface S_i, as i goes to $+\infty$. In the next paragraph, we make this idea precise: a simple closed curve of slope q in S_i will meet the pleating lines of S_i in a perfectly balanced (Sturmian) order (as would a straight Euclidean line of slope q drawn on a Euclidean torus equipped with the triangulation of S_i).

Choose a small $\mu_1 > 0$, and pick $k \in \mathbb{Z}$ large enough so that $\phi_k^+ \leq \mu_1$. Let q be the rational opposite the Farey edge e_k, on the same side as β^-/α^-, so that $\lambda^+(q) = \phi_k^+$. Let $m_i^{\mathcal{C}}$ be the simple closed geodesic of slope q in S_i (equipped with the intrinsic metric): for some superscript $*$ independent of i, the Weierstrass point $\omega_i = \omega_i^*$ belongs to $m_i^{\mathcal{C}}$. By Proposition 7.6 and Hausdorff convergence, there exists $\mu_2 > 0$ such that for all $i > k$, the angle between $m_i^{\mathcal{C}}$ and the pleating line l_i of S_i at ω_i is at least μ_2, and there exists $\mu_3 > 0$ such that the segment m_i of $m_i^{\mathcal{C}}$ of length $2\mu_3$, centered at ω_i, is embedded in S_i. The ends of m_i are at distance at least $\frac{1}{2}\mu_2\mu_3$ from the pleating line l_i in S_i. Observe that the simple closed curve $m_i^{\mathcal{C}}$ meets the pleating edges of S_i in a perfectly equidistributed (Sturmian) order: therefore, the algebraic sum of the

pleating angles crossed by any given subsegment of m_i does not exceed $2\mu_1$. Finally, let κ_i be a subsegment of the pleating line l_i, centered at ω_i, of length 2: if μ_3 is small enough, any pleating line L met by m_i makes an angle at least $\frac{1}{2}\mu_2$ with m_i, and comes within $3\mu_3$ of both ends of κ_i for the intrinsic metric of S_i.

We proceed following Section 8-A. Arrange the developing map $\Phi : \widetilde{V} \to \mathbb{H}^3$ in the upper half-space model so that the $\{\varphi_i(\omega)\}_{i\in\mathbb{Z}}$ lie on the line 0∞ at heights less than 1, and $\varphi_{+\infty}(l)$ is the oriented line from -1 to 1. Consider lifts $\widetilde{m}_i$ of the arcs m_i through the $\varphi_i(\omega)$. By the above (considering lifts of the κ_i), if the μ's are small enough, any pleating line of $\varphi_i(\mathbb{H}^2)$ met by $\widetilde{m}_i$ has its endpoints within distance $1/2$ from 1 and -1 in $\mathbb{C}$ (recall φ_i is 1-Lipschitz). Following Subsection 8-A, let c_i^{+1} (resp. c_i^{-1}) denote the piecewise smooth curve in $\mathbb{C}$ defined by the transverse segment $\widetilde{m}_i$ of $\varphi_i(\mathbb{H}^2)$ near 1 (resp. -1).

Let $\sigma \subset \mathbb{H}^3$ be a subsegment of $\widetilde{m}_i$ across an ideal triangle of $\varphi_i(\mathbb{H}^2)$. Let $\sigma' \subset \mathbb{C}$ be the circle arc (not reduced to a point) contributed by σ to $c_i^{\pm 1}$. By the above, for some universal $K_1 > 0$,

$$\frac{\mu_2}{K_1} \leq \frac{\text{euclidean length of } \sigma'}{\text{hyperbolic length of } \sigma} \leq K_1.$$

In particular, the $c_i^{\pm 1}$ have length at most $2K_1\mu_3$. But the regular curvature radii of $c_i^{\pm 1}$ are at least $\frac{1}{2}$ (the corresponding circles come near 1 and -1), so the total regular curvature of $c_i^{\pm 1}$ is at most $4K_1\mu_3$. By the above, the total singular curvature on any subinterval of $c_i^{\pm 1}$ is at most $2\mu_1$. If the μ's are small enough, it follows that all tangent vectors of $c_i^{\pm 1}$ have complex arguments within $[\pi/4, 3\pi/4]$. As a consequence, if τ ranges over the ideal triangles of $\varphi_i(\mathbb{H}^2)$ crossed by $\widetilde{m}_i$, and $p : \mathbb{H}^3 \to \mathbb{C}$ is the vertical projection, the different $p(\tau)$ intersect only along their edges: so $\varphi_i(\bigcup_\tau \tau)$ is an embedded surface (which we can see, say, as the graph of a function from an open set of $\mathbb{C}$ to $\mathbb{R}^+$). Moreover, define the *breadth* of $p(\tau)$ as the length of the segment $p(\tau) \cap \sqrt{-1}\mathbb{R}$ (of the imaginary axis). Then, for some universal $K_2 > 0$,

$$\frac{\text{breadth of } p(\tau)}{\text{hyperbolic length of } \tau \cap \widetilde{m}_i} \geq \frac{\mu_2}{K_2}.$$

To conclude concerning $\varphi_{+\infty}$, define a lift $m^{\mathcal{C}}$ through $\omega \in \mathbb{H}^2$ of the simple closed geodesic of slope q in $S_{+\infty}$, and a subsegment m of $m^{\mathcal{C}}$, of length $2\mu_3$, centered at ω. The angle between m and any pleating line of Λ^+ it encounters is at least $\frac{1}{2}\mu_2$. Let I be the (infinite) collection of ideal triangles of $\pi_{+\infty}^{-1}(S_{+\infty} \smallsetminus \Lambda^+)$ crossed by m. For $\tau \in I$, denote by $|\tau|$ the length of $m \cap \tau$. By convergence in the Hausdorff metric, $\varphi_{+\infty}(\tau)$ is approached by triangles of the $\varphi_i(\mathbb{H}^2)$, of breadth at least $\frac{|\tau|\mu_2}{2K_2}$: so $p(\varphi_{+\infty}(\tau))$ has nonzero breadth. Injectivity follows: if $x, x' \in \overline{\bigcup_{\tau\in I} \tau}$ do not belong to the same stratum of the lamination $\mathcal{P} = \pi_{+\infty}^{-1}(\Lambda^+)$, find τ separating x from x' to prove that $\varphi_{+\infty}(x) \neq \varphi_{+\infty}(x')$. By vertical projection to $\mathbb{C}$, we see that $\varphi_{+\infty}(\mathbb{H}^2)$ is topologically immersed in $\mathbb{H}^3$.

8-E. Pleating measure of $\varphi_{+\infty}$. These arguments can be extended to prove that the pleating measure of $\varphi_{+\infty}$, as defined in 8-A, is the limit of the pleating measures of the φ_i: the rectifiable curves $c_{+\infty}^{\pm 1} \subset \mathbb{C}$ defined near ± 1 by $\varphi_{+\infty}$

have lengths $\ell^{\pm 1}$, and for any $\varepsilon > 0$, there exists a finite disjoint union of circle arcs γ_ι in $c_{+\infty}^{\pm 1}$ whose lengths add up to at least $\ell^{\pm 1} - \varepsilon$ (moreover the direction of $c_{+\infty}^{\pm 1}$, like that of $c_i^{\pm 1}$, is everywhere within $\pi/4$ of the vertical axis). The γ_ι can be approached by (unions of) arcs of the $c_i^{\pm 1}$, and the regular curvature *not* contributed by the γ_ι is bounded by 3ε. It follows that the pleating of $\varphi_{+\infty}$ is the limit of the pleatings of the φ_i (on any transverse arc, and therefore, on any simple closed curve): that pleating is simply (a lift of) λ^+, by Corollary 5.3. By Theorem D of [**B2**], $\varphi_{+\infty}$ is completely determined by its pleating λ^+ and the intrinsic metric of $S_{+\infty}$. In particular, $\varphi_{+\infty} = \lim \varphi_{\nu(i)}$ is independent of the original subsequence ν, and the φ_i converge to a pleated map whose pleating is given by $\lambda^{\pm}$, as i goes to $\pm\infty$.

