ANOSOV REPRESENTATIONS: DOMAINS OF DISCONTINUITY AND APPLICATIONS

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ABSTRACT. The notion of Anosov representations has been introduced by Labourie in his study of the Hitchin component for $\mathrm{SL}(n, \mathbf{R})$. Subsequently, Anosov representations have been studied mainly for surface groups, in particular in the context of higher Teichmüller spaces, and for lattices in $\mathrm{SO}(1, n)$. In this article we extend the notion of Anosov representations to representations of arbitrary word hyperbolic groups and start the systematic study of their geometric properties. In particular, given an Anosov representation $\Gamma \to G$ we explicitly construct open subsets of compact G-spaces, on which Γ acts properly discontinuously and with compact quotient.

As a consequence we show that higher Teichmüller spaces parametrize locally homogeneous geometric structures on compact manifolds. We also obtain applications regarding (non-standard) compact Clifford-Klein forms and compactifications of locally symmetric spaces of infinite volume.

CONTENTS

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1. Introduction

The concept of Anosov structures has been introduced by Labourie [57], and is, in some sense, a dynamical analogue of locally homogeneous geometric structures for manifolds endowed with an Anosov flow. Holonomy representations of Anosov structures are called Anosov representations. They are defined for representations into any real semisimple Lie group G. Anosov representations have several important properties. For example, the set of Anosov representations is open in the representation variety, and any Anosov representation is a quasi-isometric embedding. The notion of Anosov representations has turned out to be very useful in the study of surface group representations, in particular in the study of higher Teichmüller spaces [2, 18, 22, 41, 57]. Here we put the concept of Anosov representations into the broader context of finitely generated word hyperbolic groups. In this framework Anosov representations are generalizations of convex cocompact subgroups of rank one Lie groups to the context of discrete subgroups of Lie groups of higher rank.

Let us recall that given a real semisimple Lie group G, a discrete subgroup $\Lambda < G$ is said to be convex cocompact if there exists a Λ-invariant convex subset in the symmetric space G/K, on which Λ acts properly discontinuously with compact quotient. A representation $\rho : \Gamma \to G$ of a group Γ into G is said to be convex cocompact if ρ has finite kernel, and $\rho(\Gamma)$ is a convex cocompact subgroup of G.

When G is of real rank one, i.e. when G/K has negative sectional curvature, the set of convex cocompact representations is an open subset of the representation variety $\mathrm{Hom}(\Gamma, G)/G$. In that case, being convex cocompact is equivalent to being a quasi-isometric embedding [14] (see Theorem 1.8). This implies in particular that Γ is a word hyperbolic group. Important examples of convex cocompact representations are discrete embeddings of surface groups into $\mathrm{PSL}(2, \mathbf{R})$, quasi-Fuchsian representations of surface groups into $\mathrm{PSL}(2, \mathbf{C})$, embeddings of free groups as Schottky groups, or embeddings of uniform lattices. Anosov representations into Lie groups of rank one are exactly convex cocompact representations (see Theorem 1.8)

For Lie groups G of real rank ≥ 2, a rigidity result of Kleiner-Leeb [52] and Quint [65] says that the class of convex cocompact subgroups of G reduces to products of uniform lattices and of convex cocompact subgroups in rank one Lie groups. On the other hand, there is an abundance of examples of Anosov representations of surface groups, uniform lattices in $\mathrm{SO}(1, n)$ or hyperbolic groups into Lie groups of higher rank.

In the first part of the article we extend the definition of Anosov representations to representations of arbitrary finitely generated word hyperbolic groups Γ (into semisimple Lie groups) and establish their basic properties. In the second part we develop a quite intriguing geometric picture for Anosov representations. In particular, given an Anosov representation $\rho : \Gamma \to G$ we construct a Γ-invariant open subset of a compact G-space, on which Γ acts properly discontinuously with compact quotient. In the third part we discuss several applications of this construction, in particular to higher Teichmüller spaces, to the existence of non-standard Clifford-Klein forms of homogeneous spaces, and to compactifications of locally symmetric spaces of infinite volume.

We now describe the contents of the three parts in more detail.

1.1. **Anosov representations.** We quickly mention the definition of Anosov representations in the context of manifolds given in [57] (see Section 2 below for details).

Let G be a semisimple Lie group and (P^+, P^-) a pair of opposite parabolic subgroups of G. Let M be a compact manifold equipped with an Anosov flow. A representation $\rho : \pi_1(M) \to G$ is said to be a (P^+, P^-)-Anosov representation if the associated $G/(P^+ \cap P^-)$-bundle over M admits a section σ which is constant along the flow and with certain contraction properties.

When N is a negatively curved compact Riemannian manifold, and $M = T^1 N$ is its unit tangent bundle equipped with the geodesic flow ϕ_t, there is a natural projection $\pi : \pi_1(M) \to \pi_1(N)$; a representation $\rho : \pi_1(N) \to G$ is said to be (P^+, P^-)-Anosov, if $\rho \circ \pi$ is (P^+, P^-)-Anosov.

In order to extend the notion of Anosov representations to arbitrary finitely generated word hyperbolic groups Γ, we replace $T^1\widetilde{N}$ by the flow space $\widehat{\Gamma}$, introduced by Gromov [37] and developed by Champetier [24], Mineyev [60] and others. The flow space $\widehat{\Gamma}$ is a proper hyperbolic metric space with an action of $\Gamma \times \mathbf{R} \rtimes \mathbf{Z}/2\mathbf{Z}$, where the $\mathbf{R}$-action (the flow) is free and such that $\widehat{\Gamma}/\mathbf{R}$ is naturally homeomorphic to $\partial_\infty \Gamma^{(2)}$. Here $\partial_\infty \Gamma^{(2)}$ denotes the space of distinct points in the boundary at infinity $\partial_\infty \Gamma$ of Γ. In analogy to the above, a representation $\rho : \Gamma \to G$ is said to be (P^+, P^-)-Anosov if the associated $G/(P^+ \cap P^-)$-bundle over $\Gamma\backslash\widehat{\Gamma}$ admits a section σ which is constant along $\mathbf{R}$-orbits and with certain contraction properties.

The conjugacy class of a pair of opposite parabolic subgroups (P^+, P^-) is completely determined by P^+. We thus say that a representation is P^+-Anosov if it is (P^+, P^-)-Anosov. We say that a representation is Anosov if it is P^+-Anosov for some proper parabolic subgroup $P^+ < G$.

Examples 1.1.

– Let G be a split real simple Lie group and let Σ be a closed connected orientable surface of genus ≥ 2. Representations $\rho : \pi_1(\Sigma) \to G$ in the Hitchin component are B-Anosov, where $B < G$ is a Borel subgroup.

– The holonomy representation $\rho : \pi_1(M) \to \mathrm{PGL}(n+1, \mathbf{R})$ of a convex real projective structure on an n-dimensional orbifold M is P-Anosov, where $P < \mathrm{PGL}(n+1, \mathbf{R})$ is the stabilizer of a line.

For more details, see Section 6.

We extend Labourie's result on the stability of Anosov representations to the more general framework of word hyperbolic groups:

Theorem 1.2. *(Theorem 5.13) Let Γ be a finitely generated word hyperbolic group. Let G be a semisimple Lie group and $P^+ < G$ be a parabolic subgroup. The set of P^+-Anosov representations is open in* $\mathrm{Hom}(\Gamma, G)$.

An immediate consequence of a representation $\rho : \Gamma \to G$ being P^+-Anosov is the existence of continuous ρ-equivariant maps $\xi^+ : \partial_\infty \Gamma \to G/P^+$ and $\xi^- : \partial_\infty \Gamma \to G/P^-$. We call these maps *Anosov maps.* They have the following properties:

(i) for all $(t, t') \in \partial_\infty \Gamma^{(2)}$, the pair $(\xi^+(t), \xi^-(t'))$ is in the (unique) open G-orbit in $G/P^+ \times G/P^-$.

(ii) for all $t \in \partial_\infty \Gamma$, the pair $(\xi^+(t), \xi^-(t))$ is contained in the (unique) closed G-orbit in $G/P^+ \times G/P^-$.
(iii) they satisfy some contraction property with respect to the flow.

A representation admitting continuous equivariant maps ξ^+, ξ^- with the above properties is easily seen to be P^+-Anosov. A consequence of this characterization of Anosov representations in terms of equivariant maps is

Corollary 1.3. *Let Γ be a finitely generated word hyperbolic group, and let $\Gamma' < \Gamma$ be of finite index. Then a representation $\rho : \Gamma \to G$ is P^+-Anosov if and only if the restriction of ρ to Γ' is P^+-Anosov.*

A corollary of this and [18] is (see also Remark 6.7)

Example 1.4 ([20]). Let G be a classical Lie group of Hermitian type, and let $\rho : \pi_1(\Sigma) \to G$ be a maximal representation, then ρ is P-Anosov, where P is the stabilizer of a point in the Shilov boundary of the bounded symmetric domain associated to G.

A pair of maps ξ^+, ξ^- satisfying the first two of the above properties is said to be *compatible*. We show that for a generic representation the contraction property involving the flow is satisfied by any pair of equivariant continuous compatible maps.

Theorem 1.5. *(Theorem 4.11) Let Γ be a finitely generated word hyperbolic group. Let G be a semisimple Lie group and $P^+ < G$ a parabolic subgroup. Let $\rho : \Gamma \to G$ be a Zariski dense representation. Suppose that ρ admits a pair of equivariant continuous compatible maps $\xi^+ : \partial_\infty \Gamma \to G/P^+$, $\xi^- : \partial_\infty \Gamma \to G/P^-$.*

Then the representation ρ is P^+-Anosov and (ξ^+, ξ^-) are the associated Anosov maps.

Note that the statement of Theorem 1.5 does not involve the flow space of the hyperbolic group, but only its boundary at infinity.

A consequence of Theorem 1.5 is that any representation admitting a pair of equivariant continuous compatible maps is Anosov as a representation into the Zariski closure of its image. This leads to the problem of determining when the composition of an Anosov representation with a Lie group homomorphism is still Anosov.

More precisely, if $\phi : G \to G'$ is an embedding and $\rho : \Gamma \to G$ is a P^+-Anosov representation, when is the composition $\phi \circ \rho : \Gamma \to G'$ an Anosov representation and with respect to which parabolic subgroup $P'^+ < G'$?

When G is a Lie group of rank one, an answer has been given by Labourie [57] (see also Proposition 4.7 below). We provide an answer to this question for general semisimple Lie groups G in Section 4.1. Here we just note one consequence, which plays an important role for the construction of domains of discontinuity.

Proposition 1.6. *A representation $\rho : \Gamma \to G$ is Anosov if and only if there exists a real vector space V with a non-degenerate indefinite quadratic form F and a homomorphism $\phi : G \to \mathrm{O}(F)$ such that $\phi \circ \rho$ is Q_0-Anosov, where $Q_0 < \mathrm{O}(F)$ is the stabilizer of an F-isotropic line in V.*

We introduce so called L-Cartan projections as a new tool to study Anosov representations (see Section 3.3 for details). These are continuous maps from $\Gamma \backslash \widehat{\Gamma} \times \mathbf{R}$ with values in the closure of a Weyl chamber of $L = P^+ \cap P^-$, which are defined whenever a representation ρ admits a continuous section σ of the G/L-bundle over $\Gamma\backslash\widehat{\Gamma}$ that is flat along $\mathbf{R}$-orbits. The L-Cartan projections control the contraction properties of the section σ. They play a central role in the discussion in Section 4.1.

We now turn our attention to the geometric properties of Anosov representations. The reader is referred to Sections 5.1 and 5.2 for definitions and details.

Theorem 1.7. *(Theorems 5.3 and 5.9) Let Γ be a finitely generated word hyperbolic group, G a real semisimple Lie group and $\rho : \Gamma \to G$ an Anosov representation. Then*

(i) the kernel of ρ is finite and the image of ρ is discrete.
(ii) the map $\rho : \Gamma \to G$ is a quasi-isometric embedding, with respect to the word-metric on Γ and any (left) G-invariant Riemannian metric on G.
(iii) the representation ρ is well-displacing.
(iv) the representation ρ is (AMS)-proximal (Definition 5.7).

As a consequence of this and [14] we obtain the following characterization of Anosov representations in Lie groups of rank one:

Theorem 1.8. *(Theorem 5.15) Let Γ be a finitely generated word hyperbolic group and G a real semisimple Lie group with $\mathrm{rk}_{\mathbf{R}} G = 1$. For a representation $\rho : \Gamma \to G$ the following statements are equivalent:*

(i) ρ is Anosov.
(ii) There exists a continuous, ρ-equivariant, and injective map $\xi : \partial_\infty\Gamma \to G/P$.
(iii) $\rho : \Gamma \to G$ is a quasi-isometric embedding, with respect to the word metric on Γ and any (left) G-invariant Riemannian metric on G.
(iv) ρ is convex cocompact.

1.2. **Domains of discontinuity.** The heart of this article is to construct, given an Anosov representation $\rho : \Gamma \to G$, an open subset Ω of a compact G-space, which is Γ-invariant, and on which Γ acts properly discontinuously and with compact quotient.

Let us recall that any real semisimple Lie group admits an Iwasa decomposition $G = KAN$, where K is a maximal compact subgroup and A the $\mathbf{R}$-split part of a Cartan subgroup and N is the unipotent radical of a minimal parabolic subgroup B containing A.

Theorem 1.9. *(Theorem 9.4) Let $\rho : \Gamma \to G$ be a P-Anosov representation for some proper parabolic subgroup $P < G$. Then there exists a Γ-invariant open set $\Omega \subset G/AN$ such that*

(i) the action of Γ on Ω is properly discontinuous, and
(ii) the quotient $\Gamma\backslash\Omega$ is compact.

Remarks 1.10.

(i) The domain of discontinuity Ω is constructed explicitly using the Anosov maps associated to ρ, and depends on some additional combinatorial data. For specific examples the domain of discontinuity Ω might be empty. One such example is an Anosov representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{R})$, where Σ is a closed connected oriented surface of genus ≥ 2. Then $\pi_1(\Sigma)$ acts minimally on $\mathrm{PSL}(2, \mathbf{R})/AN = \mathbb{P}^1(\mathbf{R})$. See Remark 8.5 for other examples.

We describe conditions for Ω to be nonempty in Section 9.2, see also Corollary 10.5, Theorem 1.11 and Theorem 1.12 below. The domain of discontinuity can always be ensured to be nonempty by embedding G into a larger Lie group G'.

(ii) The domain of discontinuity Ω is not unique. For Anosov representations into $\mathrm{SL}(n, \mathbf{K})$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, we describe in Section 10 several different domains of discontinuities in G/AN.

(iii) Recall that a minimal parabolic subgroup $B < G$ admits a decomposition as $B = MAN$ with M being the centralizer of A in K. In particular, the compact G-space G/AN is a compact fiber bundle over G/B and hence over G/Q for any parabolic subgroup $Q < G$. In many cases the domain of discontinuity Ω is indeed the pull back of a domain of discontinuity in some G/Q.

For free groups and for surface groups we obtain

Theorem 1.11. *Let F_n be the free group on n letters, and let G be a real semisimple Lie group. Assume that $\rho : F_n \to G$ is an Anosov representation. Then there exists a nonempty F_n-invariant open subset $\Omega \subset G/AN$ such that the action of F_n on Ω is properly discontinuous and cocompact.*

Theorem 1.12. *Let $\pi_1(\Sigma)$ be the fundamental group of a closed connected orientable surface of genus ≥ 2, and let G be a real semisimple Lie group with no (almost) factor being locally isomorphic to $\mathrm{SL}(2, \mathbf{R})$. Assume that $\rho : \pi_1(\Sigma) \to G$ is an Anosov representation. Then there exists a nonempty $\pi_1(\Sigma)$-invariant open subset $\Omega \subset G/AN$ such that the action of $\pi_1(\Sigma)$ on Ω is properly discontinuous and cocompact.*

Remarks 1.13.

(i) In view of Remark 1.10.(i) the condition on G in Theorem 1.12 is optimal.
(ii) Theorem 1.12 was announced in [40] in the form that $\Omega \subset G/B$. This is true in many cases, but in general our construction provides only a domain of discontinuity in G/AN (which is a compact fiber bundle over G/B).
(iii) Theorem 1.11 holds more generally for hyperbolic groups whose virtual cohomological dimension is one, respectively Theorem 1.12 holds for hyperbolic groups whose virtual cohomological dimension is two, see Theorem 9.10.

We shortly describe the general strategy for the construction of the domain of discontinuity Ω.

A basic example is when $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{C})$ is a quasi-Fuchsian representation. Then, there is an equivariant continuous curve $\xi : \partial_\infty \pi_1(\Sigma) \cong S^1 \to \mathbb{P}^1(\mathbf{C})$, whose image is a Jordan curve. The action of $\pi_1(\Sigma)$ on the complement $\Omega = \mathbb{P}^1(\mathbf{C}) \smallsetminus \xi(S^1)$ is free and properly discontinuous. The quotient $\pi_1(\Sigma)\backslash\Omega$ has two connected components, both of which are homeomorphic to the surface Σ.

When $\rho : \Gamma \to G$ is an Anosov representation into a Lie group G of rank one, the construction is a straightforward generalization of this procedure. We consider the equivariant continuous Anosov map $\xi : \partial_\infty(\Gamma) \to G/P$, where P is (up to conjugation) the unique proper parabolic subgroup of G. In that case $\Omega = G/P \smallsetminus \xi(\partial_\infty \Gamma)$.

In general, when G is a Lie group of higher rank, the situation becomes more complicated. We will have to consider the Anosov map $\xi : \partial_\infty(\Gamma) \to G/P$ in order to construct a domain of discontinuity in G/P' with $P' < G$ being a different parabolic subgroup.

We start first with the case when $\rho : \Gamma \to \mathrm{O}(F)$ is a Q_0-Anosov representation, where Q_0 is the stabilizer of an F-isotropic line in V. Let $\mathcal{F}_0 = G/Q_0$ be the space of F-isotropic lines in V, let $\mathcal{F}_1 = G/Q_1$ be the space of maximal F-isotropic subspaces of V, and let $\mathcal{F}_{01}$ be the space of pairs consisting of an F-isotropic line and an incident maximal F-isotropic space. There are two projections $\pi_i : \mathcal{F}_{01} \to \mathcal{F}_i$, $i = 0, 1$. Starting from the Anosov map $\xi : \partial_\infty \Gamma \to \mathcal{F}_0$, we consider the subset $K_\xi := \pi_1(\pi_0^{-1}(\xi(\partial_\infty \Gamma))) \subset \mathcal{F}_1$ and define $\Omega_\rho = \mathcal{F}_1 \smallsetminus K_\xi$.

That the action of Γ on Ω_ρ is properly discontinuous follows from the fact that $\rho(\Gamma)$ has special dynamical properties. A more general construction of domains of discontinuity for discrete subgroups of $\mathrm{O}(F)$ with special dynamical behavior is described in Theorem 7.4. We deduce the compactness of the quotient $\Gamma \backslash \Omega_\rho$ from computations in homology.

Given an Anosov representation $\rho : \Gamma \to G$ into an arbitrary semisimple Lie group, the first step is to choose an appropriate finite dimensional (irreducible) representation of G on a real vector space V preserving a non-degenerate indefinite symmetric bilinear form F, such that the composition of ρ with the representation $\phi : G \to \mathrm{O}(F)$ is a Q_0-Anosov representation $\phi \circ \rho : \Gamma \to \mathrm{O}(F)$; this is made possible by Proposition 1.6. Let $\Omega_{\phi\circ\rho} \subset \mathcal{F}_1$ denote the domain of discontinuity whose construction we just described. We show that there is always a maximal isotropic subspace $W \in \Omega_{\phi\circ\rho} \subset \mathcal{F}_1$ which is invariant by AN. The intersection of $\Omega_{\phi\circ\rho}$ with the G-orbit of W is a domain of discontinuity $\Omega' \subset G/\mathrm{Stab}_G(W)$, which can be lifted to obtain a domain of discontinuity $\Omega \subset G/AN$.

1.3. **Applications.**

1.3.1. *Hitchin components, maximal representations and deformation spaces of geometric structures.* By the uniformization theorem, the Teichmüller space of a surface Σ can be identified with the moduli space of marked hyperbolic structures on Σ, and consequently with a connected component of the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbf{R}))/\mathrm{PSL}(2,\mathbf{R})$. In 1992 Hitchin defined the Teichmüller component, now called the Hitchin component, a connected component of the space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ of representations into a split real adjoint simple Lie group, which he proved to be homeomorphic to a ball [46]. Choi and Goldman [25] showed that the Hitchin component for $\mathrm{PSL}(3,\mathbf{R})$ can be realized as the moduli space of convex real projective structures on Σ.

Hitchin's work, using methods from the theory of Higgs bundles, does not provide much information on the geometric significance of representations in the Hitchin component. In recent years, due to work of Labourie [57, 58, 59] and of Fock and Goncharov [29], it has been shown that representations in Hitchin components have beautiful geometric properties, which generalize properties of classical Teichmüller space. Parallel to this, the study of maximal representations of $\pi_1(\Sigma)$ into Lie groups of Hermitian type [16, 17, 18, 21, 32, 36, 45, 73] showed that spaces of maximal representations also share several properties with classical Teichmüller space, which itself is identified with the space of maximal representations into $\mathrm{PSL}(2,\mathbf{R})$ [34]. Therefore, Hitchin components and spaces of maximal representations are also referred to as higher Teichmüller spaces.

Using the construction of domains of discontinuity for Anosov representations we can show that Hitchin components parametrize connected components of deformation spaces of locally homogeneous (G, X)-structures on compact manifolds.

Theorem 1.14. *(Theorem 11.5) Let Σ be a closed connected orientable surface of negative Euler characteristic. Assume that G is $\mathrm{PSL}(2n, \mathbf{R})$ $(n \geq 2)$, $\mathrm{PSp}(2n, \mathbf{R})$ $(n \geq 2)$, or $\mathrm{PSO}(n, n)$ $(n \geq 3)$, and $X = \mathbb{P}^{2n-1}(\mathbf{R})$; or that G is $\mathrm{PSL}(2n+1, \mathbf{R})$ $(n \geq 1)$, or $\mathrm{PSO}(n, n+1)$ $(n \geq 2)$, and $X = \mathcal{F}_{1,2n}(\mathbf{R}^{2n+1}) = \{(D, H) \in \mathbb{P}^{2n}(\mathbf{R}) \times \mathbb{P}^{2n}(\mathbf{R})^* \mid D \subset H\}$.*

Then there exist a compact manifold M and a connected component $\mathcal{D}$ of the deformation space of (G, X)-structures on M which is parametrized by the Hitchin component in $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

Remarks 1.15. The Lie groups given in Theorem 1.14 comprise (up to local isomorphism) all classical split real simple Lie groups. An analogous statement holds also for exceptional split real simple Lie groups with $X = G/B$, where $B < G$ is the Borel subgroup.

We expect M to have the homeomorphism type of a bundle over Σ with compact fibers. In the case when $G = \mathrm{PSL}(2n, \mathbf{R})$ $(n \geq 2)$ or $\mathrm{PSp}(2n, \mathbf{R})$ $(n \geq 2)$ we prove in [42] that M is homeomorphic to the total space of an $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle over Σ. For $\mathrm{PSL}(4, \mathbf{R})$ we refer the reader to [39]. The known topological relation between M and Σ is the existence of a homomorphism $\pi_1(M) \to \pi_1(\Sigma)$; this homomorphism is in fact used to relate the component $\mathcal{D}$ to the Hitchin component, we refer to Theorem 11.5 for a precise statement.

We also associate locally homogeneous (G, X)- structures on compact manifolds to all maximal representations. We state the result in the case of the symplectic group.

Theorem 1.16. *(Theorem 11.6) Let Σ be a closed connected orientable surface of genus ≥ 2.*

Then for any connected component component $\mathcal{C}$ of the space of maximal representations, there exists a compact manifold M and a connected component $\mathcal{D}$ of the deformation space of $(\mathrm{Sp}(2n, \mathbf{R}), \mathbb{P}^{2n-1}(\mathbf{R}))$-structures on M, which is parametrized by a Galois cover of $\mathcal{C}$.

Remarks 1.17. The components of the space of maximal representations can have nontrivial fundamental group (see Section 11.5 for details).

There are similar statements for components of the space of maximal representations of $\pi_1(\Sigma)$ into other Lie groups G of Hermitian type, with

- $G = \mathrm{SO}(2, n)$, $X = \mathcal{F}_1(\mathbf{R}^{2+n})$ the space of isotropic 2-planes.
- $G = \mathrm{SU}(p, q)$, $X \subset \mathbb{P}^{p+q-1}(\mathbf{C})$ is the null cone for the Hermitian form.
- $G = \mathrm{SO}^*(2n)$, X is the null cone for the skew-Hermitian form.

Recall that $\mathrm{SO}^*(2n)$ can be realized as the automorphism group of a skew-Hermitian form $V \times V \to \mathbf{H}$ on an n-dimensional right $\mathbf{H}$-vector space V.

1.3.2. *Compactifications of actions on symmetric spaces.* The construction of domains of discontinuity we give is very flexible as it applies to Anosov representations into arbitrary semisimple Lie groups, in particular into complex groups. Using this flexibility one can apply the construction to obtain natural compactifications of non-compact quotients of symmetric spaces or of other homogeneous spaces. We illustrate this in the case when $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbf{R})$, $n \geq 2$, is a Q_0-Anosov representation, where Q_0 is the stabilizer of a line in $\mathbf{R}^{2n}$.

Let $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$ denote the bounded symmetric domain associated to $\mathrm{Sp}(2n, \mathbf{R})$. Since $\rho(\Gamma)$ is discrete, the action of Γ on $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$ is properly discontinuous. The quotient $M = \Gamma \backslash \mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$ is not compact.

Theorem 1.18. *(Theorem 12.1) Let $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbf{R})$ be a Q_0-Anosov representation. Then there exists a compactification $\overline{M}$ of $M = \Gamma \backslash \mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$ such that $\overline{M}$ carries a $(\mathrm{Sp}(2n, \mathbf{R}), \overline{\mathcal{H}}_{\mathrm{Sp}(2n,\mathbf{R})})$-structure, where $\overline{\mathcal{H}}_{\mathrm{Sp}(2n,\mathbf{R})}$ is the bounded symmetric domain compactification, and the inclusion $M \subset \overline{M}$ is an $\mathrm{Sp}(2n, \mathbf{R})$-map.*

The proof of this theorem relies on the fact that $\Gamma \to \mathrm{Sp}(2n, \mathbf{R}) \to \mathrm{Sp}(2n, \mathbf{C})$ is a Q_0-Anosov representation, where Q_0 is the stabilizer of a line in $\mathbf{C}^{2n}$. With this, one constructs a domain of discontinuity Ω in the space of complex Lagrangians $\mathcal{L}(\mathbf{C}^{2n})$, which contains the image of $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$ under the Borel embedding into $\mathcal{L}(\mathbf{C}^{2n})$, then $\overline{M} = \Gamma \backslash (\Omega \cap \overline{\mathcal{H}}_{\mathrm{Sp}(2n,\mathbf{R})})$.

1.3.3. *Compact Clifford-Klein forms.* As we already mentioned, for some Anosov representations $\rho : \Gamma \to G$ the domain of discontinuity turns out to be empty. Nevertheless embedding G into a bigger group G' one can obtain a nonempty domain of discontinuity Ω for $\rho' : \Gamma \to G \to G'$. In some cases this nonempty domain of discontinuity coincides with a G-orbit, $\Omega = G/H$. Then $\Gamma \backslash \Omega = \Gamma \backslash G/H$ is a compact Clifford-Klein form.

With this we recover several examples of compact Clifford-Klein forms (see Proposition 13.1), including non-standard ones. In particular, using results of Barbot [3] we prove the following.

Theorem 1.19. *(Theorem 13.3) Let $\Gamma < \mathrm{SO}(1, 2n)$ be a cocompact lattice, let $\mathrm{SO}(1, 2n) < \mathrm{SO}(2, 2n)$ be the standard embedding, and consider $\rho : \Gamma \to \mathrm{SO}(1, 2n) < \mathrm{SO}(2, 2n)$. Then any representation ρ' in the connected component of ρ in $\mathrm{Hom}(\Gamma, \mathrm{SO}(2, 2n))$ gives rise to a properly discontinuous and cocompact action on the homogeneous space $\mathrm{SO}(2, 2n)/\mathrm{U}(1, n)$.*

This extends a recent result of Kassel [51, Theorem 1.1].

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Part 1. Anosov representations

2. Definition

In this section we recall the notion of Anosov representations, a concept introduced by Labourie [57, Section 2], and we generalize it to representations of arbitrary finitely generated word hyperbolic groups. A variety of examples of Anosov representations are discussed in Section 6 (see also Remark 8.5).

2.1. **For Riemannian manifolds.** Let (N, g) be a closed negatively curved Riemannian manifold and let $M = T^1N$ be its unit tangent bundle, equipped with the geodesic flow ϕ_t for the metric g. The geodesic flow is an Anosov flow. We denote by $\widehat{M} = T^1\tilde{N}$ the $\pi_1(N)$-cover of M and by ϕ_t again the geodesic flow on $\widehat{M}$.

Let G be a semisimple Lie group and (P^+, P^-) be a pair of opposite parabolic subgroups[1] of G and set $\mathcal{F}^\pm = G/P^\pm$. The subgroup $L = P^+ \cap P^-$ is the Levi subgroup of both P^+ and P^-. The homogeneous space $\mathcal{X} = G/L$ is the unique open G-orbit in the product $\mathcal{F}^+ \times \mathcal{F}^- = G/P^+ \times G/P^-$. From this product structure $\mathcal{X}$ inherits two G-invariant distributions E^+ and E^-: $(E^\pm)_{(x_+,x_-)} = T_{x_\pm}\mathcal{F}^\pm$. As a consequence any $\mathcal{X}$-bundle is equipped with two distributions which are denoted also by E^+ and E^-.

Notation 2.1. Let M be a topological space, Γ be a group, and let $\widehat{M}$ be the Γ-cover of M. Let $\rho : \Gamma \to G$ be a representation and let $\mathcal{S}$ be a G-space. We set

$$\mathcal{S}_\rho = \widehat{M} \times_\rho \mathcal{S} = \Gamma\backslash(\widehat{M} \times \mathcal{S}),$$

where Γ acts diagonally, as deck transformations on $\widehat{M}$ and via the representation ρ on $\mathcal{S}$. The projection onto the first factor gives $\widehat{M} \times_\rho \mathcal{S}$ the structure of a flat $\mathcal{S}$-bundle over M.

Definition 2.2. A representation $\rho : \pi_1(N) \to G$ is said to be (P^+, P^-)-Anosov if

(i) the flat bundle $\mathcal{X}_\rho$ admits a section $\sigma : M \to \mathcal{X}_\rho$ which is flat along flow lines (i.e. the restriction of σ to any geodesic leaf is flat).

(ii) The (lifted) action of ϕ_t on σ^*E^+ (resp. σ^*E^-) is dilating (resp. contracting).

The section σ will be called *Anosov section*.

Remarks 2.3.

(a) The second condition means more precisely that there exists a continuous family of norms $(\|\cdot\|_m)_{m\in M}$ on the fibers of the vector bundle $\sigma^*E^+ \to M$ (resp. $\sigma^*E^- \to M$) and positive constants A, a such that, for any $t \in \mathbf{R}_{\geq 0}$, $e \in \sigma^*E^+$ (resp. $e \in \sigma^*E^-$), with $\pi(e) = m$, one has $\|\phi_{-t}e\|_{\phi_{-t}m} \leq Ae^{-at}\|e\|_m$ (resp. $\|\phi_t e\|_{\phi_t m} \leq Ae^{-at}\|e\|_m$).

(b) Due to the compactness of M, the definition does not depend on the particular choice of $\|\cdot\|_m$ or the particular parametrization of the flow on M.

(c) The Anosov section is uniquely determined (see Lemma 3.3).

2.2. **In terms of equivariant maps.** Let N, $M = T^1N$ and $\widehat{M}$ be as before.

We denote by $\partial_\infty\tilde{N}$ the boundary at infinity of the universal cover $\tilde{N}$ of N and by $\partial_\infty\pi_1(N)$ the boundary at infinity of $\pi_1(N)$. We can identify $\partial_\infty\tilde{N}$ and $\partial_\infty\pi_1(N)$. Since N is negatively curved, $\partial_\infty\tilde{N}$ is homeomorphic to a sphere.

The space of geodesic leaves $\widehat{M}/\{\phi_t\}$ in $\widehat{M}$ is canonically identified with

$$\partial_\infty\tilde{N}^{(2)} = \partial_\infty\tilde{N} \times \partial_\infty\tilde{N} \smallsetminus \{(t,t) \mid t \in \partial_\infty\tilde{N}\};$$

the identification associates to a geodesic its endpoints in $\partial_\infty\tilde{N}$.

[1] We review the structure theory of parabolic subgroups in Section 3.2. Pairs of opposite parabolic subgroups arise as follows: choose a semisimple element $g \in G$ and set $\mathrm{Lie}(P^+)$ (resp. $\mathrm{Lie}(P^-)$) to be the sum of eigenspaces of $\mathrm{Ad}(g)$ associated with eigenvalues of modulus ≥ 1 (resp. ≤ 1).

