NONCOMMUTATIVE COORDINATES FOR SYMPLECTIC REPRESENTATIONS

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Abstract. We introduce coordinates on the spaces of framed and decorated representations of the fundamental group of a surface with boundary into the symplectic group $\mathrm{Sp}(2n,\mathbf{R})$. These coordinates provide a noncommutative generalization of the parameterizations of the spaces of representations into $\mathrm{SL}(2,\mathbf{R})$ or $\mathrm{PSL}(2,\mathbf{R})$ given by Thurston, Penner, Kashaev and Fock–Goncharov. On the space of decorated symplectic representations the coordinates give a geometric realization of the noncommutative cluster-like structures introduced by Berenstein–Retakh. The locus of positive coordinates maps to the space of framed maximal representations. We use this to determine an explicit homeomorphism between the space of framed maximal representations and a quotient by the group $\mathrm{O}(n)$. This allows us to describe the homotopy type and, when $n=2$, to give an exact description of the singularities. Along the way, we establish a complete classification of pairs of nondegenerate quadratic forms.

Contents

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1. Introduction

In their seminal paper [9], Fock and Goncharov introduced a pair of moduli spaces, the $\mathcal{X}$-space and the $\mathcal{A}$-space, which are closely related to the representations variety of the fundamental group of a surface S with nonempty boundary and of negative Euler characteristic into a split real simple Lie group G. They introduced on these spaces explicit cluster $\mathcal{X}$-coordinates and $\mathcal{A}$-coordinates associated to an ideal triangulation of S. Changing the triangulation, the coordinates change by positive rational functions. Thus the locus of positive coordinates is independent of the choice of triangulation. When G is $\mathrm{SL}(2,\mathbf{R})$, the positive locus in the $\mathcal{X}$-space is closely related to the Teichmüller space, and the positive locus in the $\mathcal{A}$-space to the decorated Teichmüller space of S, therefore the Fock–Goncharov coordinates are extensions of Thurston's shear coordinates, respectively Penner's λ-lengths [20, 17]. When G is a split real group of higher rank, these moduli spaces give higher Teichmüller spaces, and the positive locus of the $\mathcal{X}$-space is closely related to the Hitchin component in the representation variety.

The set of positive representations of Fock–Goncharov and the Hitchin components account only for one family of higher Teichmüller spaces, another family is given by maximal representations into Lie groups of Hermitian type. The symplectic groups $\mathrm{Sp}(2n,\mathbf{R})$ form essentially the only family of Lie groups that are both split real forms and of Hermitian type. In this article we generalize the work of Fock–Goncharov in the following way. We introduce two new moduli spaces, an $\mathcal{X}$-space and an $\mathcal{A}$-space of representations of the fundamental group of S into the symplectic group $\mathrm{Sp}(2n,\mathbf{R})$, and describe noncommutative A_1-type cluster coordinates on them. We show, on the one hand, that the positive locus of the $\mathcal{X}$-space corresponds precisely to maximal representations into $\mathrm{Sp}(2n,\mathbf{R})$; we use this to determine the homotopy type of the space of framed maximal representations, and for $\mathrm{Sp}(4,\mathbf{R})$ also its homeomorphism type. On the other hand, we show that the $\mathcal{A}$-space gives a geometric realization of the noncommutative cluster-like structures introduced by Berenstein–Retakh [2].

In Fock–Goncharov's work, an important role is played by Lusztig's total positivity, in our work, a similar role is played by positivity related to the Maslov index. As such, our work fits well in the framework of Θ-positivity, recently introduced by Guichard and Wienhard [26, 14, 13, 12], that generalizes Lusztig's total positivity and provides a unifying framework for the different higher Teichmüller spaces. There are four families of Lie groups which admit a Θ-positive structure, where Θ is a subset of the set of simple restricted roots. One is the family of split real Lie groups, the second one is the family of Hermitian Lie groups of tube type, the third family consists of the groups $\mathrm{SO}(p,q)$ with $p<q$, and the fourth is an exceptional family consisting of four groups which are real forms of real rank 4 of the complex simple Lie groups of type F_4, E_6, E_7, E_8. In the case of Hermitian Lie groups of tube type, positivity is governed by a Weyl group of type A_1, giving Θ-positivity in that case the flavor of a noncommutative A_1-theory. This is precisely what is reflected in the structure of the coordinates we define here. This analogy will be made even more clear for any classical Hermitian Lie group of tube type in forthcoming work, by showing that any such group can be realized as Sp_2 over a noncommutative ring. In other forthcoming work we will define coordinates for appropriately framed and decorated representations into $\mathrm{SO}(p,q)$, such that the positive locus corresponds to the set of Θ-positive representations.