8-F. Completeness. The construction of 8-D further allows us to embed the universal cover $\widetilde{V}$ of $V = \bigcup_{i\in\mathbb{Z}} \Delta_i$ into a (topological) manifold with boundary $\widetilde{V}_\partial$, as follows. For each $x \in \mathbb{H}^2$, consider a neighborhood U_x of x such that $\varphi_{+\infty}$ is an embedding on U_x. Then $\varphi_{+\infty}(U_x)$, which has a well-defined transverse ("outward") orientation, splits a small ball B_x centered at $\varphi_{+\infty}(x)$ into two (topological) hemispheres, which we can call "inner" and "outer", referring to the transverse orientation. The inner hemispheres H_x, for x ranging over $\mathbb{H}^2$, can be patched together to obtain a manifold with boundary H. Without loss of generality, the balls B_x can be chosen small enough so that, by the construction of Section 8-D, each $H_x \smallsetminus \varphi_{+\infty}(U_x)$ is identified with a subset of $\widetilde{V}$, embedded in $\mathbb{H}^3$. Then, H can further be patched to $\widetilde{V}$. Since each H_x is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}^+$, the space $\widetilde{V}_\partial = \widetilde{V} \cup H$ is a (possibly non-complete) topological manifold with boundary.

Proposition 8.1. *The action of the fundamental group Γ of the punctured torus S on $\widetilde{V}$ extends to a properly discontinuous action on $\widetilde{V}_\partial$.*

Proof. Consider the representation $\rho : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^3)$ given by the developing map Φ from (16), and the representations $\rho_n : \Gamma \to \mathrm{Isom}^+(\mathbb{H}^2)$ which satisfy $\varphi_n \circ \rho_n(g) = \rho(g) \circ \varphi_n$ for all $g \in \Gamma$. By Lemma 7.6, the ρ_n converge to some $\rho_{+\infty}$. Convergence of the φ_n immediately implies $\varphi_{+\infty} \circ \rho_{+\infty}(g) = \rho(g) \circ \varphi_{+\infty}$. Therefore, the hemispheres H_x can be chosen in an equivariant fashion, and the action of Γ on $\widetilde{V}_\partial$ is well-defined. The action is already properly discontinuous at every point x of $\widetilde{V}$ (namely, as g ranges over Γ, the gx do not accumulate at x). But if Γ acts without fixed points and by isometries on a locally compact metric space X, the set of $x \in X$ such that the action is properly discontinuous at x is open (obviously) and closed: if $g_n x \to x$ for some sequence (g_n) of $\Gamma \smallsetminus \{1\}$ and U is a compact neighborhood of x, then U contains a ball of radius ε centered at x, and whenever $d(x, x') \leq \varepsilon/2$, the $g_n x'$ accumulate at some point of U, so the action is not totally discontinuous at x'. The Proposition follows by connectedness of $\widetilde{V}_\partial$. q.e.d.

As a consequence, the space $V_\partial := \widetilde{V}_\partial/\Gamma$ is a (topological) manifold with boundary, containing $V = \bigcup_{i\in\mathbb{Z}} \Delta_i$; and ∂V_∂ consists of two pleated punctured tori (intrinsically isometric to $S_{+\infty}$ and another surface $S_{-\infty}$), with pleatings λ^+ and λ^-.

Proposition 8.2. *The manifold with boundary V_∂ is complete.*

Proof. Consider the metric completion $\overline{V} \supset V_\partial$ and assume the inclusion is strict. Define a continuous function $f : V_\partial \to \mathbb{R}^{>0}$ by $f(x) = d(x, \overline{V} \smallsetminus V_\partial)$. By assumption, $\inf(f) = 0$.

Consider the immersion of $\widetilde{V}_\partial$ into the upper half-space model of $\mathbb{H}^3$ obtained by sending a lift of the cusp to infinity (Figures 4 and 5-6). The image of $\partial \widetilde{V}_\partial$ contains, in particular, vertical half-planes (interrupted at some height above $\mathbb{C}$). Therefore, any geodesic of $\widetilde{V}_\partial$ starting high enough above $\mathbb{C}$ is defined for all times $t \leq 1$ (unless it hits $\partial \widetilde{V}_\partial$). As a result, if $H \subset V_\partial$ denotes a small enough open horoball neighborhood of the cusp, we have $f \geq 1$ on H.

For $i \in \mathbb{Z}$, consider the compact set $K_i := S_i \smallsetminus H$ in V_∂. There exists a ball B of $\mathbb{H}^2$ centered at the base point ω, with radius independent of i, such that $K_i \subset \pi_i(B)$: by convergence of (φ_i), the $\pi_i(B)$ converge metrically to a compact subset K' of ∂V_∂, on which f is positive. Therefore f is bounded away from 0 on some neighborhood U of K' in V_∂, and $K_i \subset U$ for large enough i. So f is bounded away from 0 on $\bigcup_{i \in \mathbb{Z}} K_i$, and therefore on $\bigcup_{i \in \mathbb{Z}} S_i$.

However, assume $\gamma(t)$ is a rectifiable 1-Lipschitz arc of V_∂, defined for $t < M$, with no limit at M. For any $\varepsilon > 0$, the restriction $\gamma_{|[M-\varepsilon, M)}$ meets $V_\partial \smallsetminus \partial V_\partial$ (because $\partial V_\partial = S_{+\infty} \sqcup S_{-\infty}$ is complete), but then $\gamma_{|[M-\varepsilon, M)}$ must meet Δ_i for an unbounded set of indices i (any finite union of tetrahedra is complete). Therefore, we can find a sequence $t_n \to M$ such that $\gamma(t_n) \in \bigcup_{i \in \mathbb{Z}} S_i$. Clearly, $f(\gamma(t_n)) \leq M - t_n$ which goes to 0: a contradiction. So V_∂ is complete. q.e.d.

8-G. A quasifuchsian punctured-torus group. The end of the argument is now quite standard: recall the complete manifold with locally convex boundary $\widetilde{V}_\partial$, which is a universal cover of $\overline{V} = V_\partial$. Given distinct points $x, x' \in \widetilde{V}_\partial$, consider a shortest possible path γ from x to x'. If γ has an interior point in $\partial \widetilde{V}_\partial$, by local convexity, we must have $\gamma \subset \partial \widetilde{V}_\partial$, and γ is a geodesic segment of $\partial \widetilde{V}_\partial$. If not, γ is (the closure of) a geodesic segment of $\widetilde{V}$. At any rate, the extended developing map $\Phi : \widetilde{V}_\partial \to \mathbb{H}^3$ is an embedding (it sends γ to a segment with distinct endpoints) and has a closed convex image C, endowed with a properly discontinuous action of the fundamental group Γ of S (Proposition 8.1). The action extends properly discontinuously to $\mathbb{H}^3$ (this can be seen by projecting any point of $\mathbb{H}^3$ to C). The manifold $\mathbb{H}^3/\Gamma$ contains $\overline{V} \simeq C/\Gamma$, which has the desired boundary pleatings $\lambda^\pm$. Clearly, C is the smallest closed, convex set containing all parabolic fixed points of Γ; therefore $\overline{V}$ is the convex core, and Γ is quasifuchsian, with pleating data $\lambda^\pm$. Theorem 1.1 is proved.

9. Application: the EPH theorem

Let us quickly recall the correspondence between the horoballs of $\mathbb{H}^3$ and the vectors in the positive isotropic (light) cone of Minkowski space $\mathbb{R}^{3+1}$. Endow $\mathbb{R}^4$ with the Lorentzian product $\langle (x, y, z, t) | (x', y', z', t') \rangle := xx' + yy' + zz' - tt'$. Define
$$X := \{v = (x, y, z, t) \in \mathbb{R}^4 \mid t > 0 \text{ and } \langle v | v \rangle = -1\}.$$
Then $\langle . | . \rangle$ restricts to a Riemannian metric on X and there is an isometry $X \simeq \mathbb{H}^3$, with $\mathrm{Isom}^+(X)$ a component of $SO_{3,1}(\mathbb{R})$. This isometry takes hyperbolic planes in $\mathbb{H}^3$ to intersections of X with (linear) hyperplanes. We will identify the point (x, y, z, t) of X with the point at Euclidean height $\frac{1}{t+z}$

above the complex number $\frac{x+iy}{t+z}$ in the Poincaré upper half-space model. Under this convention, the closed horoball $H_{d,\zeta}$ of Euclidean diameter d centered at $\zeta = \xi + i\eta \in \mathbb{C}$ in the upper half-space model is identified with the set $\{v \in X \mid \langle v | v_{d,\zeta} \rangle \geq -1\}$, where $v_{d,\zeta} = \frac{1}{d}(2\xi, 2\eta, 1 - |\zeta|^2, 1 + |\zeta|^2)$. We therefore identify $H_{d,\zeta}$ with the point $v_{d,\zeta}$ of the light cone. Similarly, the closed horoball $H_{h,\infty}$ of points at Euclidean height no less than h in the half-space model corresponds to $\{v \in X \mid \langle v | v_{h,\infty} \rangle \geq -1\}$ where $v_{h,\infty} = (0, 0, -h, h)$, so we identify $H_{h,\infty}$ with $v_{h,\infty}$.