Given a representation $\rho : \pi_1(N) \to G$, a section σ of $\mathcal{X}_\rho = \widehat{M} \times_\rho \mathcal{X}$ is completely determined by its pullback $\hat{\sigma}$ to $\widehat{M}$, which is a ρ-equivariant map:

$$\hat{\sigma} : \widehat{M} \longrightarrow \mathcal{X}.$$

Conversely, such a ρ-equivariant map $\hat{\sigma}$ descends to a section σ. The section σ is flat along flow lines if and only if $\hat{\sigma}$ is ϕ_t-invariant. In this case, one can consider $\hat{\sigma}$ as being defined on $\widehat{M}/\{\phi_t\} \cong \partial_\infty \tilde{N}^{(2)}$:

$$\hat{\sigma} = (\xi^+, \xi^-) : \partial_\infty \tilde{N}^{(2)} \longrightarrow \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-.$$

The contraction property implies immediately (see [57, Proposition 3.2] and [41, Proposition 2.5]) that $\xi^+ : \partial_\infty \tilde{N}^{(2)} \to \mathcal{F}^+$ factors through the projection to the first factor $\pi_1 : \partial_\infty \tilde{N}^{(2)} \to \partial_\infty \tilde{N}$, i.e. ξ^+ is a map from $\partial_\infty \tilde{N}$ to $\mathcal{F}^+$. Similarly ξ^- factors through the projection to the second factor. Thus we get a pair of maps $\xi^+ : \partial_\infty \tilde{N} \to \mathcal{F}^+$ and $\xi^- : \partial_\infty \tilde{N} \to \mathcal{F}^-$.

Remark 2.4. In definition 2.2 the dilatation property and contraction property on E^+ and E^- have been exchanged compared to the original definition [57, Section 2.0.1]. The convention chosen here are such that ξ^+ factors through the projection onto the first factor and also such that $\xi^+(t_\gamma^+)$ is the attracting fixed point of γ (see Lemma 3.1). These two properties seem natural to us.

Definition 2.5. The maps $\xi^\pm : \partial_\infty \tilde{N} \to \mathcal{F}^\pm$ are said to be the *Anosov maps* associated to the Anosov representation $\rho : \pi_1(N) \to G$.

Let $\tau^+, \tau^- : \widehat{M} \to \partial_\infty \tilde{N}$ be the maps associating to a tangent vector the endpoints at $+\infty$ and $-\infty$ of the corresponding geodesic. The dilatation/contraction property in Definition 2.2 translates into a dilatation property for ϕ_t on the family of tangent spaces $(T_{\xi^+(\tau^+(\hat{m}))}\mathcal{F}^+)_{\hat{m}\in\widehat{M}}$ (resp. contraction on $(T_{\xi^-(\tau^-(\hat{m}))}\mathcal{F}^-)_{\hat{m}\in\widehat{M}}$).

Conversely, one can use the maps ξ^+, ξ^- to express the Anosov property.

Definition 2.6. A pair of points $(x^+, x^-) \in \mathcal{F}^+ \times \mathcal{F}^-$ is said to be *transverse*, if $(x^+, x^-) \in \mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-$.

Given $x \in \mathcal{F}^+$ and $y \in \mathcal{F}^-$ we say that y is *transverse* to x (and x is transverse to y) if (x, y) is transverse.

Proposition 2.7. *Let $\rho : \pi_1(N) \to G$ be a representation. Suppose that there exist maps $\xi^+ : \partial_\infty \tilde{N} \to \mathcal{F}^+$ and $\xi^- : \partial_\infty \tilde{N} \to \mathcal{F}^-$ such that:*

(i) ξ^+ and ξ^- are continuous and ρ-equivariant.
(ii) For all $(t^+, t^-) \in \partial_\infty \tilde{N}^{(2)}$ the pair $(\xi^+(t^+), \xi^-(t^-))$ is transverse.
(iii) For one (and hence any) continuous and equivariant family of norms $(\|\cdot\|_{\hat{m}})_{\hat{m}\in\widehat{M}}$ on $(T_{\xi^+(\tau^+(\hat{m}))}\mathcal{F}^+)_{\hat{m}\in\widehat{M}}$ (resp. $(T_{\xi^-(\tau^-(\hat{m}))}\mathcal{F}^-)_{\hat{m}\in\widehat{M}}$), the following property holds:
   - *there exist positive constants A, a such that for all $t \in \mathbf{R}_{\geq 0}$, $\hat{m} \in \widehat{M}$ and $e \in T_{\xi^+(\tau^+(\hat{m}))}\mathcal{F}^+$ (resp. $e \in T_{\xi^-(\tau^-(\hat{m}))}\mathcal{F}^-$):*

$$\|e\|_{\phi_{-t}\hat{m}} \leq Ae^{-at}\|e\|_{\hat{m}} \quad (\textit{resp. } \|e\|_{\phi_t\hat{m}} \leq Ae^{-at}\|e\|_{\hat{m}}).$$

Then the representation ρ is (P^+, P^-)-Anosov and the pull-back to $\widehat{M}$ of the Anosov section σ of $\mathcal{X}_\rho$ is the map $\hat{\sigma} : \widehat{M} \to \mathcal{X}$, $\hat{m} \mapsto (\xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m})))$.

Remark 2.8. We will see later (Theorem 4.11) that in the case of Zariski dense representations the existence of ξ^+ and ξ^- satisfying (i) and (ii) is sufficient to ensure the Anosov property.

2.3. **For hyperbolic groups.** In order to define the notion of Anosov representations $\rho : \Gamma \to G$ for an arbitrary finitely generated word hyperbolic group Γ we need a replacement for the space $(\widehat{M}, \phi_t)$. Geodesic flows for hyperbolic groups were introduced by Gromov [37] and later developed by Champetier [24], Mineyev [60] and others. We recall the results necessary for our purpose.

Let Γ be a finitely generated word hyperbolic group and let $\partial_\infty\Gamma$ denote its boundary at infinity ([26, Chapitre 2]). We set $\partial_\infty\Gamma^{(2)} = \partial_\infty\Gamma \times \partial_\infty\Gamma \smallsetminus \{(t,t) \mid t \in \partial_\infty\Gamma\}$.

Theorem 2.9. [37, Theorem 8.3.C], [60, Theorem 60]

Let Γ be a finitely generated word hyperbolic group. Then there exists a proper hyperbolic metric space $\widehat{\Gamma}$ such that:

(i) $\Gamma \times \mathbf{R} \rtimes \mathbf{Z}/2\mathbf{Z}$ *acts on* $\widehat{\Gamma}$.
(ii) The $\Gamma \times \mathbf{Z}/2\mathbf{Z}$*-action is isometric.*
(iii) Every orbit $\Gamma \to \widehat{\Gamma}$ *is a quasi-isometry. In particular,* $\partial_\infty\widehat{\Gamma} \cong \partial_\infty\Gamma$.
(iv) The $\mathbf{R}$*-action is free, and every orbit* $\mathbf{R} \to \widehat{\Gamma}$ *is a quasi-isometric embedding. The induced map* $\widehat{\Gamma}/\mathbf{R} \to \partial_\infty\widehat{\Gamma}^{(2)}$ *is a homeomorphism.*

In fact $\widehat{\Gamma}$ is unique up to a $\Gamma \times \mathbf{Z}/2\mathbf{Z}$-equivariant quasi-isometry sending $\mathbf{R}$-orbits to $\mathbf{R}$-orbits. We shall denote by ϕ_t the $\mathbf{R}$-action on $\widehat{\Gamma}$ and by $(\tau^+, \tau^-) : \widehat{\Gamma} \to \widehat{\Gamma}/\mathbf{R} \cong \partial_\infty\Gamma^{(2)}$ the maps associating to a point the endpoints of its $\mathbf{R}$-orbit.

Definition 2.10. A representation $\rho : \Gamma \to G$ is said to be (P^+, P^-)-*Anosov* if there exist continuous ρ-equivariant maps $\xi^+ : \partial_\infty\Gamma \to \mathcal{F}^+$, $\xi^- : \partial_\infty\Gamma \to \mathcal{F}^-$ such that:

(i) For all $(t^+, t^-) \in \partial_\infty\Gamma^{(2)}$ the pair $(\xi^+(t^+), \xi^-(t^-))$ is transverse.
(ii) For one (and hence any) continuous and equivariant family of norms $(\|\cdot\|_{\hat{m}})_{\hat{m}\in\widehat{\Gamma}}$ on $(T_{\xi^+(\tau^+(\hat{m}))}\mathcal{F}^+)_{\hat{m}\in\widehat{\Gamma}}$ (resp. $(T_{\xi^-(\tau^-(\hat{m}))}\mathcal{F}^-)_{\hat{m}\in\widehat{\Gamma}}$), there exist $A, a > 0$ such that for all $t \geq 0$, $\hat{m} \in \widehat{\Gamma}$ and $e \in T_{\xi^+(\tau^+(\hat{m}))}\mathcal{F}^+$ (resp. $e \in T_{\xi^-(\tau^-(\hat{m}))}\mathcal{F}^-$):
$$\|e\|_{\phi_{-t}\hat{m}} \leq Ae^{-at}\|e\|_{\hat{m}} \quad (\text{resp. } \|e\|_{\phi_t\hat{m}} \leq Ae^{-at}\|e\|_{\hat{m}}).$$

The maps $\xi^\pm$ are said to be the *Anosov maps* associated to ρ.

Remark 2.11. As explained in Section 2.2, the definition here is equivalent to the existence of a section σ of the $\mathcal{X}$-bundle $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho \mathcal{X}$ over $\Gamma\backslash\widehat{\Gamma}$ that is flat along $\mathbf{R}$-orbits and such that the action of ϕ_t on the vector bundle σ^*E^+ (resp. σ^*E^-) is dilating (resp. contracting).

3. Controlling the Anosov section

In this section we first recall some well known properties of Anosov representations. Then we introduce L-Cartan projections which are Γ-invariant continuous maps from $\widehat{\Gamma} \times \mathbf{R}$ with values in a closed Weyl chamber of $L = P^+ \cap P^-$. These L-Cartan projections provide a simple criterion for a section to satisfy the contraction property (see Definition 2.10.(ii)).

3.1. **Holonomy and uniqueness.** Any non-torsion element $\gamma \in \Gamma$ has two fixed points in $\partial_\infty\Gamma$. We denote the attracting fixed point by t_γ^+ and the repelling fixed point by t_γ^-. From the definition of Anosov representation, one deduces easily.

Lemma 3.1. *Let $\rho : \Gamma \to G$ be a (P^+, P^-)-Anosov representation and let $\xi^\pm : \partial_\infty\Gamma \to \mathcal{F}^\pm$ be the associated Anosov maps. Let $\gamma \in \Gamma$ be a non-torsion element.*

Then $\xi^+(t^+_\gamma)$ is the unique attracting fixed point of $\rho(\gamma) \in \mathcal{F}^+$. The basin of attraction is the set of all points in $\mathcal{F}^+$ that are transverse to $\xi^-(t^-_\gamma)$. In particular the eigenvalues of $\rho(\gamma)$ acting on $T_{\xi^+(t^+_\gamma)}\mathcal{F}^+$ are all of modulus less than 1.

An analogous statement holds for the action on $\mathcal{F}^-$.

Corollary 3.2. *Let $\rho : \Gamma \to G$ be a (P^+, P^-)-Anosov representation and $\gamma \in \Gamma$ a non-torsion element. Then $\rho(\gamma)$ is conjugate to an element of L, whose action is dilating on $T_{P^-}\mathcal{F}^-$ and contracting on $T_{P^+}\mathcal{F}^+$.*

A refinement of this corollary will be given in Lemma 3.9, providing a quantitative statement for the contraction.

Lemma 3.1, together with the density of the fixed points $\{t^+_\gamma\}_{\gamma\in\Gamma}$ in $\partial_\infty\Gamma$, has the following consequence:

Lemma 3.3. [41, Proposition 2.5] *Let $\rho : \Gamma \to G$ be a (P^+, P^-)-Anosov representation. Then the maps $\xi^+ : \partial_\infty\Gamma \to \mathcal{F}^+$ and $\xi^- : \partial_\infty\Gamma \to \mathcal{F}^-$ satisfying the properties of Definition 2.10 are unique.*

The uniqueness of the Anosov maps (ξ^+, ξ^-) gives the following corollaries:

Corollary 3.4. [41, Proposition 2.8] *Let $\Gamma' < \Gamma$ be a finite index subgroup. A representation $\rho : \Gamma \to G$ is (P^+, P^-)-Anosov if and only if $\rho|_{\Gamma'}$ is (P^+, P^-)-Anosov. Furthermore with the canonical identification $\partial_\infty\Gamma \cong \partial_\infty\Gamma'$ the Anosov maps ξ^+ and ξ^- are the same for ρ and for $\rho|_{\Gamma'}$.*

Corollary 3.5. *Let $\rho : \Gamma \to G$ be a (P^+, P^-)-Anosov representation and let $\xi^+ : \partial_\infty\Gamma \to \mathcal{F}^+$ and $\xi^- : \partial_\infty\Gamma \to \mathcal{F}^-$ be the corresponding Anosov maps.*

Then any element $z \in Z_G(\rho(\Gamma))$ in the centralizer of $\rho(\Gamma)$ fixes $\xi^\pm(\partial_\infty\Gamma)$ pointwise, i.e. for any $t \in \partial_\infty\Gamma$, $z \cdot \xi^\pm(t) = \xi^\pm(t)$.

Corollary 3.6. *Let $\pi : \widehat{G} \to G$ be a covering of Lie groups, (P^+, P^-) a pair of opposite parabolic subgroups of G so that $(\widehat{P}^+, \widehat{P}^-) = (\pi^{-1}(P^+), \pi^{-1}(P^-))$ is a pair of opposite parabolic subgroups of $\widehat{G}$.*

Then a representation $\rho : \Gamma \to \widehat{G}$ is $(\widehat{P}^+, \widehat{P}^-)$-Anosov if and only if $\pi \circ \rho$ is (P^+, P^-)-Anosov.

3.1.1. *Lifting.* Even though we do not use it in this article, for future reference we describe when the maps $\xi^\pm : \partial_\infty\Gamma \to G/P^\pm$ can be lifted to maps $\xi'^\pm : \partial_\infty\Gamma \to G/P'^\pm$ where $P'^\pm \subset P^\pm$ are finite index subgroups.

Recall that P^+ (and P^-) is the semi-direct product of its unipotent radical by L. Hence $\pi_0(P^+) \cong \pi_0(L) \cong \pi_0(P^-)$, and a finite index subgroup $P'^+ \subset P^+$ corresponds to a finite index subgroup $\pi_0(P'^+) \subset \pi_0(P^+)$, and hence to a finite index subgroup $L' \subset L$ as well as to a finite index subgroup $P'^- \subset P^-$. Using that the unipotent radical of P^+ is contractible (and the classical fact that the space of sections of a bundle with contractible fibers is nonempty and contractible) one deduces:

Lemma 3.7. *Let $\rho : \Gamma \to G$ be a (P^+, P^-)-Anosov representation with Anosov maps $\xi^+, \xi^- : \partial_\infty\Gamma \to \mathcal{F}^+, \mathcal{F}^-$. Let σ be the corresponding section of $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho G/L$. Then the following are equivalent:*

(i) ξ^+ *lifts to a continuous ρ-equivariant map* $\partial_\infty\Gamma \to G/P'^+$.

(ii) σ *lifts to a continuous section* σ' *of* $\mathcal{X}'_\rho = \widehat{\Gamma} \times_\rho G/L'$.

(iii) ξ^- *lifts to a continuous ρ-equivariant map* $\partial_\infty\Gamma \to G/P'^-$.

In that case the ρ-equivariant lift $\hat\sigma' : \widehat{\Gamma} \to G/L'$ *of the section* σ' *is of the form* $(\xi'^+ \circ \tau^+, \xi'^- \circ \tau^-)$*, with maps* $\xi'^+, \xi'^- : \partial_\infty\Gamma \to G/P'^+, G/P'^-$.

Note that the maps ξ'^+, ξ'^- in the above lemma are not necessarily unique.

Proof. Note that ξ^+ naturally defines a section ζ^+ of $\mathcal{F}^+_\rho = \widehat{\Gamma} \times_\rho \mathcal{F}^+$. This section is the image of σ under the map $\mathcal{X}_\rho \to \mathcal{F}^+_\rho$ induced by the projection to the first factor. The spaces $\mathcal{X}'_\rho$ and $\mathcal{F}'^+_\rho$ are defined similarly, and it is easy to see that ξ^+ lifts if and only if ζ^+ lifts. Furthermore $\mathcal{X}'_\rho$ is the fiber product $\mathcal{X}_\rho \times_{\mathcal{F}^+_\rho} \mathcal{F}'^+_\rho$ so that σ lifts if and only if ζ^+ does. This proves the equivalence of (i) and (ii); the remaining statements follow. □

3.2. **Structure of parabolic subgroups.** In the following sections we will use the finer structure of parabolic subgroups. In order to fix notation we recall here the classical structure theory of parabolic subgroups.

– Let G be a semisimple Lie groups and let $\mathfrak{g}$ be its Lie algebra. Let K be a maximal compact subgroup of G and $\mathfrak{k}$ its Lie algebra; the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ is orthogonal with respect to the Killing form.

– Let $\mathfrak{a}$ be a maximal abelian subalgebra contained in $\mathfrak{k}^\perp$; its action on $\mathfrak{g}$ gives rise to a decomposition into eigenspaces:

$$\mathfrak{g} = \bigoplus_{\alpha \in \Sigma \cup \{0\}} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha = \bigcap_{a \in \mathfrak{a}} \ker(\mathrm{ad}(a) - \alpha(a)),$$

where $\Sigma = \{\alpha \in \mathfrak{a}^* \smallsetminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is the system of restricted roots of $\mathfrak{g}$.

– Let $N_K(\mathfrak{a})$ and $Z_K(\mathfrak{a})$ be the normalizer and the centralizer of $\mathfrak{a}$ in K. The *Weyl group* $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ acts on $\mathfrak{a}$ and also on Σ.

– Let $<_{\mathfrak{a}^*}$ (or simply $<$) be a total ordering on the group $\mathfrak{a}^*$. The sets $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha > 0\}$ and $\Sigma^- = \{\alpha \in \Sigma \mid \alpha < 0\}$ are the *positive* roots and the *negative* roots, $\Sigma^- = -\Sigma^+$.

– A positive root is *decomposable* if it is the sum of two positive roots; it is called *simple* otherwise. The set $\Delta \subset \Sigma^+$ is the set of simple roots.

– The unique element $w_0 \in W$ sending Σ^- to Σ^+ induces an involution $\iota : \Sigma^+ \to \Sigma^+$, $\alpha \mapsto -w_0(\alpha)$, called the *opposition involution*; $\iota(\Delta) = \Delta$. The opposition involution is also defined on $\mathfrak{a}$.

– A *Weyl chamber* is $\mathfrak{a}^+ = \{a \in \mathfrak{a} \mid \alpha(a) > 0, \forall \alpha \in \Sigma^+\} = \{a \in \mathfrak{a} \mid \alpha(a) > 0, \forall \alpha \in \Delta\}$. The involution ι sends $\mathfrak{a}^+$ into itself.

– Any element $g \in G$ can be written as a product

$$g = k \exp(\mu(g)) l, \quad \text{with } k, l \in K \text{ and } \mu(g) \in \overline{\mathfrak{a}}^+;$$

$\mu(g)$ is uniquely determined by g, it is called the *Cartan projection* of g. The map $\mu : G \to \overline{\mathfrak{a}}^+$ is continuous. Another continuous projection $\lambda : G \to \overline{\mathfrak{a}}^+$ comes from the Jordan decomposition. The two projections are related by $\lambda(g) = \lim_{n\to\infty} \frac{\mu(g^n)}{n}$ [6, Corollaire in Paragraph 2.5.].

– The subalgebra $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ is nilpotent, $N = \exp(\mathfrak{n}^+)$ is unipotent.

The subgroup $A \ltimes N = \exp(\mathfrak{a}) \ltimes \exp(\mathfrak{n}^+) < G$ will be denoted by AN.

– The set $B^+ = Z_K(\mathfrak{a})AN$ is a subgroup of G called the *minimal parabolic subgroup*. Its Lie algebra is $\mathfrak{b}^+ = \mathfrak{g}_0 \oplus \mathfrak{n}^+$.

– Similarly one defines $\mathfrak{n}^-$, N^-, B^-, $\mathfrak{b}^-$. The group B^- is conjugate to B^+.

Parabolic subgroups of G are conjugate to subgroups containing B^+. A pair of parabolic subgroups is said to be *opposite* if their intersection is a reductive group.

Conjugacy classes of parabolic subgroups are in one to one correspondence with subsets $\Theta \subset \Delta$. Given such a subset we set $\mathfrak{a}_\Theta = \bigcap_{\alpha\in\Theta} \ker\alpha$ and denote by $M_\Theta = Z_K(\mathfrak{a}_\Theta)$ its centralizer in K. Then $P^+_\Theta = M_\Theta \exp(\mathfrak{a})N$ and $P^-_\Theta = M_\Theta \exp(\mathfrak{a})N^-$ are parabolic subgroups, which are opposite. A parabolic subgroup containing B^+ is of the form P^+_Θ for a uniquely determined Θ. Any pair of opposite parabolic subgroups is conjugate to (P^+_Θ, P^-_Θ) for some $\Theta \subset \Delta$.

Remark 3.8. The conjugacy class of (P^+, P^-) is determined by the conjugacy class of P^+. In view of this, we will sometimes say that a representation is P^+-Anosov (or L-Anosov).

Note also that P^-_Θ is conjugate to $P^+_{\iota(\Theta)}$. In particular P^+_Θ is conjugate to its opposite if and only if $\Theta = \iota(\Theta)$.

The intersection $L_\Theta = P^+_\Theta \cap P^-_\Theta$ is the common Levi component of P^+_Θ and P^-_Θ. The group M_Θ is a maximal compact subgroup of L_Θ. We denote the Weyl chamber of L_Θ by $\mathfrak{a}^+_{L_\Theta} = \{a \in \mathfrak{a} \mid \alpha(a) > 0, \text{ for all } \alpha \in \Theta\}$. The Cartan decomposition for L_Θ is $L_\Theta = M_\Theta \exp(\overline{\mathfrak{a}}^+_{L_\Theta}) M_\Theta$.

We denote by Σ_Θ the roots in the span of Θ: $\Sigma_\Theta = \mathrm{Span}_{\mathbf{R}}(\Theta) \cap \Sigma$. The Lie algebras of P^+_Θ, P^-_Θ and L_Θ are:

$$\mathfrak{p}^+_\Theta = \bigoplus_{\alpha\in\Sigma^+\cup\Sigma_\Theta\cup\{0\}} \mathfrak{g}_\alpha,\ \mathfrak{p}^-_\Theta = \bigoplus_{\alpha\in\Sigma^-\cup\Sigma_\Theta\cup\{0\}} \mathfrak{g}_\alpha,\ \mathfrak{l}_\Theta = \bigoplus_{\alpha\in\Sigma_\Theta\cup\{0\}} \mathfrak{g}_\alpha.$$

The nilpotent radicals of $\mathfrak{p}^+_\Theta$ and $\mathfrak{p}^-_\Theta$ are hence:

$$\mathfrak{n}^+_\Theta = \bigoplus_{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta} \mathfrak{g}_\alpha \ , \quad \mathfrak{n}^-_\Theta = \bigoplus_{\alpha\in\Sigma^-\smallsetminus\Sigma_\Theta} \mathfrak{g}_\alpha \ .$$

There are L_Θ-equivariant identifications of the tangent space of G/P^+_Θ at P^+_Θ with $\mathfrak{n}^-_\Theta$, and respectively of the tangent space of G/P^-_Θ at P^-_Θ with $\mathfrak{n}^+_\Theta$.

An element $\exp(a)$ with $a \in \overline{\mathfrak{a}}^+_{L_\Theta}$ contracts on $\mathfrak{n}^-_\Theta$ (resp. dilates on $\mathfrak{n}^+_\Theta$) if and only if $\alpha(a) > 0$ for all $\alpha \in \Delta \smallsetminus \Theta$. Moreover one has the quantitative statement:

Lemma 3.9. *There is a positive constant C such that:*

- *For any k and $a \in \overline{\mathfrak{a}}^+_{L_\Theta}$, if $\exp(a)$ is k-Lipschitz on $T_{P^+_\Theta} G/P^+_\Theta$ then, for all $\alpha \in \Delta \smallsetminus \Theta$, one has $\alpha(a) \geq -\log k$.*
- *For any $M \geq 0$ and $a \in \overline{\mathfrak{a}}^+_{L_\Theta}$, if for all $\alpha \in \Delta \smallsetminus \Theta$, $\alpha(a) \geq M$ then $\exp(a)$ is Ce^{-M}-Lipschitz on $T_{P^+_\Theta} G/P^+_\Theta$.*

In particular this lemma implies that an element of $\overline{\mathfrak{a}}^+_{L_\Theta}$ whose action is contracting on $T_{P^+_\Theta} G/P^+_\Theta$ is contained in the closed Weyl chamber $\overline{\mathfrak{a}}^+$ of G.

3.3. **Lifting sections and L-Cartan projections.** Let $(P^+, P^-) = (P^+_\Theta, P^-_\Theta)$ be a pair of opposite parabolic subgroups, and let L_Θ, M_Θ, etc. be as in the preceding section. We occasionally drop the subscript Θ. We set $\mathcal{Y} = G/M$, this is an L/M-bundle over $\mathcal{X} = G/L$.

Let $\rho : \Gamma \to G$ be a representation and let $\mathcal{X}_\rho = \widehat{\Gamma} \times_\rho \mathcal{X}$ and $\mathcal{Y}_\rho = \widehat{\Gamma} \times_\rho \mathcal{Y}$ be the associated flat bundles over $\Gamma \backslash \widehat{\Gamma}$. Then $\pi : \mathcal{Y}_\rho \to \mathcal{X}_\rho$ is an L/M-bundle and hence has contractible (even convex) fibers. This implies

Lemma 3.10. *Let σ be a section of $\mathcal{X}_\rho$, then there exists a section β of $\mathcal{Y}_\rho$ such that $\pi \circ \beta = \sigma$.*

Proof. Indeed it is equivalent to finding a section of the L/M-bundle $\sigma^* \mathcal{Y}_\rho$ over $\Gamma \backslash \widehat{\Gamma}$ (the pull back by σ of the L/M-bundle $\mathcal{Y}_\rho \to \mathcal{X}_\rho$). This is a locally trivial bundle, thus local sections exist. Moreover $\Gamma \backslash \widehat{\Gamma}$ is a compact metric space and admits partitions of unity, so that local sections can be glued together to a global section. □

Two different lifts are equal up to finite distance:

Lemma 3.11. *Let β and β' be two lifts of σ, and denote by $\hat{\beta}, \hat{\beta}' : \widehat{\Gamma} \to \mathcal{Y}$ their ρ-equivariant pull-backs. Then there exists $R > 0$ such that $\forall m \in \widehat{\Gamma}$, $d_{G/M}(\hat{\beta}(m), \hat{\beta}'(m)) \leq R$, where $d_{G/M}$ is a G-invariant Riemannian distance $\mathcal{Y} = G/M$.*

Suppose now that σ is flat along $\mathbf{R}$-orbits. The section β corresponds to an equivariant continuous map $\hat{\beta} : \widehat{\Gamma} \to \mathcal{Y}$ lifting $\hat{\sigma} : \widehat{\Gamma} \to \mathcal{X}$, the equivariant map corresponding to σ. As $\hat{\sigma}$ is $\mathbf{R}$-invariant, for any $\hat{m} \in \widehat{\Gamma}$ and any $t \in \mathbf{R}$, $\hat{\beta}(\hat{m})$ and $\hat{\beta}(\phi_t \hat{m})$ project to the same point in $\mathcal{X} = G/L$. Thus the pair $(\hat{\beta}(\hat{m}), \hat{\beta}(\phi_t \hat{m}))$ is in the G-orbit of a unique pair of the form $(\exp(\mu_{+,\Theta}(\hat{m}, t))M, M)$ with $\mu_{+,\Theta}(\hat{m}, t) \in \overline{\mathfrak{a}}^+_{L_\Theta}$; similarly it is in the G-orbit of a unique pair of the form $(M, \exp(\mu_{-,\Theta}(\hat{m}, t))M)$ with $\mu_{-,\Theta}(\hat{m}, t) \in \overline{\mathfrak{a}}^+_{L_\Theta}$.

Definition 3.12. Let $\sigma : \Gamma \backslash \widehat{\Gamma} \to \mathcal{X}_\rho$ be a section which is flat along $\mathbf{R}$-orbits, and let β be a lift of σ to a section of $\mathcal{Y}_\rho$. The maps

$$\mu_{+,\Theta}, \mu_{-,\Theta} : \widehat{\Gamma} \times \mathbf{R} \to \overline{\mathfrak{a}}^+_{L_\Theta}$$

are called L_Θ*-Cartan projections*. They are continuous and Γ-invariant, and hence well defined on $\Gamma \backslash \widehat{\Gamma} \times \mathbf{R}$.

Remarks 3.13.

(a) The L_Θ-Cartan projections take values in the closed Weyl chamber $\overline{\mathfrak{a}}^+_{L_\Theta} = \{a \in \mathfrak{a} \mid \alpha(a) \geq 0, \text{ for all } \alpha \in \Theta\}$ which is a closed cone of $\mathfrak{a}$ with nonempty interior.

(b) It is possible to define maps into the closed Weyl chamber $\overline{\mathfrak{a}}^+$ of $\mathfrak{g}$, however doing this will lead to a loss of information: we need to understand G-orbits of pairs of points in G/M that project to the same element in G/L; this is the same as understanding L-orbits of pairs of points in L/M which are ultimately completely classified by the closed Weyl chamber of L. The maps into $\overline{\mathfrak{a}}^+$ would amount to classifying G-orbits of pairs in G/K and cannot keep track of the action on $\mathfrak{n}^\pm_\Theta$.

(c) The idea of lifting the section σ of $\mathcal{X}_\rho$ to a section of $\mathcal{Y}_\rho$ is already implicit in [57] and [18], where a specific metric on the bundle $\sigma^* E^\pm$ (see Section 2.1) is chosen in order to prove the contraction property. The choice of this specific metric corresponds to a choice of a lift β.

(d) The classical Cartan projection $\mu : G \to \mathfrak{a}^+$ can be used to define a refined Weyl chamber valued distance function: if δ denotes the Weyl chamber valued distance on the symmetric space G/K, then $\mu(g) = \delta(K, gK)$, see for example [49, 64] for an account on this subject and references therein.

The dependence of the L-Cartan projection on the section σ is crucial. However, their asymptotic behavior does not depend on the choice of the lift β:

Lemma 3.14. *Let $\sigma : \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho$ be a section which is flat along $\mathbf{R}$-orbits, and let β and β' be two lifts. Let $\mu_{\pm,\Theta}$ and $\mu'_{\pm,\Theta}$ be the L-Cartan projections associated with β and β' respectively.*

Then there exists a constant $C > 0$ such that for all $(m, t) \in \widehat{\Gamma} \times \mathbf{R}$

$$d(\mu_{\pm,\Theta}(m, t), \mu'_{\pm,\Theta}(m, t)) \leq C,$$

where d is the distance for some norm on the vector space $\mathfrak{a}_{L_\Theta}$.

Proof. Denote by $\hat{\sigma}$, $\hat{\beta}$, and $\hat{\beta}'$ the ρ-equivariant lifts of σ, β and β'. Let $(m, t) \in \widehat{\Gamma} \times \mathbf{R}$, since $\pi(\hat{\beta}(m)) = \pi(\hat{\beta}(\phi_t m)) = \hat{\sigma}(m) = \pi(\hat{\beta}'(\phi_t m)) = \pi(\hat{\beta}'(m))$, the four points $\hat{\beta}(m)$, $\hat{\beta}(\phi_t m)$, $\hat{\beta}'(\phi_t m)$ and $\hat{\beta}'(m)$ lie in one fiber, which we can assume to be L/M.

Moreover by Lemma 3.11 there exists $R > 0$ such that $\forall m \in \widehat{\Gamma}$: $d(\hat{\beta}(m), \hat{\beta}'(m)) \leq R$ and $d(\hat{\beta}(\phi_t m), \hat{\beta}'(\phi_t m)) \leq R$.