When the Fock–Goncharov approach is applied to the group $\mathrm{Sp}(2n,\mathbf{R})$, they define a positive locus in the space of symplectic representations. It is important to remark that the positive locus that our approach gives in the space of symplectic representations is larger than the Fock–Goncharov's one. This is because the two theories are based on two different Θ-positive structures on $\mathrm{Sp}(2n,\mathbf{R})$: respectively the one for split groups and the one for groups of Hermitian type. The perspective chosen in the present paper is the one which is suitable for describing the spaces of maximal representations.

We now describe our results in more detail.

1.1. **The pair of moduli spaces.** Let S be a compact surface with nonempty boundary and negative Euler characteristic (we refer to Section 4.1 for the wider generality that can be allowed for S, for example marked disks that will lead to space of configurations of Lagrangians).

We introduce two moduli spaces, the space of *framed symplectic representations* (i.e. a representation $\pi_1(S) \to \mathrm{Sp}(2n, \mathbf{R})$ together with a k-tuples of Lagrangian subspaces $(L_1, \dots, L_k)$, that are fixed by peripheral elements $c_1, \dots, c_k$ in $\pi_1(S)$) which serves as our $\mathcal{X}$-space, and the space of *decorated symplectic representations* (i.e. $\pi_1(S) \to \mathrm{Sp}(2n, \mathbf{R})$ together with a k-tuple of decorated Lagrangian subspaces $((L_1, \mathbf{v}_1), \dots, (L_k, \mathbf{v}_k))$, where $\mathbf{v}_i$ is a basis of L_i, and the isomorphism of L_i induced by c_i is $-\mathrm{Id}_{L_i}$, cf. Corollary 4.18) which serves as our $\mathcal{A}$-space.

Fixing an ideal triangulation $\mathcal{T}$ of S, we introduce systems of $\mathcal{X}$-coordinates, using invariants of triples, quadruples, and quintuples of Lagrangian subspaces. A system of $\mathcal{X}$-coordinates consists of a triangle invariant for each triangle, which is given by the Maslov index of the triple of Lagrangians associated to the vertices of the triangle, an edge invariant for every edge of the triangulation, which can be seen as a cross ratio function of four Lagrangians, and an angle invariant, associated to each corner of a triangle, which comes from an invariant of quintuples of Lagrangians. Along the way, we show that the (opposite of the) Toledo number of the associated representation is the half sum of the triangle invariants (cf. Theorem 4.26). We then describe in detail a map denoted by $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$ from the set $\mathcal{X}(\mathcal{T}, n)$ of $\mathcal{X}$-coordinates to the space of framed representations. A special role is played by the set $\mathcal{X}^+_\Delta(\mathcal{T}, n)$ of positive diagonal $\mathcal{X}$-coordinates, those for which the triangle invariants are equal to n, the edge invariants are just n-tuples of positive real numbers, and the angle invariants take values in $\mathrm{O}(n)$.

Theorem 1.1. *The map $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$ induces a proper generically finite-to-one surjection from $\mathcal{X}^+_\Delta(\mathcal{T}, n)$ to the space of framed maximal representations.*

Maximal representations into Lie groups of Hermitian type have been extensively studied in [7], and further considered in [6, 25]. All maximal representation are discrete embeddings, and spaces of maximal representations are examples of higher Teichmüller spaces.

Let us emphasize that the correspondence between positive $\mathcal{X}$-coordinates and framed maximal representations is not one-to-one. To every framed maximal representation corresponds a system of positive $\mathcal{X}$-coordinates, but in general only the edge invariants are uniquely determined, the angle invariants involve some choices. We also explicitly describe the fibers of the map $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$ (cf. Theorem 6.4).