Consider the following objects: a complete oriented hyperbolic 3-manifold M with one cusp, a horoball neighborhood H of the cusp, a universal covering $\pi : \mathbb{H}^3 \to M$, and the group $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^3)$ of deck transformations of π. Then H lifts to a family of horoballs $(H_i)_{i \in \mathcal{I}}$ in $\mathbb{H}^3$, corresponding to a family of isotropic vectors $(v_i)_{i \in \mathcal{I}}$ in Minkowski space. The closed (linear) convex hull C of $\{v_i\}_{i \in \mathcal{I}}$ is Γ-invariant, and its boundary ∂C comes with a natural decomposition $\widetilde{\mathcal{D}}$ into polyhedral *facets*. Let V be the interior of the convex core of M. In [**AS**], Akiyoshi and Sakuma extended the Epstein-Penner convex hull construction to prove that $\widetilde{\mathcal{D}}$ defines an H-independent decomposition $\mathcal{D}^{Geom}$ of V into ideal hyperbolic polyhedra, typically tetrahedra, allowing for a few clearly defined types of degeneracies. In [**ASWY2**], with Wada and Yamashita, they also conjectured

Theorem 9.1. *If M is quasifuchsian, homeomorphic to the product of the punctured torus with the real line, and has irrational pleating laminations of slopes β^+/α^+ and β^-/α^-, the decomposition $\mathcal{D}^{Geom}$ of V is combinatorially the triangulation $\{\Delta_i\}_{i \in \mathbb{Z}}$ defined in Section 1.*

Proof. It is known (see [**Se**] or Corollary 10.8 below) that quasifuchsian punctured-torus groups are fully determined by their measured pleating laminations. It follows that the manifold constructed in Sections 1-8, with the same pleating data as M, is isometric to M. Recall the triangulated space $\widetilde{V} \subset \mathbb{H}^3$ of (16), which is a universal cover of the union $\bigcup_{i \in \mathbb{Z}} \Delta_i$ of all tetrahedra Δ_i. Given a horoball neighborhood H of the cusp of M, consider its lifts $(H_i)_{i \in \mathcal{I}}$ in $\mathbb{H}^3$ and the corresponding isotropic vectors $(v_i)_{i \in \mathcal{I}}$ in Minkowski space $\mathbb{R}^4$. Then each tetrahedron Δ of $\widetilde{V}$ has its vertices at the centers of four horoballs $H_1, \ldots, H_4$: we introduce the convex hull τ_Δ of $v_1, \ldots, v_4$, and define $D := \bigcup_\Delta \tau_\Delta$. The central projection to the hyperboloid X with respect to the origin sends τ_Δ homeomorphically to the lift of Δ in X, so the interiors of the τ_Δ are pairwise disjoint, and each τ_Δ comes with a transverse "upward" orientation $\overrightarrow{u}$ (given by any vector of τ_Δ, i.e. with head in τ_Δ and foot at the origin). The theorem claims exactly that $D \subset \partial C$ as polyhedral complexes (the inclusion is expected to be strict, since ∂C also contains faces projecting, say, to the boundary of the convex core — they are analyzed in detail in [**AS**]).

Suppose the codimension-1 complex $D \subset \mathbb{R}^4$ is locally convex, with $\overrightarrow{u}$ pointing inward and nonzero bending angles. We can then prove $D \subset \partial C$: for any subset Y of a real vector space, define $IY := \{ty \mid t \geq 1, y \in Y\}$. Since the interior of the convex core is convex, ID is a convex set (for any non-collinear $x, y \subset ID$, the 2-dimensional quadrant $P = \mathbb{R}^+ x + \mathbb{R}^+ y$ intersects the hyperboloid X along a hyperbolic segment of the universal cover $\widetilde{V}$ of the

convex core, and $P \cap ID$ is convex by local convexity of D: so the segment $[x, y]$ is in ID). Moreover, ID contains all the isotropic vectors $\{v_i\}_{i \in \mathcal{I}}$, so its closure $\overline{ID}$ contains their convex hull C. Conversely, D is clearly included in C and it is easy to check that $IC \subset C$ (because $I\{v_i\} \subset C$ for all i, see [**AS**]). So $ID \subset C$ and, by closedness, $\overline{ID} = C$. Clearly, $D \subset \partial \overline{ID}$ because no two points of D are collinear. So $D \subset \partial C$ as sets, and hence as polyhedral complexes because the bending angles of D are nonzero. So we only need to prove

Lemma 9.2. *The codimension-1 simplicial complex $D \subset \mathbb{R}^4$ is locally convex ($\overrightarrow{u}$ pointing inward).*

Proof — Consider adjacent ideal tetrahedra Δ, Δ' in $\mathbb{H}^3$ which are lifts from tetrahedra Δ_{i-1}, Δ_i of the manifold. We must prove that the dihedral angle in $\mathbb{R}^4$ between τ_Δ and $\tau_{\Delta'}$ points "downward". We will assume that the letter between $i - 1$ and i is an L belonging to a subword $RL^n R$ of Ω. In the link of the cusp, the pleated surface S_i between Δ_{i-1} and Δ_i contributes a broken line $(-1, \zeta, \zeta', 1)$ in $\mathbb{C}$ together with its iterated images under $u \mapsto u \pm 2$, as in Figure 10 (we assume that the vertices $-1, 1$ both belong to the base segments of the Euclidean triangles just below and just above the broken line, in the sense of Figure 5). We use the notation

$$
\begin{aligned}
\zeta + 1 &= \overrightarrow{a} = a\,e^{iA} \\
\zeta' - \zeta &= \overrightarrow{b} = b\,e^{iB} \\
1 - \zeta' &= \overrightarrow{c} = c\,e^{iC}
\end{aligned}
$$

(so far A, B, C are only defined modulo 2π). Above this broken line in the upper half-space model lives a lift of S_i, containing the ideal triangles $(-1, \zeta, \infty,)$ and (ζ, ∞, ζ') and $(\infty, \zeta', 1)$, which admits as a deck transformation

$$
f : u \mapsto 1 + \frac{(\zeta + 1)(\zeta' - 1)}{u + 1} \ ,
$$

because f maps the first of the above mentioned triangles to the last (respecting the order of the vertices). Therefore, $f(H_{1,\infty}) = H_{|\zeta+1||\zeta'-1|,1} = H_{ac,1}$. Note that a and c are the lengths of the segments of the broken line adjacent to the vertex 1. Similarly, the following horoballs are all images of one another by deck transformations:

$$
H_{1,\infty} \ ; \ H_{ac,-1} \ ; \ H_{ab,\zeta} \ ; \ H_{bc,\zeta'} \ ; \ H_{ac,1} \ .
$$

If $\zeta = \xi + \eta\sqrt{-1}$ and $\zeta' = \xi' + \eta'\sqrt{-1}$, the corresponding isotropic vectors in Minkowski space are respectively

$$
(18) \quad
\begin{aligned}
v_\infty &= & (& 0, & 0, & -1, & 1 &) \\
v_{-1} &= \tfrac{1}{ac} & (& -2, & 0, & 0, & 2 &) \\
v_\zeta &= \tfrac{1}{ab} & (& 2\xi, & 2\eta, & 1 - |\zeta|^2, & 1 + |\zeta|^2 &) \\
v_{\zeta'} &= \tfrac{1}{bc} & (& 2\xi', & 2\eta', & 1 - |\zeta'|^2, & 1 + |\zeta'|^2 &) \\
v_1 &= \tfrac{1}{ac} & (& 2, & 0, & 0, & 2 &).
\end{aligned}
$$

Observe that the hyperplane through $0, v_\zeta, v_{\zeta'}, v_\infty$ separates v_1 from v_{-1} because the hyperbolic plane through the ideal points $\zeta, \zeta', \infty \in \mathbb{P}^1\mathbb{C}$ separates the ideal points 1 and -1. Also, observe that the imaginary parts η, η' of ζ, ζ'
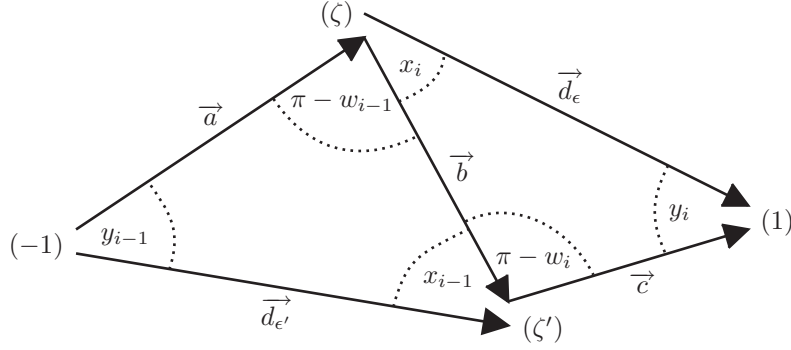
Figure 10. Adjacent tetrahedra Δ_{i-1}, Δ_i (cusp view).

satisfy $\eta > \eta'$ because the Euclidean triangles $-1\zeta'\zeta$ and $1\zeta\zeta'$ are counterclockwise oriented.