The statement now follows from the triangle inequality for the Weyl chamber valued distance function on the symmetric space L/M [49, 64]. □

Recall that $\lambda(g)$ denotes the hyperbolic part of the Jordan decomposition of g (Section 3.2):

Lemma 3.15. *Let $\gamma \in \Gamma$ be a non-torsion element and denote by $\hat{\gamma} \subset \widehat{\Gamma}$ the corresponding γ-invariant $\mathbf{R}$-orbit in $\widehat{\Gamma}$. Let T be the period of this orbit (i.e. γ acts as ϕ_T on $\hat{\gamma}$) and $\hat{m} \in \hat{\gamma}$. Assume that $\rho : \Gamma \to G$ is an Anosov representation. Then*

$$\mu_{+,\Theta}(\hat{m}, T) = \lambda(\rho(\gamma)), \text{ and } \mu_{-,\Theta}(\hat{m}, T) = \lambda(\rho(\gamma)^{-1}).$$

For $(m, t) \in \Gamma\backslash\widehat{\Gamma} \times \mathbf{R}$ we set

$$A_+(m, t) = \min_{\alpha \in \Delta \smallsetminus \Theta} \alpha(\mu_{+,\Theta}(\hat{m}, t)), \quad A_-(m, t) = \min_{\alpha \in \Delta \smallsetminus \Theta} \alpha(\mu_{-,\Theta}(\hat{m}, t)).$$

Note that $\mu_{-,\Theta}(\hat{m}, t) = \iota_\Theta(\mu_{+,\Theta}(\hat{m}, t))$ where ι_Θ is the opposition involution for L_Θ, and also that $\mu_{+,\Theta}(\hat{m}, t) = \mu_{-,\Theta}(\phi_t \hat{m}, -t)$. This means that dilatation on $\sigma^* E^+$, which is governed by $\mu_{+,\Theta}$, is equivalent to contraction on $\sigma^* E^-$, which is governed by $\mu_{-,\Theta}$. Hence, in the next proposition, only one of the L-Cartan projections needs to be considered:

Proposition 3.16. *Let $\rho : \Gamma \to G$ be a representation, let σ be a section of $\mathcal{X}_\rho$ which is flat along $\mathbf{R}$-orbits. Let A_+, A_- be as above. The following are equivalent:*

- *(i) σ is an Anosov section (and hence ρ is (P^+, P^-)-Anosov).*
- *(ii) There exist positive constants C and c such that for all $t \geq 0$ and all $m \in \Gamma\backslash\widehat{\Gamma}$, one has $A_+(m, t) \geq ct - C$ and $A_-(m, -t) \geq ct - C$.*
- *(iii) There exist positive constants C and c such that for all $t \geq 0$ and all $m \in \Gamma\backslash\widehat{\Gamma}$, one has $A_+(m, t) \geq ct - C$.*
- *(iv) $\lim_{t\to+\infty} \inf_{m \in \Gamma\backslash\widehat{\Gamma}} A_+(m, t) = +\infty$.*

Remark 3.17. Note that this implies that $\mu_{+,\Theta}(m, t)$ belongs to $\overline{\mathfrak{a}}^+$ for t big enough.

Proof. Indeed (i) ⇒ (ii) follows from Lemma 3.9; the implications (ii) ⇒ (iii) ⇒ (iv) are immediate. By Lemma 3.9, condition (iv) implies weak dilatation of the flow on the bundle $\sigma^* E^+$, and thus also weak contraction for the flow on the bundle $\sigma^* E^-$, from the relations between $\mu_{+,\Theta}$ and $\mu_{-,\Theta}$ mentioned above. Uniform estimates follow from the compactness of $\Gamma\backslash\widehat{\Gamma}$. For details on this last argument see [57, Section 6.1]. ☐

3.4. **Consequences.** Proposition 3.16 reduces the property of being Anosov to the control of a few eigenvalues (or rather principal values). From this one immediately deduces the following

Lemma 3.18. *Let $\rho : \Gamma \to G$ be a representation.*

(i) If ρ is P_Θ-Anosov, then it is $P_{\iota(\Theta)}$-Anosov.
(ii) If ρ is P_Θ-Anosov, then it is $P_{\Theta'}$-Anosov for any $\Theta' \supset \Theta$.
(iii) If ρ is P_{Θ_1}-Anosov and P_{Θ_2}-Anosov, then it is $P_{\Theta_1 \cap \Theta_2}$-Anosov.
(iv) If ρ is Anosov, then it is P_Θ-Anosov for some Θ satisfying $\iota(\Theta) = \Theta$.

The first statement is a consequence of the fact that the $\mathbf{Z}/2\mathbf{Z}$-action on $\widehat{\Gamma}$ anticommutes with the action of $\mathbf{R}$. The other consequences follow immediately from Proposition 3.16.

4. Lie group homomorphisms and equivariant maps

In this section we first describe how the property of being Anosov behaves with respect to compositions with Lie group homomorphisms. Then we show that for irreducible or Zariski dense representations the existence of equivariant maps readily implies the contraction property.

4.1. **Lie group homomorphisms.** Let $\phi : G \to G'$ be a homomorphism of semisimple Lie groups and let $\rho : \Gamma \to G$ be a P_Θ-Anosov representation. For several arguments we give later it will be essential to determine the parabolic subgroup $P' < G'$ such that the composition $\phi \circ \rho : \Gamma \to G'$ is P'-Anosov. Whereas this has a rather simple answer when G is of rank one (see Proposition 4.7), it is more complicated when G is of higher rank (see Section 4.3 for examples). In this section we give an explicit construction of P'.

Let $\phi : G \to G'$ be a homomorphism between semisimple Lie groups. We can assume that the maximal compact subgroup $K' < G'$ and the Cartan algebra $\mathfrak{a}'$ are chosen to be compatible with ϕ, i.e. $\phi(K) \subset K'$, $\phi_*(\mathfrak{a}) \subset \mathfrak{a}'$ ([50], [62, Theorem 6]). The set of simple roots of G' relative to $\mathfrak{a}'$ is denoted by Δ'. We shall usually denote with primes the objects associated with G'.

Proposition 4.1. *Let $\rho : \Gamma \to G$ be a representation. Suppose that $\phi \circ \rho$ is $P'_{\Theta'}$-Anosov then ρ is P_Θ-Anosov where $\Theta = \{\alpha \in \Delta \mid \alpha|_{\phi^{-1}(\mathfrak{a}'_{\Theta'})}$ is zero$\}$.*

For the other direction we first describe in detail the case when $G' = \mathrm{SL}(V)$ for some irreducible G-module V.

Lemma 4.2. *Let V be an irreducible G-module and denote by $\phi : G \to \mathrm{SL}(V)$ the corresponding homomorphism and by $\phi_* : \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism.*

Suppose that there exists a line $D \subset V$ that is P_Θ-invariant for some $\Theta \subset \Delta$.

Then the following holds:

(i) there exists a basis $(e_1, \dots, e_n)$ of V with $e_1 \in D$ and consisting of eigenvectors for $\mathfrak{a}$:

$$\phi_*(a) \cdot e_i = \lambda_i(a) e_i, \ \forall a \in \mathfrak{a}, \ \textit{with } \lambda_i \in \mathfrak{a}^*.$$

(ii) For all $i > 1$, $\lambda_1 - \lambda_i$ is a linear combination with integer coefficients of the elements of Δ:

$$\lambda_1 - \lambda_i = \sum_{\alpha \in \Delta} n_{\alpha,i} \alpha,\ n_{\alpha,i} \in \mathbf{N},$$

(iii) with the property that $(n_{\alpha,i})_{\alpha \in \Delta \smallsetminus \Theta}$ are not simultanously zero.

Proof. For that proof we use standard tools from representation theory (see [31] for example). By irreducibility $V = \mathfrak{n}^- \cdot D$. Also if $v \in V$ is an eigenvector for $\mathfrak{a}$ with eigenvalue λ and if $n \in \mathfrak{n}_\beta$, for $\beta \in \Sigma$, then the vector $w = n \cdot v$, if nonzero, is an eigenvector with eigenvalue $\lambda + \beta$. These two remarks give the two first conclusions of the lemma (starting with $e_1 = v \in D$). To prove the third conclusion, one only has to note that the hypothesis implies $\mathfrak{n}_{-\alpha} \cdot e_1 = 0$ for all $\alpha \in \Theta$. □

Proposition 4.3. *Let $\phi : G \to G' = \mathrm{SL}(V)$ be an irreducible finite dimensional linear representation of G. Let $V = D \oplus H$ be a decomposition of V into a line and a hyperplane, and set $Q_0^+ = \mathrm{Stab}(D)$ and $Q_0^- = \mathrm{Stab}(H)$. Suppose that $(P^+, P^-) = (\mathrm{Stab}_G(D), \mathrm{Stab}_G(H))$ is a pair of opposite parabolic subgroups.*

Then a representation $\rho : \Gamma \to G$ is (P^+, P^-)-Anosov if and only if $\phi \circ \rho : \Gamma \to \mathrm{SL}(V)$ is (Q_0^+, Q_0^-)-Anosov. Furthermore the Anosov maps satisfy $\phi^\pm \circ \xi_\rho^\pm = \xi_{\phi\circ\rho}^\pm$, where $\phi^+ : \mathcal{F}^+ \to \mathbb{P}(V)$ and $\phi^- : \mathcal{F}^- \to \mathbb{P}(V^*)$ are the maps induced by ϕ.

Proof. The if part is a consequence of Proposition 4.1.

For the converse statement, we can assume (up to conjugating in G) that $P^+ = P_\Theta^+$. Let $(e_1, \dots, e_n)$ denote the basis obtained in Lemma 4.2 and λ_i the corresponding eigenvalues. The Cartan subalgebra $\mathfrak{a}' \subset \mathfrak{sl}(V)$ is chosen to be the set of matrices that are diagonal with respect to the basis (e_i) and the Weyl chamber $\mathfrak{a}'^+$ is the set of diagonal matrices $\mathrm{diag}(t_1, \dots, t_n)$ with $t_1 > \cdots > t_n$.

With these choices the root $\alpha' \in \Delta'$ such that $Q_0^+ = P'^+_{\Delta' \smallsetminus \alpha'}$ is $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_1 - t_2$. The Weyl chamber $\mathfrak{a}'^+_{L'_{\Delta' \smallsetminus \alpha'}}$ of the Levi component $L'_{\Delta' \smallsetminus \alpha'}$ is the set of diagonal matrices $\mathrm{diag}(t_1, \dots, t_n)$ with $t_2 > \cdots > t_n$, the Weyl group $W'_{\Delta' \smallsetminus \alpha'}$ of $L'_{\Delta' \smallsetminus \alpha'}$ acts on $\mathfrak{a}'_{L'_{\Delta' \smallsetminus \alpha'}} \subset \mathfrak{a}'$ as the group of permutations on the last $(n-1)$-th diagonal coefficients.

The maps $\phi^+ : \mathcal{F} \to \mathbb{P}(V)$ and $\phi^- : \mathcal{F}^- \to \mathbb{P}(V^*)$ induced by ϕ, give rise to a map $\phi^\mathcal{X} : \mathcal{X} \to \mathcal{X}'$ with

$$\mathcal{X}' = \{(L, P) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid V = L \oplus P\} = \mathrm{SL}(V)/L'_{\Delta' \smallsetminus \alpha'}.$$

Let $\hat{\sigma} : \widehat{\Gamma} \to \mathcal{X}$ be the lift of the Anosov section for ρ. We want to prove that $\phi^\mathcal{X} \circ \hat{\sigma}$ is the lift of an Anosov section for $\phi \circ \rho$.

For this let $\hat{\beta}' : \widehat{\Gamma} \to \mathcal{Y}'$ be an equivariant lift of $\hat{\sigma}' = \phi^\mathcal{X} \circ \hat{\sigma}$ where $\mathcal{Y}' = \mathrm{SL}(V)/M'_{\Delta' \smallsetminus \alpha'}$, with $M'_{\Delta' \smallsetminus \alpha'} = L'_{\Delta' \smallsetminus \alpha'} \cap \mathrm{SO}(V)$ the maximal compact subgroup of $L'_{\Delta' \smallsetminus \alpha'}$. Let $\mu'_+ : \Gamma \backslash \widehat{\Gamma} \times \mathbf{R} \to \mathfrak{a}'^+_{L'_{\Delta' \smallsetminus \alpha'}}$ the Cartan projection associated to $\hat{\beta}'$ (Section 3.3). By Proposition 3.16 it is enough to prove that $\lim_{t \to +\infty} \alpha'(\mu'_+(m,t)) = +\infty$.

There is also a natural map $\phi^\mathcal{Y} : \mathcal{Y} \to \mathcal{Y}'$ induced by ϕ. If $\hat{\beta}$ is an equivariant lift $\hat{\sigma}$, one can suppose that $\hat{\beta}' = \phi^\mathcal{Y} \circ \hat{\beta}$. Let μ_+ the Cartan projection associated with $\hat{\beta}$. By Proposition 3.16 we have that, for all $\alpha \in \Delta \smallsetminus \Theta$, $\lim_{t \to +\infty} \alpha(\mu_+(m,t)) = +\infty$. Furthermore $\mu'_+(m,t)$ is the unique element of the orbit of $\phi_*(\mu_+(m,t))$ under the Weyl group $W'_{\Delta' \smallsetminus \alpha'}$ of $L'_{\Delta' \smallsetminus \alpha'}$ that belongs to the Weyl chamber $\mathfrak{a}'^+_{L'_{\Delta' \smallsetminus \alpha'}}$. From this fact and the above description of the Weyl chamber and the Weyl group we get that

$$\alpha'(\mu'_+(m,t)) = \min_{i=2,\dots,n} (\lambda_1 - \lambda_i)(\mu_+(m,t)).$$

The third conclusion of the previous lemma implies then:

$$\alpha'(\mu'_+(m,t)) \geq \min_{\alpha\in\Delta\smallsetminus\Theta} \alpha(\mu_+(m,t)).$$

This last inequality proves: $\lim_{t\to+\infty} \alpha'(\mu'_+(m,t)) = +\infty$. □

The above proof relies on the fact that we are able to estimate μ'_+ in terms of μ_+. More precisely, we used the fact that $\mu_+(m,t)$ belongs to $\overline{\mathfrak{a}}^+ \smallsetminus \bigcup_{\alpha\in\Theta} \ker(\alpha)$ and that the image of this last cone by ϕ_* is contained in $W'_{\Theta'} \cdot \big(\overline{\mathfrak{a}}'^+ \smallsetminus \bigcup_{\alpha'\in\Theta'} \ker(\alpha')\big)$, which is a reformulation of the properties deduced in Lemma 4.2. This is precisely the condition in the general statement:

Proposition 4.4. *Let $\phi : G \to G'$ be a Lie group homomorphism as above. Let $\Theta \subset \Delta$ and suppose that there exist w' in W' and $\Theta' \subset \Delta'$ such that*

$$\phi_*\big(\overline{\mathfrak{a}}^+ \smallsetminus \bigcup_{\alpha\in\Theta} \ker(\alpha)\big) \subset w' \cdot W'_{\Theta'} \cdot \big(\overline{\mathfrak{a}}'^+ \smallsetminus \bigcup_{\alpha'\in\Theta'} \ker(\alpha')\big).$$

Then, for any P^+_Θ-Anosov representation $\rho : \Gamma \to G$, the representation $\phi\circ\rho$ is $P'^+_{\Theta'}$-Anosov. Furthermore $\phi(P^\pm_\Theta) \subset w' P'^\pm_{\Theta'} w'^{-1}$ and hence there are maps $\phi^+ : \mathcal{F}^+_\Theta \to \mathcal{F}'^+_{\Theta'}$ and $\phi^- : \mathcal{F}^-_\Theta \to \mathcal{F}'^-_{\Theta'}$. If $\xi^\pm$ are the Ansosov maps associated to ρ, the Anosov maps for $\phi\circ\rho$ are $\phi^\pm \circ \xi^\pm$.

Remark 4.5. It can happen that $\Theta' = \Delta'$, i.e. that $P'^+_{\Theta'} = G'$.

Proof. Up to changing the Weyl chamber of G' one can suppose that $w' = 1$.

We first prove that $\phi(P^\pm_\Theta) \subset P'^\pm_{\Theta'}$. For this note that P^+_Θ contains the stabilizer of any point of the visual compactification of the symmetric space G/K that is the endpoint at infinity of a geodesic ray $(\exp(ta)\cdot K)_{t\in\mathbf{R}_{\geq 0}}$ with $a \in \overline{\mathfrak{a}}^+ \smallsetminus \bigcup_{\alpha\in\Theta}\ker(\alpha)$. This geometric characterization and the hypothesis on ϕ_* imply that $\phi(P^+_\Theta)$ is contained in $\omega P'^+_{\Theta'} \omega^{-1}$ for some $\omega \in W'_{\Theta'}$. Here $\omega \in W'_{\Theta'}$ is such that $\omega \cdot \big(\overline{\mathfrak{a}}'^+ \smallsetminus \bigcup_{\alpha'\in\Theta'} \ker(\alpha')\big)$ contains $\phi_*(a)$ with $a \in \overline{\mathfrak{a}}$ satisfying $\alpha(a) = 0$ for $\alpha \in \Delta\smallsetminus\Theta$ and $\alpha(a) > 0$ for $\alpha \in \Theta$. In conclusion $\phi(P^+_\Theta) \subset \omega P'^+_{\Theta'}\omega^{-1} = P'^+_{\Theta'}$. Similarly $\phi(P^-_\Theta) \subset P'^-_{\Theta'}$.

Now the proof follows the same lines as the proof of the Proposition 4.3, where the hypothesis on ϕ_* replaces the use of Lemma 4.2. □

Notice that in view of establishing Proposition 3.16 one could equally work with a continuous function $\mu_{+,\Theta} : \Gamma\backslash\widehat{\Gamma} \times \mathbf{R} \to C + \overline{\mathfrak{a}}^+_{L_\Theta}$, where $C \subset \mathfrak{a}$ is a compact subset, satisfying that, for all $(\hat{m}, t)$, the pair $(\hat{\beta}(\hat{m}), \hat{\beta}(\phi_t\hat{m}))$ is in the G-orbit of $(\exp(\mu_{+,\Theta}(\hat{m},t))M, M)$. This gives a refined version of Proposition 4.4:

Proposition 4.6. *Let $\Theta \subset \Delta$ and let $\rho : \Gamma \to G$ be a P_Θ-Anosov representation; let $\mu_{+,\Theta} : \Gamma\backslash\widehat{\Gamma} \times \mathbf{R} \to \overline{\mathfrak{a}}^+_{L_\Theta}$ be the L_Θ-Cartan projection defined in Section 3.3. Suppose that there exist w' in W' and $\Theta' \subset \Delta'$ and a compact $C' \subset \mathfrak{a}'$ such that*

$$\phi(\mathrm{Im}(\mu_{+,\Theta})) \subset C' + w' \cdot W'_{\Theta'} \cdot \big(\overline{\mathfrak{a}}'^+ \smallsetminus \bigcup_{\alpha'\in\Theta'} \ker(\alpha')\big).$$

Then $\phi\circ\rho$ is Θ'-Anosov.

4.2. **Groups of rank one.** When G is of rank one, for any homomorphism $\phi : G \to G'$ one can arrange that the closed Weyl chamber $\overline{\mathfrak{a}}'^+$ contains $\phi_*(\mathfrak{a}^+)$. Thus Proposition 4.4 implies the following (see also [57, Proposition 3.1])

Proposition 4.7. *Let G be a Lie group of real rank one. Let $\rho : \Gamma \to G$ be an Anosov representation and $\phi : G \to G'$ a homomorphism of Lie groups. Assume that the Weyl chambers of G and G' are arranged so that $\phi(\mathfrak{a}^+) \subset \overline{\mathfrak{a}}'^+$.*

Then $\phi \circ \rho$ is $P_{\Theta'}$-Anosov where $\Theta' = \{\alpha' \in \Delta' \mid \phi^*\alpha' = 0\}$, where $\phi^* : \mathfrak{a}'^* \to \mathfrak{a}^*$ is the map induced by ϕ..

4.3. **Injection of Lie groups: examples and counterexamples.** We describe an example that shows that composing an Anosov representation $\rho : \Gamma \to G$ with an injective Lie group homomorphism $\phi : G \to G'$ does not always give rise to a (nontrivial) Anosov representation.

Let G_1 and G_2 be two copies of $\mathrm{SL}(2, \mathbf{R})$, $G = G_1 \times G_2$ and $G' = \mathrm{SL}(4, \mathbf{R})$ with the natural injection $\phi : G_1 \times G_2 \to G'$. Let P_1 and P_2 be parabolic subgroups of G_1 and G_2. Up to conjugation the proper parabolic subgroups of $G_1 \times G_2$ are $P_1 \times G_2$, $P_1 \times P_2$ and $G_1 \times P_2$.

Let Q_0 be the stabilizer in G' of a line in $\mathbf{R}^4$ and let Q_2 be the stabilizer of a 2-plane. Let $\iota_1, \iota_2 : \Gamma \to G_1, G_2$ be non-conjugate discrete and faithful representations of a surface group Γ; ι_1 is P_1-Anosov, ι_2 is P_2-Anosov. Define the two representations

$$\rho = (\iota_1, 1) : \Gamma \to G_1 \times G_2, \quad \rho' = (\iota_1, \iota_2) : \Gamma \to G_1 \times G_2;$$

then ρ and ρ' are $P_1 \times G_2$-Anosov. The representation $\phi \circ \rho : \Gamma \to G$ is Q_0-Anosov. However the representation $\phi \circ \rho'$ is not Q_0-Anosov. If it were, this would imply that for all $\gamma \in \Gamma$ one has $|\mathrm{tr}(\iota_1(\gamma))| \geq |\mathrm{tr}(\iota_2(\gamma))|$. This is impossible unless ι_1 and ι_2 are conjugate (see [72, Theorem 3.1]).

The representation ρ' is $P_1 \times P_2$-Anosov and, as a consequence, the composition $\phi \circ \rho'$ is Q_2-Anosov. However, by choosing ι_2 appropriately (not discrete and faithful), one can ensure that the composition $\phi \circ \rho'$ is also not Q_2-Anosov, and hence not Anosov with respect to any proper parabolic subgroup of G'.

4.4. **Equivariant maps.** In this section we consider representations $\rho : \Gamma \to G$ that admit a pair of continuous transverse equivariant maps (ξ^+, ξ^-), $\xi^\pm : \partial_\infty \Gamma \to \mathcal{F}^\pm$ without requiring any contraction property. We conclude that such representations are Anosov, at least up to considering them into a subgroup of G.

Recall that a pair $(x^+, x^-) \in \mathcal{F}^+ \times \mathcal{F}^-$ is transverse if it belongs to $\mathcal{X} \subset \mathcal{F}^+ \times \mathcal{F}^-$.

Definition 4.8. A pair $(x^+, x^-) \in \mathcal{F}^+ \times \mathcal{F}^-$ is said to be *singular* if $\mathrm{Stab}(x^+) \cap \mathrm{Stab}(x^-)$ is a parabolic subgroup. There is one G-orbit of singular pairs, namely the orbit of $(P_\Theta^+, P_{\iota(\Theta)}^+)$; we denote the set of singular pairs by $\mathcal{S} \subset \mathcal{F}^+ \times \mathcal{F}^-$.

Definition 4.9. A pair of maps (ξ^+, ξ^-), $\xi^\pm : \partial_\infty \Gamma \to \mathcal{F}^\pm$, is said to be *compatible* if

$$\forall t \in \partial_\infty \Gamma,\ (\xi^+(t), \xi^-(t)) \in \mathcal{S}, \quad \forall t^+ \neq t^- \in \partial_\infty \Gamma,\ (\xi^+(t^+), \xi^-(t^-)) \in \mathcal{X}.$$

Due to Proposition 4.3 the main case we have to consider is when $G = \mathrm{SL}(V)$.

Proposition 4.10. *Let $V = D \oplus H$ be a decomposition of a vector space V into a line and a hyperplane. $Q_0^+ = \mathrm{Stab}(D)$ and $Q_0^- = \mathrm{Stab}(H)$, and denote by $\mathcal{F}^+ =$*

$G/Q_0^+ = \mathbb{P}(V)$ and $\mathcal{F}^- = G/Q_0^- = \mathbb{P}(V^*)$ be the corresponding homogeneous spaces. Let $\rho : \Gamma \to \mathrm{SL}(V)$ be a representation.

Suppose that:

- *ρ is irreducible, and*
- *ρ admits a compatible pair (ξ^+, ξ^-) of continuous equivariant maps.*

Then ρ is (Q_0^+, Q_0^-)-Anosov and $\xi^\pm$ are its Anosov maps.

Proof. Only the contraction property needs to be proved. The basic observation is that the action of the group Γ on its boundary at infinity $\partial_\infty\Gamma$ already exhibits some contraction property and hence one gets contraction along the image of $\partial_\infty\Gamma$ by ξ^+. We will use the maps $\xi^\pm$ as much as possible to define an equivariant family of norms $(\|\cdot\|_{\hat m})_{\hat m \in \widehat\Gamma}$ and prove the contraction property.

The projection $\widehat\Gamma \to \Gamma\backslash\widehat\Gamma$ is denoted by π.

Definition of $\|\cdot\|_{\hat m}$.

We already observed earlier, that it is enough to prove dilatation on $(\sigma^* E^+)_{\hat m} = T_{\xi^+(\tau^+(\hat m))}\mathbb{P}(V)$, thus we define norms only on these spaces.

The irreducibility of ρ implies that for any $\hat p \in \widehat\Gamma$:

- there exist an open neighborhood $V_{\hat p}$ of $\hat p$ and $t^1_{\hat p}, \dots, t^{n-1}_{\hat p} \in \partial_\infty\Gamma$, such that
- for all $\hat m \in V_{\hat p}$, the sum $\xi^+(\tau^+(\hat m)) + \xi^+(t^1_{\hat p}) + \cdots + \xi^+(t^{n-1}_{\hat p})$ is direct,
- for all $\hat m \in V_{\hat p}$ and all i, the sum $\xi^-(\tau^-(\hat m)) + \xi^+(t^i_{\hat p})$ is direct,
- π is injective in restriction to $V_{\hat p}$.

In particular, for all $\hat m \in V_{\hat p}$ and all i, $\tau^\pm(\hat m) \neq t^i_{\hat p}$. Furthermore, the set $\{(\tau^+(\hat m), t^i_{\hat p}, \tau^-(\hat m)) \mid \hat m \in V_{\hat p}\}$ is contained in a compact subset of the set of pairwise distinct triples of $\partial_\infty\Gamma$, which we denote by $\partial_\infty\Gamma^{(3)}$.

We first construct a basis $(e^i_{\hat p}(\hat m))_{i=1,\dots,n-1}$ of $T_{\xi^+(\tau^+(\hat m))}\mathbb{P}(V)$, for any $\hat m \in V_{\hat p}$, that varies continuously with $\hat m$. The vector $e^i_{\hat p}(\hat m)$ is defined by the property that the corresponding map $\phi : \xi^+(\tau^+(\hat m)) \to \xi^-(\tau^-(\hat m))$ (under the isomorphism $T_{\xi^+(\tau^+(\hat m))}\mathbb{P}(V) \cong \mathrm{Hom}(\xi^+(\tau^+(\hat m)), \xi^-(\tau^-(\hat m))))$ is such that $\xi^+(t^i_{\hat p}) = \{v + \phi(v) \mid v \in \xi^+(\tau^+(\hat m))\}$. We say that $e^i_{\hat p}(\hat m)$ corresponds to the line $\xi^+(t^i_{\hat p})$.

In turn, when $\hat m$ is in $V_{\hat p}$, we can define a norm on $T_{\xi^+(\tau^+(\hat m))}\mathbb{P}(V)$:

$$\|v\|_{\hat p} = \sum |\lambda^i| \text{ if } v = \sum \lambda^i e^i_{\hat p}(\hat m).$$

By compactness, there exist $\hat p_1, \dots, \hat p_K$ such that $\Gamma\backslash\widehat\Gamma = \bigcup \pi(V_{\hat p_k})$. For ease of notation, we will write $V_k = V_{\hat p_k}$, $e^i_k(\hat m) = e^i_{\hat p_k}(\hat m)$, $\|\cdot\|_k = \|\cdot\|_{\hat p_k}$. There exist continuous functions $f_1, \dots, f_K : \Gamma\backslash\widehat\Gamma \to \mathbf{R}_{\geq 0}$ such that $\sum f_k = 1$ and $\mathrm{Supp}(f_k) \subset \pi(V_k)$ for all k.

For all $\hat m \in \widehat\Gamma$, we now define a norm $\|\cdot\|_{\hat m}$ on $T_{\xi^+(\tau^+(\hat m))}\mathbb{P}(V)$ in a Γ-equivariant way using the $\|\cdot\|_k$ and the f_k. For all k, if $\hat m \in \Gamma \cdot V_k = \pi^{-1}(\pi(V_k))$, there exists a unique $\gamma^{\hat m}_k \in \Gamma$ such that $\gamma^{\hat m}_k \cdot \hat m$ belongs to V_k. We then set

$$\|v\|_{\hat m} = \sum f_k(\pi(\hat m)) \|\rho(\gamma^{\hat m}_k) v\|_k = \sum f_k(\pi(\hat m)) \|v\|_{\hat m, k}, \text{ for } v \in T_{\xi^+(\tau^+(\hat m))}\mathbb{P}(V),$$

this is well defined and continuous in $\hat m$.

The relation $\gamma^{\hat m}_k = \gamma^{\gamma\hat m}_k \gamma$ is easy to check and implies the equivariance:

$$\|\rho(\gamma)v\|_{\gamma\cdot\hat m} = \sum f_k(\pi(\gamma\hat m)) \|\rho(\gamma^{\gamma\hat m}_k)\rho(\gamma)v\|_k = \sum f_k(\pi(\hat m)) \|\rho(\gamma^{\hat m}_k)v\|_k = \|v\|_{\hat m}.$$

The contraction property.

To check the contraction property, as in the proof of Proposition 3.16, we only need to prove weak dilatation, that is

– $\forall \hat{m} \in \widehat{\Gamma}$, $v \in T_{\xi^+(\tau^+(\hat{m}))}\mathbb{P}(V)$, and sequence $(x_l)_{l\in\mathbf{N}}$ in $\mathbf{R}$ with $x_l \mapsto +\infty$, one has $\lim \|v\|_{\phi_{x_l}\hat{m}} = +\infty$.

Note that it is enough to have $\lim \|v\|_{\phi_{x_l}\hat{m},k} = +\infty$ for some k. Hence we can assume (up to passing to a subsequence) that, for all l, $\phi_{x_l}\hat{m} \in \Gamma \cdot V_k$. Let γ_l be such that $\gamma_l \cdot \phi_{x_l}\hat{m} \in V_k$. Thus, by definition,

$$\|v\|_{\phi_{x_l}\hat{m},k} = \sum |\lambda_l^i| \text{ with } \rho(\gamma_l)v = \sum \lambda_l^i e_k^i(\gamma_l \cdot \phi_{x_l}\hat{m});$$

hence $v = \sum \lambda_l^i \epsilon_l^i$ where $\epsilon_l^i = \rho(\gamma_l^{-1})e_k^i(\gamma_l \cdot \phi_{x_l}\hat{m})$ is the vector of $T_{\xi^+(\tau^+(\hat{m}))}\mathbb{P}(V) \cong \mathrm{Hom}(\xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m})))$ which corresponds to the line $\xi^+(\gamma_l^{-1}\cdot t_k^i)$. It is therefore enough to prove that $\epsilon_l^i \mapsto 0$ which is equivalent to $\xi^+(\gamma_l^{-1}\cdot t_k^i) \mapsto \xi^+(\tau^+(\hat{m}))$. From the continuity of ξ^+, it suffices to prove $\gamma_l^{-1}\cdot t_k^i \mapsto \tau^+(\hat{m})$.

For this, note first that, since the sequence $(\gamma_l \cdot \phi_{x_l}\hat{m})_{l\in\mathbf{N}}$ is contained in V_k, the triples $(\tau^+(\gamma_l\cdot\phi_{x_l}\hat{m}), t_k^i, \tau^-(\gamma_l\cdot\phi_{x_l}\hat{m}))$ belong to a compact subset of $\partial_\infty\Gamma^{(3)}$. The action of Γ on $\partial_\infty\Gamma^{(3)}$ is proper and cocompact (see [15]), thus $\gamma_l \mapsto \infty$ implies that the sequence $(\tau^+(\phi_{x_l}\hat{m}), \gamma_l^{-1}\cdot t_k^i, \tau^-(\phi_{x_l}\hat{m})) = (\tau^+(\hat{m}), \gamma_l^{-1}\cdot t_k^i, \tau^-(\hat{m}))$ diverges in $\partial_\infty\Gamma^{(3)}$. This means either that $\gamma_l^{-1}\cdot t_k^i \mapsto \tau^+(\hat{m})$ or that $\gamma_l^{-1}\cdot t_k^i \mapsto \tau^-(\hat{m})$. The second possibility is easily eliminated (as it would contradict that $\phi_{x_l}\hat{m} = \gamma_l^{-1}\cdot\hat{m}_l$ tends to $\tau^+(\hat{m})$ with $\hat{m}_l = \gamma_l\cdot\phi_{x_l}\hat{m}$ being bounded). Thus we conclude that $\gamma_l^{-1}\cdot t_k^i \mapsto \tau^+(\hat{m})$. □

From Proposition 4.10 and Proposition 4.3, we deduce the following

Theorem 4.11. *Let $\rho : \Gamma \to G$ be a Zariski dense representation and $P^+, P^- < G$ opposite parabolic subgroups of G. Suppose that ρ admits a pair of equivariant continuous compatible maps (Definition 4.9) $\xi^+ : \partial_\infty\Gamma \to \mathcal{F}^+$, $\xi^- : \partial_\infty\Gamma \to \mathcal{F}^-$.*

Then the representation ρ is (P^+, P^-)-Anosov and (ξ^+, ξ^-) are the associated Anosov maps.

Proof. By classical representation theory there exists an irreducible G-module V admitting a decomposition $V = D \oplus H$ into a line and a hyperplane and such that $P^+ = \mathrm{Stab}_G(D)$ and $P^- = \mathrm{Stab}_G(H)$. The result then follows from Proposition 4.10 and Proposition 4.3. □

Remark 4.12. A G-module satisfying the hypothesis of Proposition 4.3 is easy to find; e.g. $\bigwedge^p \mathfrak{g}$ where $p = \dim \mathfrak{p}^+$. Taking the irreducible factor containing the line $\bigwedge^p \mathfrak{p}^+$ gives an irreducible module V satisfying the requirements of the above proof.

Remark 4.13. As a conclusion of Theorem 4.11, a representation $\rho : \Gamma \to G$ admitting a pair of compatible ρ-equivariant and continuous maps (ξ^+, ξ^-) is Anosov when considered as a representation into its *Zariski closure* H (or more precisely into the quotient of H by its radical, since we define Anosov representations only into semisimple Lie groups). Proposition 4.4 then gives sufficient conditions for the representation $\rho : \Gamma \to G$ to be Anosov.