In general (i.e. not restricting to the positive locus), the space $\mathcal{X}(\mathcal{T}, n)$ can parameterize only certain framed representations that we call $\mathcal{T}$-transverse (in particular, maximal framed representations are $\mathcal{T}$-transverse). Still, we show that the map $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$ is onto the space of $\mathcal{T}$-transverse framed representations, generically finite-to-one, and we have an explicit description of its fibers (Theorem 9.12). In turn, topological conclusions are drawn concerning the space of $\mathcal{T}$-transverse framed representations (cf. Corollary 9.21).

The $\mathcal{X}$-coordinates are quite geometric, and can be used to determine the topology of the space of maximal representations, but they do not have nice algebraic properties. For example, we did not include in this paper the explicit formulas for the change of coordinates for $\mathcal{X}$-coordinates under a flip of the triangulation as they involve some unpleasant operations such as diagonalizing symmetric matrices. The $\mathcal{A}$-coordinates have better and cleaner algebraic properties.

To define the $\mathcal{A}$-coordinates on the space of decorated symplectic representations, we introduce the symplectic Λ-length, which is an invariant of pairs of decorated Lagrangians. Let ω denote the symplectic form, and let $(L, \mathbf{v})$, $(M, \mathbf{w})$ be a pair of decorated Lagrangians, where $\mathbf{v} = (v_1, \dots, v_n)$ is a basis of L and $\mathbf{w} = (w_1, \dots, w_n)$ a basis of M, then the symplectic Λ-length is $\Lambda_{\mathbf{v},\mathbf{w}} :=$

$(\omega(v_i, w_j))_{i,j=1,\dots,n}$. It takes values in $\mathrm{GL}(n, \mathbf{R})$ if and only if the pair (L, M) is transverse, and its square provides a noncommutative generalization of Penner's λ-lengths, which are the special case when $n = 1$. We show that the symplectic Λ-lengths satisfy a noncommutative analogue of the Ptolemy relation, as well as special triangle relations (which are trivially satisfied for $n = 1$). A system of $\mathcal{A}$-coordinates associates to every oriented edge the symplectic Λ-length of the two decorated Lagrangians at the vertices of the edge. The noncommutative Ptolemy equation translates into an explicit formula for the changes of $\mathcal{A}$-coordinates under a flip.

Theorem 1.2. *Let $(L_1, \mathbf{v}_1), (L_2, \mathbf{v}_2), (L_3, \mathbf{v}_3), (L_4, \mathbf{v}_4)$ be four pairwise transverse decorated Lagrangians. Let Λ_{ij} be the symplectic Λ-length associated to the pair $(\mathbf{v}_i, \mathbf{v}_j)$, then*

$$\Lambda_{24} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.$$

When $n = 1$, this formula reduces to the formula for the flip in the $\mathrm{SL}(2, \mathbf{R})$-situation, and in general it is a noncommutative generalization of it. This lets us view the theory of decorated symplectic representations as a noncommutative A_1-theory, and it gives a geometric realization of the noncommutative algebras introduced by Berenstein–Retakh [2] (see Theorem 11.11).

There is a natural map from the $\mathcal{A}$-space to the $\mathcal{X}$-space. Under this map, the formula for the flip for $\mathcal{A}$-coordinates leads to a formula for the flip of the cross ratios in the $\mathcal{X}$-space (see Proposition 11.9), which provides a noncommutative generalization of the well-known formula for the coordinate change of shear coordinates under a flip.

1.2. **Topology of the space of maximal representations.** We now discuss the applications to the topology of the space of (framed) maximal representations. Let us point out that contrary to the space of positive representations or the Hitchin component, which are contractible, the space of maximal representations has nontrivial topology. In the case of maximal representations of fundamental groups of closed surfaces, the topology of the space of maximal representations has been studied using the theory of Higgs bundles in [1, 5, 10, 11]. These techniques do not apply easily to the case of maximal representations of fundamental groups of surface with punctures, in particular since we do not fix the holonomy along peripheral curves on the surface.

Here we rely on Theorem 1.1 to determine topological properties of the space of maximal framed representations. Note that the positive locus of the $\mathcal{X}$-coordinates does not parameterize the space of framed maximal representations, but maps surjectively to it. The fibers of this surjection are complicated to describe, because they depend on the shape of the edge invariants. However, they are generically finite.