To prove that the dihedral angle at the codimension-2 face projecting to $(\zeta\zeta'\infty)$ is convex, it is enough to show that if $Pv_\zeta + Qv_{\zeta'} + Rv_\infty = \lambda v_1 + (1-\lambda)v_{-1}$ (for some $0 < \lambda < 1$) then $P + Q + R > 1$ (moreover, this will in fact take care of *all* codimension-2 faces of the simplicial complex D). One easily finds the unique solution

$$P = \frac{-b\eta'}{c(\eta - \eta')} \ ; \ Q = \frac{b\eta}{a(\eta - \eta')} \ ; \ R = \frac{\eta(1 - |\zeta'|^2) - \eta'(1 - |\zeta|^2)}{ac(\eta - \eta')}$$

hence

$$P+Q+R = 1 + \frac{Z}{ac(\eta - \eta')} \quad \text{where} \quad Z = bc\eta - ab\eta' + \eta(1 - |\zeta'|^2) - \eta'(1 - |\zeta|^2) + ac(\eta' - \eta).$$

So it is enough to prove that $Z > 0$. Endow $\mathbb{C} \simeq \mathbb{R}^2$ with the usual scalar product "$\bullet$" and observe that $1 - |\zeta|^2 = \overrightarrow{a} \bullet (\overrightarrow{b} + \overrightarrow{c})$ and $1 - |\zeta'|^2 = (\overrightarrow{a} + \overrightarrow{b}) \bullet \overrightarrow{c}$. Hence

$$
\begin{aligned}
Z &= \eta(bc + \overrightarrow{b} \bullet \overrightarrow{c}) - \eta'(ab + \overrightarrow{a} \bullet \overrightarrow{b}) + (\eta' - \eta)(ac - \overrightarrow{a} \bullet \overrightarrow{c}) \\
&= abc \left[\frac{\eta}{a}(1 + \cos(B - C)) - \frac{\eta'}{c}(1 + \cos(A - B)) + \frac{\eta' - \eta}{b}(1 - \cos(A - C)) \right] \\
&= abc[\sin A(1 + \cos(B - C)) + \sin C(1 + \cos(A - B)) + \sin B(1 - \cos(A - C))] \\
&= 4abc \sin \frac{A + C}{2} \cos \frac{A - B}{2} \cos \frac{B - C}{2}.
\end{aligned}
$$

by standard trigonometric formulae. Observe that the last expression is a well-defined function of $A, B, C \in \mathbb{R}/2\pi\mathbb{Z}$ (although each factor is defined only up to sign). Next, however, we shall carefully pick representatives of A, B, C in $\mathbb{R}$.

OBSERVATION 9.3. There exists a unique triple of representatives $(A, B, C) \in \mathbb{R}^3$ such that $\{A, B, C\} \subset J$ where J is an open interval of length less than π, containing 0. (Indeed, the broken line $(\ldots, -1, \zeta, \zeta', 1, \ldots)$ invariant under $u \mapsto u \pm 2$ has no self-intersection, and this fact is equivalent to the existence of

an open half-plane H containing the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$: the observation follows since $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = 2$ belongs to H).

We pick the representatives $A, B, C \in \mathbb{R}$ given by Observation 9.3, and do the same for the broken line contributed by each surface S_i for $i \in \mathbb{Z}$ (each broken line, being included in an infinite horizontal strip, can be oriented from the $-\infty$ to $+\infty$, so each edge ϵ of the cusp link inherits an orientation, as in Figure 10: we write its complex coordinate $\overrightarrow{d_\epsilon} = d_\epsilon e^{D_\epsilon}$). By Observation 9.3, all the complex arguments D_ϵ are in $(-\pi, \pi)$. But since $0 \leq w \leq \pi$, the existence of the interval J also implies (in the notation of Figure 10):

$$A = B + w_{i-1} \quad \text{and} \quad C = B + w_i$$

hence $B = \inf J$ and $B \in (-\pi, 0)$. It follows that $B + x_i \in (-\pi, \pi)$, hence (writing $\epsilon = [\zeta, 1]$ and $\epsilon' = [-1, \zeta']$ as in Figure 10):

$$D_\epsilon = B + x_i \quad \text{and similarly} \quad D_{\epsilon'} = B + x_{i-1}.$$

In other words (by transitivity), for any two edges ϵ_1, ϵ_2 of the cusp link, the difference of arguments $D_{\epsilon_1} - D_{\epsilon_2} \in \mathbb{R}$ can be read off "naively" as a linear combination of the $\{w_i\}_{i \in \mathbb{Z}}$, with no multiple of 2π added.

Now that all arguments are fixed in the real interval $(-\pi, \pi)$, we can see that $\cos \frac{A-B}{2} = \cos(w_{i-1}/2)$ and $\cos \frac{B-C}{2} = \cos(w_i/2)$ are positive. So to prove $Z > 0$ it remains to see that $0 < A + C < 2\pi$. This in turn follows from a small variant of Lemma 16 in [**GF**], which we prove now (it implies an empirical observation of Jørgensen which was Conjecture 8.6 in [**ASWY2**]).

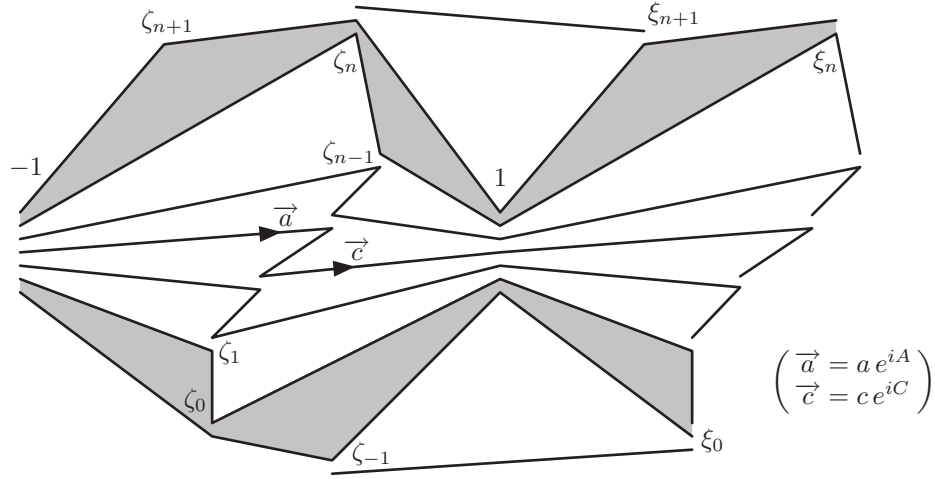**Proposition 9.4.** *With the above notation, one has $0 < A + C < 2\pi$.*

Figure 11.   Cusp view: a subword RL^nR, bounded by hinge indices 0 and n (grey).

Consider the full subword RL^nR, with the broken line $(-1, \zeta, \zeta', 1)$ corresponding to some L. There is in fact a sequence of points $(\zeta_0, \zeta_1, \ldots, \zeta_n)$ in $\mathbb{C}$ such that the broken line corresponding to the j-th letter L is $(-1, \zeta_j, \zeta_{j-1}, 1)$

for all $1 \leq j \leq n$. The broken line corresponding to the initial (resp. final) R is $(-1, \zeta_0, \zeta_{-1}, 1)$ for some ζ_{-1} (resp. $(-1, \zeta_{n+1}, \zeta_n, 1)$ for some ζ_{n+1}). There exists $1 \leq k \leq n$ such that $(\zeta, \zeta') = (\zeta_k, \zeta_{k-1})$.