Remark 4.14. Recently Sambarino [67] established counting theorems for representations of fundamental groups of negatively curved manifolds into $\mathrm{SL}(V)$ satisfying the assumptions of Proposition 4.10. Using Proposition 3.16 his results should extend to all (Q_0^+, Q_0^-)-Anosov representations of fundamental groups of negatively curved manifolds into $\mathrm{SL}(V)$.

4.5. **Parabolic subgroups conjugate to their opposite.** A parabolic subgroup P_Θ^+ is conjugate to P_Θ^- if and only if $\Theta = \iota(\Theta)$. Lemma 3.18 states that any Anosov representation is P_Θ-Anosov for some Θ satisfying $\Theta = \iota(\Theta)$. In that case the two homogeneous spaces $\mathcal{F}_\Theta^+ = G/P_\Theta^+$ and $\mathcal{F}_\Theta^- = G/P_\Theta^-$ are canonically identified. Hence, by uniqueness (Lemma 3.3), there is a single Anosov map

$$\xi = \xi^+ = \xi^- : \partial_\infty\Gamma \to \mathcal{F}_\Theta^+ = \mathcal{F}_\Theta^- = \mathcal{F}_\Theta.$$

Definition 4.15. A map $\xi : \partial_\infty\Gamma \to \mathcal{F}_\Theta$ is said to be *transverse* if for all $t^+ \neq t^- \in \partial_\infty\Gamma$, the pair $(\xi(t^+), \xi(t^-)) \subset \mathcal{F}_\Theta \times \mathcal{F}_\Theta$ is transverse.

A special case of Theorem 4.11 is the following

Corollary 4.16. *Let $\rho : \Gamma \to G$ be a Zariski dense representation. Suppose that $\Theta = \iota(\Theta)$ and assume that there exists a continuous ρ-equivariant map $\xi : \partial_\infty\Gamma \to \mathcal{F}_\Theta$ such that for all $t^+ \neq t^- \in \partial_\infty\Gamma$, the pair $(\xi(t^+), \xi(t^-))$ is transverse.*

Then the representation ρ is P_Θ-Anosov.

When $\Theta = \iota(\Theta)$, there exists an irreducible G-module V with a G-invariant non-degenerate bilinear form F, and an isotropic line D in V such that $P_\Theta^+ = \mathrm{Stab}_G(D)$. We denote by G_F the automorphism group of (V, F). The irreducibility implies that F can be supposed to be either symmetric or skew-symmetric, i.e. $G_F = \mathrm{O}(V, F)$ or $\mathrm{Sp}(V, F)$. We denote by Q_0 the stabilizer in G_F of the line D. This discussion together with Proposition 4.3 implies

Proposition 4.17. *A representation $\rho : \Gamma \to G$ is Anosov if and only if there is a self-dual G-module (V, F) with $\phi : G \to G_F$ the corresponding homomorphism, such that $\phi \circ \rho$ is Q_0-Anosov.*

Applying the construction of Lemma 8.8.(iii) below we deduce the following

Corollary 4.18. *A representation $\rho : \Gamma \to G$ is Anosov if and only if there is a Lie group homomorphism $\phi : G \to \mathrm{O}(V, F)$ such that $\phi \circ \rho : \Gamma \to \mathrm{O}(V, F)$ is a Q_0-Anosov representation.*

5. Discreteness, metric properties and openness

5.1. **Quasi-isometric embeddings and well displacing.** The group Γ is endowed with the left invariant distance d_Γ coming from a word length ℓ_Γ. The group G is endowed with the distance d_G comming from a left invariant Riemannian metric. With this distance G is quasi-isometric to any homogeneous space G/M where M is a compact subgroup, endowed with a left invariant Riemannian metric. The translation length of an element $\gamma \in \Gamma$ (resp. g in G) is

$$t_\Gamma(\gamma) = \inf_{x\in\Gamma} d_\Gamma(x, \gamma x) \quad (\text{resp. } t_G(g) = \inf_{x\in G} d_G(x, gx)).$$

Definition 5.1. A representation $\rho : \Gamma \to G$ is a *quasi-isometric embedding* if there exist positive constants K, C such that, for every $\gamma \in \Gamma$,

$$K^{-1}\ell_\Gamma(\gamma) - C \leq d_G(1, \rho(\gamma)) \leq K\ell_\Gamma(\gamma) + C$$

(for generalities on quasi-isometries, quasi-geodesics, etc. see [26, Chapitre 3]). A representation $\rho : \Gamma \to G$ is said to be *well displacing* [27, 59] if, for all $\gamma \in \Gamma$,

$$K^{-1} t_\Gamma(\gamma) - C \leq t_G(\rho(\gamma)) \leq K t_\Gamma(\gamma) + C.$$

Remark 5.2. Note that, since Γ is finitely generated, the upper bound is automatically satisfied. Furthermore, from the classical equality $t_{G/K}(g) = \lim d_{G/K}(K, g^n K)/n$, where G/K is the symmetric space associated with G, endowed with a left invariant Riemannian metric, it follows that any representation ρ which is a quasi-isometric embedding is also well displacing.

Theorem 5.3. *Let ρ be an Anosov representation, then $\rho : \Gamma \to G$ is a quasi-isometric embedding. In particular:*
(i) $\ker \rho$ is finite, (ii) $\rho(\Gamma) < G$ is discrete, and (iii) ρ is well displacing.

Proof. We consider $\hat{\sigma}$, $\hat{\beta}$, $\mu_{+,\Theta}$, $\mu_{-,\Theta}$, which were introduced in Section 3.3. Since Γ and $\widehat{\Gamma}$ are quasi-isometric, it is enough to show that $\hat{\beta} : \widehat{\Gamma} \to G/M$ is a quasi-isometric embedding. As a matter of fact Proposition 3.16 already shows that there are constants (K, C) such that the restriction of $\hat{\beta}$ to any $\mathbf{R}$-orbit is a (K, C)-quasi-geodesic. To conclude, one has to note the following property of $\widehat{\Gamma}$, which is a consequence of its hyperbolicity: there exists $D \geq 0$ such that for any m, p in $\widehat{\Gamma}$ there exist $m_0, p_0 \in \widehat{\Gamma}$ that are on the same $\mathbf{R}$-orbit and such that $d(m, m_0) \leq D$ and $d(p, p_0) \leq D$. □

In the case when $\Gamma = \pi_1(\Sigma)$ is the fundamental group of a closed connected oriented surface of genus ≥ 2, following arguments of [59, Section 6.3], or when Γ is a free group, following the arguments of [61, Theorem 3.3] one can deduce from Theorem 5.3 and from Theorem 5.14 that the action of the outer automorphism group of Γ on the set of Anosov representations is proper. We expect that the arguments of [61, Theorem 3.3] can be generalized to arbitrary word hyperbolic groups.

For this let $\mathrm{Hom}_{\mathrm{Anosov}}(\Gamma, G)$ denote the set of Anosov representations and denote by $(\mathrm{Hom}_{\mathrm{Anosov}}(\Gamma, G)/G)^{red}$ its Hausdorff quotient (i.e. two elements $x, y \in \mathrm{Hom}_{\mathrm{Anosov}}(\Gamma, G)/G$ are identified if every neighborhood of x meets any neighborhood of y).

Corollary 5.4. *Let Σ be a connected orientable surface of negative Euler characteristic and $\Gamma = \pi_1(\Sigma)$. Then the outer automorphism group of Γ acts properly on $(\mathrm{Hom}_{\mathrm{Anosov}}(\Gamma, G)/G)^{red}$.*

5.2. **Proximality.** In this section we show that images of Anosov representations have strong proximality properties.

Recall that (P^+, P^-) is a fixed pair of opposite parabolic subgroups in G and $\mathcal{F}^\pm = G/P^\pm$ denote the corresponding homogeneous spaces.

For $x^- \in \mathcal{F}^-$, set $V^-(x^-) = \{x \in \mathcal{F}^+ \mid x \text{ and } x^- \text{ are not transverse}\}$.

Definition 5.5. An element $g \in G$ is said to be *proximal* relative to $\mathcal{F}^+$ (or $\mathcal{F}^+$-proximal) if g has two fixed points, $x^+ \in \mathcal{F}^+$ and $x^- \in \mathcal{F}^-$ with $x^+ \notin V^-(x^-)$ and such that for all $x \notin V^-(x^-)$, $\lim_{n\to+\infty} g^n \cdot x = x^+$.

A subgroup $\Lambda < G$ is proximal if it contains at least one proximal element.

When g is proximal, the fixed points x^+ and x^- are uniquely determined, we denote them by x_g^+ and x_g^-.

When $\Lambda < G$ is a subgroup which is proximal with respect to $\mathcal{F}^{\pm}$, then there exists a well defined closed Γ-invariant minimal set $\mathcal{L}_{\Lambda}^{\pm} \subset \mathcal{F}^{\pm}$, which is called the *limit set* of Γ, see [6]; it is the closure of the set of attracting fixed points of proximal elements in Λ.

Let d be a (continuous) distance on $\mathcal{F}^{+}$ and define

- for $x^{+} \in \mathcal{F}^{+}$, $b_{\epsilon}(x^{+}) = \{x \in \mathcal{F}^{+} \mid d(x, x^{+}) \leq \epsilon\}$, and
- for $x^{-} \in \mathcal{F}^{-}$, $B_{\epsilon}(x^{-}) = \{x \in \mathcal{F}^{+} \mid d(x, V^{-}(x^{-})) \geq \epsilon\}$.

Definition 5.6. An element g is (r,ϵ)-*proximal* (or (r,ϵ)-$\mathcal{F}^{+}$-proximal) if g has two fixed points $x^{+} \in \mathcal{F}^{+}$ and $x^{-} \in \mathcal{F}^{-}$ such that $d(x^{+}, V^{-}(x^{-})) \geq r$, $g \cdot B_{\epsilon}(x^{-}) \subset b_{\epsilon}(x^{+})$ and $g|_{B_{\epsilon}(x^{-})}$ is ϵ-contracting.

In [1] Abels, Margulis and Soĭfer investigated proximality properties of strongly irreducible subgroups of $\mathrm{GL}(V)$. In order to restate their result let us make the following

Definition 5.7. A subgroup $\Lambda < G$ is said to be *(AMS)-proximal* (or proximal in the sense of Abels, Margulis ans Soĭfer) relative to $\mathcal{F}^{+}$ if there exist constants $r > 0$ and $\epsilon_0 > 0$ such that, for any $\epsilon < \epsilon_0$ the following holds:

- there exists a finite subset $S \subset \Lambda$ with the property that, for any $\delta \in \Lambda$, there is $s \in S$ such that $s\delta$ is (r,ϵ)-proximal.

A representation $\rho : \Gamma \to G$ is said to be *(AMS)-proximal* if $\ker \rho$ is finite and $\rho(\Gamma)$ is (AMS)-proximal.

With this, the result of Abels, Margulis and Soĭfer can be reformulated as follows.

Theorem 5.8. [1, Theorem 4.1]
If $\Lambda < \mathrm{SL}(V)$ is strongly irreducible (i.e. any finite index subgroup acts irreducibly on V) then Λ is (AMS)-proximal relative to $\mathbb{P}(V)$.

For Anosov representations we have the following:

Theorem 5.9. *If $\rho : \Gamma \to G$ is P^{+}-Anosov, then ρ is (AMS)-proximal relative to $\mathcal{F}^{+}$.*

Proof. Lemma 3.1 already shows that $\rho(\gamma)$ is proximal relative to $\mathcal{F}^{\pm}$, when $\gamma \neq 1$. The following theorem thus implies the statement. □

Theorem 5.10. *Let $\Lambda < G$ be a subgroup. If Λ is proximal relative to $\mathcal{F}^{\pm}$, then Λ is (AMS)-proximal relative to $\mathcal{F}^{\pm}$.*

Proof. The strategy is to reduce to the situation of a strongly irreducible subgroup of $\mathrm{SL}(V)$ and then to apply Theorem 5.8.

Let V be an irreducible G-module with a decomposition $V = D \oplus H$ into a line and a hyperplane, such that $P^{+} = \mathrm{Stab}_G(D)$ and $P^{-} = \mathrm{Stab}_G(H)$, and denote by $\phi : G \to \mathrm{SL}(V)$ the embedding. The induced maps $\phi^{+} : \mathcal{F}^{+} \to \mathbb{P}(V)$ and $\phi^{-} : \mathcal{F}^{-} \to \mathbb{P}(V^{*})$ satisfy the following property: for $x^{-} \in \mathcal{F}^{-}$ we have $V^{-}(x^{-}) = (\phi^{+})^{-1}(\mathbb{P}(\phi^{-}(x^{-})))$, where one considers $\phi^{-}(x^{-})$ as a hyperplane in V. Thus, if $g \in G$ is proximal with respect to $\mathcal{F}^{\pm}$, then $\phi(g)$ is proximal with respect to $\mathbb{P}(V)$ and $\mathbb{P}(V^{*})$. Moreover, if $\phi(g)$ is (r,ϵ)-proximal on $\mathbb{P}(V)$ then g is (r',ϵ')-proximal on $\mathcal{F}^{+}$ for some functions $r' = r'(r)$ and $\epsilon' = \epsilon'(r,\epsilon)$ satisfying $\epsilon'(r,\epsilon) \xrightarrow{\epsilon \to 0} 0$. Thus from now on we suppose that $G = \mathrm{SL}(V)$.

We consider the Λ-invariant subspace $W = \bigcap_{x \in \mathcal{L}_\Lambda^-} x$, where x is regarded as hyperplane in V, and assume that W is non-empty. Using the minimality of the action of Λ on $\mathcal{L}_\Lambda^+$ we have $\mathcal{L}_\Lambda^+ \cap \mathbb{P}(W) = \emptyset$. Moreover $\Lambda < \mathrm{Stab}(W)$ and thus, Λ defines a subgroup $\bar{\Lambda} < \mathrm{SL}(V/W)$ which is $\mathbb{P}(V/W)$ and $\mathbb{P}(V/W^*)$ proximal. Lemma 5.11 below (with $\delta = d(\mathcal{L}_\Lambda^+, \mathbb{P}(W)) > 0$) implies that if $\bar{\Lambda}$ is (AMS)-proximal then so is Λ.

Therefore, we can assume $\bigcap_{x \in \mathcal{L}_\Lambda^-} x = \{0\}$. Analogously we can assume $\sum_{x \in \mathcal{L}_\Lambda^+} x = V$. By Lemma 5.12 below this implies that Λ is strongly irreducible.

Thus Theorem 5.8 implies that Λ is (AMS)-proximal. □

Lemma 5.11. *Let $\delta > 0$ be a real number and $W \subset V$ two vector spaces. Then there are functions $r' : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ and $\epsilon' : \mathbf{R}_{>0}^2 \to \mathbf{R}_{>0}$ satisfying $\epsilon'(r, \epsilon) \xrightarrow{\epsilon \to 0} 0$ and such that:*

– If $g \in \mathrm{Stab}(W)$ is proximal with $W \subset x_g^-$ and $d(W, x_g^+) \geq \delta$ and $\pi(g) \in \mathrm{SL}(V/W)$ is (r, ϵ)-proximal, then g is (r', ϵ')-proximal.

Proof. The statement follows from a direct compactness argument. □

Lemma 5.12. *Let $\Lambda < \mathrm{SL}(V)$ be a $\mathcal{F}^{\pm}$-proximal subgroup. Suppose that*

$$\bigcap_{x \in \mathcal{L}_\Lambda^-} x = \{0\} \quad \text{and} \quad \sum_{x \in \mathcal{L}_\Lambda^+} x = V.$$

Then Λ is strongly irreducible.

Proof. Let Λ' be a finite index subgroup of Λ. Any closed Λ'-invariant subset of $\mathcal{L}_\Lambda^{\pm}$ is either $\emptyset$ or $\mathcal{L}_\Lambda^{\pm}$. Suppose that $W \subset V$ is Λ'-invariant. For any proximal element $g \in \Lambda'$ one has

$$W = x_g^+ \cap W \oplus x_g^- \cap W,$$

where $x_g^{\pm}$ are the attractive fixed points of g in $\mathbb{P}(V)$ and $\mathbb{P}(V)^*$ respectively. Hence $x_g^+ \subset W$ or $x_g^- \supset W$, and consequently one of the following two closed Λ'-invariant subset is nonempty:

$$\{x \in \mathcal{L}_\Lambda^+ \mid x \subset W\} \quad \text{or} \quad \{x \in \mathcal{L}_\Lambda^- \mid x \supset W\}.$$

If the first is nonempty, one concludes that $W \supset \sum_{x \in \mathcal{L}_\Lambda^+} x = V$; if the second set is nonempty, $W \subset \bigcap_{x \in \mathcal{L}_\Lambda^-} x = \{0\}$. In either case W is trivial, proving the strong irreducibility of Λ. □

5.3. **Openness.** In this section we prove that the set of Anosov representations is open. In the case of fundamental groups of negatively curved Riemannian manifolds, this is proven in [57, Proposition 2.1].

Let P be a parabolic subgroup of G. Denote by $\mathrm{Hom}_{P\text{-Anosov}}(\Gamma, G)$ the set of P-Anosov representations.

Theorem 5.13. *The set $\mathrm{Hom}_{P\text{-Anosov}}(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$.*

Furthermore the map $\mathrm{Hom}_{P\text{-Anosov}}(\Gamma, G) \to C^0(\partial_\infty \Gamma, \mathcal{F})$ associating to a representation ρ its Anosov map is continuous.

As a corollary a small adaptation of the proof of Theorem 5.3 gives the following uniformity statement:

Theorem 5.14. *Let $\rho : \Gamma \to G$ be a P-Anosov representation. Then there exist constants $K, C > 0$ and an open neighborhood U of ρ in $\mathrm{Hom}(\Gamma, G)$ such that every representation $\rho' \in U$ is a (K,C)-quasi-isometric embedding.*

Proof of Theorem 5.13. By Proposition 4.3 we can reduce to the case when $G = \mathrm{SL}(V)$ and $P = Q_0^+$. Let $\mathcal{F}^+ = \mathbb{P}(V)$ and $\mathcal{F}^- = \mathbb{P}(V^*)$. Given a representation $\rho : \Gamma \to G$ consider the bundles $\mathcal{F}_\rho^+ = \widehat{\Gamma} \times_\rho \mathcal{F}^+$ and $\mathcal{F}_\rho^- = \widehat{\Gamma} \times_\rho \mathcal{F}^-$. We denote by $(d_m)_{m \in \Gamma\backslash\widehat{\Gamma}}$ a (continuous) family of distances on the fibers of $\mathcal{F}_\rho^+$, i.e. d_m is a distance on $(\mathcal{F}_\rho^+)_m \cong \mathbb{P}(V)$. We will regard an element in $(\mathcal{F}_\rho^-)_m \cong \mathbb{P}(V^*)$ as a hyperplane in $(\mathcal{F}_\rho^+)_m$.

Suppose now that ρ is Anosov. Let ξ^+ and ξ^- be the Anosov maps, and $\sigma : \Gamma\backslash\widehat{\Gamma} \to \mathcal{X}_\rho \subset \mathcal{F}_\rho^+ \times \mathcal{F}_\rho^-$ the section defined by (ξ^+, ξ^-). For all $\epsilon > 0$ consider (topological) subbundles B_ϵ and b_ϵ defined fiberwise, i.e. for all $m \in \Gamma\backslash\widehat{\Gamma}$,

$$(B_\epsilon)_m = \{l \in (\mathcal{F}_\rho^+)_m \mid d_m(l, \xi^-(m)) > \epsilon\},$$
$$(b_\epsilon)_m = \{l \in (\mathcal{F}_\rho^+)_m \mid d_m(l, \xi^+(m)) < \epsilon\}.$$

The space of continuous sections of those bundles are denoted by $\Gamma(\mathcal{F}_\rho^+)$, $\Gamma(B_\epsilon)$ and $\Gamma(b_\epsilon)$. Note that $\Gamma(\mathcal{F}_\rho^+)$ is a complete metric space. Furthermore there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, $b_\epsilon \subset B_\epsilon$.

The flow ϕ_t acts naturally on $\mathcal{F}_\rho^+$ and on the space of sections $\Gamma(\mathcal{F}_\rho^+)$. The contraction property implies:

- for all $\epsilon < \epsilon_0$ there exists T_ϵ such that for all $t \geq T_\epsilon$, and for all $f \in \Gamma(B_\epsilon)$ one has $\phi_{-t} \cdot f \in \Gamma(b_\epsilon)$. Moreover, for all $t \geq T_\epsilon$, the map $\phi_{-t} : \Gamma(B_\epsilon) \to \Gamma(B_\epsilon)$ is ϵ-contracting.

Now let U be a neighborhood of ρ in $\mathrm{Hom}(\Gamma, G)$. Consider the bundles $\mathcal{F}_U^\pm$ over $U \times \Gamma\backslash\widehat{\Gamma}$:

$$\mathcal{F}_U^\pm = (U \times \widehat{\Gamma}) \times_\rho \mathcal{F}^\pm,$$

where the action of Γ on $U \times \widehat{\Gamma}$ is trivial on the first factor, i.e. $\gamma \cdot (\rho', \hat{m}) = (\rho', \gamma \cdot \hat{m})$.

For U small enough the bundles $(\mathcal{F}_{\rho'}^\pm)_{\rho' \in U}$ are all isomorphic, i.e. there exists a bundle isomorphism $\psi : \mathcal{F}_U^+ \to U \times \mathcal{F}_\rho^+$ with $\psi|_{\mathcal{F}_\rho^+} = \mathrm{Id}$ ([70, p. 53]). Note that the flow ϕ_t acts on $\mathcal{F}_U^+$ and hence on the space of sections $\Gamma(\mathcal{F}_U^+)$.

By continuity (and again for U small enough) there exists $\epsilon_1 > 0$ such that for all $\epsilon < \epsilon_1$ there exists T_0 such that for all $t \geq T_0$ the following holds:

- for any section $f_U \in \Gamma(\psi^{-1}(U \times B_\epsilon))$, its image $\phi_{-t} \cdot f_U$ by the flow belongs to $\Gamma(\psi^{-1}(U \times b_\epsilon))$; moreover the map $\phi_{-t} : \Gamma(\psi^{-1}(U \times B_\epsilon)) \to \Gamma(\psi^{-1}(U \times B_\epsilon))$ is 2ϵ-contracting.

Since $\{\phi_{-t}\}_{t \geq T_0}$ is a commuting family of contracting maps, they have a unique common fixed point ξ_U^+ in $\Gamma(\psi^{-1}(U \times B_\epsilon))$. Certainly ξ_U^+ is also fixed by ϕ_t for any $t \in \mathbf{R}$. Furthermore, the contraction property implies that $\xi_U^+|_{\{\rho\} \times \Gamma\backslash\widehat{\Gamma}} = \xi^+$. Similarly one finds ξ_U^- extending ξ^-.

Since ξ_U^+ and ξ_U^- are transverse in restriction to $\{\rho\} \times \Gamma\backslash\widehat{\Gamma}$, for U small enough, ξ_U^+ and ξ_U^- are transverse on $U \times \Gamma\backslash\widehat{\Gamma}$. Therefore $\mathcal{X}_U = (U \times \widehat{\Gamma}) \times_\rho \mathcal{X} \subset \mathcal{F}_U^+ \times \mathcal{F}_U^-$ admits a section $\sigma_U = (\xi_U^+, \xi_U^-)$ that is flat along flow lines.

The action of ϕ_t on $(\sigma_U^* E^+)|_{\{\rho\} \times \Gamma\backslash\widehat{\Gamma}}$ (resp. $(\sigma_U^* E^-)|_{\{\rho\} \times \Gamma\backslash\widehat{\Gamma}}$) is dilating (resp. contracting) (see Section 2.1 for the definition of $E^\pm$). Hence, again for U small enough, this implies that the action of ϕ_t is dilating on $\sigma_U^* E^+$ (resp. contracting on $\sigma_U^* E^-$).

This shows that there exists a neighborhood U of ρ in $\mathrm{Hom}(\Gamma, G)$ such that any $\rho' \in U$ is Q_0^+-Anosov, and moreover the Anosov map varies continuously with ρ'. □

5.4. **Groups of rank one.** When G is of rank one, there is only one conjugacy class of parabolic subgroups. Henceforth we can talk of Anosov representations unambiguously. Furthermore two points in $\mathcal{F} = G/P$ are transverse if and only if there are distinct.

A subgroup $\Lambda < G$ is said to be *convex cocompact* if it acts properly discontinuously and cocompactly on a convex subset $\mathcal{C}$ of the symmetric space G/K. In that case Λ is hyperbolic, and $\partial_\infty \Lambda \cong \partial_\infty \mathcal{C}$ injects into $\partial_\infty(G/K) \cong G/P$ and the injection $\Lambda \to G$ is a quasi-isometric embedding. Conversely, [14, Corollaire 1.8.4, Proposition 1.8.6] if the injection $\Lambda \to G$ is a quasi-isometric embedding then Λ is convex cocompact. Thus from [14] and the characterizations of Anosov representations one has:

Theorem 5.15. *Let G be a Lie group of real rank one. Let $\rho : \Gamma \to G$ be a representation. Then the following are equivalent:*

(i) ρ is Anosov.
(ii) There exists $\xi : \partial_\infty \Gamma \to G/P$ a continuous, injective and equivariant map.
(iii) ρ is a quasi-isometric embedding.
(iv) $\ker \rho$ is finite and $\Lambda = \rho(\Gamma)$ is convex cocompact.

6. Examples

In this section we give various examples of Anosov representations.

6.1. **Groups of rank one.** If G is a semisimple Lie group of rank one and $\Gamma < G$ is a convex cocompact subgroup, then the injection $\iota : \Gamma \to G$ is Anosov by Theorem 5.15. This gives the following examples:

(i) Inclusion of uniform lattices.
(ii) Embeddings of free groups as Schottky groups.
(iii) Embeddings of Fuchsian groups into $\mathrm{PSL}(2, \mathbf{R})$.
(iv) Embeddings of quasi-Fuchsian groups into $\mathrm{PSL}(2, \mathbf{C})$.

Composing the representation $\iota : \Gamma \to G$ with an embedding $\phi : G \to G'$ of G into a Lie group of higher rank G', we obtain an Anosov representation $\phi \circ \iota : \Gamma \to G'$ (Proposition 4.7). By Theorem 5.13 any small enough deformation of $\phi \circ \iota$ is also an Anosov representation. In many cases there exist small deformations with Zariski dense image. In some particular cases all deformations of $\phi \circ \iota$ remain Anosov representations.

We list some examples, several of which will be discussed in more detail below:

(i) Holonomies of convex real projective structures: Let $\iota : \Gamma \to \mathrm{SO}(1, n)$ be the embedding of a uniform lattice. Consider $\phi \circ \iota : \Gamma \to \mathrm{PGL}(n + 1, \mathbf{R})$, where $\phi : \mathrm{SO}(1, n) \to \mathrm{PGL}(n + 1, \mathbf{R})$ is the standard embedding; this is a Q_0-Anosov representation, where $Q_0 < \mathrm{PGL}(n + 1, \mathbf{R})$ is the stabilizer of a line. Moreover, $\iota(\Gamma)$ preserves the quadric in $\mathbb{P}^n(\mathbf{R})$ (the Klein model for hyperbolic space) and acts on it properly discontinuously with compact quotient. In particular, $\phi \circ \iota$ is an example of a holonomy representation of a convex real projective structure. In [8] Benoist shows that the entire connected component of $\mathrm{Hom}(\Gamma, \mathrm{PGL}(n+1, \mathbf{R}))$ containing $\phi \circ \iota$ consists of holonomies of convex real projective structures, hence of Anosov representations. More details are given in Section 6.2.

(ii) Hitchin component: Let Γ be the fundamental group of a closed connected oriented surface Σ of genus ≥ 2, and $\iota : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ a discrete embedding. Let $\phi : \mathrm{PSL}(2, \mathbf{R}) \to \mathrm{PSL}(n, \mathbf{R})$ be the n-dimensional irreducible representation. Then $\phi \circ \iota$ is B-Anosov, where $B < \mathrm{PSL}(n, \mathbf{R})$ is the Borel subgroup. The connected component of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(n, \mathbf{R}))/\mathrm{PSL}(n, \mathbf{R})$ containing $\phi \circ \iota$ is called the Hitchin component, it is known that every representation in the Hitchin component is B-Anosov. More details are given in Section 6.3.

By a theorem of Choi and Goldman [25], for $n = 3$ representations in the Hitchin component are precisely the holonomy representations of convex real projective structures on Σ.

(iii) Quasi-Fuchsian groups in $\mathrm{SO}(2, n)$: In [4] Barbot and Mérigot introduced the notion of quasi-Fuchsian representations $\rho : \Gamma \to \mathrm{SO}(2, n)$ of a uniform lattice $\Gamma < \mathrm{SO}(1, n)$, the basic example being the injection of a lattice $\Gamma < \mathrm{SO}(1, n)$ composed with the natural embedding $\mathrm{SO}(1, n) < \mathrm{SO}(2, n)$. They showed that quasi-Fuchsian representations are precisely Q_0-Anosov, where $Q_0 < \mathrm{SO}(2, n)$ is the stabilizer of an isotropic line. In unpublished work, Barbot shows furthermore that the entire connected component in $\mathrm{Hom}(\Gamma, \mathrm{SO}(2, n))$ of the injection $\Gamma \to \mathrm{SO}(1, n) \to \mathrm{SO}(2, n)$ consists of quasi-Fuchsian representations [3].

6.2. **Holonomies of convex projective structures.** A discrete group $\Gamma < \mathrm{SL}(V)$ is said to *divide* an open convex set $\mathcal{C}$ in $\mathbb{P}(V)$ (i.e. the projectivization of a convex cone in V) if Γ acts properly discontinuously on $\mathcal{C}$ with compact quotient (see [11, 66] for surveys on this subject). The cone $\mathcal{C}$ is said to be *strictly convex* if $\partial \mathcal{C}$ intersects every projective line in at most two points.

A discrete group $\Gamma < \mathrm{SL}(V)$ dividing a convex set $\mathcal{C}$, is hyperbolic if and only if $\mathcal{C}$ is strictly convex [7, Théorème 1.1]. In that case $\partial_\infty \Gamma$ is naturally homeomorphic to $\partial \mathcal{C} \subset \mathbb{P}(V)$. This identification gives an equivariant map $\xi^+ : \partial_\infty \Gamma \to \mathbb{P}(V)$. Since Γ also divides the dual cone $\mathcal{C}^*$ in $\mathbb{P}(V^*)$ (see [66, Lemme 2.10]) one gets a second equivariant map $\xi^- : \partial_\infty \Gamma \to \mathbb{P}(V^*)$. Strict convexity of $\mathcal{C}$ easily implies that (ξ^+, ξ^-) is compatible (Definition 4.9). Furthermore by [74] the action of Γ on V is irreducible. Hence Proposition 4.10 applies and we have

Proposition 6.1. *Let $\Gamma < \mathrm{SL}(V)$ be a discrete subgroup dividing a strictly convex set $\mathcal{C} \subset \mathbb{P}(V)$. Then the inclusion $\Gamma \to \mathrm{SL}(V)$ is a Q_0-Anosov representation, where Q_0 is the stabilizer of a point in $\mathbb{P}(V)$.*

The quotient $\Gamma \backslash \mathcal{C}$ is an orbifold with a convex real projective structure. A representation $\rho : \Gamma \to \mathrm{SL}(V)$ whose image divides a strictly convex set is thus the holonomy of a convex real projective structure. Koszul [56] showed that for any finitely generated group Γ, the set of such holonomy representations in $\mathrm{Hom}(\Gamma, \mathrm{PGL}(n, \mathbf{R}))$ is open. Benoist [8] showed that this set is a connected component if and only if the virtual center of Γ is trivial.

Examples of convex real projective structures on manifolds whose fundamental groups are not isomorphic to a lattice in a Lie group have been constructed by

Benoist [9, 10]. Kapovich [48] provides several examples of convex real projective structures on Gromov-Thurston manifolds, i.e. compact manifolds which carry a metric of negative curvature pinched arbitrarily close to -1 but which do not admit a metric of constant negative curvature. The corresponding holonomy representations thus give examples of Anosov representations of hyperbolic groups Γ into $\mathrm{SL}(V)$, with Γ not being isomorphic to a lattice in a Lie group.

6.3. **Hitchin components.** Let G be the split real form of an adjoint simple algebraic group, i.e. $G = \mathrm{PSL}(n, \mathbf{R})$, $\mathrm{PSO}(n, n)$, $\mathrm{PSO}(n, n+1)$, $\mathrm{PSp}(2n, \mathbf{R})$ or a split real form of an exceptional group. Any such group admits a principal three dimensional subgroup, that is an (up to conjugation) well defined homomorphism $\phi_p : \mathrm{PSL}(2, \mathbf{R}) \to G$ that generalizes the n-dimensional irreducible representation $\mathrm{PSL}(2, \mathbf{R}) \to \mathrm{PSL}(n, \mathbf{R})$. The principal $\mathrm{PSL}(2, \mathbf{R})$ (or, more accurately, its Lie algebra) was discovered simultaneously by Dynkin and de Siebenthal, later Kostant studied its connection with the representation theory of G. The relevant results, as well as references to Kostant's papers, are summarized in [46, Sections 4 and 6].

Let Γ be the fundamental group of a closed connected oriented surface of genus ≥ 2. The Hitchin components are the connected components of $\mathrm{Hom}(\Gamma, G)/G$ containing representation of the form $\phi_p \circ \iota$ where $\iota : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ is a discrete embedding.