There is a variant of $\mathcal{X}_\Delta^+(\mathcal{T}, n)$ from which we deduce a description of the space of maximal representations as the quotient of an $\mathrm{O}(n)$-action, see Theorem 6.10. To state the theorem we denote by $\mathrm{Sym}^+(n, \mathbf{R})$ the space of positive definite symmetric matrices and let $\mathrm{O}(n)$ act on it by conjugation on each factor:

Theorem 1.3. *The space of framed maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$ is homeomorphic to the quotient of $\mathrm{Sym}^+(n, \mathbf{R})^{3|\chi(S)|+1} \times \mathrm{O}(n)^{|\chi(S)|+1}$ by the diagonal $\mathrm{O}(n)$-action.*

We furthermore determine an explicit description of the space of framed maximal representations into any connected Lie group isogenic to $\mathrm{PSp}(2n, \mathbf{R})$, cf. Theorem 10.18.

As a corollary, we obtain a different proof of [25, Theorem 4] for the value of the number of connected components.

Corollary 1.4. *The space of maximal representations and the space of framed maximal representations into $\mathrm{Sp}(2n, \mathbf{R})$ have $2^{|\chi(S)|+1}$ connected components. The space of maximal representations and the space of framed maximal representations into $\mathrm{PSp}(2n, \mathbf{R})$ have $2^{|\chi(S)|+1}$ connected components when n is even; they are connected when n is odd.*

There is a special subset of framed representations (Section 7.1), for which the edge invariants are "totally singular". It corresponds to the subset $\{(\mathrm{Id},\dots,\mathrm{Id})\}\times \mathrm{O}(n)^{|\chi(S)|+1}$ of $\mathrm{Sym}^+(n,\mathbf{R})^{3|\chi(S)|+1}\times \mathrm{O}(n)^{|\chi(S)|+1}$. From this, we obtain

Theorem 1.5. *The space of totally singular framed representations is a strong deformation retract of the space of framed maximal representations into* $\mathrm{Sp}(2n,\mathbf{R})$*; it is homeomorphic to* $\mathrm{O}(n)^{|\chi(S)|+1}/\,\mathrm{O}(n)$ (*where the action of* $\mathrm{O}(n)$ *is by simultaneous conjugation*).

When $n=2$, we analyze the quotient of Theorem 1.3 in more detail and show that all connected components except one are orbifolds, one connected component contains a non-orbifold singularity, see Theorem 8.1.

Structure of the paper: In Section 2 we recall classical facts on the symplectic group, the Maslov index and the Souriau index. In Section 3 we introduce the invariants of Lagrangians and decorated Lagrangians which are used to define coordinates. In Section 4, we introduce the spaces of framed and decorated local systems, recall the definition and key properties of maximal local systems. Section 5 gives an interpretation of these spaces in terms of local systems on a quiver embedded in the surface. In Section 6 we introduce positive $\mathcal{X}$-coordinates, and construct the map to framed maximal representations. Variants of $\mathcal{X}$-coordinates are introduced in Section 6.4. The applications for the topology of the space of maximal representations are proven in Section 7 (homotopy type) and Section 8 (singularities when n is 2). The general $\mathcal{X}$-coordinates are introduced in Section 9, and in Section 10 we generalize them to local systems on connected Lie groups isogenic to $\mathrm{Sp}(2n,\mathbf{R})$. Finally, in Section 11 we introduce $\mathcal{A}$-coordinates, describe the relations to the noncommutative algebras of Berenstein and Retakh, and give formulas for the coordinate changes under a flip of the triangulation. The Appendix A contains a description of the invariants of pairs of nondegenerate symmetric bilinear forms that are used in Section 9.

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2. Symplectic group, Lagrangians

This section introduces the symplectic group and the Lagrangian Grassmannian. We also recall facts on the Maslov and Souriau indices, and the translation number.

2.1. Lagrangian Grassmannian and decorated Lagrangian Grassmannian.

We consider the symplectic vector space $(\mathbf{R}^{2n},\omega)$ where ω is the standard symplectic form on $\mathbf{R}^{2n}$, i.e.

$$\omega(x,y) = {}^{T}x \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} y, \tag{2.1}$$

for x and y in $\mathbf{R}^{2n}$.