Set $\xi_j := \zeta_j + 2$ for all j. By construction, the rays issued from 1 through $\zeta_{-1}, \zeta_0, \ldots, \zeta_{n-1}, \zeta_n, \xi_{n+1}, \xi_n, \ldots, \xi_1, \xi_0, \zeta_{-1}$ (in that cyclic order) divide $\mathbb{C}$ clockwise into angular sectors whose measures are less than π and add up to 2π (these $2n+4$ rays realize the link of the vertex 1 in the cusp triangulation, see Figure 11). Comparing the angles at 1 and at -1, and denoting segments ϵ of the cusp link by their endpoints (and their complex arguments by D_ϵ), we see immediately that $D_{-1\zeta_{k+1}} - D_{-1\zeta_k} = D_{\zeta_k 1} - D_{\zeta_{k-1} 1}$. Therefore,

$$(19) \quad A + C = D_{-1\zeta_k} + D_{\zeta_{k-1}1} = D_{-1\zeta_{k+1}} + D_{\zeta_k 1} = \cdots = D_{-1\zeta_j} + D_{\zeta_{j-1}1}$$

(for all $0 \leq j \leq n+1$). Since the triangles $\zeta_0 \zeta_{-1} 1$ and $\xi_0 1 \zeta_{-1}$ are counterclockwise oriented (and $\xi_0 = \zeta_0 + 2$), we have $\mathrm{Im}(\zeta_{-1}) < \mathrm{Im}(1)$ i.e. $\mathrm{Im}(\zeta_{-1}) < 0$. Similarly, $\zeta_n 1 \xi_{n+1}$ and $\xi_n \xi_{n+1} 1$ are counterclockwise oriented, so $\mathrm{Im}(\xi_{n+1}) > 0$. So there exists $0 \leq j \leq n+1$ such that $\mathrm{Im}(\zeta_{j-1}) < 0$ and $\mathrm{Im}(\zeta_j) \geq 0$. This implies $D_{-1\zeta_j} \in (0, \pi)$ and $D_{\zeta_{j-1}1} \in [0, \pi)$. By (19), this implies $A + C \in (0, 2\pi)$. Theorem 9.1 is proved.   q.e.d.

In [**GF**], we studied punctured-torus bundles over the circle by indexing the tetrahedra Δ_i in $\mathbb{Z}/m\mathbb{Z}$ (instead of $\mathbb{Z}$): note that the proof of Theorem 9.1 applies without alteration to that context.

10. Generalizations

In this section, we extend all the previous results to punctured-torus groups with rational pleatings and/or infinite ends. By an *admissible* lamination λ, we mean any of the following objects:

- An ordered pair (up to sign) $\lambda = \pm(\alpha, \beta) \in \mathbb{R}^{*2}/\pm$ such that β/α is irrational;
- An ordered pair $\lambda = \pm(\theta\alpha, \theta\beta)$ where $\alpha, \beta \in \mathbb{Z}$ are coprime and $0 < \theta < \pi$ (we call θ the *weight* of λ and write $|\lambda| = \theta$);
- An ordered pair $\lambda = \pm(\pi\alpha, \pi\beta)$ where $\alpha, \beta \in \mathbb{Z}$ are coprime (we call π the *weight* of λ and write $|\lambda| = \pi$);
- A real projective class $\lambda = \{(\mu\alpha, \mu\beta) \mid \mu \neq 0\}$ where β/α is irrational.

The *slope* of the admissible lamination is the number $s(\lambda) = \beta/\alpha \in \mathbb{P}^1\mathbb{R}$. Each end ω of a punctured-torus group can be associated a unique (ending or pleating) admissible lamination: the four cases above correspond respectively to ω being

- Finite, with irrational *pleating* lamination;
- Finite, with rational *pleating* lamination and pleating angle θ;
- Finite, with a rational *ending* lamination (an accidental cusp);
- Infinite, with an irrational *ending* lamination (see [**B1**] and Section 3 of [**Min**] for a definition).

Theorem 10.1. *Let λ^+, λ^- be admissible laminations of distinct slopes. There exists a punctured-torus group Γ with ending and/or pleating laminations $\lambda^\pm$, and the interior V of the convex core of $\mathbb{H}^3/\Gamma$ has an ideal decomposition $\mathcal{D} = \mathcal{D}^{Comb}$ into polyhedral cells (of positive volume) whose combinatorics are*

given by $\lambda^{\pm}$ in the following sense: if Λ is the line from $s(\lambda^-)$ to $s(\lambda^+)$ across the Farey diagram in $\mathbb{H}^2$, then

i – *If $s(\lambda^+)$ and $s(\lambda^-)$ are irrational, $\mathcal{D}$ consists of ideal tetrahedra $(\Delta_i)_{i \in \mathbb{Z}}$ in natural bijection with the Farey edges crossed by Λ, as in Section 1.*

ii – *If only $s(\lambda^+)$ is rational and $|\lambda^+| < \pi$, then $\mathcal{D}$ has one ideal tetrahedron per Farey edge crossed by D, and one cell T whose interior is non-contractible, homeomorphic to a solid torus: $\partial_1 = \overline{\partial T \cap \partial V}$ is a punctured torus pleated along a simple closed curve of slope $s(\lambda^+)$, and $\partial_2 = \overline{\partial T \smallsetminus \partial V}$ is a punctured torus pleated along the ideal triangulation associated to the Farey triangle with vertex $s(\lambda^+)$ crossed by Λ. Finally, $\partial_1 \cap \partial_2$ is a line from the puncture to itself of slope $s(\lambda^+)$. See the left panel of Figure 12.*

iii – *If only $s(\lambda^+)$ is rational and $|\lambda^+| = \pi$, all the statements of the previous case apply, except that $\partial_1 = \overline{\partial T \cap \partial V}$ becomes a thrice-punctured sphere (the simple closed curve of slope $s(\lambda^+)$ has been "pinched" to become a cusp): see the right panel of Figure 12.*

iv – *If only $s(\lambda^-)$ is rational, the situation is similar to the two previous cases, exchanging λ^- and λ^+.*

v – *If $s(\lambda^+), s(\lambda^-)$ are rationals but not Farey neighbors, the situation is again similar, with two solid torus cells T^+ and T^-.*

vi – *If $s(\lambda^+), s(\lambda^-)$ are Farey neighbors, $\mathcal{D}$ only consists of two solid tori T^+ and T^- as above, glued along a punctured torus S pleated along only two lines from the cusp to itself, of slopes $s(\lambda^+)$ and $s(\lambda^-)$.*

Moreover, $\mathcal{D}$ agrees with the geometrically canonical decomposition $\mathcal{D}^{Geom}$ of V given by the Epstein-Penner convex hull construction.

Note that the combinatorics of $\mathcal{D}^{Geom}$ does not depend on the nature (finite or infinite) of the ends of $\mathbb{H}^3/\Gamma$. At this point, we have treated the case of two irrational pleatings (finite ends). We proceed to prove the remaining cases of Theorem 10.1.
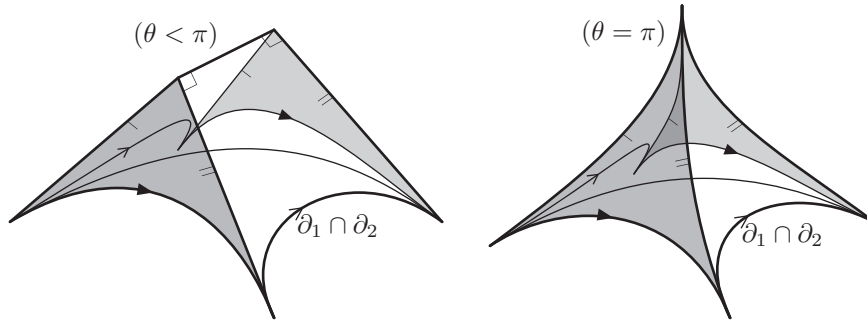

Figure 12. Toric cells: the exterior dihedral angle θ is the weight $|\lambda^{\pm}|$. Shaded faces are identified, as well as edges carrying identical arrows.