Theorem 6.2. [57, Theorems 4.1, 4.2], [29, Theorem 1.15] *Every representation ρ in the Hitchin component is (B^+, B^-)-Anosov, where (B^+, B^-) is a pair of opposite Borel subgroups of G.*

Remark 6.3. In fact, Fock and Goncharov in [29] provide a continuous equivariant map $\xi : \partial_\infty \Gamma \to G/B^+$ that satisfies the transversality property of Definition 4.15. Thus Corollary 4.16, together with an analysis of the potential Zariski closures of Hitchin representations, implies the above theorem. One can also use the positivity property of the equivariant curve, see [29, Definition 1.10], to obtain directly the control on $A_+(m, t)$ required in Proposition 3.16.

Remark 6.4. When G is a split real simple Lie group, which is not adjoint (e.g. $\mathrm{SL}(n, \mathbf{R})$ of $\mathrm{Sp}(2n, \mathbf{R})$), we call a connected component of $\mathrm{Hom}(\Gamma, G)/G$ a Hitchin component if and only it its image in $\mathrm{Hom}(\Gamma, G^{ad}) G^{ad}$ is the Hitchin component, where G^{ad} is the adjoint group of G.

6.4. **Maximal representations.** Let G be a Lie group of Hermitian type, i.e. G is connected, semisimple with finite center and has no compact factors and the symmetric space $\mathcal{H} = G/K$ admits a G-invariant complex structure. Let Γ be the fundamental group of a closed connected oriented surface of genus ≥ 2. There is a characteristic number, often called the Toledo invariant, $\tau_G : \mathrm{Hom}(\Gamma, G) \to \mathbf{Z}$, which satisfies a Milnor-Wood type inequality [21, Section 3]: $|\tau_G(\rho)| \leq (2g - 2)c(G)$, where $c(G)$ is a constant depending only on G.

Definition 6.5. A representation is said to be *maximal* if $\tau_G(\rho) = (2g-2)c(G)$.

Let $\check{S} = G/\check{P}$ be the Shilov boundary of G; it is the closed G-orbit in the boundary of the bounded symmetric domain realization of $\mathcal{H}$. (see [75, Chapter 4] and [68]).

Theorem 6.6. [20] *Any maximal representation $\rho : \Gamma \to G$ is $\check{P}$-Anosov.*

The symplectic group $\mathrm{Sp}(2n, \mathbf{R})$ is of Hermitian type. Its Shilov boundary $\check{S}$ is the space $\mathcal{L}$ of Lagrangian (i.e. maximal isotropic) subspaces of $\mathbf{R}^{2n}$. In that case the map $\xi : \partial_\infty \Gamma \to \mathcal{L}$ associated to a maximal representation was constructed in [18]; it satisfies the following transversality condition: for all $t^+ \neq t^-$ in $\partial_\infty \Gamma$, $\xi(t^+) \oplus \xi(t^-) = \mathbf{R}^{2n}$.

Remark 6.7. In many cases Theorem 6.6 can be deduced from the corresponding result for symplectic groups.

Indeed note that, up to passing to a finite index subgroup, the image of a maximal representation $\rho : \pi_1(\Sigma) \to G$ is contained in a Lie group H of Hermitian type, which is of tube type [21, Theorem 5]. Furthermore if G is a classical Lie group, then H is also a classical group and therefore (up to taking finite covers) admits a tight embedding $\phi : H \to \mathrm{Sp}(2n, \mathbf{R})$, which extends to an equivariant map of the Shilov boundary $\check{S}$ of H into the space of Lagrangians $\mathcal{L}$, (this was already used in [76], see definitions and references therein, in particular [19] for the notion of tight embeddings). Up to passing to a finite index subgroup, the composition $\phi \circ \rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ is a maximal representation into the symplectic group. Thus, as a consequence of Corollary 3.4 and Proposition 4.1 one can deduce Theorem 6.6 in the case when G is a classical Lie group from the case of the symplectic group.

6.5. **Projective Schottky groups.** In [63] Nori constructed Schottky groups $\Gamma \subset \mathrm{PGL}(2n, \mathbf{C})$, which act properly discontinuously and cocompactly on an open subset $\Omega \subset \mathbb{P}(\mathbf{C}^{2n})$. These examples have been generalized by Seade and Verjovsky in [69]. Their construction also gives discrete free subgroups of $\mathrm{PGL}(2n, \mathbf{R})$. The embeddings $\rho : \Gamma \to \mathrm{PGL}(2n, \mathbf{K})$, $\mathbf{K} = \mathbf{R}, \mathbf{C}$, of these projective Schottky groups are P_n-Anosov representations, where P_n is the stabilizer of an n-dimensional $\mathbf{K}$-vector subspace in $\mathbf{K}^{2n}$ (see [38]). Such Schottky groups do not exist in $\mathrm{PGL}(2n+1, \mathbf{K})$, see [23].

Part 2. Domains of discontinuity

7. Automorphism groups of sesquilinear forms

In this section we construct domains of discontinuity for discrete subgroups of automorphism groups of non-degenerate sesquilinear forms, which exhibit special dynamical properties. We then apply this construction to Anosov representations of hyperbolic groups.

7.1. **Notation.** Let (V, F) be a (right) $\mathbf{K}$-vector space (with $\mathbf{K} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$) with a non-degenerate form $F : V \otimes_{\mathbf{R}} V \to \mathbf{K}$, linear in the second variable ($F(x, y\lambda) = F(x, y)\lambda$)[2] and such that:

- If $\mathbf{K} = \mathbf{R}$, F is an indefinite symmetric form or a skew-symmetric form.
- If $\mathbf{K} = \mathbf{C}$, F is a symmetric form, a skew-symmetric form or an indefinite Hermitian form.
- If $\mathbf{K} = \mathbf{H}$, F is an indefinite Hermitian form or a skew-Hermitian form.

Let $G_F < \mathrm{GL}(V)$ be the automorphism group of (V, F). Then G_F is $\mathrm{O}(p, q)$ $(0 < p \leq q)$, $\mathrm{Sp}(2n, \mathbf{R})$; $\mathrm{O}(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$, $\mathrm{U}(p, q)$, $(0 < p \leq q)$; $\mathrm{Sp}(p, q)$ $(0 < p \leq q)$ or $\mathrm{SO}^*(2n)$ respectively.

[2] the order in the equation matters only for the case $\mathbf{K} = \mathbf{H}$.

We denote by

$$\mathcal{F}_0 = G_F/Q_0 = \mathbb{P}(\{x \in V \mid F(x,x) = 0\}) \subset \mathbb{P}(V)$$

the space of isotropic lines and by

$$\mathcal{F}_1 = G_F/Q_1 = \{P \in \mathrm{Gr}_l(V) \mid F|_P = 0\}$$

the space of maximal isotropic subspaces of V. Here $\mathrm{Gr}_l(V)$ denotes the Grassmannian of l-planes with $l = p$ if $G_F = \mathrm{O}(p,q)$ $(0 < p \leq q)$, $l = n$ if $G_F = \mathrm{Sp}(2n, \mathbf{R})$, etc. When we explicitly want to refer to the vector space we will use the notation $\mathcal{F}_0(V)$ and $\mathcal{F}_1(V)$. In the case of G_F, transversality of points in $\mathcal{F}_i$ can be put concretely:

Lemma 7.1. *A pair $(P_1, P_2) \in \mathcal{F}_i \times \mathcal{F}_i$ is* transverse *if and only if $P_1 + P_2^{\perp_F} = V$ (or equivalently $P_1 \cap P_2^{\perp_F} = \{0\}$, or $P_1^{\perp_F} + P_2 = V$, etc.).*

The closed G-orbit in $\mathcal{F}_0 \times \mathcal{F}_1$ is

$$\mathcal{F}_{01} = \{(D,P) \in \mathcal{F}_0 \times \mathcal{F}_1 \mid D \subset P\}.$$

There are two projections $\pi_i : \mathcal{F}_{01} \to \mathcal{F}_i$, $i = 0, 1$.

For any subset $A \subset \mathcal{F}_i$ we define a subset K_A in $\mathcal{F}_{1-i}$ by

$$\begin{aligned} K_A = \pi_{1-i}(\pi_i^{-1}(A)) &= \{D \in \mathcal{F}_0 \mid \exists P \in A, D \subset P\} \text{ if } i = 1, \\ &= \{P \in \mathcal{F}_1 \mid \exists D \in A, D \subset P\} \text{ if } i = 0. \end{aligned}$$

If A is closed, K_A is closed.

7.2. **Subgroups with special dynamical properties.** We denote the Lie algebra of G_F by $\mathfrak{g}$ and use the notation introduced in Section 4.5. One can choose a basis of V such that $\mathfrak{a} \subset \mathfrak{g} \subset \mathfrak{gl}(V)$ is the set of diagonal matrices $\mathrm{diag}(t_1, \dots, t_l, 0, \dots, 0, -t_l, \dots, -t_1)$ with $t_i \in \mathbf{R}$ for all i (here $l = \mathrm{rk}_{\mathbf{R}} G_F$), and such that $\mathfrak{a}^+$ are those matrices satisfying $t_1 > t_2 > \cdots > t_l > 0$. Let α_1 denote the simple root of $\mathfrak{a}$ such that $Q_1 = P_{\Delta \smallsetminus \{\alpha_1\}}$ (see Section 4.5). Then α_1 is $\mathrm{diag}(t_1, \dots, t_l, \dots) \mapsto t_l$ (or $\mathrm{diag}(t_1, \dots, t_l, \dots) \mapsto 2t_l$ if there are no "zeroes"). The root α_0 such that $Q_0 = P_{\Delta \smallsetminus \{\alpha_0\}}$ is $\mathrm{diag}(t_1, \dots, t_l, \dots) \mapsto t_1 - t_2$.

Definition 7.2. A discrete subgroup $\Gamma < G_F$ is *α_i-divergent*, $i = 0, 1$, if:

– any sequence $(g_n)_{n \in \mathbf{N}}$ in G diverging to infinity has a subsequence $(g_{\phi(n)})_{n \in \mathbf{N}}$ such that $\lim_{n \to \infty} \alpha_i(\mu(g_{\phi(n)})) = \infty$.

Lemma 7.3. *Let $\Gamma < G_F$ be a discrete α_i-divergent subgroup. Then Γ is proximal with respect to $\mathcal{F}_i$.*

Proof. This is a direct consequence of [6, Section 3.2, Lemme]. □

Theorem 7.4. *Let $\Gamma < G_F$ be a discrete subgroup, and let i be 0 or 1. Assume that Γ is α_i-divergent. Let $\mathcal{L}_\Gamma < \mathcal{F}_i$ denote the limit set of Γ. Set*

$$\Omega_\Gamma = \mathcal{F}_{1-i} \smallsetminus K_{\mathcal{L}_\Gamma}.$$

Then $\Omega_\Gamma \subset \mathcal{F}_{1-i}$ is a Γ-invariant open subset. Moreover, Γ acts properly discontinuously on Ω_Γ.

Proof. We consider the case when $\Gamma < G_F$ is α_1-divergent. The proof for the other case is entirely analogous. We consider a point $x \in \mathcal{F}_1$ as a subspace of V, in particular $\mathbb{P}(x) \subset \mathcal{F}_0 \subset \mathbb{P}(V)$.

Let $\mathcal{L}_\Gamma < \mathcal{F}_1$ be the limit set. We want to show the properness of the action of Γ on Ω_Γ, where

$$\Omega_\Gamma = \mathcal{F}_0 \smallsetminus K_{\mathcal{L}_\Gamma} = \mathcal{F}_0 \smallsetminus \bigcup_{x \in \mathcal{L}_\Gamma} \mathbb{P}(x) \subset \mathbb{P}(V).$$

We argue by contradiction.

Suppose that there exist compact subsets A and B of Ω_Γ and a sequence $(\gamma_n)_{n\in\mathbf{N}}$ in Γ such that: $\gamma_n \xrightarrow{n\to+\infty} \infty$, and for all n, $\gamma_n A \cap B \neq \emptyset$.

(i) By Theorem 5.10 there is a finite set $S \subset \Gamma$ such that, for all n, there is $s_n \in S$ such that $s_n\gamma_n$ is (r,ϵ)-proximal relative to $\mathcal{F}_1$. Also $s_n\gamma_n A \cap s_n B$ is nonempty. Hence, up to replacing (γ_n) by $(s_n\gamma_n)$ and B by $\bigcup_{s\in S} sB$, we can suppose that γ_n is (r,ϵ)-proximal for all n.

(ii) Let $x_n^+, x_n^- \in \mathcal{L}_\Gamma$ be the attracting and repelling fixed points of $\gamma_n \in \Gamma$. Up to extracting a subsequence we can suppose that $x_n^\pm$ converge to $x^\pm \in \mathcal{L}_\Gamma$. Since, for all n the element γ_n is (r,ϵ)-proximal we have, for all n, that $d(x_n^+, V^-(x_n^-)) \geq r$, hence also $d(x^+, V^-(x^-)) \geq r$ (see Section 5.2 for notation). This shows that x^+ and x^- are transverse.

(iii) Without loss of generality, we can assume that $L = \mathrm{Stab}(x^+, x^-)$ is the Levi component of Q_1, and that x^+ is the attracting fixed point of $\exp(a)$ when a is in the Weyl chamber $\mathfrak{a}_L^+ \subset \mathfrak{a}$.

(iv) As $\lim_{n\to\infty}(x_n^+, x_n^-) = (x^+, x^-)$ in $\mathcal{X} = G/L$, there exists a sequence $(g_n)_{n\in\mathbf{N}}$ in G_F converging to 1 and such that, for all n, we have $(x_n^+, x_n^-) = g_n(x^+, x^-)$. Hence, for all n, $g_n\gamma_n g_n^{-1}$ fixes (x^+, x^-) and hence belongs to L. We can thus write $g_n\gamma_n g_n^{-1} = k_n \exp(a_n) l_n$ with $a_n \in \overline{\mathfrak{a}}_L^+$ and $k_n, l_n \in M$. Up to passing to a subsequence we can assume that (k_n) and (l_n) converge to k and l. Since Γ is α_1-divergent we have that $(\alpha_1(a_n))$ tends to $+\infty$.

(v) Now consider the set $\bigcup_{n\in\mathbf{N}} l_n g_n^{-1} A \cup lA$. It is a compact subset of $\mathcal{F}_0 \smallsetminus \mathbb{P}(x^-)$. Therefore there exists $\eta > 0$ such that

$$\bigcup_{n\in\mathbf{N}} l_n g_n^{-1} A \cup lA \subset B_\eta := \{y \in \mathcal{F}_0 \mid d(y, \mathbb{P}(x^-)) \geq \eta\}.$$

A simple calculation shows that, for all $\epsilon > 0$, there exists R such that if $a \in \overline{\mathfrak{a}_L}^+$ satisfies $\alpha_1(a) \geq R$, then

$$\exp(a) \cdot C \subset \{y \in \mathcal{F}_0 \mid d(y, \mathbb{P}(x^+)) \leq \epsilon\}.$$

This implies that, for any sequence $(y_n)_{n\in\mathbf{N}}$ in A, any accumulation point of $(\exp(a_n) l_n g_n^{-1} y_n)$ is contained in $\mathbb{P}(x^+)$. Since $\lim_{n\to\infty} g_n = 1$ and since $k = \lim_{n\to\infty} k_n$ stabilizes $\mathbb{P}(x^+)$, also any accumulation point of $\gamma_n y_n = g_n k_n \exp(a_n) l_n g_n^{-1} y_n$ is contained in $\mathbb{P}(x^+)$.

Now we are ready to conclude. If $\gamma_n A \cap B \neq \emptyset$ for all n, then there exists an accumulation point of $\gamma_n y_n$ which is contained in B. With the above this means in particular that $B \cap \mathbb{P}(x^+)$ is nonempty. This contradicts the assumption that $B \subset \Omega_\Gamma \subset \mathcal{F}_0 \smallsetminus \mathbb{P}(x^+)$. ∎

Remarks 7.5.

– In the special case when $G = \mathrm{SO}(2, n)$ Frances [30] constructed domains of discontinuity in $\mathcal{F}_0$ for discrete subgroups with special dynamical properties. In this case, α_1-divergent groups in the sense of Definition 7.2 are called groups of the first type in Frances' paper, see [30, Definition 4 and Proposition 6].

– Benoist criterion for the properness of actions on homogeneous spaces [5, Section 1.5, Proposition] implies that a discrete subgroup $\Gamma < \mathrm{O}(p, q)$ is α_1-divergent if and only if Γ acts cocompactly on $\mathrm{O}(p, q)/\mathrm{O}(p-1, q)$; Proposition 6 in [30] is a special case of this.

7.3. **Other Lie groups.** The following proposition allows to use the above construction of the domain of discontinuity to obtain domains of discontinuity for discrete subgroups of more general Lie groups.

Proposition 7.6. *Let $\Gamma < G$ be a subgroup and $\phi : G \to G'$ be an injective homomorphism. Suppose that $\mathcal{F}$ is a closed G-invariant subset of a G'-space $\mathcal{F}'$. Let $U' \subset \mathcal{F}'$ be an open $\phi(\Gamma)$-invariant subset such that $U = U' \cap \mathcal{F}$ is nonempty.*

(i) If Γ acts (via ϕ) properly on U' then Γ acts properly on U.
(ii) If furthermore the quotient of U' by Γ is compact, then the quotient of U by Γ is also compact.

In Section 9 we will use Proposition 7.6 to reduce the discussion of a general Anosov representation to the case of a Q_0-Anosov representation into a symplectic group or an orthogonal group. We will also use this proposition in the applications discussed in Sections 12 and 13.

8. Anosov representations into orthogonal or symplectic groups

Here we apply the constructions of Section 7 in order to obtain cocompact domains of discontinuity for Anosov representations. We first describe the structure of the domain of discontinuity in more detail and deduce the properness of the action. Then we introduce some reduction steps, which allow us to simplify the proof for the compactness of the quotient.

8.1. **Structure of the domain of discontinuity.** Recall that given a Q_i-Anosov representation $\rho : \Gamma \to G_F$, the image $\rho(\Gamma) < G_F$ is a discrete subgroup which is (AMS)-proximal relative to $\mathcal{F}_i$. In particular, its limit set $\mathcal{L}_{\rho(\Gamma)} < \mathcal{F}_i$ is well defined and equals the image of the Anosov map associated to ρ, $\mathcal{L}_{\rho(\Gamma)} = \xi(\partial_\infty \Gamma)$.

Proposition 8.1. *Let $\rho : \Gamma \to G_F$ be a Q_i-Anosov representation with associated Anosov map $\xi : \partial_\infty \Gamma \to \mathcal{F}_i$. Set*

$$\Omega_\rho := \mathcal{F}_{1-i} \smallsetminus K_{\xi(\partial_\infty \Gamma)}.$$

Then:

(i) Ω_ρ is an open $\rho(\Gamma)$-invariant subset of $\mathcal{F}_{1-i}$.
(ii) The map $\pi_{1-i} : \pi_i^{-1}(\xi(\partial_\infty \Gamma)) \to K_{\xi(\partial_\infty \Gamma)}$ is a homeomorphism. In particular,

$$K_{\xi(\partial_\infty \Gamma)} \cong \pi_i^{-1}(\xi(\partial_\infty \Gamma)) \xrightarrow{\pi_i} \xi(\partial_\infty \Gamma) \cong \partial_\infty \Gamma$$

is a locally trivial bundle over $\partial_\infty \Gamma$ whose fiber over a point t is $\mathbb{P}(\xi(t))$ when $i = 1$, and $\mathcal{F}_1(P^{\perp_F}/P)$ when $i = 0$.

Proof. The first statement is obvious.

For the second statement note that the map ξ is injective, thus $\partial_\infty\Gamma \cong \xi(\partial_\infty\Gamma)$. The transversality condition $P_1 \cap P_2^{\perp_F} = \{0\}$ implies $K_{P_1} \cap K_{P_2} = \emptyset$. Since ξ is transverse this implies that $K_{\xi(\partial_\infty\Gamma)}$ is the disjoint union $\coprod_{t\in\partial_\infty\Gamma} K_{\xi(t)}$. □

Recall that the cohomological dimension of a group Γ is the smallest n such that every cohomology group with coefficient in any Γ-module vanishes in degree $> n$. The virtual cohomological dimension $\mathrm{vcd}(\Gamma)$ is the infimum of the cohomological dimensions of finite index subgroups. Dimensions of topological spaces in the following statements are also cohomological dimensions (for Čech cohomology) or, what amounts to the same, covering dimensions. In the next proposition,We will replace the dimension of $\partial_\infty\Gamma$ by the virtual cohomological dimension of Γ thanks to the following result of Bestvina and Mess. Formally, this replacement is not necessary in our proofs, however it gives a hint why the quotient should be compact.

Lemma 8.2. [13, Corollary 1.4] *Let Γ be a word hyperbolic group, then*

$$\dim \partial_\infty\Gamma = \mathrm{vcd}(\Gamma) - 1.$$

Proposition 8.3. *Let $\rho : \Gamma \to G_F$ be a Q_i-Anosov representation and i be 0 or 1. Let $\mathrm{vcd}(\Gamma)$ be the virtual cohomological dimension of Γ. Set $\delta = \dim \mathcal{F}_{1-i} - \dim K_{\xi(\partial_\infty\Gamma)}$. Then*

(i) – *If $G_F = \mathrm{O}(p,q)$, $\mathrm{U}(p,q)$ or $\mathrm{Sp}(p,q)$ $(0 < p \le q)$, then $\delta = q - \mathrm{vcd}(\Gamma)$, $2q - \mathrm{vcd}(\Gamma)$ or $4q - \mathrm{vcd}(\Gamma)$ respectively.*
– *If $G_F = \mathrm{O}(2n, \mathbf{C})$ or $\mathrm{O}(2n-1, \mathbf{C})$ then $\delta = 2n - \mathrm{vcd}(\Gamma)$.*
– *If $G_F = \mathrm{Sp}(2n, \mathbf{R})$ or $\mathrm{Sp}(2n, \mathbf{C})$ then $\delta = n + 1 - \mathrm{vcd}(\Gamma)$ or $2n + 2 - \mathrm{vcd}(\Gamma)$ respectively.*
– *If $G_F = \mathrm{SO}^*(2n)$ then $\delta = 4n - 2 - \mathrm{vcd}(\Gamma)$.*

(ii) If $\partial_\infty\Gamma$ is a topological manifold and $\delta = 0$, then $K_{\xi(\partial_\infty\Gamma)} = \mathcal{F}_{1-i}$. In particular, in this case, Ω_ρ is empty.

Proof. By Proposition 8.1 the dimension of $K_{\xi(\partial_\infty\Gamma)}$ is:

$$\begin{aligned}\dim K_{\xi(\partial_\infty\Gamma)} &= \dim \pi_i^{-1}(\xi(\partial_\infty\Gamma)) = \dim \pi_i^{-1}(\{P\}) + \dim \xi(\partial_\infty\Gamma)\\ &= \dim \pi_i^{-1}(\{P\}) + \dim \partial_\infty\Gamma,\end{aligned}$$

and $\pi_i^{-1}(\{P\}) \cong \mathcal{F}_1(P^{\perp_F}/P)$ if $i = 0$ and $\pi_i^{-1}(\{P\}) \cong \mathbb{P}(P)$ if $i = 1$. With Lemma 8.2 we thus have

$$\begin{aligned}\delta &= \dim \mathcal{F}_1 - \dim \mathcal{F}_1(P^{\perp_F}/P) - \mathrm{vcd}(\Gamma) + 1, \text{ if } i = 0,\\ &= \dim \mathcal{F}_0 - \dim \mathbb{P}(P) - \mathrm{vcd}(\Gamma) + 1, \text{ if } i = 1.\end{aligned}$$

The first statement follows now by calculating the dimensions of the homogeneous spaces $\mathcal{F}_0$ and $\mathcal{F}_1$.

When $\partial_\infty\Gamma$ is a topological manifold, then $K_{\xi(\partial_\infty\Gamma)}$ is also a manifold (by Proposition 8.1). If $\delta = 0$ this implies that $K_{\xi(\partial_\infty\Gamma)}$ is an open submanifold of $\mathcal{F}_{1-i}$, since it is also closed, the equality $K_{\xi(\partial_\infty\Gamma)} = \mathcal{F}_{1-i}$ follows. □

Remark 8.4. The coincidence for the values of δ for Q_0 and Q_1-Anosov representations is explained by the following observation: if $p_1 : M \to M_1$ and $p_2 : M \to M_2$ are two submersions such that $p_1 \times p_2 : M \to M_1 \times M_2$ is an immersion, then, for any $m_1 \in M_1$ and $m_2 \in M_2$, the codimensions of $p_1(p_2^{-1}(m_2))$ in M_1 and of $p_2(p_1^{-1}(m_1))$ in M_2 are equal to $\dim M_1 + \dim M_2 - \dim M$.

Remark 8.5. The control on the codimension of $K_{\xi(\partial_\infty \Gamma)}$ given by Proposition 8.3 allows to deduce nonemptyness for many examples of Anosov representations. Here we give some examples where the domain of discontinuity is empty; we come back to these examples in Section 13.

(i) Let $\iota : \Gamma \hookrightarrow \mathrm{SO}(1,q)$ be a convex cocompact representation, i.e. ι is Q_0-Anosov. The composition of ι with embedding $\phi : \mathrm{SO}(1,q) \to \mathrm{SO}(1+p', q+q')$ gives a Q_0-Anosov representation $\phi \circ \iota$ (see Lemma 8.8.(i)). When $i(\Gamma)$ is a cocompact lattice in $\mathrm{SO}(1,q)$ and $\phi : \mathrm{SO}(1,q) \to \mathrm{SO}(p,q)$, $p \leq q$, the equality case of Proposition 8.3 is attained and $\Omega_{\phi\circ\iota} = \emptyset$.

(ii) Let $\iota : \Gamma \to G$ be a convex cocompact representation into $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ or $G_{\mathcal{O}}$ (the isometry group of the Cayley hyperbolic plane). If $G = \mathrm{SU}(1,n)$ consider the natural injection $\phi : \mathrm{SU}(1,n) \to \mathrm{SO}(2,2n)$ be the natural injection; if $G = \mathrm{Sp}(1,n)$ consider $\phi : \mathrm{Sp}(1,n) \to \mathrm{SO}(4,4n)$; if G is $G_{\mathcal{O}}$ consider $\phi : G \to \mathrm{SO}(8,8)$. In any case $\phi \circ \iota$ is Q_1-Anosov and $\Omega_{\phi\circ\iota}$ is empty when $\iota(\Gamma)$ is a cocompact lattice.

Theorem 8.6. *Let $\rho : \Gamma \to G_F$ be a Q_i-Anosov representation.* if $i = 1$ G_F should not be $\mathrm{PSO}(n,n)$ *If $\Omega_\rho \subset \mathcal{F}_{1-i}(V)$ is nonempty (e.g. if $\delta > 0$), then*

(i) Γ acts properly discontinuously on Ω_ρ

(ii) The quotient $\Gamma \backslash \Omega_\rho$ is compact.

The proof of statement (i) is a direct consequence of Theorem 7.4, Theorem 5.9, and the following lemma.

Lemma 8.7. *Let $\rho : \Gamma \to G_F$ be a Q_i-Anosov representation, then $\rho(\Gamma)$ is α_i-divergent.*

Proof. This is a direct application of Proposition 3.16, which gives control on the contraction rate in terms of the L-Cartan projections. ☐

The proof of statement (ii) is deferred to Section 8.3.

8.2. **Reduction steps.** Before we turn to the proof of compactness of the quotient $\Gamma \backslash \Omega_\rho$ we introduce some reduction steps, which will allow us to restrict our attention mainly to Q_i-Anosov representations into $\mathrm{O}(p,q)$.

Lemma 8.8. *Let (V, F) be a vector space with a sesquilinear form that satisfies the conditions of Section 8.1. Let $\rho : \Gamma \to G_F$ be a representation.*

(i) Let (V', F') be of the same type as (V, F). Then the injection $V \to V \oplus V'$ induces a homomorphism $\phi : G_F \to G_{F+F'}$. If ρ is Q_0-Anosov then $\phi \circ \rho$ is Q_0-Anosov.

Let (V, F) be a complex orthogonal space of dimension $2n-1$, and (V', F') a complex orthogonal space of dimension one, then if ρ is Q_1-Anosov, then $\phi \circ \rho$ is Q_1-Anosov.

(ii) Let k a positive integer. Consider the vector space V^k, which is endowed with the form F^k, and let $\phi : G_F \to G_{F^k}$ be the diagonal embedding. If ρ is Q_1-Anosov then $\phi \circ \rho$ is Q_1-Anosov.

(iii) Suppose F is skew-symmetric (i.e. $G_F = \mathrm{Sp}(2n, \mathbf{R})$ or $\mathrm{Sp}(2n, \mathbf{C})$) then the form $F \otimes F$ on $V \otimes V$ is symmetric. Let $\phi : G_F \to G_{F\otimes F}$ the corresponding homomorphism. If ρ is Q_i-Anosov then $\phi \circ \rho$ is Q_i-Anosov.

(iv) If F is Hermitian, $\Re F$ is a non-degenerate symmetric bilinear form on $V_{\mathbf{R}}$ the real space underlying V. Hence there is a homomorphism $\phi : G_F \to G_{\Re F}$. If ρ is Q_1-Anosov then $\phi \circ \rho$ is Q_1-Anosov.

In all of the above cases, if ρ is Q_i-Anosov, one has

$$\Omega_\rho = \mathcal{F}_{1-i}(V) \cap \Omega_{\phi\circ\rho}.$$

Proof. The cases (i) through (iv) follow from the general proposition 4.4. In order to illustrate the ideas we give a direct proof for (iii) and ρ being a Q_0-Anosov representation. Consider the ϕ-equivariant maps

$$\phi_0 : \mathcal{F}_0(V) \to \mathcal{F}_0(V \otimes V),\ D \mapsto D \otimes D$$

and

$$\phi_1 : \mathcal{F}_1(V) \to \mathcal{F}_1(V \otimes V),\ L \mapsto L \otimes V.$$

If $D, D' \in \mathcal{F}_0(V)$ are transverse then $\phi_0(D)$ and $\phi_0(D')$ are also transverse: indeed $D^\perp \oplus D' = V$ implies $V \otimes V = (D \otimes D)^\perp \oplus D' \otimes D'$. So if $\xi : \partial_\infty \Gamma \to \mathcal{F}_0(V)$ is transverse, then $\phi_0 \circ \xi$ is transverse (see Definition 4.15). Concerning the contraction property required for an Anosov representation (see Definition 2.10) it is enough to remark that if a diagonal element g in G_F fixes D in $\mathcal{F}_0(V)$ and contracts $T_D\mathcal{F}_0(V)$ then $\phi(g)$ contracts $T_{\phi_0(D)}\mathcal{F}_0(V \otimes V)$.

The equality $\Omega_\rho = \mathcal{F}_1(V) \cap \Omega_{\phi\circ\rho}$ results from the fact that, for $D \in \mathcal{F}_0(V)$ and $L \in \mathcal{F}_1(V)$, $D \subset L$ if and only if $D \otimes D \subset L \otimes V$. □

Remarks 8.9.

(a) Due to this lemma the proof of Proposition 8.3 reduces in many cases to the case of orthogonal groups $\mathrm{O}(p, q)$. However, this reduction does not seem to work for Q_0-Anosov representations into $G_F(V) = \mathrm{O}(n, \mathbf{C})$, $\mathrm{U}(p, q)$ or $\mathrm{Sp}(p, q)$. Indeed in these cases one would consider the orthogonal space $W = \bigwedge^2_{\mathbf{R}} V$ (or $\bigwedge^4_{\mathbf{R}} V$) and the corresponding embedding $\phi : G_F \to G_F(W)$ sends Q_0-Anosov representations to Q_0-Anosov representations. Yet there is no corresponding embedding of $\mathcal{F}_1(V)$ into $\mathcal{F}_1(W)$, so that the final conclusion of the lemma does not hold.

(b) Cases (i) and (ii) of the above lemma together with the formula for the codimension given in Theorem 8.6 will allow us to assume that the open Ω_ρ has high connectedness properties when proving the compactness of $\Gamma\backslash\Omega_\rho$.

8.3. **Compactness.** In this section we prove the compactness of $\Gamma\backslash\Omega_\rho$, claimed in Theorem 8.6.(ii).

For clarity of the exposition we will suppose in the following that $G_F = \mathrm{O}(p, q)$. Applying Lemma 8.8 and Proposition 7.6 this proves Theorem 8.6 in all cases except for Q_0-Anosov representations into $\mathrm{O}(n, \mathbf{C})$, $\mathrm{U}(p, q)$ or $\mathrm{Sp}(p, q)$. In the remaining cases the proof is a straightforward adaptation of the arguments presented here.

8.3.1. *Homological formulation.* Let $\rho : \Gamma \to G_F = \mathrm{O}(p, q)$ be a Q_i-Anosov representation. In view of Lemma 8.8 we can assume without loss of generality that $\min(p - 2, q - p) > \max(m + 1, \mathrm{vcd}(\Gamma))$, for some $m \in \mathbf{N}$ which will be fixed later.

Furthermore, up to passing to a finite index subgroup, we can assume that Γ is torsion-free, and that ρ is injective.