10-A. One rational pleating. We focus on the case of two finite ends, with only β^+/α^+ rational. We can choose to end the word $\Omega \in \{R, L\}^{\mathbb{Z}}$ with an

infinite suffix $LRR...R...$ (or of $RLL...L...$: that is an arbitrary choice) and proceed from Section 1 onward. We shall assume that $i = 0$ is the greatest hinge index. Sections 2 through 3-A are unchanged: the sequence (w_i) is just concave (thus convergent and non-decreasing) on $\mathbb{N}$. In Subsection 3-B, we find that the sequence (ϕ_i^+) is constant on $\mathbb{N}$, equal to some positive real. By (9), this real is the weight θ of the rational lamination λ^+, so we assume $\theta < \pi$. Section 4 goes through essentially unchanged: by the computation of Sublemma 4.2 (and with the same notation), the sum of the volumes of all tetrahedra Δ_i for $i \geq 2^n$ is at most

$$\sum_{k>n} \Sigma_{2^{k-1}}^{2^k} \leq \sum_{k>n} 2^{-k}[1 + (2k-1)\log 2] = O(2^{-n/2}).$$

Therefore the volume functional $\mathcal{V}$ is bounded, continuous for the product topology, and concave. We can find a maximizer w of $\mathcal{V}$, and it still satisfies Propositions 4.4 and 4.5: in particular, all tetrahedra Δ_i for $i > 0$ have positive angles.

Something new is required in Section 5: we must prove that $\lim_{i \to +\infty} w_i = \theta$. The $\{w_i\}_{i \geq 2}$ contribute only to the angles of the $\{\Delta_i\}_{i \geq 1}$, which are positive: so the volume $\mathcal{V}$ is critical with respect to each w_i for $i \geq 2$. By Sublemma 6 of [**GF**], it follows that the cusp triangles of the $\{\Delta_i\}_{i \geq 1}$ fit together correctly and can be drawn in the Euclidean plane $\mathbb{C}$. More precisely, there exists a sequence of complex numbers $(\zeta_i)_{i \geq 0}$ such that for all $i \geq 1$, the triangles contributed by Δ_i have vertices at $(-1, \zeta_i, \zeta_{i-1})$ and $(1, \zeta_i, \zeta_{i+1})$ (Figure 13, left). These triangles being similar, we have $(\zeta_i + 1)(\zeta_{i+1} - 1) = (\zeta_{i-1} + 1)(\zeta_i - 1) = ... = (\zeta_0 + 1)(\zeta_1 - 1)$, hence

$$\zeta_{i+1} = \frac{\zeta_i + \kappa}{\zeta_i + 1} =: \varphi(\zeta_i)$$

for some complex number $\kappa \neq 1$ independent of i. Observe that the complex length ℓ of the hyperbolic isometry (extending) φ satisfies $\cosh \ell = \frac{1+\kappa}{1-\kappa}$. In the end, φ will be a lift of the loop along the rational pleating line of the convex core.

Proposition 10.2. *The number κ lies in the real interval $(0, 1)$.*

Proof. Let Z_i be the $(u \mapsto u \pm 2)$-invariant broken line $(\ldots, -1, \zeta_{i-1}, \zeta_i, 1, \ldots)$ contributed by the pleated surface S_i. First, κ is real: if not, the ζ_i have a limit in $\mathbb{C} \smallsetminus \mathbb{R}$ (a square root of κ), so $w_i - w_{i-1}$ (the angle of Z_i at 1) cannot go to 0. If $\kappa < 0$, then φ is a pure rotation: the ζ_i all belong to a circle of the lower half-plane, which also contradicts $w_i - w_{i-1} \to 0$. If $\kappa > 1$, then φ is a gliding axial symmetry of $\mathbb{H}^3$: the ζ_i go to $\pm\sqrt{\kappa}$ and belong alternatively to the upper and lower half-plane, so Z_i has self-intersection for large i. If $\kappa = 0$, then φ is a parabolic transformation fixing 0: the ζ_i go to 0 along a circle tangent to $\mathbb{R}$, and one can see that $w_i = \widehat{\zeta_i \zeta_{i+1} 1}$ goes to $\pi > \lim_{+\infty} \phi^+ = \theta$. The only remaining possibility is $\kappa \in (0, 1)$, where φ is a pure translation.   q.e.d.

Thus, the ζ_i lie on a circle arc C of the lower half-plane which meets the real line at $\pm\sqrt{\kappa}$, and $\lim_{+\infty} \zeta_i = \sqrt{\kappa}$. We denote by θ^* the angle between C and the segment $[-\sqrt{\kappa}, \sqrt{\kappa}]$ (more precisely, the angle between their half-tangents

at $\sqrt{\kappa}$). It is easy to see that

$$\theta^* = \lim_{i \to +\infty} w_i$$
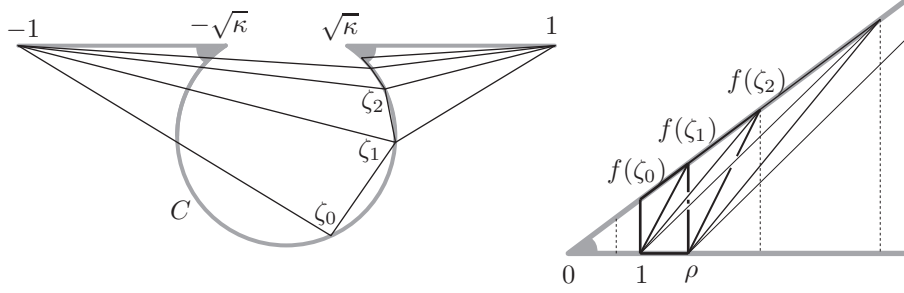
(see Figure 13). Hence, $\theta^* \leq \theta$.

Figure 13. All marked angles (grey) are $\pi - \theta^*$, and $\zeta_{j+1} = \varphi(\zeta_j) = \frac{\zeta_j + \kappa}{\zeta_j + 1}$.

Proposition 10.3. *One has $\theta^* = \theta$.*

Proof. First, it is easy to check that any data $0 \leq w_0 < w_1 < \theta^* < \pi$ smoothly determines a *unique* pair of complex numbers ζ_0, ζ_1 such that:

  i – the broken line $Z_1 = (\dots, -1, \zeta_0, \zeta_1, 1, \dots)$ has angles $(w_0, -w_1, w_1 - w_0)$ as in (2) above;
  ii – the number κ such that $\zeta_1 = \frac{\zeta_0 + \kappa}{\zeta_0 + 1}$ lies in $(0, 1)$;
  iii – the circle through ζ_0 and ζ_1 centered on the imaginary axis intersects the real axis at an angle θ^*.

These ζ_0, ζ_1 in turn define all $\{\zeta_j\}_{j \geq 2}$ *via* $\zeta_{j+1} = \varphi(\zeta_j) = \frac{\zeta_j + \kappa}{\zeta_j + 1}$, and we can read off the angle $w_j = \widehat{1\zeta_{j+1}\zeta_j} \leq \theta^*$ and construct the associated ideal tetrahedron Δ_j. In what follows, we investigate the shape of the space $U := \bigcup_{j \geq 1} \Delta_j$, whose boundary (the punctured torus S_1) has pleating angles $(w_0, -w_1, w_1 - w_0)$ (see (2) above).

Define $f(\zeta) := \frac{\zeta + \sqrt{\kappa}}{\zeta - \sqrt{\kappa}}$, so that $f(\varphi(\zeta)) = \rho f(\zeta)$ where $\rho := \frac{1 + \sqrt{\kappa}}{1 - \sqrt{\kappa}}$. The hyperbolic convex hull of $(\infty, 1, \zeta_j, \zeta_{j+1})$ is isometric to the tetrahedron Δ_j: pushing forward by f, we obtain a tetrahedron Δ_j', isometric to Δ_j, with vertices $(1, \rho, \rho^j f(\zeta_0), \rho^{j+1} f(\zeta_0))$ (Figure 13, right). Moreover, all the $f(\zeta_j) = \rho^j f(\zeta_0)$ lie on the half-line $e^{i(\pi - \theta^*)} \mathbb{R}^+$. By reasoning in a fundamental domain of the loxodromy $\Phi : u \mapsto \rho u$, it is then easy to see that U has the same volume as $D := D_1 \cup D_2$, where D_1, D_2 are ideal tetrahedra of vertices $(\infty, f(\zeta_1), 1, \rho)$ and $(\infty, f(\zeta_1), 1, f(\zeta_0))$ respectively. Moreover, Φ identifies the faces $(\infty, 1, f(\zeta_0))$ and $(\infty, \rho, f(\zeta_1))$ of D, so that D/Φ is a manifold with polyhedral boundary, homeomorphic to a solid torus, with interior dihedral angles

$$\left(\pi - w_1, \pi + w_0, \frac{w_1 - w_0}{2}, \frac{w_1 - w_0}{2}, \pi - \theta^* \right).$$

The edge of D/Φ with dihedral angle $\pi - \theta^*$ is a simple closed curve of length $\log \rho$, toward which D_1, D_2 spiral. A picture of D/Φ is obtained by replacing θ with θ^* in the left panel of Figure 12.

Using the smooth dependence on θ^* (and fixing w_0, w_1), the Schläfli volume formula then gives $\partial \mathcal{V}(D/\Phi)/\partial \theta^* = \frac{1}{2} \log \rho > 0$: so D/Φ (and therefore U) has largest volume when θ^* is largest. Regard the $\{w_j\}_{j \leq 1}$ as fixed, and the $\{w_j\}_{j \geq 2}$ as unknowns: then $(w_2, w_3, \dots)$ is clearly the solution to the maximization problem for the volume of U (with fixed pleating angles on S_1). Therefore, the w_i will choose the largest possible limit θ^* at $+\infty$, namely $\theta^* = \theta$. q.e.d.

The above proof does more than determining $\lim_{+\infty} w_j$: as in Sections 5-8, it gives a full description of $U = \bigcup_{j \geq 1} \Delta_j$ and of its boundary (whose pleating on the outer side turns out to be λ^+). Here is, however, an important observation:

OBSERVATION 10.4. Proposition 9.4 above, "$A + C > 0$", does not hold for the family of pleated surfaces $(S_i)_{i \geq 0}$. Instead, we have $A + C = 0$. This simply means that $\overrightarrow{-1, \zeta_i}$ and $\overrightarrow{\zeta_{i+1}, 1}$ make opposite angles with the real line. Indeed, $(\zeta_i + 1)(1 - \zeta_{i+1}) = 1 - \kappa^2$ is a positive real.

We can now establish

Proposition 10.5. *All the (strict) inequalities of (11) are true.*

Proof. This is Proposition 4.6 (in the new context where β^+/α^+ is rational). The proof is the same, with the following caveat: in ruling out $w_j = 0$ for j hinge, we call upon [**GF**] (especially Lemma 16 and the argument of Section 9 there). The strategy is to assume $w_j = 0$, and then perturb w to a well-chosen w^ε so as to make the volume increase: $\partial \mathcal{V}/\partial \varepsilon > 0$. The latter inequality holds essentially because the inequality of Proposition 9.4 ("$A + C > 0$") is true, both in the R^n-word preceding j and in the L^m-word following j. (Note: in Section 9 of [**GF**], the inequality "$A + C > 0$" is formulated in terms of lengths and takes the guise "$Q < P + T$".) More exactly, $\partial \mathcal{V}/\partial \varepsilon$ will be positive when at least one of the two instances of "$A + C \geq 0$" is strict. But it is always strict, except in a single case (the infinite suffix $LRR...R...$): so we can conclude. q.e.d.

As a result, all the tetrahedra Δ_i have positive angles and fit together correctly: Sections 5-8 carry through for the λ^--end, and $\bigcup_{i \in \mathbb{Z}} \Delta_i$ is the interior of the convex core of a quasifuchsian punctured-torus group, with the prescribed pleatings $\lambda^\pm$.

The results of Section 9 extend readily: the only modification is that tetrahedra $(\Delta_i)_{i \geq 0}$ lift to a family of *coplanar* cells in Minkowski space, because the key inequality of Proposition 9.4 has become an equality. Therefore the geometrically canonical decomposition of the interior of the convex core contains the non-contractible cell D/Φ (Figure 12, left).

10-B. Two rational pleatings. When both β^+/α^+ and β^-/α^- are rational, the encoding word $\Omega \in \{R, L\}^{\mathbb{Z}}$ can be chosen with an infinite prefix $...R...RRL$ and an infinite suffix $LRR...R...$ We apply the same argument as above to both ends simultaneously. Again, Proposition 10.5 holds because no hinge index j belongs both to the prefix and to the suffix.

If the rational pleating slopes $s(\lambda^\pm)$ are not Farey neighbors, we can convert the prefix to $...LLR$ and/or the suffix to $RLL...$, obtaining different triangulations of (the interior of) the convex core of the same quasifuchsian group.

If $s(\lambda^+), s(\lambda^-)$ are Farey neighbors, then $\Omega = ...RRLRR...$ (observe that prefix and suffix overlap, so the two word conversions do *not* commute). If the indices i before and after the central L are 0 and 1, we obtain $w_0 = w_1$ by applying Observation 10.4 to prefix and suffix. In other words, the points a, b, c in Figure 14 (left) are collinear. It is therefore possible to triangulate the same convex core according to a word $\Omega = ...LLLRRR...$, provided that we allow the hinge tetrahedron Δ_0 to become flat (black in Figure 14, right). In any case, the sequence $(w_i)_{i\in\mathbb{Z}}$ maximizes the total volume.
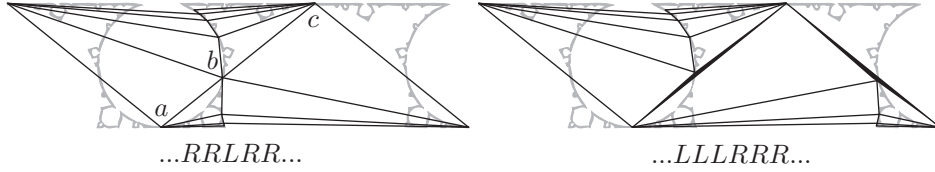

$...RRLRR...$ $...LLLRRR...$

Figure 14. Two triangulations seen against the same limit set.

10-C. Pinching. The case where one (or both) of the pleatings $\lambda^\pm$ has weight π is a straightforward limit case of Subsections 10-A and 10-B (the term "pinching" refers to the fact that the pleating curve becomes shorter and shorter, and eventually turns into a cusp as the pleating angle reaches π).

Suppose $|\lambda^+| = \pi$ (note that the conditions of (11) involving ϕ^+ become vacuous, because $\phi^+ \geq \pi$). Subsection 10-A carries through with $\theta = \pi$ and $\kappa = 0$. The circle arc C of Figure 13 becomes a full circle, tangent to $\mathbb{R}$ at 0. The analysis of Proposition 10.3 (existence and uniqueness of $\zeta_0, \zeta_1 \in \mathbb{C}$) extends smoothly to $\theta^* = \pi$. One then finds that the tetrahedron with vertices $(\infty, 1, \zeta_j, \zeta_{j+1})$ is sent by $f : u \mapsto 1/u$ to a tetrahedron Δ'_j of $\mathbb{H}^3$, isometric to Δ_j, with vertices $(0, 1, \tau + j, \tau + j + 1)$ for a certain $\tau \in \mathbb{C}$ independent of j (in the right panel of Figure 13, the grey bounding rays are replaced by parallel lines). It is straightforward to check that $\bigcup_{j\geq 1} \Delta_j$ is the solid torus pictured in Figure 12 (right), and the Schläfli formula again implies that $\theta^* = \pi$ realizes the maximum volume.

After Proposition 10.3, the argument is unchanged.

10-D. Infinite ends. By [**Min**], quasifuchsian groups are dense in the set of all discrete, faithful, type-preserving representations of $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$. In fact, a geometrically infinite end of such a representation comes with an *ending lamination*, namely an irrational projective measured lamination which should be thought of as an "infinitely strong pleating". In this section we will assume $\beta^+/\alpha^+ \notin \mathbb{P}^1\mathbb{Q}$ and consider w^T, the solution of the volume maximization problem for $(\phi^-, T\phi^+)$, where $T > 0$ (namely, w is subject to conditions (11), where ϕ^+ is replaced by $T\phi^+$; and Table (3) still expresses x_i, y_i, z_i).

In a discrete, type-preserving representation of a surface group, each compact measured lamination λ on the surface receives a *length*, which can be

computed by measuring the (weighted) lengths of weighted curves converging to λ. It is known [**Br**] that the lengths of the pleating laminations of a quasi-fuchsian group are bounded by a constant depending only on the underlying surface (here the punctured torus). By Thurston's double limit theorem (Theorems 4.1 and 6.3 of [**Th**]; see also [**Ot**]), the space of discrete, type-preserving representations in which two fixed measured laminations have length bounded by a given constant is *compact*. Therefore, up to taking a subsequence, the groups Γ^T corresponding to w^T converge algebraically to a certain Γ. By Theorem A of [**Min**], $\mathbb{H}^3/\Gamma$ is homeomorphic to $S \times \mathbb{R}$ and must have an infinite end (otherwise, Γ would be quasifuchsian and the volumes would stay bounded).