To simplify some of the cohomological arguments we consider the following 2-fold covers of $\mathcal{F}_0$ and $\mathcal{F}_1$:

$$\mathcal{F}_0^{or} = \{D \in \mathbb{S}(V) \mid D \text{ oriented line with } F|_D = 0\} \cong \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$$
$$\mathcal{F}_1^{or} = \{P \in \mathrm{Gr}_p^{or}(V) \mid P \text{ oriented } p\text{-plane with } F|_P = 0\} \cong \mathrm{SO}(q)/\mathrm{SO}(q-p).$$

The homogeneous spaces $\mathcal{F}_i^{or}$ are $\min(p-2, q-p)$-connected. The lift Ω_ρ^{or} of Ω_ρ to $\mathcal{F}_{1-i}^{or}$ is a 2-fold cover of Ω_ρ and its complement $K_{\xi(\partial_\infty\Gamma)}^{or}$ fibers over $\partial_\infty\Gamma$ with fibers isomorphic to $\mathbb{S}^{p-1}$ if $i = 0$, and to $\mathrm{SO}(q-1)/\mathrm{SO}(q-p)$ if $i = 1$; in both case the fibers are also $\min(p-2, q-p)$-connected.

Up to passing to a finite index subgroup we suppose that Γ preserves the orientation on Ω_ρ^{or} hence $\Gamma\backslash\Omega_\rho^{or}$ is an oriented manifold.

For the rest of this section we denote Ω_ρ^{or}, $K_{\xi(\partial_\infty\Gamma)}^{or}$ and $\mathcal{F}_{1-i}^{or}$ by Ω, K and $\mathcal{F}$ respectively.

Let $l = \dim \Omega = \dim \mathcal{F}$. By our assumption $q - p > \mathrm{vcd}(\Gamma)$, and hence $\delta = \mathrm{codim} K = q - \mathrm{vcd}(\Gamma) \geq 2$, and Ω is connected. Therefore $\Gamma\backslash\Omega$ is a l-dimensional connected oriented manifold. Thus $\Gamma\backslash\Omega$ is compact if and only if $H_c^0(\Gamma\backslash\Omega)$ (cohomology with compact support and coefficients in $\mathbf{R}$) is nonzero. By Poincaré duality this is equivalent to the top-dimensional homology group $H_l(\Gamma\backslash\Omega) = H_c^0(\Gamma\backslash\Omega)^*$ being nonzero.

Therefore Theorem 8.6.(ii) follows from the following

Proposition 8.10. *With the notation above*

$$H_l(\Gamma\backslash\Omega) \cong \mathbf{R}.$$

We will now first prove Proposition 8.10 in the case when Γ is the fundamental group of a negatively curved closed manifold, then we consider the case when Γ is an arbitrary word hyperbolic group.

8.3.2. *Fundamental groups.* Let $\Gamma = \pi_1(N)$ be the fundamental group of a negatively curved closed manifold N of dimension m.

The fibration $\Gamma\backslash(\tilde{N} \times \Omega) \to \Gamma\backslash\Omega$ has contractible fibers (isomorphic to $\tilde{N}$), hence induces an isomorphism in homology

$$H_l(\Gamma\backslash\Omega) = H_l(\Gamma\backslash(\tilde{N} \times \Omega)).$$

Applying Poincaré duality to this $(l+m)$-dimensional manifold gives

$$H_l(\Gamma\backslash(\tilde{N} \times \Omega))^* \cong H_c^m(\Gamma\backslash(\tilde{N} \times \Omega)).$$

By definition, this last group is the direct limit

$$H_c^m(\Gamma\backslash(\tilde{N} \times \Omega)) = \varinjlim_{\mathcal{O} \supset \Gamma\backslash(\tilde{N} \times K)} H^m(\Gamma\backslash(\tilde{N} \times \mathcal{F}), \mathcal{O}),$$

where the limit is taken over the open neighborhoods of $\Gamma\backslash(\tilde{N} \times K)$ in $\Gamma\backslash(\tilde{N} \times \mathcal{F})$. The long exact sequence of the pair $(\Gamma\backslash(\tilde{N} \times \mathcal{F}), \mathcal{O})$ reads as

$$H^m(\mathcal{O}) \longrightarrow H^m(\Gamma\backslash(\tilde{N} \times \mathcal{F})) \longrightarrow H^m(\Gamma\backslash(\tilde{N} \times \mathcal{F}), \mathcal{O}) \longrightarrow H^{m+1}(\mathcal{O}) \longrightarrow H^{m+1}(\Gamma\backslash(\tilde{N} \times \mathcal{F})).$$

Passing to the limit one gets

$$\check{H}^m(\Gamma\backslash(\tilde{N}\times K)) \longrightarrow H^m(\Gamma\backslash(\tilde{N}\times \mathcal{F})) \longrightarrow H^m_c(\Gamma\backslash(\tilde{N}\times \Omega)) \longrightarrow \check{H}^{m+1}(\Gamma\backslash(\tilde{N}\times K)) \longrightarrow H^{m+1}(\Gamma\backslash(\tilde{N}\times \mathcal{F})),$$

where $\check{H}^*(\Gamma\backslash(\tilde{N}\times K))$ is the Čech cohomology of $\Gamma\backslash(\tilde{N}\times K)$. Furthermore the fibrations $\Gamma\backslash(\tilde{N}\times K)\to\Gamma\backslash(\tilde{N}\times\partial_\infty\Gamma)$ and $\Gamma\backslash(\tilde{N}\times\mathcal{F})\to\Gamma\backslash\tilde{N}$ induce isomorphisms in cohomology up to degree $m+1$, since the fibers are $\min(p-2,q-p)$-connected and $\min(p-2,q-p)>m+1$. Therefore the last long exact sequence reads as:

$$\check{H}^m(\Gamma\backslash(\tilde{N}\times \partial_\infty\Gamma)) \longrightarrow H^m(\Gamma\backslash\tilde{N}) \longrightarrow H^m_c(\Gamma\backslash(\tilde{N}\times \Omega)) \longrightarrow \check{H}^{m+1}(\Gamma\backslash(\tilde{N}\times \partial_\infty\Gamma)) \longrightarrow H^{m+1}(\Gamma\backslash\tilde{N}).$$

When we replace Ω, K and $\mathcal{F}$ by $\tilde{N}$, $\partial_\infty\tilde{N}\cong\partial_\infty\Gamma$ and $\overline{\tilde{N}}=\tilde{N}\cup\partial_\infty\tilde{N}$ respectively, the same argument leads to the long exact sequence

$$\check{H}^m(\Gamma\backslash(\tilde{N}\times \partial_\infty\Gamma)) \longrightarrow H^m(\Gamma\backslash\tilde{N}) \longrightarrow H^m_c(\Gamma\backslash(\tilde{N}\times \tilde{N})) \longrightarrow \check{H}^{m+1}(\Gamma\backslash(\tilde{N}\times \partial_\infty\Gamma)) \longrightarrow H^{m+1}(\Gamma\backslash\tilde{N}).$$

Comparing these two exact sequences shows that

$$H^m_c(\Gamma\backslash(\tilde{N}\times\Omega))\cong H^m_c(\Gamma\backslash(\tilde{N}\times\tilde{N})).$$

Poincaré duality implies $H^m_c(\Gamma\backslash(\tilde{N}\times\tilde{N}))^*\cong H_m(\Gamma\backslash(\tilde{N}\times\tilde{N}))$; the fibers $\Gamma\backslash(\tilde{N}\times\tilde{N})\to\Gamma\backslash\tilde{N}$ being contractible, one has $H_m(\Gamma\backslash(\tilde{N}\times\tilde{N}))\cong H_m(\Gamma\backslash\tilde{N})$. Recapitulating, we obtain the following chain of isomorphisms:

$$H_l(\Gamma\backslash\Omega)\cong H_l(\Gamma\backslash(\tilde{N}\times\Omega))\cong H^m_c(\Gamma\backslash(\tilde{N}\times\Omega))^*\cong H^m_c(\Gamma\backslash(\tilde{N}\times\tilde{N}))^*\cong H_m(\Gamma\backslash(\tilde{N}\times\tilde{N}))\cong H_m(\Gamma\backslash\tilde{N})\cong H_m(N)\cong\mathbf{R}. \tag{8.11}$$

8.3.3. *Hyperbolic groups.* The previous proof is deeply based on the fact that we are calculating (co)homology groups of manifolds. In order to adapt the proof to the case of a general finitely generated word hyperbolic group Γ we need a replacement for N and $\tilde{N}$.

Let $R_d(\Gamma)$ be a Rips complex for Γ. This is the simplicial complex whose k-simplices are given by $(k+1)$-tuples $(\gamma_0,\dots,\gamma_k)$ of Γ satisfying $d_\Gamma(\gamma_i,\gamma_j)\leq d$ for all i,j. For d big enough $R_d(\Gamma)$ is contractible [26, Chapitre 5]. Let $\tilde{R}$ denote such a contractible Rips complex $R_d(\Gamma)$ and let $R=\Gamma\backslash\tilde{R}$ be its quotient. Then R is a finite simplicial complex and as such admits an embedding $R\hookrightarrow\mathbf{R}^m$ into Euclidean space [44, Corollary A.10] [28, II.9]. A small (and regular) neighborhood of R in $\mathbf{R}^m$ gives an m-dimensional manifold with boundary $(U,\partial U)$ such that R is a retract of U.

In particular $\tilde{R}$ is a retract of $\tilde{U}$ which is hence a contractible manifold. Note also that $\partial_\infty\tilde{U}\cong\partial_\infty\tilde{R}\cong\partial_\infty\Gamma$ and that $\tilde{U}\cup\partial_\infty\Gamma$ retracts to $\tilde{R}\cup\partial_\infty\Gamma$ which is a contractible space ([13, Theorem 1.2]), therefore $\tilde{U}\cup\partial_\infty\Gamma$ is also contractible.

The same argument as in Section 8.3.2 (working with manifolds with boundary) gives the following sequences of isomorphisms, with $m=\dim U$ and $l=\dim\Omega$,

$$H_l(\Gamma\backslash\Omega)\cong H_l(\Gamma\backslash(\Omega\times\tilde{U}))\cong H^m_c(\Gamma\backslash(\Omega\times\tilde{U}))^*\cong H^m_c(\Gamma\backslash(\mathring{\tilde{U}}\times\tilde{U}))^*\cong H_m(\Gamma\backslash(\mathring{\tilde{U}}\times\tilde{U}),\Gamma\backslash(\mathring{\tilde{U}}\times\partial\tilde{U}))\cong H_m(U,\partial U)\cong\mathbf{R}, \tag{8.12}$$

where the last two isomorphisms are given by Poincaré duality for manifolds with boundary and by considering the fibration $\Gamma\backslash(\mathring{\tilde{U}} \times \tilde{U}) \to U$ of manifolds with boundary (the fibers are isomorphic to $\mathring{\tilde{U}}$, thus contractible).

9. General groups

We now turn to the case of Anosov representations into general semisimple Lie groups. We explicitly construct an open subset of G/AN, on which Γ acts properly discontinuously and with compact quotient. The set we construct will depend on the choice of an irreducible G-module (V, F), hence a homomorphism $\phi : G \to G_F$, such that the composition $\phi \circ \rho : \Gamma \to G_F$ is Q_0-Anosov. We consider then the domain of discontinuity $\Omega_{\phi\circ\rho} \subset \mathcal{F}_1(V)$ (constructed in Section 8.1) and apply Proposition 7.6 in order to obtain a domain of discontinuity in G/AN on which Γ acts with compact quotient. The dependence of the domain of discontinuity on the choice of G-module (V, F) is illustrated by several examples in Section 10.

We give an explicit description of the domain of discontinuity in terms of the Bruhat decomposition of G. This allows us to describe sufficient conditions for the domain of discontinuity to be nonempty. In the case when Γ is a free group or the fundamental group of a closed surface of genus ≥ 2 this leads to the statements of Theorem 1.11 and Theorem 1.12.

9.1. G-Modules.

We use the notation introduced in Section 3.2.

Let $\Theta \subset \Delta$ with $\iota(\Theta) = \Theta$, and $P = P_\Theta^+ < G$ the corresponding parabolic subgroup. Let (V, F) be an irreducible G-module, where F is a non-degenerate bilinear form on V, (indefinite) symmetric or skew-symmetric, and such that there is an F-isotropic line $D \subset V$ with $P = \mathrm{Stab}_G(D)$. In this section, we use standard theory for decomposition of a G-module V into weight spaces V_χ. The reader who is not familiar with basic representation theory is referred to [31] and [43, Chapter IV].

The G-module V decomposes under the action of $\mathfrak{a}$ into weight spaces:

$$V = \bigoplus_{\mu\in\mathcal{C}} V_\mu,\ V_\mu = \{v \mid \forall a \in \mathfrak{a},\, a\cdot v = \mu(a)v\},\ \mathcal{C} = \{\mu \in \mathfrak{a}^* \mid V_\mu \neq \{0\}\}.$$

The set $\mathcal{C}$ is a finite subset of $\mathfrak{a}^*$ that may contain 0; it is invariant by the action of the Weyl group W; it is also invariant by $\mu \mapsto -\mu$ since V is isomorphic to V^*.

We set $V_+ = \bigoplus_{\mu>0} V_\mu$ and $V_- = \bigoplus_{\mu<0} V_\mu$.

Let λ be the highest weight of V, i.e. the highest element of $\mathcal{C}$ with respect to the order $<$ on $\mathfrak{a}^*$. Then V_λ is in the kernel of every element $n \in \mathfrak{n}^+$ and is in fact equal to the intersection of the kernels $\ker(n)$, $n \in \mathfrak{n}^+$, (this follows from the fact that, for $n \in \mathfrak{g}_\alpha$ and for $v \in V_\mu$, $n \cdot v \in V_{\mu+\alpha}$ and is nonzero when $\mu, \mu+\alpha \in \mathcal{C}$ and $v \neq 0$).

Lemma 9.1.

(i) *The highest weight space V_λ is one-dimensional, $V_\lambda = D$.*
(ii) *The spaces V_+ and V_- are F-isotropic subspaces of V.*
(iii) *There exists a maximal F-isotropic AN-invariant subspace T containing V_+. For any such T, $T \cap V_- = \{0\}$.*

Proof. The line D is $\mathfrak{p}_\Theta^+$-invariant, hence $D \subset V_\lambda$. The $\mathfrak{n}^-$-module generated by D is $\mathfrak{g}$-invariant. By irreducibility of V this implies $D = V_\lambda$, in particular $\dim V_\lambda = 1$.

The map given by the bilinear form $F : V \otimes V \to \mathbf{R}$ is a morphism of G-modules, where $\mathbf{R}$ is the trivial G-module. Hence, for any μ in $\mathfrak{a}^*$, $F((V \otimes V)_\mu) \subset \mathbf{R}_\mu$ and $\mathbf{R}_\mu = \{0\}$ unless $\mu = 0$. It follows that $F(V_\mu \otimes V_{\mu'}) = \{0\}$ whenever $\mu + \mu' \neq 0$. Therefore the decomposition

$$V = V_0 \oplus \bigoplus_{\mu > 0} (V_\mu \oplus V_{-\mu})$$

is F-orthogonal and V_+ and V_- are isotropic. This proves that the restriction of F to V_0 and to $V_+ \oplus V_-$ is non-degenerate and that $(V_+)^\perp = V_+ \oplus V_0$.

As a consequence any isotropic space containing V_+ is contained in $V_+ \oplus V_0$ and henceforth intersects V_- trivially.

Note now that V_+ and V_0 are $\mathfrak{a}$-invariant and that $\mathfrak{n}^+ \cdot V_0 \subset V_+$ and that $\mathfrak{a}$ acts trivially on V_0. Thus for any maximal isotropic space W of $(V_0, F|_{V_0})$, the space $T = V_+ \oplus W$ is maximal isotropic in V and is $(\mathfrak{a} + \mathfrak{n}^+)$-invariant. □

Remark 9.2. In the announcement [40] we claimed that V_+ is itself maximal isotropic, this is false. A counter-example is the $\mathrm{O}(p, p+k)$-module $V = \bigwedge^2 \mathbf{R}^{2p+k}$, where the restriction of F to V_0 has signature $(k(k-1)/2, p)$.

We do not know if it is always possible to choose some G-module V that has a B-invariant maximal isotropic subspace.

Corollary 9.3. *Let (V, F) be an irreducible G-module as in Section 9.1 and let $\phi : G \to G_F = G(V, F)$ be the corresponding homomorphism. It induces the following ϕ-equivariant maps*

$$\begin{aligned} \phi_0 : G/P &\longrightarrow \mathcal{F}_0(V) & \qquad \phi_1 : G/AN &\longrightarrow \mathcal{F}_1(V) \\ gP &\longmapsto \phi(g)V_\lambda & gH &\longmapsto \phi(g)T, \end{aligned}$$

where $T \subset V$ is a maximal F-isotropic subspace, whose existence is guaranteed by Lemma 9.1.

9.2. **Domains of discontinuity in G/AN.** Let now $\rho : \Gamma \to G$ be a P-Anosov representation, and $\phi : G \to G_F$ the homomorphism considered in Section 9.1. Then $\phi \circ \rho : \Gamma \to G_F$ is Q_0-Anosov, and hence admits a cocompact domain of discontinuity $\Omega_{\phi\circ\rho} \subset \mathcal{F}_1(V)$ (see Section 8.1).

Applying Proposition 7.6 to the G-orbit $\phi_1(G/AN) \subset \mathcal{F}_1(V)$ we obtain

Theorem 9.4. *Let G be a semisimple Lie group and $P < G$ a proper parabolic subgroup. Let $\rho : \Gamma \to G$ be a P-Anosov representation. Let $\phi : G \to G_F(V)$ such that $\phi \circ \rho$ is Q_0-Anosov and let $T \subset V$ be a maximal isotropic AN-invariant subspace.*

Then Γ acts properly discontinuously and with compact quotient on the $\rho(\Gamma)$-invariant subset

$$\Omega = \Omega_{\rho,V,T} = \phi_1^{-1}(\Omega_{\phi\circ\rho}) \subset G/AN,$$

where $\Omega_{\phi\circ\rho}$ is the domain of discontinuity constructed in Section 8.1.

Remark 9.5. Examples in Section 10 illustrate that different choices of irreducible representations ϕ can lead to different domains of discontinuity $\Omega \subset G/AN$.

The main problem is that the domain of discontinuity Ω might be empty (see the examples in Remark 8.5). In order to get criteria for Ω to be nonempty, we describe its complement $K = (G/AN \smallsetminus \Omega) \subset G/AN$. Note that

$$K = K_{\rho,V,T} = \phi_1^{-1}(K_{\phi_0\circ\xi(\partial_\infty\Gamma)}),$$

where $\xi : \partial_\infty \Gamma \to G/P$ is the Anosov map associated with ρ.

The compact K is the union:

$$K = \bigcup_{t \in \partial_\infty \Gamma} \phi_1^{-1}(K_{\phi_0 \circ \xi(t)}). \tag{9.6}$$

The ϕ-equivariance of ϕ_1 and ϕ_0 implies that for any element $gP \in G/P$

$$\phi_1^{-1}(K_{\phi_0(gP)}) = \phi_1^{-1}(\phi(g) K_{\phi_0(P)}) = g\phi_1^{-1}(K_{V_\lambda}).$$

We now describe in more detail the set $\phi_1^{-1}(K_{V_\lambda})$ which is P-invariant and in particular B-invariant where $B = Z_K(\mathfrak{a})AN$ is a minimal parabolic subgroup contained in P. Let W be the Weyl group, it is generated by reflections $(s_\alpha)_{\alpha \in \Delta}$ corresponding to the simple roots.

Lemma 9.7. *Let $T \subset V$ be a maximal isotropic subspace given by Lemma 9.1. Consider the subset $S_\phi = \{w \in W \mid V_{w\cdot\lambda} \subset T\}$ of the Weyl group. Then:*

(i) The set $\phi_1^{-1}(K_{V_\lambda})$ is the disjoint union:

$$\phi_1^{-1}(K_{V_\lambda}) = \bigcup_{w \in S_\phi} BwAN \subset G/AN.$$

(ii) The set $\{w \in W \mid w \cdot \lambda < 0\}$ is contained in $W \smallsetminus S_\phi$.

(iii) The subset S_ϕ is right invariant under the action of the subgroup $W_P < W$, which is generated by the s_α, $\alpha \in \Theta$, if $P = P_\Theta$.

Proof. The set $\phi_1^{-1}(K_{V_\lambda})$ is a union of B-orbits. The Bruhat decomposition for G [53, Theorem 7.40] reads as $G = \bigcup_{w \in W} BwB$ (disjoint union). Since $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ one has, for w in W, $Z_K(\mathfrak{a})wZ_K(\mathfrak{a}) = Z_K(\mathfrak{a})w$ and since $B = Z_K(\mathfrak{a})AN$ one has also $BwB = BwAN$. The space G/AN is therefore the disjoint union of finitely many B-orbits: $BwAN$ ($w \in W$).

It is hence enough to understand when wAN belongs to $\phi_1^{-1}(K_{V_\lambda})$. This happens precisely when $\phi_1(wAN) \in K_{V_\lambda}$ or, equivalently, when $V_\lambda \subset \phi(w)T$. This is equivalent to $\phi(w)^{-1}V_\lambda = V_{w\cdot\lambda} \subset T$, which precisely means that $w \in S_\phi$. This proves the first claim. The second follows from the fact that $V_- \cap T = 0$; the third claim from the fact that $\phi_1^{-1}(K_{V_\lambda})$ is not only B-invariant but P-invariant. ☐

Corollary 9.8. *Let $w_0 \in W$ be the longest element with respect to the generating set $(s_\alpha)_{\alpha \in \Delta}$. Then w_0 is not in S_ϕ.*

Proof. The highest weight of V^* is $-w_0 \cdot \lambda$, thus $-w_0 \cdot \lambda = \lambda$ in view of the isomorphism $V^* \cong V$. Hence w_0 is not in S_ϕ. ☐

Remark 9.9. The orbit Bw_0B is the unique open orbit; for w in W one has $\operatorname{codim}(Bw_0wAN) \geq \ell(w)$, where $\ell(w)$ is the length of w.

9.3. **Groups of small virtual cohomological dimension.** The following theorem implies Theorems 1.11 and 1.12 in the introduction.

Theorem 9.10. *Let G be a semisimple Lie group, $\rho : \Gamma \to G$ be a P-Anosov representation of a finitely generated word hyperbolic group and let $\phi : G \to G_F(V)$ be such that $\phi \circ \rho$ is Q_0-Anosov. Let $\Omega = \Omega_{\rho,V,T}$ be the domain of discontinuity constructed above (Section 9.2).*

(i) If $\operatorname{vcd}(\Gamma) = 1$, then Ω is nonempty.

(ii) If $\mathrm{vcd}(\Gamma) = 2$ *and* $P < G$ *is a proper parabolic subgroup which contains every factor of* G *that is locally isomorphic to* $\mathrm{SL}(2, \mathbf{R})$*, then* Ω *is nonempty.*

In any case Γ *acts properly discontinuously and cocompactly on* Ω*.*

Proof. We only have to prove that Ω is nonempty. In the cases under consideration $\partial_\infty \Gamma$ is 0 or 1 dimensional. Using the description of $K = G/AN \smallsetminus \Omega$ given above (Equation 9.6) it is thus enough to establish that:

– $\operatorname{codim} \phi_1^{-1}(K_{V_\lambda}) \geq 1$ if $\mathrm{vcd}(\Gamma) = 1$, and
– $\operatorname{codim} \phi_1^{-1}(K_{V_\lambda}) \geq 2$ if $\mathrm{vcd}(\Gamma) = 2$.

If G is of rank one, this is immediate. If G is of rank ≥ 2 let $S_\phi \subset W$ be the set defined in Lemma 9.7. Using the fact that $\operatorname{codim} B w_0 w AN$ is at least $\ell(w)$, Corollary 9.8 implies the claim in the case when $\mathrm{vcd}(\Gamma) = 1$.

When $\mathrm{vcd}(\Gamma) = 2$, we need to prove furthermore that, for all $\alpha \in \Delta$, either $\operatorname{codim} B w_0 s_\alpha AN \geq 2$ or $w_0 s_\alpha \notin S_\phi$. If α is a root corresponding to a rank one factor of G, the bound for the codimension is satisfied by the hypothesis on P. If α corresponds to a higher rank factor, the following lemma and the point (ii) of Lemma 9.7 insure that $w_0 s_\alpha \notin S_\phi$. □

Lemma 9.11. *If* $\mathrm{rk}_{\mathbf{R}} G \geq 2$ *and* G *is simple and* $\lambda \in \mathfrak{a}^*$ *is the highest weight of a nontrivial* G*-module, then there exist a total order* $<_{\mathfrak{a}^*}$ *on* $\mathfrak{a}^*$ *such that* $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha >_{\mathfrak{a}^*} 0\}$ *and such that, for all* α *in* Δ*,* $s_\alpha \cdot \lambda >_{\mathfrak{a}^*} 0$*.*

In other words, we can modify the order on $\mathfrak{a}^*$ without changing the set of positive roots and such that, every $s_\alpha \cdot \lambda$ becomes positive.

Proof. Let ρ be the half sum of positive roots, $\rho = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with $m_\alpha \in \frac{1}{2}\mathbf{N}^*$. Let $\langle \cdot, \cdot \rangle$ be a W-invariant scalar product on $\mathfrak{a}^*$. Hence $\langle \lambda, \alpha \rangle \geq 0$ for all α and there exists α such that $\langle \lambda, \alpha \rangle > 0$. Using the fact that for any positive roots α_1, α_2 such that $\langle \alpha_1, \alpha_2 \rangle < 0$, $\alpha_1 + \alpha_2$ is a positive root, one easily gets that $m_\alpha \geq 3/2$ for all α unless $\mathfrak{g} \cong \mathfrak{sl}(3, \mathbf{R})$.

We see that $\langle \rho, \lambda \rangle > 0$ and that for any α'

$$\langle \rho, s_{\alpha'} \cdot \lambda \rangle = \langle s_{\alpha'} \cdot \rho, \lambda \rangle = \langle \rho - \alpha', \lambda \rangle = \sum_\alpha (m_\alpha - \delta_{\alpha, \alpha'}) \langle \alpha, \lambda \rangle > 0.$$

Hence a lexicographic order on $\mathfrak{a}^* \cong \mathbf{R}^k$ where the first coordinate is $\langle \rho, \cdot \rangle$ gives the desired order. The case of $\mathfrak{sl}(3, \mathbf{R})$ can be treated directly. □

9.4. **Homotopy invariance.** In general it is not easy to determine the topology of the quotient manifolds $\Gamma \backslash \Omega_{\rho, V, T}$. The following theorem shows that the homeomorphism type of $\Gamma \backslash \Omega_{\rho, V, T}$ only depends on the connected component of $\mathrm{Hom}_{P\text{-Anosov}}(\Gamma, G)$ the representation ρ lies in. This allows to restrict the computation to representations $\rho : \Gamma \to G$ of a particularly nice form.

Theorem 9.12. *Let* $\rho : \Gamma \to G$ *be a* P*-Anosov representation. Let* V, T *be as in Section 9.1, and let* $\Omega_{\rho, V, T} \subset G/AN$ *be the domain of discontinuity constructed in Section 9.3.*

Suppose that $\Omega_{\rho, V, T}$ *is nonempty.*

Then there exists a neighborhood U *of* ρ *in* $\mathrm{Hom}_{P\text{-Anosov}}(\Gamma, G)$ *such that, for any* $\rho' \in U$*,* $\Omega_{\rho', V, T}$ *is nonempty. Furthermore there exists a trivialization:*

$$\Gamma \backslash \bigcup_{\rho' \in U} \Omega_{\rho', V, T} \cong U \times \Gamma \backslash \Omega_{\rho, V, T}$$

as a bundle over U. In particular, the homotopy type of $\Gamma\backslash\Omega_{\rho,V,T}$ is locally constant.

Proof. The proof is a variation of the openness of the holonomy map for geometric structures [12, 33].

We can suppose without loss of generality that Γ is torsion-free. Hence the quotient $X = \Gamma\backslash\Omega_{\rho,V,T}$ is a compact manifold. We shall denote by $\widehat{X} = \Omega_{\rho,V,T}$ the corresponding Γ-cover of X.

Denote by $\mathcal{B}_\rho = \widehat{X} \times_\rho G/AN$ the flat G/AN-bundle over X associated with the representation $\rho : \Gamma \to G$. It comes with a canonical section $\sigma : X \to \mathcal{B}_\rho$ that is transverse to the horizontal distribution so that the corresponding ρ-equivariant map $\hat{\sigma} : \widehat{X} \to G/AN$ is a local diffeomorphism (in fact $\hat{\sigma}$ is the injection $\Omega_{\rho,V,T} \hookrightarrow G/AN$).

The complement K_ρ of $\Omega_{\rho,V,T}$ in G/AN defines a closed subset $\mathcal{K}_\rho = \widehat{X} \times_\rho K_\rho \subset \mathcal{B}_\rho$. By construction $\sigma(X) \cap \mathcal{K}_\rho = \emptyset$, thus there exists $\epsilon > 0$ such that $d(\sigma(X), \mathcal{K}_\rho) > 2\epsilon$, where d is a fixed continuous distance on $\mathcal{B}_\rho$.

Let U be a neighborhood of ρ contained in the space of P-Anosov representations. For U sufficiently small, we can suppose that the (topological) G/AN-bundle $\mathcal{B}_U = \coprod_{\rho' \in U} \mathcal{B}_{\rho'}$ is trivial, i.e. there exists $\psi : \mathcal{B}_U \cong U \times \mathcal{B}_\rho$ with $\psi|_{\mathcal{B}_\rho} = \mathrm{Id}$. Note that $\mathcal{K}_U = \coprod \mathcal{K}_{\rho'}$ is a closed subset of $\mathcal{B}_U$ (this follows from the fact that $\xi_{\rho'}$ varies continuously with ρ', see Theorem 5.13). Hence, for U sufficiently small, the section $\sigma_U = \psi^{-1}(\sigma)$ is transverse to the flat horizontal distribution and $d(\sigma_U(U \times X), \mathcal{K}_U) > \epsilon$. This means that for any $\rho' \in U$ there is a ρ'-equivariant local diffeomorphism $\hat{\sigma}_{\rho'} : \widehat{X} \to G/AN$ whose image is contained in $G/AN \smallsetminus K_{\rho'} = \Omega_{\rho',V,T}$; in particular this last set is nonempty. Furthermore, passing to the quotient, this gives a local diffeomorphism $\beta_{\rho'} : X \to \Gamma\backslash\Omega_{\rho',V,T}$ that varies continuously with ρ' and such that $\beta_\rho = \mathrm{Id}$. From this we get that $\beta_{\rho'}$ is a diffeomorphism for any $\rho' \in U$ and that

$$\coprod \beta_{\rho'} : U \times X \to \coprod \Gamma\backslash\Omega_{\rho',V,T}$$

gives a trivialization. ☐

10. Explicit descriptions of some domains of discontinuity

In this section we describe in more detail some domains of discontinuity which are obtained by applying the construction of Section 9.2.

10.1. **Lie groups of rank one.** Let $\Gamma < G$ be a convex cocompact subgroup in a rank one Lie group. The group Γ is word hyperbolic and by Theorem 5.15 the injection $\Gamma \hookrightarrow G$ is an Anosov representation; the corresponding ρ-equivariant map is the identification of $\partial_\infty\Gamma$ with the limit set $\mathcal{L}_\Gamma \subset G/P$ of Γ (Section 6.1).

Theorem 8.6 implies that Γ acts properly discontinuously and with compact quotient on the complement of the limit set $\Omega_\Gamma = G/P \smallsetminus \mathcal{L}_\Gamma$.

Note that $\Omega_\Gamma = \emptyset$ if and only if Γ is a uniform lattice in G.

10.2. **Representations into $\mathrm{SL}(n, \mathbf{K})$.** We now describe domains of discontinuity for representations $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{K})$, $\mathbf{K} = \mathbf{R}, \mathbf{C}$ of an arbitrary word hyperbolic group, which are P-Anosov with P being

- the minimal parabolic B (or Borel subgroup), i.e. B is the stabilizer of a complete flag in $\mathbf{K}^n$.

– a maximal parabolic subgroup P_k of $\mathrm{SL}(n, \mathbf{K})$, $k = 1, \ldots, n-1$, i.e. P_k is the stabilizer of a k-plane in $\mathbf{K}^n$.

Note that the parabolic subgroup opposite to P_k is (conjugate to) the maximal parabolic subgroup P_{n-k}. By Lemma 3.18 the study of P_k-Anosov representations thus reduces to the study of $P_{k,n-k}$-Anosov representations, $k = 1, \ldots, \lfloor n/2 \rfloor$, where $P_{k,n-k}$ is the stabilizer of a partial flag consisting of a k-plane and an incident $(n-k)$-plane.

In order to fix notation we consider

– $\mathcal{F} = G/B = \{(F_1, \ldots, F_{n-1}) \mid F_i \subset F_{i+1}, \dim F_i = i\}$,
– $\mathcal{F}_{k,n-k} = G/P_{k,n-k} = \{(F_k, F_{n-k}) \mid F_k \subset F_{n-k}, \dim F_i = i\}$,

and we set $F_0 = \{0\}$ and $F_n = \mathbf{K}^n$.

10.2.1. *The modules.* We introduce now the $\mathrm{SL}(n, \mathbf{K})$-modules we use to apply the construction of Section 9.2. In this section, $\perp$ will be used for the duality between a vector space and its dual: for $F \subset V$, $F^\perp$ is the space of linear forms canceling on F, $F^\perp = \{\psi \in V^* \mid \psi(f) = 0, \forall f \in F\}$.