Proposition 10.6. *The λ^--end of $\mathbb{H}^3/\Gamma$ is finite, with pleating lamination λ^-. The λ^+-end is infinite, and its ending lamination is the projective class of λ^+.*

Proof. By [**B2**] (Theorem D), the space of type-preserving representations of the abstract group Γ is smoothly (in fact, holomorphically) parametrized by the data (τ, ω) of a point τ of Teichmüller space $\mathcal{T}$, and a *transverse $\mathbb{R}/2\pi\mathbb{Z}$-valued cocycle* relative to a fixed topological lamination μ (such cocycles include pleating measures as special cases). Taking for μ the support of λ^-, we see that the moduli of the λ^--boundaries of the convex cores of the Γ^T must converge in $\mathcal{T}$. Therefore, $\mathbb{H}^3/\Gamma$ contains a locally convex pleated surface H with pleating λ^-, which must be a boundary component of the convex core (∂H contains all parabolic fixed points).

The λ^+-end of $\mathbb{H}^3/\Gamma$ must therefore be infinite. The parabolic fixed points of the limit group Γ determine a version of Figure 4 (a Euclidean cusp link), and therefore a family of *non-negative* angle assignments for the tetrahedra Δ_i. By algebraic convergence, the w^T converge to w in the product topology. For any $i \leq j$, the total volume of $\Delta_{i-1}, \Delta_i, \ldots, \Delta_j, \Delta_{j+1}$ is maximal with respect to $w_i, \ldots, w_j$: in particular, Propositions 4.4 and 4.5 are still true. The techniques of [**GF**] (Lemma 16 and Section 9 there) show that all Δ_i have positive angles, and Proposition 9.4, hence also Lemma 9.2, still hold: $\{\Delta_i\}_{i \in \mathbb{Z}}$ is the geometrically canonical decomposition of $\mathbb{H}^3/\Gamma$. In particular, the family of all edges of all tetrahedra $\{\Delta_i\}_{i \geq 0}$ forms a sequence of laminations which exits $\mathbb{H}^3/\Gamma$: therefore the projective class of λ^+ is the end invariant.   q.e.d.

The case of two infinite ends is already treated in [**Ak**]. Theorem 10.1 is proved.

10-E. The Pleating Lamination Theorem for punctured-torus groups.

Proposition 10.7. *The group Γ constructed at the end of Section 4 is continuously parametrized by (λ^+, λ^-).*

Proof. Our first observation is that if β^+/α^+ is rational, the initial choice of infinite prefix/suffix in Subsection 10-A does not change the resulting group Γ: it just induces different triangulations of the toric piece of Figure 12, whose deformation space is still the same.

Define the open set
$$\mathcal{U} := \mathbb{R}^2 \smallsetminus \bigcup_{m,n \in \mathbb{Z}} [\pi, +\infty) \cdot \{(m,n)\}$$

(note that $0 \notin \mathcal{U}$). An admissible pleating lamination λ can be identified with an element $\pm(\alpha, \beta)$ of $\mathcal{U}/\pm$. Suppose $(\alpha_n^+, \beta_n^+) \rightarrow (\alpha^+, \beta^+)$ and $(\alpha_n^-, \beta_n^-) \rightarrow (\alpha^-, \beta^-)$ in $\mathcal{U}$, and define the oriented line Λ_n from β_n^-/α_n^- to β_n^+/α_n^+ across the Farey diagram. Also define the associated functions $\phi^{\pm,n}$ as in (9), domains W^n as in Definition 3.4, and solutions $(w_i^n)_{i \in \mathbb{Z}}$ to the volume maximization problem over W^n. If β^+/α^+ is rational, we may assume (up to restricting to two subsequences) that the (α_n^+, β_n^+) converge to (α^+, β^+) in the clockwise direction for the natural orientation of $\mathbb{P}^1\mathbb{R}$. We also make a similar assumption for β^-/α^-.

A priori, the sequences $(w_i^n)_{i \in \mathbb{Z}}$ are defined only up to a shift of the index i. However, we can choose these shifts in a consistent way: there exists a Farey edge e which is crossed by all the lines Λ_n for n large enough, so we decide that w_0^n always lives on e (namely, $e_0^n = e$). By compactness of $[0, \pi]^{\mathbb{Z}}$, some subsequence of $(w^n)_{n \in \mathbb{N}}$ converges to some w^* in the product topology. It is enough to show that $w^* = w$: indeed, the group Γ is completely determined by the shapes of a finite number of tetrahedra Δ_i.

The main observation is that the words $\Omega^n \in \{R, L\}^{\mathbb{Z}}$ converge pointwise to Ω, and $\phi^{\pm,n} \rightarrow \phi^{\pm}$ (pointwise in $\mathbb{R}^{\mathbb{Z}}$), by definition (9). Therefore, w^* belongs to the space W, hence the volume inequality $\mathcal{V}(w^*) \leq \mathcal{V}(w)$. Since $\max_W \mathcal{V}$ is achieved at a unique point (the volume is a strictly concave function), it is enough to prove the reverse inequality. We proceed by contradiction.

Suppose $\mathcal{V}(w^*) < \mathcal{V}(w)$. There exist integers $m < 0 < M$ such that the tetrahedra $\{\Delta_i\}_{m < i < M}$ defined by w have total volume larger than $\mathcal{V}(w^*)$. If we can extend $(w_i)_{m \leq i \leq M}$ to a sequence $(v_i)_{i \in \mathbb{Z}}$ of W^n for large enough n, we will obtain a contradiction.

By Corollary 5.2, we can assume $\frac{w_m}{\phi_m^-} > \frac{w_{m+1}}{\phi_{m+1}^-}$ and $\frac{w_M}{\phi_M^-} > \frac{w_{M-1}}{\phi_{M-1}^-}$. Since w satisfies (11) (strong inequalities), for n large enough, the restricted sequence $(w_m, \ldots, w_M)$ satisfies the corresponding inequalities defining W^n, by convergence of the $\phi^{\pm,n}$. Pick such a large n and define

$$v_i := \begin{cases} \phi_i^{-,n} \frac{w_m}{\phi_m^{-,n}} & \text{if } i \leq m \text{ ;} \\ w_i & \text{if } m \leq i \leq M \text{ ;} \\ \phi_i^{+,n} \frac{w_M}{\phi_M^{+,n}} & \text{if } M \leq i \text{ .} \end{cases}$$

If n is large enough, it is straightforward to check that $(v_i)_{i \in \mathbb{Z}}$ belongs to W^n.

<div align="right">q.e.d.</div>

Corollary 10.8. (C. Series) *A quasifuchsian punctured-torus group Γ is determined up to conjugacy in $\mathrm{Isom}(\mathbb{H}^3)$ by its pleating measures $\lambda^{\pm}$.*

Proof. We will use the well-known fact that the space $\mathcal{QF}$ of quasifuchsian, non-fuchsian (punctured-torus) groups is a connected real manifold of dimension 4. Recall the open set $\mathcal{U}$ from the proof of Proposition 10.7, and consider the map

$$f : \mathcal{U} \times \mathcal{U} \longrightarrow \mathcal{QF}$$

defined by the construction of the group Γ (end of Section 4). We know that f is well-defined and injective (Theorem 1.1), and continuous (Proposition 10.7). Since $\mathcal{U}^2$ has dimension 4, the theorem of domain invariance states that the image $\mathrm{Im}(f)$ is open. It remains to show that $\mathrm{Im}(f)$ is closed.

Consider pairs $(\lambda_n^+, \lambda_n^-)$ such that the corresponding groups $\Gamma_n = f(\lambda_n^+, \lambda_n^-)$ converge to some Γ in $\mathcal{QF}$. The function which to a group associates its pleatings is continuous (see [**KS**]), so the $\lambda_n^\pm$ converge to some $\lambda^\pm$ in $\mathcal{U}$. Proposition 10.7 then implies that $\Gamma = f(\lambda^+, \lambda^-)$.   q.e.d.

Theorem 0.1 now follows from Theorem 10.1.

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