Adjoint Representation. The adjoint representation provides a homomorphism $\mathrm{SL}(n, \mathbf{K}) \to \mathrm{O}(\mathfrak{sl}(n, \mathbf{K}), q_K)$ where q_K is the Killing form. Note that $\mathfrak{sl}(n, \mathbf{K}) \subset \mathrm{End}(\mathbf{K}^n)$ and $\mathrm{O}(\mathfrak{sl}(n, \mathbf{K}), q_K) \subset \mathrm{O}(\mathrm{End}(\mathbf{K}^n), \mathrm{tr})$ is a natural injection which, by Lemma 8.8, gives the same domain of discontinuity. Therefore we can use the latter $\mathrm{SL}(n, \mathbf{K})$-module and denote by

$$\phi^{\mathrm{Ad}} : \mathrm{SL}(n, \mathbf{K}) \longrightarrow \mathrm{O}(\mathrm{End}(\mathbf{K}^n), \mathrm{tr})$$

the corresponding homomorphism. The maps:

$$\begin{aligned} \phi_0^{Ad} : \mathcal{F}_{1,n-1} &\longrightarrow \mathcal{F}_0(\mathrm{End}(\mathbf{K}^n)) \\ (F_1, F_{n-1}) &\longmapsto \{h \in \mathrm{End}(\mathbf{K}^n) \mid h(F_{n-1}) = \{0\}, h(\mathbf{K}^n) \subset F_1\} \\ &\qquad = \{h \mid h(\mathbf{K}^n) \subset F_1, h^t(\mathbf{K}^{n*}) \subset F_{n-1}^\perp\} \\ \phi_1^{Ad} : \mathcal{F} &\longrightarrow \mathcal{F}_1(\mathrm{End}(\mathbf{K}^n)) \\ (F_1, \ldots, F_{n-1}) &\longmapsto \{h \in \mathrm{End}(\mathbf{K}^n) \mid h(F_{i+1}) \subset F_i, i = 0, \ldots, n-1\} \end{aligned}$$

are ϕ^{Ad}-equivariant.

Endomorphisms of $\bigwedge^k \mathbf{K}^n$. There is a natural homomorphism

$$\phi^k : \mathrm{SL}(n, \mathbf{K}) \to \mathrm{O}(\mathrm{End}(\textstyle\bigwedge^k \mathbf{K}^n), \mathrm{tr}).$$

The map

$$\begin{aligned} \phi_0^k : \mathcal{F}_{k,n-k} &\longrightarrow \mathcal{F}_0(\mathrm{End}(\textstyle\bigwedge^k \mathbf{K}^n)) \\ (F_k, F_{n-k}) &\longmapsto \{h \in \mathrm{End}(\textstyle\bigwedge^k \mathbf{K}^n) \mid \mathrm{Im}(h) \subset \textstyle\bigwedge^k F_k, \mathrm{Im}(h^t) \subset \textstyle\bigwedge^k F_{n-k}^\perp\} \end{aligned}$$

is ϕ^k-equivariant.

In order to define the map $\phi_1^k : \mathcal{F} \to \mathcal{F}_1(\mathrm{End}(\bigwedge^k \mathbf{K}^n))$, we need to introduce some notation.

Let $(F_1, \ldots, F_{n-1})$ be a complete flag of $\mathbf{K}^n$ and let $(e_i)_{i \in \{1, \ldots, n\}}$ be an adapted basis, i.e. $(e_i)_{i \in \{1, \ldots, l\}}$ is a basis of F_l, for each $l = 1, \ldots, n$. For an ordered k-tuple $I = (i_1 < \cdots < i_k)$ of integers between 1 and n, we set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$; then $(e_I)_I$ is a basis of $\bigwedge^k \mathbf{K}^n$. The flag $(F_I)_I$, where $F_I = \langle e_J \mid J \leq_{lex} I \rangle$ and $\leq_{lex}$ is the lexicographic order on the k-tuples, depends only on the initial flag $(F_i)_i$. We define:

$$\begin{aligned} \phi_1^k : \mathcal{F} &\longrightarrow \mathcal{F}_1(\mathrm{End}(\textstyle\bigwedge^k \mathbf{K}^n)) \\ (F_i) &\longmapsto \phi_1^k(F_i) = \{h \in \mathrm{End}(\textstyle\bigwedge^k \mathbf{K}^n) \mid \forall I,\ h(F_I) \subset F_I,\ \text{and } h \text{ is nilpotent}\} \\ &\qquad = \{h \in \mathrm{End}(\textstyle\bigwedge^k \mathbf{K}^n) \mid \forall I,\ h(F_I) \subset \bigcup_{J <_{lex} I} F_J\}. \end{aligned}$$

Remark 10.1. Other self-dual $\mathrm{SL}(n, \mathbf{K})$-modules that could be used are $V \oplus V^*$ with its natural orthogonal structure or with its natural symplectic structure and $\bigwedge V = \bigoplus_k \bigwedge^k V$ where V is any $\mathrm{SL}(n, \mathbf{K})$-module.

10.2.2. *Anosov representation with respect to the minimal parabolic.* Let Γ be a finitely generated word hyperbolic group and $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{K})$ be a B-Anosov representation. Let $\xi = (\xi_1, \dots, \xi_{n-1}) : \partial_\infty\Gamma \to \mathcal{F}$ be the corresponding Anosov map.

Adjoint representation. The representation $\phi^{\mathrm{Ad}} \circ \rho : \Gamma \to \mathrm{O}(\mathrm{End}(\mathbf{K}^n), \mathrm{tr})$ is a Q_1-Anosov representation with Anosov map $\phi_1^{Ad} \circ \xi : \partial_\infty\Gamma \to \mathcal{F}_1(\mathrm{End}(\mathbf{K}^n))$.

Let $\Omega_{\mathrm{Ad}\circ\rho} \subset \mathcal{F}_0(\mathrm{End}(\mathbf{K}^n))$ be the domain of discontinuity given by Theorem 8.6. Intersecting $\Omega_{\mathrm{Ad}\circ\rho}$ with the image of ϕ_0^{Ad} gives a domain of discontinuity Ω_ρ^{Ad} in $\mathcal{F}_{1,n-1}$. Then

$$\Omega_\rho^{Ad} = \mathcal{F}_{1,n-1} \smallsetminus K_\xi^{Ad}$$

with $K_\xi^{Ad} = \bigcup_{t \in \partial_\infty\Gamma} K_{\xi(t)}^{Ad}$ and, for $t \in \partial_\infty\Gamma$,

$$K_{\xi(t)}^{Ad} = \{(F_1, F_{n-1}) \in \mathcal{F}_{1,n-1} \mid \exists k \in \{1, \dots, n-1\},\ \text{with } F_1 \subset \xi_k(t) \subset F_{n-1}\}.$$

Endomorphisms of $\bigwedge^k \mathbf{K}^n$. Analogously, for any $k = 1, \dots, \lfloor n/2 \rfloor$ the representation $\phi^k \circ \rho : \Gamma \to \mathrm{O}(\mathrm{End}(\bigwedge^k \mathbf{K}^n), \mathrm{tr})$ is Q_1-Anosov and we obtain a domain of discontinuity in $\mathcal{F}_{k,n-k}$:

$$\Omega_\rho^k = \mathcal{F}_{k,n-k} \smallsetminus K_\xi^k$$

with $K_\xi^k = \bigcup_{t \in \partial_\infty\Gamma} K_{\xi(t)}^k$, where

$$K_{\xi(t)}^k = \{(F_k, F_{n-k}) \in \mathcal{F}_{k,n-k} \mid \exists I \text{ with } \textstyle\bigwedge^k F_k \subset \xi_I(t) \subset (\bigwedge^k F_{n-k}^\perp)^\perp\}.$$

The representations $\phi^k \circ \rho : \Gamma \to \mathrm{O}(\mathrm{End}(\bigwedge^k \mathbf{K}^n), \mathrm{tr})$ are also Q_0-Anosov; the corresponding domains of discontinuity are described in the next paragraph.

10.2.3. *Anosov representation with respect to maximal parabolics.*

Adjoint representation. Let $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{K})$ be a $P_{1,n-1}$-Anosov representation and $\xi = (\xi_1, \xi_{n-1}) : \partial_\infty\Gamma \to \mathcal{F}_{1,n-1}$ the corresponding Anosov map. The composition $\phi^{\mathrm{Ad}} \circ \rho$ is a Q_0-Anosov representation, and $\phi_0^{Ad} \circ \xi : \partial_\infty\Gamma \to \mathcal{F}_0(\mathrm{End}(\mathbf{K}^n))$ is the corresponding Anosov map. The intersection of the domain of discontinuity in $\mathcal{F}_1(\mathrm{End}(\mathbf{K}^n))$ with the image of ϕ_1^{Ad} gives a domain of discontinuity $\Omega_\rho'^{Ad}$ in $\mathcal{F}$;

$$\Omega_\rho'^{Ad} = \mathcal{F} \smallsetminus K_\xi'^{Ad} \text{ with } K_\xi'^{Ad} = \bigcup_{t \in \partial_\infty\Gamma} K_{\xi(t)}'^{Ad},$$

and for t in $\partial_\infty\Gamma$,

$$K_{\xi(t)}'^{Ad} = \{(F_1, \dots, F_{n-1}) \in \mathcal{F} \mid \exists k \in \{1, \dots, n-1\},\ \xi_1(t) \subset F_k \subset \xi_{n-1}(t)\}.$$

Endomorphisms of $\bigwedge^k \mathbf{K}^n$. Let $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{K})$ be a $P_{k,n-k}$-Anosov representation (or, what amounts to the same, a P_k-Anosov representation), and $\xi = (\xi_k, \xi_{n-k}) : \partial_\infty \Gamma \to \mathcal{F}_{k,n-k}$ the corresponding Anosov map. Then the composition $\phi^k \circ \rho$ is Q_0-Anosov with Anosov map $\phi_0^k \circ \xi : \partial_\infty \Gamma \to \mathcal{F}_0(\mathrm{End}(\bigwedge^k \mathbf{K}^n))$. We obtain a domain of discontinuity Ω'^k_ρ in $\mathcal{F}$ which is the complement of $K'^k_\xi = \bigcup_{t \in \partial_\infty \Gamma} K'^k_{\xi(t)}$ where

$$K'^k_{\xi(t)} = \{(F_1, \dots, F_{n-1}) \in \mathcal{F} \mid \exists I, \bigwedge^k \xi_k(t) \subset F_I \subset (\bigwedge^k \xi_{n-k}(t)^\perp)^\perp\}$$

A flag $(F_1, \dots, F_{n-1})$ belongs to $K'^k_{\xi(t)}$ if and only if the following holds:

$$(i_1, \dots, i_k) \leq_{lex} (j_1, \dots, j_k)$$

where the two sequences (i_l) and (j_l) are defined by:

$$\begin{aligned} \forall l \in \{1, \dots, k\}, i_l &= \min\{i \mid \dim F_i \cap \xi_k(t) = l\} \\ j_l &= \max\{j \mid \dim F_j + \xi_{n-k}(t) = n - k - 1 + l\}. \end{aligned} \tag{10.2}$$

The sequence $I = (i_l)$ satisfies in fact $I = \min\{I' \mid \bigwedge^k \xi_k(t) \subset F_{I'}\}$ where the min is taken with respect to the lexicographic order, and similarly $J = \max\{J' \mid F_{J'}^\perp \subset \bigwedge^k \xi_{n-k}(t)^\perp\}$.

Remark 10.3. The explicit descriptions of the domains of discontinuity given here show that different choices of G-modules in the construction of Section 9 can lead to different domains of discontinuity in the same flag variety. For example, let $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{K})$ be a B-Anosov representation. Then we can consider ρ as P_k-Anosov representation for any $k = 1, \dots, \lfloor n/2 \rfloor$. The domains of discontinuity $\Omega^k_\rho \subset \mathcal{F}$, if nonempty, are different.

10.2.4. *Codimension.* We give bounds for the codimensions of the sets K^{Ad}_ξ, K^k_ξ, K'^{Ad}_ξ and K'^k_ξ.

Proposition 10.4. *Let $k \leq n/2$.*

(i) Let $E = (E_1, \dots, E_{n-1})$ be a complete flag in $\mathbf{K}^n$ and let $(E_I)_I$ be the flag of $\bigwedge^k \mathbf{K}^n$ constructed in Section 10.2.1. Then the codimension of

$$K^k_E = \{(F_k, F_{n-k}) \in \mathcal{F}_{k,n-k} \mid \exists I, \text{ with } \bigwedge^k F_k \subset E_I \subset (\bigwedge^k F_{n-k}^\perp)^\perp\}$$

in $\mathcal{F}_{k,n-k}$ is a least $n-k$.

(ii) Let $F = (F_k, F_{n-k}) \in \mathcal{F}_{k,n-k}$, then the codimension of

$$K'^k_F = \{(E_1, \dots, E_{n-1}) \in \mathcal{F} \mid \exists I, \text{ with } \bigwedge^k F_k \subset E_I \subset (\bigwedge^k F_{n-k}^\perp)^\perp\}$$

in $\mathcal{F}$ is a least $n-k$.

Proof. We discuss here only the last case. The treatment of the other case is similar. If $(E_1, \dots, E_{n-1}) \in K'^k_F$, then $i_1 \leq j_1$ where $i_1 = \min\{i \mid \dim E_i \cap F_k = 1\}$ and $j_1 = \max\{j \mid E_j \subset F_{n-k}(t)\}$ (see Equation (10.2)). Hence K'^k_F is the union of the $L_{s,u}$, for $1 \leq s \leq u$, with

$$L_{s,u} = \{E \in K'^k_F \mid E_s \cap F_k \text{ is a line and } E_u \subset F_{n-k}\}.$$

In particular it is enough to calculate the codimension of the projection $\bar{L}_{s,u}$ of $L_{s,u}$ to the partial flag manifold

$$\mathcal{F}_{s,u} = \{E_s \subset E_u \mid \dim E_s = s, \dim E_u = u\}.$$

The dimension of $\mathcal{F}_{s,u}$ is $s(n-s)+(u-s)(n-u)$ whereas the dimension of $\bar{L}_{s,u} = \{E_s \subset E_u \mid E_s \cap F_k \text{ is a line and } E_u \subset F_{n-k}\}$ is $k-1+(s-1)(n-k-s)+(u-s)(n-k-u)$. Thus, the codimension is $n-k+(k-1)(s-1)+k(u-s) \geq n-k$. □

Corollary 10.5. *Let $k \leq n/2$. Let Γ be a finitely generated word hyperbolic group and let $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{K})$ be a P_k-Anosov representation (respectively a B-Anosov representation). Let Ω be the domain of discontinuity constructed in Section 10.2.3 (respectively Section 10.2.2). Set $\epsilon_{\mathbf{R}} = 1$ and $\epsilon_{\mathbf{C}} = 2$. If the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$ is less than or equal to $\epsilon_{\mathbf{K}}(n-k)$, then Ω is nonempty.*

Proof. Indeed by Proposition 10.4 the real codimension of the complement of Ω is at least $\epsilon_{\mathbf{K}}(n-k) - \dim \partial_\infty \Gamma = \epsilon_{\mathbf{K}}(n-k) - \mathrm{vcd}(\Gamma) + 1$ (Lemma 8.2). □

10.2.5. *The case of* $\mathrm{SL}(2n, \mathbf{K})$. The $\mathrm{SL}(2n, \mathbf{K})$-module $V = \bigwedge^n \mathbf{K}^{2n}$ has a natural invariant non-degenerate bilinear form $F : V \otimes V \to \bigwedge^{2n} \mathbf{K}^{2n} \cong \mathbf{K}$, $v \otimes w \mapsto v \wedge w$ that is symmetric when n is even and symplectic when n is odd. Let us denote by $\phi^\wedge : \mathrm{SL}(2n, \mathbf{K}) \to G_F(V)$ the corresponding homomorphism.

$$\begin{aligned} \phi_1^\wedge : \mathbb{P}(\mathbf{K}^{2n}) &\longrightarrow \mathcal{F}_1(V),\ [v] \longmapsto \ker(v \wedge \cdot : \textstyle\bigwedge^n \mathbf{K}^{2n} \to \bigwedge^{n+1} \mathbf{K}^{2n}) \\ \phi_0^\wedge : \mathrm{Gr}_n(\mathbf{K}^{2n}) &\longrightarrow \mathcal{F}_0(V),\ P \longmapsto \textstyle\bigwedge^n P. \end{aligned}$$

Given a P_1-Anosov representation $\rho : \Gamma \to \mathrm{SL}(2n, \mathbf{K})$ with Anosov map $\xi_1 : \partial_\infty \Gamma \to \mathbb{P}(\mathbf{K}^{2n})$, the composition $\phi^\wedge \circ \rho$ is Q_1-Anosov with Anosov map $\phi_1^\wedge \circ \xi_1$. Let $\Omega_{\phi^\wedge \circ \rho} \subset \mathcal{F}_0(V)$ be the domain of discontinuity constructed in Section 8. The preimage under $\phi_0^\wedge$ is a domain of discontinuity $\Omega_\rho^{\phi^\wedge, 1} = {\phi_0^\wedge}^{-1}(\Omega_{\phi^\wedge \circ \rho}) \subset \mathrm{Gr}_n(\mathbf{K}^{2n})$ for Γ. It can be described more explicitly by setting

$$K_{\xi_1} = \bigcup_{t \in \partial_\infty \Gamma} K_{\xi_1(t)} = \bigcup_{t \in \partial_\infty \Gamma} \{P \in \mathrm{Gr}_n(\mathbf{K}^{2n}) \,|\, \xi_1(t) \subset P\}.$$

Then

$$\Omega_\rho^{\phi^\wedge, 1} = \mathrm{Gr}_n(\mathbf{K}^{2n}) \smallsetminus K_{\xi_1}.$$

Similarly, given a P_n-Anosov representation $\rho : \Gamma \to \mathrm{SL}(2n, \mathbf{K})$ with Anosov map $\xi_n : \partial_\infty \Gamma \to \mathrm{Gr}_n(\mathbf{K}^{2n})$ one constructs a domain of discontinuity $\Omega_\rho^{\phi^\wedge, n} \subset \mathbb{P}(\mathbf{K}^{2n})$. It satisfies

$$\Omega_\rho^{\phi^\wedge, n} = \mathbb{P}(\mathbf{K}^{2n}) \smallsetminus K_{\xi_n}, \text{ where } K_{\xi_n} = \bigcup_{t \in \partial_\infty \Gamma} \mathbb{P}(\xi_n(t)).$$

In order to compare those open sets with the domains of discontinuity $\Omega_\rho^1, \Omega_\rho^n$ constructed in Section 10.2.3 we denote by $\pi_1 : \mathcal{F}(\mathbf{K}^{2n}) \to \mathbb{P}(\mathbf{K}^{2n})$ and by $\pi_n : \mathcal{F}(\mathbf{K}^{2n}) \to \mathrm{Gr}_n(\mathbf{K}^{2n})$ the natural projections.

Proposition 10.6. *With the notations introduced above:*

(i) *Let $\rho : \Gamma \to \mathrm{SL}(2n, \mathbf{K})$ be P_1-Anosov, then $\pi_n^{-1}(\Omega_\rho^{\phi^\wedge, 1}) = \Omega_\rho^1$.*

(ii) *Let $\rho : \Gamma \to \mathrm{SL}(2n, \mathbf{K})$ be P_n-Anosov, then $\pi_1^{-1}(\Omega_\rho^{\phi^\wedge, n}) = \Omega_\rho^n$.*

Proof. We prove (ii). One has $\Omega_\rho^n = \mathcal{F} \smallsetminus K_\xi^n$ with $K_\xi^n = \bigcup_{t \in \partial_\infty \Gamma} K_{\xi_n(t)}$ and $K_{\xi_n(t)} = \{(E_1, \dots, E_{2n-1}) \in \mathcal{F} \mid \exists I, \bigwedge^n \xi_n(t) \subset E_I \subset (\bigwedge^n \xi_n(t)^\perp)^\perp\}$. Also $\pi_1^{-1}(\Omega_\rho^{\phi^\wedge, n}) = \mathcal{F} \smallsetminus K'_\xi$ with $K'_\xi = \bigcup_{t \in \partial_\infty \Gamma} K'_{\xi_n(t)}$ and $K'_{\xi_n(t)} = \pi_1^{-1}(K^{\phi^\wedge, n}_{\xi_n(t)})$ and $K^{\phi^\wedge, n}_{\xi_n(t)} = \{D \mid D \subset \xi_n(t)\}$. It is easy to see that (using e.g. Equation 10.2) that $K_{\xi_n(t)} \subset K'_{\xi_n(t)}$. Hence

$K_\xi^n \subset K'_\xi$ and $\pi_1^{-1}(\Omega_\rho^{\phi^\wedge,n}) \subset \Omega_\rho^n$. Since the action of Γ is proper and cocompact on both these open sets, this last inclusion implies the equality $\pi_1^{-1}(\Omega_\rho^{\phi^\wedge,n}) = \Omega_\rho^n$. $\square$

Remark 10.7. For the projective Schottky groups, discussed in Section 6.5, the domain of discontinuity constructed in $\mathbb{P}(\mathbf{K})$, $\mathbf{K} = \mathbf{R}, \mathbf{C}$, is precisely the domain of discontinuity given in [63, 69].

10.2.6. *The Case of* $\mathrm{SL}(3,\mathbf{R})$. By Lemma 3.18.(iv) an Anosov representation $\rho : \Gamma \to \mathrm{SL}(3,\mathbf{R})$ is automatically B-Anosov (because $B = P_{1,2}$ is the only parabolic subgroup of $\mathrm{SL}(3,\mathbf{R})$ conjugate to its opposite). The corresponding Anosov map is a pair of compatible maps $\xi_1 : \partial_\infty\Gamma \to \mathbb{P}^2(\mathbf{R})$, $\xi_2 : \partial_\infty\Gamma \to \mathbb{P}^2(\mathbf{R})^*$ and the domain of discontinuity $\Omega_\rho \subset \mathcal{F}(\mathbf{R}^3)$ is the following open set (see Section 10.2.2):

$$\Omega_\rho = \{(E_1, E_2) \in \mathcal{F}(\mathbf{R}^3) \mid E_1 \notin \xi_1(\partial_\infty\Gamma) \text{ and } E_2 \notin \xi_2(\partial_\infty\Gamma)\}.$$

When $\Gamma = \pi_1(\Sigma)$ is a surface group, B-Anosov representations and the above domain of discontinuity have been studied by Barbot [2]. He proved the following dichotomy:

(i) If ρ is in the Hitchin component, then Ω_ρ has three connected components. One component Ω_1 is the pull-back of the invariant convex set $\mathcal{C} \subset \mathbb{P}^2(\mathbf{R})$ (see Goldman [35]), another component Ω_2 is the pull-back of the invariant convex set $\mathcal{C}^* \subset \mathbb{P}^2(\mathbf{R})^*$. The third component Ω_3 is "de-Sitter like", it is the set of flags (D, P) with $D \notin \overline{\mathcal{C}}$ and $P \notin \overline{\mathcal{C}}^*$. For any i, $\pi_1(\Sigma)\backslash\Omega_i$ is homeomorphic to the projectivized tangent bundle of Σ.

(ii) If ρ is not in the Hitchin component, then Ω_ρ is connected and $\pi_1(\Sigma)\backslash\Omega_\rho$ is diffeomorphic to a circle bundle over Σ; Barbot asked if it is always homeomorphic to the double cover $\pi_1(\Sigma)\backslash\mathrm{SL}(2,\mathbf{R})$ of $T^1\Sigma \cong \pi_1(\Sigma)\backslash\mathrm{PSL}(2,\mathbf{R})$. This is known to be true in some explicit examples.

10.2.7. *Holonomies of convex projective structures.* If $\Gamma \subset \mathrm{SL}(n+1,\mathbf{R})$ divides a strictly convex set $\mathcal{C}$ in $\mathbb{P}^n(\mathbf{R})$ (see Section 6.2) then the injection $\iota : \Gamma \to \mathrm{SL}(n+1,\mathbf{R})$ is $P_{1,n}$-Anosov. Thus the construction in Section 10.2.3 provides a domain of discontinuity Ω in the full flag variety $\mathcal{F}(\mathbf{R}^{n+1})$. The pull-back of $\mathcal{C}$ to $\mathcal{F}(\mathbf{R}^{n+1})$ and the pull-back of the dual convex set $\mathcal{C}^*$ are components of Ω. However, Ω has in general other components, for example for a lattice in $\mathrm{SO}(1,n)$, Ω has $n+1$ components.

10.3. **Representations into** $\mathrm{Sp}(2n,\mathbf{K})$**.** Section 8 gives a direct construction of domains of discontinuity for Q_i-Anosov representations $\rho : \Gamma \to \mathrm{Sp}(2n,\mathbf{K})$, $i = 0, 1$. By embedding $\mathrm{Sp}(2n,\mathbf{K})$ into $\mathrm{SL}(2n,\mathbf{K})$, the construction in Section 10.2.3 can be applied to representations $\rho : \Gamma \to \mathrm{Sp}(2n,\mathbf{K})$ which are P_k-Anosov, where P_k is the stabilizer of an isotropic k-plane. This gives domains of discontinuity in the complete flag variety of $\mathrm{Sp}(2n,\mathbf{K})$:

$$\mathcal{F}_{\mathrm{Sp}} = \{(E_1, \dots, E_{2n-1}) \in \mathcal{F}(\mathbf{K}^{2n}) \mid \forall i, E_{2n-i} = E_i^{\perp,\omega}\}.$$

For Q_i-Anosov representations this is the same as applying the general construction of Section 9 using the representations $\phi^1 : \mathrm{Sp}(2n,\mathbf{K}) \to \mathrm{O}(\mathrm{End}(\mathbf{K}^{2n}), \mathrm{tr})$ when $i = 0$ and $\phi^n : \mathrm{Sp}(2n,\mathbf{K}) \to \mathrm{O}(\mathrm{End}(\bigwedge^n \mathbf{K}^{2n}), \mathrm{tr})$ when $i = 1$. Thus, for a Q_0-Anosov representation, we constructed three domains of discontinuity: $\Omega_\rho^0 \subset \mathcal{L}$, $\Omega_{\phi^1,\rho}^{0,\mathrm{Sp}} \subset \mathcal{F}_{\mathrm{Sp}}$, and $\Omega_{\phi^n,\rho}^{0,\mathrm{Sp}} \subset \mathcal{F}_{\mathrm{Sp}}$. Similarly, for a Q_1-Anosov representation, we obtain $\Omega_\rho^1 \subset \mathbb{P}(\mathbf{K}^{2n})$, $\Omega_{\phi^1,\rho}^{1,\mathrm{Sp}} \subset \mathcal{F}_{\mathrm{Sp}}$, and $\Omega_{\phi^n,\rho}^{1,\mathrm{Sp}} \subset \mathcal{F}_{\mathrm{Sp}}$.

Proposition 10.8. *Let* $\pi_{1,\mathrm{Sp}} : \mathcal{F}_{\mathrm{Sp}} \to \mathbb{P}(\mathbf{K}^{2n})$ *and* $\pi_{n,\mathrm{Sp}} : \mathcal{F}_{\mathrm{Sp}} \to \mathcal{L}$ *be the natural projection.*

(i) *If* ρ *is* Q_0*-Anosov, then* $\Omega^{0,\mathrm{Sp}}_{\phi^1,\rho} = \Omega^{0,\mathrm{Sp}}_{\phi^n,\rho} = \pi^{-1}_{n,\mathrm{Sp}}(\Omega^0_\rho)$.

(ii) *If* ρ *is* Q_1*-Anosov, then* $\Omega^{1,\mathrm{Sp}}_{\phi^1,\rho} = \Omega^{1,\mathrm{Sp}}_{\phi^n,\rho} = \pi^{-1}_{1,\mathrm{Sp}}(\Omega^1_\rho)$.

Proof. The equalities $\Omega^{0,\mathrm{Sp}}_{\phi^n,\rho} = \pi^{-1}_{n,\mathrm{Sp}}(\Omega^0_\rho)$ and $\Omega^{1,\mathrm{Sp}}_{\phi^n,\rho} = \pi^{-1}_{1,\mathrm{Sp}}(\Omega^1_\rho)$ follow from Proposition 10.6. To prove, for example, the second equality in (i): $\Omega^{0,\mathrm{Sp}}_{\phi^1,\rho} = \pi^{-1}_{n,\mathrm{Sp}}(\Omega^0_\rho)$, it is enough to note that, for $L \in \mathbb{P}(\mathbf{K}^{2n})$, $\{(E_1, \ldots, E_{2n-1}) \in \mathcal{F}_{\mathrm{Sp}} \mid \exists k, L \subset E_k \subset L^{\perp\omega}\} = \{(E_1, \ldots, E_{2n-1}) \in \mathcal{F}_{\mathrm{Sp}} \mid L \subset E_n\}$. □

Remark 10.9. As mentioned in Section 6.4 any maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ is Q_1-Anosov. The quotient manifolds $M = \pi_1(\Sigma)\backslash\Omega^1_\rho$ will be investigated in more detail in [42], using results on topological invariants for maximal representations from [41]. In particular, we deduce that the manifolds M are homeomorphic to the total space of $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundles over Σ. This implies also that for representations $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2n, \mathbf{R})$ in the Hitchin component the quotient manifold $\pi_1(\Sigma)\backslash\Omega^{\phi^\wedge,n}_\rho$ is homeomorphic to the total space of a $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle over Σ.

10.4. **Representations into** $\mathrm{SO}(p, q)$**.** The construction in Section 8 gives an explicit description of domains of discontinuity $\Omega^{1-i}_\rho \subset \mathcal{F}_{1-i}$ for Q_i-Anosov representations $\rho : \Gamma \to \mathrm{SO}(p, q)$, $i = 0, 1$.

(i) Let $\Gamma < \mathrm{SO}(1, n)$ be a convex cocompact subgroup. Consider the Q_0-Anosov representation $\rho : \Gamma \to \mathrm{SO}(1, n + 1)$, obtained by naturally embedding $\mathrm{SO}(1, n) < \mathrm{SO}(1, n + 1)$. Then the quotient $\Gamma\backslash\Omega_\rho$ is the union of two copies of $\Gamma\backslash\mathbf{H}^n$ when Γ is cocompact or the double of the compact manifold with boundary whose interior is $\Gamma\backslash\mathbf{H}^n$ otherwise. This is similar to what happens for Fuchsian groups embedded in $\mathrm{PSL}(2, \mathbf{C})$.

(ii) When $\Gamma < \mathrm{SO}(1, n) < \mathrm{SO}(p, q)$ $(p \leq q)$ is a convex cocompact subgroup of $\mathrm{SO}(1, n)$, we have $\Omega^{1-i}_\rho = \emptyset$ if $q = n$ and Γ is a lattice, but one gets interesting domains of discontinuity for $q = n + 1$, see Section 13.

(iii) A representation $\rho : \pi_1(\Sigma) \to \mathrm{SO}(2, 3)$ in the Hitchin component is Q_1-Anosov. In this case we obtain a nonempty domain of discontinuity $\Omega^0_\rho \subset \mathcal{F}_0$ in the Einstein space. The quotient $\pi_1(\Sigma)\backslash\Omega^0_\rho$ consists of two connected components, which are both homeomorphic to the unit tangent bundle of the surface. Considering the representation ρ as Q_0-Anosov representation, one obtains a domain of discontinuity in $\mathbb{P}^3(\mathbf{R})$.

Part 3. **Applications**

11. Higher Teichmüller spaces

In Part 2 we gave a construction of domains of discontinuity for Anosov representations $\rho : \Gamma \to G$. The domains of discontinuity are open subsets Ω_ρ of a homogeneous space $X = G/H$ for some subgroup H containing AN. The quotient $W = \Gamma\backslash\Omega_\rho$ is naturally equipped with a (G, X)-structure. Thus one can rephrase the result of Theorem 9.4 as associating a (G, X)-structure to an Anosov representation. In this section we make this statement precise for representations in a Hitchin component as well as for maximal representations.

11.1. **Geometric structures.** Let G be a Lie group and X be a manifold with a smooth G-action. (The definition below can be adapted to treat more general G-spaces.)

A (G, X)-*variety* W is a topological space together with a (maximal) G-atlas on W, that is

(i) an open cover $\mathcal{U}$ and, for each U in $\mathcal{U}$ a homeomorphism $\phi_U : U \to \phi_U(U)$ onto an open subset of X, such that
(ii) for any U and U' in $\mathcal{U}$, $\phi_{U'} \circ \phi_U^{-1} : \phi_U(U \cap U') \to \phi_{U'}(U \cap U')$ is (locally) the restriction of an element of G.

The maps ϕ_U are called *charts*. A map $\psi : W \to W'$ between (G, X)-varieties is a G-*map* if ψ is (locally) in the charts an element of G.

A (G, X)-*structure* on a manifold M is a pair (W, f) of a (G, X)-variety W and a diffeomorphism $f : M \to W$. Two (G, X)-structures (W, f) and (W', f') on M are said to be *equivalent* if there exists a G-map $\psi : W \to W'$. They are said to be *isotopic* if there exists a G-map $\psi : W \to W'$ such that $\psi \circ f'$ is isotopic to f.

The space of isotopy classes of (G, X)-structures on M is denoted by $\mathcal{D}_{G,X}(M)$. The space of equivalence classes is denoted by $\mathcal{M}_{G,X}(M)$. There is a natural action of $\mathrm{Diff}(M)$ on $\mathcal{D}_{G,X}(M)$ (by precomposition) that factors through the mapping class group $\mathrm{Mod}(M) = \pi_0(\mathrm{Diff}(M))$; and $\mathcal{M}_{G,X}(M) = \mathcal{D}_{G,X}(M)/\mathrm{Mod}(M)$.

11.2. **The holonomy theorem.** In this section we recall some background on locally homogeneous (G, X)-structures, we refer the reader to [33, Section 3] for more details.

Every (G, X)-structure (W, f) on M induces a $\pi_1(M)$-invariant (G, X)-structure on the universal cover $\widetilde{M}$. As $\widetilde{M}$ is simply connected, the (G, X)-structure on $\widetilde{M}$ can be encoded in *one* map $\mathrm{dev} : \widetilde{M} \to X$ that is a local diffeomorphism (the charts ϕ_U can be "patched" together). The map dev is called the *developing map* and is unique up to postcomposition with elements of G.

This uniqueness means that there exists a representation $\rho : \pi_1(M) \to G$ such that $\mathrm{dev} \circ \gamma = \rho(\gamma) \cdot \mathrm{dev}$ for any $\gamma \in \pi_1(M)$. The homomorphism ρ is called the *holonomy representation*. If dev is changed to $g \cdot \mathrm{dev}$ then ρ is changed to the conjugate homomorphism $\gamma \mapsto g\rho(\gamma)g^{-1}$, in particular only the conjugacy class of ρ is well defined by the (G, X)-structure (W, f). This conjugacy class is denoted by $\mathrm{hol}(W, f) \in \mathrm{Hom}(\pi_1(M), G)/G$.

The space of isotopy classes $\mathcal{D}_{G,X}(M)$ is thus identified with equivalence classes of pairs $(\mathrm{dev}, \mathrm{hol})$, of a local diffeomorphism dev that is equivariant with respect to a representation hol, and can be topologized using the compact open topology on these spaces of maps.

If $\psi : M \to M$ is a diffeomorphism, then the holonomy representations for (W, f) and $(W, f \circ \psi)$ are related by $\mathrm{hol}(W, f \circ \psi) = \mathrm{hol}(W, f) \circ \psi_*$ where $\psi_* : \pi_1(M) \to \pi_1(M)$ is the induced homomorphism. Hence the holonomy map descends to a map $\mathrm{hol} : \mathcal{D}_{G,X}(M) \to \mathrm{Hom}(\pi_1(M), G)/G$ that is $\mathrm{Mod}(M)$-equivariant.

Theorem 11.1. [71, §5.3.1] *(Holonomy Theorem)*
The holonomy map $\mathrm{hol} : \mathcal{D}_{G,X}(M) \to \mathrm{Hom}(\pi_1(M), G)/G$ *is a local homeomorphism.*

Remark 11.2. In order to avoid having to deal with potential singularities in the G-quotient $\mathrm{Hom}(\pi_1(M), G)/G$, one can work with the space $\mathcal{D}^*_{G,X}(M)$ of *based*

(G, X)-structures. Here, in addition, a germ of (G, X)-structure at a point $m \in M$ is specified. Then the holonomy map is well defined as a map $\mathrm{hol} : \mathcal{D}^*_{G,X}(M) \to \mathrm{Hom}(\pi_1(M, m), G)$ that is a local homeomorphism.

11.3. **Hitchin component for $\mathrm{SL}(2n, \mathbf{R})$.**

Theorem 11.3. *Let Σ be a closed connected orientable surface of genus ≥ 2. Let $\mathcal{C}$ be the Hitchin component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2n, \mathbf{R}))/\mathrm{PSL}(2n, \mathbf{R})$. Assume that $n \geq 2$.*

Then there exist

- *a compact $(2n-1)$-dimensional manifold M,*
- *a homomorphism $\pi : \pi_1(M) \to \pi_1(\Sigma)$,*
- *and a connected component $\mathcal{D}$ of the space $\mathcal{D}_{\mathrm{PSL}(2n,\mathbf{R}),\mathbb{P}^{2n-1}(\mathbf{R})}(M)$ of $(\mathrm{PSL}(2n, \mathbf{R}), \mathbb{P}^{2n-1}(\mathbf{R}))$-structures on M.*

such that

(i) The map

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathcal{C}_M \subset \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2n, \mathbf{R}))/\mathrm{PSL}(2n, \mathbf{R}) \\ \rho &\longmapsto \rho \circ \pi \end{aligned}$$

is a homeomorphism onto a connected component $\mathcal{C}_M$.

(ii) The restriction of the holonomy map hol to $\mathcal{D}$ is a homeomorphism onto $\mathcal{C}_M$, i.e. $\mathrm{hol}|_{\mathcal{D}} : \mathcal{D} \xrightarrow{\sim} \mathcal{C}_M$.

Furthermore, there exists a homomorphism $\theta : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(M)$ such that the identification $\mathcal{C} \cong \mathcal{C}_M$ is θ-equivariant. In other words, $\mathcal{D} \cong \mathcal{C}$ is equivariant with respect to the action of the mapping class group.

Proof. For $\rho \in \mathcal{C}$, let $\Omega_\rho \subset \mathbb{S}^{2n-1}$ the lift to the sphere of the domain of discontinuity constructed in Section 10.2.5, considering $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2n, \mathbf{R})$ as a P_n-Anosov representation. When $n = 2$ the domain of discontinuity has two connected components and Ω_ρ denotes any one of them. The quotient space $W_\rho = \Gamma \backslash \Omega_\rho$ is a $(\mathrm{PSL}(2n, \mathbf{R}), \mathbb{P}^{2n-1}(\mathbf{R}))$-variety. Moreover, by Theorem 9.12, the total space $\mathcal{W} = \coprod_{\rho \in \mathcal{C}} W_\rho$ is a fiber bundle over the base $\mathcal{C}$ (i.e. locally over $\mathcal{C}$, $\mathcal{W}$ is a product). Since $\mathcal{C}$ is simply connected (actually, by [46, Theorem A], $\mathcal{C}$ is a cell), this fiber bundle is trivial, i.e. $\mathcal{W} = \mathcal{C} \times M$.

In particular, for each $\rho \in \mathcal{C}$, there is a diffeomorphism $f_\rho : M \to W_\rho$. Hence there is a continuous map $\sigma : \mathcal{C} \to \mathcal{D}_{\mathrm{PSL}(2n,\mathbf{R}),\mathbb{P}^{2n-1}(\mathbf{R})}(M)$, $\rho \mapsto (W_\rho, f_\rho)$. Moreover the Γ-cover $\Omega_\rho \to W_\rho$ gives a homomorphism $\pi : \pi_1(W_\rho) \cong \pi_1(M) \to \Gamma$ that does not depend on ρ (again using that $\mathcal{C}$ is simply connected).

The map σ fits in the diagram

$$\begin{array}{ccc} & & \mathcal{D}_{\mathrm{PSL}(2n,\mathbf{R}),\mathbb{P}^{2n-1}(\mathbf{R})}(M) \\ & \overset{\sigma}{\nearrow} & \downarrow \mathrm{hol} \\ \mathcal{C} & \longrightarrow & \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2n, \mathbf{R}))/\mathrm{PSL}(2n, \mathbf{R}) \end{array}$$

where the bottom map is $\rho \mapsto \rho \circ \pi$. Since σ is injective and hol is a local homeomorphism, the statements of the theorem follow if the map $\beta : \rho \mapsto \rho \circ \pi$ is onto a connected component $\mathcal{C}_M$ of $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2n, \mathbf{R}))/\mathrm{PSL}(2n, \mathbf{R})$. For this, it is enough to show that β is open.

When $n \geq 4$, the codimension of $K_\rho = \mathbb{S}^{2n-1} \smallsetminus \Omega_\rho$ is bigger than 3, hence $\pi_1(\Omega_\rho) = \pi_1(\mathbb{S}^{2n-1}) = \{1\}$ and $\pi_1(M) = \pi_1(\Sigma)$ and β is the identity and is obviously open.

For $n = 2$, one observes (considering a Fuchsian representation and using Theorem 9.12) that $\Omega_\rho \cong \mathrm{SL}(2, \mathbf{R})$ or $\Omega_\rho \cong \mathrm{SL}(2, \mathbf{R})/(\mathbf{Z}/3\mathbf{Z})$, $\mathbf{Z}/3\mathbf{Z} \subset \mathrm{SO}(2)$, depending on which connected component of the domain of discontinuity one considers. In particular, M is the total space of a circle bundle over Σ with Euler number $g-1$ or $3g-3$. The set $\{\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2n, \mathbf{R})) \mid Z(\rho) \text{ is finite}\}$ is open in $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2n, \mathbf{R}))$ and since $\pi_1(M) \to \pi_1(\Sigma)$ is a central extension, the image by β of $\{\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2n, \mathbf{R})) \mid Z(\rho) \text{ is finite}\}$ is open. By [57, Lemma 10.1] $\mathcal{C} \subset \{\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2n, \mathbf{R})) \mid Z(\rho) \text{ is finite}\}$, hence β is open.

For $n = 3$ the complement K_ρ of Ω_ρ in $\mathbb{S}^5$ is homeomorphic to $(\mathbb{S}^1 \times \mathbb{S}^2)/\{\pm 1\}$. Hence $\pi_1(K_\rho) \cong \mathbf{Z}$ and $H^3(K_\rho, \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$. Alexander duality implies that $H_1(\Omega_\rho, \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$. If U denotes a tubular neighborhood of K_ρ, then $U \smallsetminus K_\rho$ has abelian fundamental group. The Van Kampen theorem implies now that $\pi(U \smallsetminus K_\rho) \to \pi_1(\Omega_\rho)$ is onto (otherwise, one would get $\pi_1(\mathbb{S}^5) \neq 0$) and hence that $\pi_1(\Omega_\rho)$ is abelian. In conclusion, $\pi_1(\Omega_\rho) \cong H_1(\Omega_\rho, \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$ is finite. This is enough to show that β is open.

The mapping class group acts naturally on $\mathcal{C}$ and on $\coprod_{\rho \in \mathcal{C}} \Omega_\rho$ and hence on $\mathcal{W} = \Gamma \backslash \coprod_{\rho \in \mathcal{C}} \Omega_\rho$. Thus, for each $\psi \in \mathrm{Mod}(\Sigma)$, we get a bundle automorphism of $\mathcal{W} \cong \mathcal{C} \times M$, that is to say a family of diffeomorphisms $\{f_{\psi,\rho}\}_{\rho \in \mathcal{C}}$. Since $\mathcal{C}$ is connected, the class $\theta(\psi) \in \mathrm{Mod}(M)$ of $f_{\psi,\rho}$ is well defined independently of ρ. This defines a homomorphism $\theta : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(M)$ satisfying all the wanted properties. ☐

Remarks 11.4.

(i) Theorem 11.3 and Theorem 11.5 below solve the problem of giving a geometric interpretation of Hitchin components[3]. We obtain an embedding of the Hitchin component into the deformation space of geometric structures (e.g. real projective structures when $G = \mathrm{PSL}(2n, \mathbf{R})$), such that the image is a connected component of $\mathcal{D}_{G,X}(M)$. This implies that the Hitchin component parametrizes specific (G, X)-structures on a manifold M. However, in the general case, we do not characterize the image in $\mathcal{D}_{G,X}(M)$ in geometric terms. A geometric characterization had been obtained for $\mathrm{PSL}(3, \mathbf{R})$ by Choi and Goldman [25, 35], and for $\mathrm{PSL}(4, \mathbf{R})$ by the authors [39].

(ii) In [42] we determine the homeomorphism type of M and show that M is homeomorphic to the total space of an $\mathrm{O}(n)/\mathrm{O}(n-2)$-bundle over Σ.

11.4. **Hitchin components for classical groups.**

Theorem 11.5. *Let Σ be a closed connected orientable surface of genus ≥ 2. Assume that G is $\mathrm{PSL}(2n, \mathbf{R})$ ($n \geq 2$), $\mathrm{PSp}(2n, \mathbf{R})$ ($n \geq 2$), or $\mathrm{PSO}(n, n)$ ($n \geq 3$), and $X = \mathbb{P}^{2n-1}(\mathbf{R})$; or that G is $\mathrm{PSL}(2n+1, \mathbf{R})$ ($n \geq 1$), or $\mathrm{PSO}(n, n+1)$ ($n \geq 2$), and $X = \mathcal{F}_{1,2n}(\mathbf{R}^{2n+1}) = \{(D, H) \in \mathbb{P}^{2n}(\mathbf{R}) \times \mathbb{P}^{2n}(\mathbf{R})^* \mid D \subset H\}$.*

Let $\mathcal{C} \subset \mathrm{Hom}(\pi_1(\Sigma), G)/G$ be the Hitchin component.

[3] Hitchin did ask for such a geometric interpretation in [46].

Then there exists a compact manifold M, a homomorphism $\pi : \pi_1(M) \to \pi_1(\Sigma)$, and a connected component $\mathcal{D}$ of the deformation space $\mathcal{D}_{G,X}(M)$ such that

$$\mathcal{C} \to \mathcal{C}_M,\ \rho \mapsto \rho \circ \pi$$

is a homeomorphism onto a connected component $\mathcal{C}_M$ of $\mathrm{Hom}(\pi_1(M), G)/G$ and such that

$$\mathrm{hol}|_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}_M$$

is a homeomorphism.

Furthermore there is a homomorphism $\theta : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(M)$ such that the identification $\mathcal{C} \cong \mathcal{D}$ is θ-equivariant.

Proof. The proof proceeds along the same lines as the proof of Theorem 11.3, considering the following domains of discontinuity $\Omega_\rho \subset X$.

(i) When $X = \mathbb{P}(\mathbf{R}^{2n})$, Ω_ρ is defined in Section 10.2.5, regarding $\rho : \pi_1(\Sigma) \to G \to \mathrm{PSL}(2n, \mathbf{R})$ as a P_n-Anosov representation.

(ii) When $X = \mathcal{F}_{1,2n}(\mathbf{R}^{2n+1})$, Ω_ρ^{Ad} is defined in Section 10.2.2, regarding $\rho : \pi_1(\Sigma) \to G \to \mathrm{PSL}(2n+1, \mathbf{R})$ as a B-Anosov representation.

The central ingredients are that the Hitchin component $\mathcal{C}$ is simply connected and that the fundamental group of the domain of discontinuity is finite (or centralized by $\pi_1(\Sigma)$). □

11.5. **Components of the space of maximal representations.** Components of the space of maximal representations might have nontrivial topology, in particular their fundamental groups can be nontrivial.

For example, work of Gothen [36, Propositions 5.11, 5.13 and 5.14] implies that the components of the space of maximal representations of $\pi_1(\Sigma)$ into $\mathrm{Sp}(4, \mathbf{R})$, which are not Hitchin components, have fundamental groups which are isomorphic to surface groups, $(\mathbf{Z}/2\mathbf{Z})^{2g}$, or to $\mathbf{Z}^{2g}$.

Theorem 11.6. *Let Σ be a closed connected orientable surface of genus ≥ 2. Let $\mathcal{C} \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Sp}(2n, \mathbf{R}))/\mathrm{Sp}(2n, \mathbf{R}))$, $n \geq 2$, be a component of the space of maximal representations.*

Then there exists a compact manifold M of dimension $2n-1$ and a homomorphism $\pi : \pi_1(M) \to \pi_1(\Sigma)$, such that $\rho \mapsto \rho \circ \pi$ gives an identification of $\mathcal{C}$ with a connected component $\mathcal{C}_M$ of $\mathrm{Hom}(\pi_1(M), \mathrm{Sp}(2n, \mathbf{R}))/\mathrm{Sp}(2n, \mathbf{R})$.

Furthermore there exists a connected component $\mathcal{D}$ of the deformation space $\mathcal{D}_{\mathrm{Sp}(2n,\mathbf{R}),\mathbb{P}^{2n-1}(\mathbf{R})}(M)$ and a homomorphism $\kappa : \pi_1(\mathcal{C}) \to \mathrm{Mod}(M)$ such that $\mathrm{hol} : \mathcal{D} \to \mathcal{C}_M \cong \mathcal{C}$ is the Galois cover associated with $\ker \kappa$. The corresponding isomorphism of universal covers induces a local homeomorphism $\widetilde{\mathcal{D}} \cong \widetilde{\mathcal{C}} \to \mathcal{D}$, that is equivariant with respect to the subgroup $\mathrm{Mod}_{\mathcal{C}}$ of $\mathrm{Mod}(\Sigma)$ stabilizing $\mathcal{C}$.

Remark 11.7. Work of García-Prada, Gothen and Mundet i Riera [32] seems to imply that components of the space of maximal representations of $\pi_1(\Sigma)$ into $\mathrm{Sp}(2n, \mathbf{R})$ with $n \geq 3$ are always simply connected. If this holds true the statement of the theorem can be simplified when $n \geq 3$.

Of course, there are similar statements for components of the space of maximal representations of $\pi_1(\Sigma)$ into other Lie groups G of Hermitian type. For classical Lie groups we list here the homogeneous space X the geometric structure is modeled on:

– $G = \mathrm{SO}(2, n)$, $X = \mathcal{F}_1(\mathbf{R}^{2+n})$ the space of isotropic 2-planes.

– $G = \mathrm{SU}(p,q)$, $X \subset \mathbb{P}^{p+q-1}(\mathbf{C})$ is the null cone for the Hermitian form.
– $G = \mathrm{SO}^*(2n)$, $X \subset \mathbb{P}^{n-1}(\mathbf{H})$ is the null cone for the skew-Hermitian form.

12. Compactifying quotients of symmetric spaces

Let Γ be a word hyperbolic group and $\rho : \Gamma \to G$ an Anosov representation. Then $\rho(\Gamma) < G$ is a discrete subgroup, and the action of $\rho(\Gamma)$ on the symmetric space $\mathcal{H} = G/K$ by isometries is properly discontinuous. In most cases, the quotient $M = \rho(\Gamma)\backslash\mathcal{H}$ will not be compact nor of finite volume.

In this section we will describe how the construction of domains of discontinuity, together with Proposition 7.6 can be applied in order to describe compactifications of $\Gamma\backslash\mathcal{H}$. More precisely, in a suitable compactification $\overline{\mathcal{H}}$ of $\mathcal{H}$ we will describe a $\rho(\Gamma)$-invariant subset $\overline{\mathcal{H}}_\rho$ containing $\mathcal{H}$ such that the following holds: $\rho(\Gamma)$ acts properly discontinuously on $\overline{\mathcal{H}}_\rho$ with compact quotient $\overline{M} = \Gamma\backslash\overline{\mathcal{H}}_\rho$, and $\overline{M}$ contains M as an open dense set.

We will now describe the construction in detail in the case when $\rho : \Gamma \to \mathrm{Sp}(2n,\mathbf{R})$ is a Q_0-Anosov representation, and $\overline{\mathcal{H}}$ is the bounded symmetric domain compactification of the symmetric space $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$. We then list other examples, to which an analogous construction applies.

12.1. Quotients of the Siegel space.

Let us recall the geometric realization of the Borel embeddings for the Siegel space $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$. Let $(\mathbf{R}^{2n},\omega)$ be a symplectic vector space and $(\mathbf{C}^{2n},\omega_{\mathbf{C}})$ be its complexification, and $\phi : \mathrm{Sp}(2n,\mathbf{R}) \to \mathrm{Sp}(2n,\mathbf{C})$ the corresponding embedding. Let h be the non-degenerate Hermitian form of signature (n,n) on $\mathbf{C}^{2n}$ defined by $h(v,w) = i\omega_{\mathbf{C}}(\overline{v},w)$. Then h is preserved by $\phi(\mathrm{Sp}(2n,\mathbf{R}))$. The symmetric space $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$ admits a ϕ-equivariant embedding into the complex Lagrangian Grassmannian $\mathcal{L}(\mathbf{C}^{2n})$, namely

$$\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})} \cong \mathcal{H} = \{W \in \mathcal{L}(\mathbf{C}^{2n}) \,|\, h|_W > 0\},$$

where $h|_W > 0$ means that h restricted to W is positive definite. The natural compactification

$$\overline{\mathcal{H}}_{\mathrm{Sp}(2n,\mathbf{R})} = \overline{\mathcal{H}} = \{W \in \mathcal{L}(\mathbf{C}^{2n}) \,|\, h|_W \geq 0\},$$

where $h|_W \geq 0$ means that h restricted to W is positive semi-definite, is isomorphic to the bounded symmetric domain compactification of $\mathcal{H}_{\mathrm{Sp}(2n,\mathbf{R})}$.

The compactification $\overline{\mathcal{H}}$ decomposes into $\mathrm{Sp}(2n,\mathbf{R})$-orbits $\mathcal{H}_k$, $k = 0,\dots,n$ with $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_n \cong \mathcal{L}(\mathbf{R}^{2n})$. The other G-orbits $\mathcal{H}_k$ have the structure of a fiber bundle over $\mathcal{F}_k = \{V \subset \mathbf{R}^{2n} \,|\, \dim(V) = k,\, \omega|_V = 0\}$, the space of isotropic k-dimensional subspace in $\mathbf{R}^{2n}$. The fiber over $V \in \mathcal{F}_k$ is

$$\{W \in \mathcal{L}(\mathbf{C}^{2n}) \,|\, h|_W \geq 0,\, W \cap \overline{W} = V \otimes_{\mathbf{R}} \mathbf{C}\}.$$

Theorem 12.1. *Let $\Gamma \to \mathrm{Sp}(2n,\mathbf{R})$ be a Q_0-Anosov representation. Then there exists a compactification $\overline{M}$ of $M = \Gamma\backslash\mathcal{H}$ such that $\overline{M}$ carries a $(\mathrm{Sp}(2n,\mathbf{R}),\overline{\mathcal{H}})$-structure and the inclusion $M \subset \overline{M}$ is an $\mathrm{Sp}(2n,\mathbf{R})$-map.*

More precisely, there exists an open $\rho(\Gamma)$-invariant subset $\overline{\mathcal{H}}_\rho$ of $\overline{\mathcal{H}}$ containing $\mathcal{H}$ such that Γ acts properly discontinuously and cocompactly on $\overline{\mathcal{H}}_\rho$, and $\overline{M} = \Gamma\backslash\overline{\mathcal{H}}_\rho$.

In conclusion, $\overline{M}$ is a manifold with corners that is locally modelled on $\overline{\mathcal{H}}$.

Proof. Let $\rho : \Gamma \to \mathrm{Sp}(2n, \mathbf{R})$ be the Q_0-Anosov representation with associated Anosov map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbf{R}^{2n})$. Then $\phi \circ \rho : \Gamma \to \mathrm{Sp}(2n, \mathbf{C})$ is a Q_0-Anosov representation with Anosov map $\xi_{\mathbf{C}} : \partial_\infty\Gamma \to \mathbb{P}(\mathbf{C}^{2n})$. Let $\Omega_{\phi\circ\rho} \subset \mathcal{L}(\mathbf{C}^{2n})$ be the domain of discontinuity associated to $\phi \circ \rho$ in Section 8.

Then, by Proposition 7.6, $\rho(\Gamma)$ acts properly discontinuously and cocompactly on

$$\overline{\mathcal{H}}_\rho := \overline{\mathcal{H}} \cap \Omega_{\phi\circ\rho}.$$

Recall that

$$\Omega_{\phi\circ\rho} = \{W \in \mathcal{L}(\mathbf{C}^{2n}) \,|\, \forall t \in \partial_\infty\Gamma, W \cap \xi_{\mathbf{C}}(t) = 0\}.$$

Since $\xi(t)$ is a real line, if $\xi_{\mathbf{C}}(t) \subset W$, then $\xi_{\mathbf{C}}(t) \subset \overline{W}$. This implies that $\Omega_{\phi\circ\rho}$ contains the set $\{W \in \mathcal{L}(\mathbf{C}^{2n}) \,|\, W \cap \overline{W} = 0\}$, which contains $\mathcal{H}$. □

We can describe $\overline{\mathcal{H}}_\rho$ more explicitly. For this we set

$$K_k(\xi) := \bigcup_{t\in\partial_\infty\Gamma} \{V \in \mathcal{F}_k \,|\, \xi(t) \subset V\}.$$

Then

$$\overline{\mathcal{H}}_\rho = \bigcup_{i=0}^{n} (\mathcal{H}_k \smallsetminus \mathcal{H}_k|_{K_k(\xi)}),$$

where $\mathcal{H}_k|_{K_k(\xi)}$ is the restriction of the bundle to $K_k(\xi) \subset \mathcal{F}_k$.

Corollary 12.2. *Let $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ be a Hitchin representation, then there is a natural compactification of $M = \pi_1(\Sigma)\backslash\mathcal{H}$ as $(\mathrm{Sp}(2n, \mathbf{R}), \overline{\mathcal{H}})$-manifold.*

An analogous construction applies for example to:

(i) Q_0-Anosov representations $\rho : \Gamma \to \mathrm{SU}(n, n)$, where $G_{\mathbf{C}} = \mathrm{SL}(2n, \mathbf{C})$. Then $\phi \circ \rho : \Gamma \to \mathrm{SL}(2n, \mathbf{C})$ is a P_1-Anosov representation and $\Omega_{\phi\circ\rho} \subset \mathrm{Gr}_n(\mathbf{C}^{2n})$ is the domain of discontinuity described in Section 10.2.5.

(ii) Q_0-Anosov representations $\rho : \Gamma \to \mathrm{SO}(n, n)$ where we consider $\phi : \mathrm{SO}(n, n) \to \mathrm{SL}(2n, \mathbf{R})$. Then $\phi \circ \rho : \Gamma \to \mathrm{SL}(2n, \mathbf{R})$ is a P_1-Anosov representation and $\Omega_{\phi\circ\rho} \subset \mathrm{Gr}_n(\mathbf{R}^{2n})$ is the domain of discontinuity described in Section 10.2.5.

Remark 12.3. In the case of an arbitrary Anosov representation $\rho : \Gamma \to G$ we do not know how to construct a natural compactification of the quotient manifolds $M = \rho(\Gamma)\backslash\mathcal{H}$, where $\mathcal{H} = G/K$ is the symmetric space.

However, we propose to investigate the following approach. Recall that for a general semisimple Lie group G the symmetric space $\mathcal{H} = G/K$ can be embedded into the space of probability measures $\mathcal{M}(G/Q)$ on G/Q, where $Q < G$ is any proper parabolic subgroup. The closure of the image $\overline{\mathcal{H}}$ is a Furstenberg-compactification of $\mathcal{H}$.

Assume that $\rho : \Gamma \to G$ is an Anosov representation which admits a domain of discontinuity $\Omega_\rho \subset G/Q$ with compact quotient. Let $K_\rho = G/Q \smallsetminus \Omega_\rho$, and denote by $\mathcal{K}_\rho \subset \mathcal{M}(G/Q)$ the set of probability measures with support on K_ρ. Then the action of $\rho(\Gamma)$ on $\mathcal{M}(G/Q) \smallsetminus \mathcal{K}_\rho$ is proper. Furthermore $\mathcal{M}(G/Q) \smallsetminus \mathcal{K}_\rho$ contains the image of $\mathcal{H}$.

When is the action of $\rho(\Gamma)$ on $\overline{\mathcal{H}} \cap (\mathcal{M}(G/Q) \smallsetminus \mathcal{K}_\rho)$ cocompact ?

13. Compact Clifford-Klein forms

In this section we apply the construction of domains of discontinuity to construct compact Clifford-Klein forms of homogeneous spaces. The examples indicate that a more systematic treatment would be very interesting. For a recent survey on compact Clifford-Klein forms we refer the reader to [55].

13.1. **Q_1-Anosov representations.** Let $G = \mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ or $G_{\mathcal{O}}$ (the isometry group of the Cayley hyperbolic plane) and let $\Gamma < G$ be a convex cocompact subgroup. Denote by $G \to \mathrm{SO}(p,q)$ the natural embedding, i.e. $(p,q) = (2,2n)$, $(4,4n)$, or $(8,8)$ respectively (see Remark 8.5). Let $\rho : \Gamma \to \mathrm{SO}(p,q)$ be the corresponding embedding of Γ. Then ρ is a Q_1-Anosov representation. The domain of discontinuity $\Omega_\rho \subset \mathcal{F}_0(\mathbf{R}^{p,q})$ is empty if and only if Γ is a uniform lattice, see Remark 8.5. Consider $\phi : \mathrm{SO}(p,q) \to \mathrm{SO}(p,q+1)$. The composition $\phi \circ \rho : \Gamma \to \mathrm{SO}(p,q+1)$ is again a Q_1-Anosov representation, but now with a domain of discontinuity $\Omega_{\phi\circ\rho} \subset \mathcal{F}_0(\mathbf{R}^{p,q+1})$ that is always nonempty. By Theorem 8.6, the action of $\phi \circ \rho(\Gamma)$ on $\Omega_{\phi\circ\rho}$ is properly discontinuous with compact quotient.

Proposition 13.1. *Let Γ, ρ and (p,q) be as above.*

Then the embedding $\rho : \Gamma \to \mathrm{SO}(p,q)$ as well as any sufficiently small deformation of ρ leads to a properly discontinuous action on the homogeneous space $\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$.

- *(i) If Γ is a uniform lattice, then $\Gamma\backslash\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$ is compact.*
- *(ii) If Γ is not a uniform lattice the quotient $\Gamma\backslash\Omega_{\phi\circ\rho}$ is a compactification of $\Gamma\backslash\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$.*

When Γ is torsion free, this compactification is a manifold. In general, $\Gamma\backslash\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$ is an orbifold.

Proof. Let $v = (v_1, \ldots, v_{p+q+1})$ be an isotropic vector in $\mathbf{R}^{p,q+1}$ with $v_{p+q+1} \neq 0$. Then, the stabilizer of $\mathbf{R}v \in \mathcal{F}_0(\mathbf{R}^{p,q+1})$ in $\mathrm{SO}(p,q)$ is $\mathrm{SO}(p-1,q)$, and the orbit $\mathrm{SO}(p,q)(\mathbf{R}v) \subset \Omega_{\phi\circ\rho}$, with equality if and only if Γ is a uniform lattice. ☐

Remark 13.2. The Clifford-Klein forms given by ρ have been studied by Kobayashi, [54]. When Γ is a uniform lattice the only deformations that exist are deformations into the normalizer of G.

13.2. **Q_0-Anosov representations.** Let $\Gamma < \mathrm{SO}(1,2n)$ be a convex cocompact subgroup, and $\mathrm{SO}(1,2n) \to \mathrm{SO}(2,2n)$ the standard embedding. The corresponding embedding $\rho : \Gamma \to \mathrm{SO}(2,2n)$ is a Q_0-Anosov representation. The domain of discontinuity $\Omega_\rho \subset \mathcal{F}_1$ is empty if and only if Γ is a uniform lattice (Remark 8.5). Let $\phi : \mathrm{SO}(2,2n) \to \mathrm{SO}(2n+2,\mathbf{C})$ be the embedding into the complexification. Then $\phi \circ \rho$ is Q_0-Anosov. The construction of Section 8 provides a domain of discontinuity $\Omega_{\phi\circ\rho} \subset \mathcal{F}_1(\mathbf{C}^{2n+2})$, on which $\phi \circ \rho(\Gamma)$ acts properly discontinuously with compact quotient.

Barbot [3] shows that the entire connected component of ρ in $\mathrm{Hom}(\Gamma, \mathrm{SO}(2,2n))$ consists of Q_0-Anosov representations. Using this and Theorem 9.12 we deduce:

Theorem 13.3. *If Γ is a uniform lattice, then any representation ρ' in the connected component of ρ in $\mathrm{Hom}(\Gamma, \mathrm{SO}(2,2n))$ leads to a properly discontinuous, and cocompact action on the homogeneous space $\mathrm{SO}(2,2n)/\mathrm{U}(1,n)$.*

Let $\Gamma < \mathrm{SO}(1,2n)$ *be a convex cocompact subgroup and* $\rho : \Gamma \to \mathrm{SO}(1,2n) \to \mathrm{SO}(2,2n)$ *the embedding. Then* ρ *and any sufficiently small deformation of* ρ *leads to a properly discontinuous action of* Γ *on* $\mathrm{SO}(2,2n)/\mathrm{U}(1,n)$*. The quotient* $\Gamma\backslash\Omega_{\phi\circ\rho}$ *is a compactification of the quotient* $\Gamma\backslash\mathrm{SO}(2,2n)/\mathrm{U}(1,n)$*.*

Proof. Let $L \in \Omega_{\phi\circ\rho} \subset \mathcal{F}_1(\mathbf{C}^{2n+2})$ be a $n+1$-plane such that $L \cap \overline{L} = 0$. A direct calculation gives that the stabilizer of L in $\mathrm{SO}(2,2n)$ is $\mathrm{U}(1,n)$, and $\mathrm{SO}(2,2n)/\mathrm{U}(1,n) \subset \Omega_{\phi\circ\rho}$ with equality if and only if Γ is a cocompact lattice. □

Note that Theorem 13.3 extends, in the case of $\mathrm{SO}(2,2n)$, a recent result of Kassel [51, Theorem 1.1], that small deformations of ρ lead to properly discontinuous action on the homogeneous space $\mathrm{SO}(2,2n)/\mathrm{U}(1,n)$.

In particular, as is noted in [51], Johnson and Millson [47] constructed explicit bending deformations with Zariski dense image in $\mathrm{SO}(2,2n)$ when Γ is an arithmetic lattice. This allows to conclude the following

Corollary 13.4. [51, Corollary 1.2] *There exist Zariski dense subgroups* $\Lambda < \mathrm{SO}(2,2n)$ *acting properly discontinuously, freely and cocompactly on the homogeneous space* $\mathrm{SO}(2,2n)/\mathrm{U}(1,n)$*.*

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