Every basis of $\mathbf{R}^{2n}$ such that ω, expressed in that basis, has the form (2.1) is called a *symplectic basis* (hence the standard basis is a symplectic basis). We will usually write a symplectic basis as a pair $(\mathbf{e},\mathbf{f})$, where $\mathbf{e}=(e_1,\dots,e_n), \mathbf{f}=(f_1,\dots,f_n)$; thus, one has, for all i,j, $\omega(e_i,f_j)=\delta_{ij}$. We will denote this last equality more concisely by $\omega(\mathbf{e},\mathbf{f})=\mathrm{Id}$. More generally, given two families $\mathbf{v}=(v_1,\dots,v_n)$ and $\mathbf{w}=(w_1,\dots,w_m)$ we will write $\omega(\mathbf{v},\mathbf{w})$ for the $m\times n$-matrix whose coefficients are $\omega(v_i,w_j)$; one has $\omega(\mathbf{w},\mathbf{v}) = -{}^{T}\omega(\mathbf{v},\mathbf{w})$.

The group

$$\mathrm{Sp}(2n,\mathbf{R}) := \{g\in \mathrm{GL}(2n,\mathbf{R}) \mid {}^{T}g\omega g=\omega\}$$

is the *symplectic group*, and its adjoint form $\mathrm{PSp}(2n,\mathbf{R}) := \mathrm{Sp}(2n,\mathbf{R})/\{\pm\,\mathrm{Id}\}$ is the *projective symplectic group*.

Definition 2.1. A subspace L of $\mathbf{R}^{2n}$ is called *Lagrangian* if $\dim(L) = n$ and $\omega(u,v) = 0$ for all $u, v \in L$. The space of all Lagrangian subspaces of $(\mathbf{R}^{2n}, \omega)$ is called the *Lagrangian Grassmannian*, and denoted $\mathcal{L}_n$.

Definition 2.2. A *decorated Lagrangian* is a pair $(L, \mathbf{v})$, where $L \in \mathcal{L}_n$ and $\mathbf{v}$ is a basis of L. The space of all decorated Lagrangians of $(\mathbf{R}^{2n}, \omega)$ is called the *decorated Lagrangian Grassmannian*, and denoted $\mathcal{L}_n^{\mathrm{d}}$.

Our notation will often retain only the family $\mathbf{v}$ for a decorated Lagrangian as it determines the Lagrangian L.

The natural projection to $\mathcal{L}_n$ turns $\mathcal{L}_n^{\mathrm{d}}$ into a right principal $\mathrm{GL}(n,\mathbf{R})$-bundle.

Remark 2.3.
(1) Recall that for any n-dimensional real vector space L, the group $\mathrm{GL}(n,\mathbf{R})$ acts on the right on the space of bases of L: if $\mathbf{v} = (v_1, \ldots, v_n)$ is a basis of L and $g = (g_{i,j})_{i,j\in\{1,\ldots,n\}}$ is in $\mathrm{GL}(n,\mathbf{R})$ then $\mathbf{w} = \mathbf{v}g$ is the family $(w_1, \ldots, w_n)$ defined by the formula: $\forall j \in \{1, \ldots, n\}$, $w_j = \sum_{i=1}^n v_i g_{i,j}$. This is the simply transitive action behind the principal bundle structure mentioned above.
(2) In the case of a symplectic vector space of dimension $2n$, we get this way a simply transitive action of $\mathrm{Sp}(2n,\mathbf{R})$ on the space of symplectic bases.
(3) We will use also the notation $\mathbf{v} \cdot g$ (or sometimes $\mathbf{v}g$) when $\mathbf{v}$ is a family (not necessarily free, nor generating) of n elements in a vector space V and when g is a $m \times n$-matrix.
(4) A morphism $\psi \colon L \to L'$ and its matrix g with respect to bases $\mathbf{v}$ of L and $\mathbf{v}'$ of L' are related by the well-known formula $\psi(\mathbf{v}) = \mathbf{v}'g$.
(5) For two families $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$, the family $(v_1 + w_1, \ldots, v_n + w_n)$ is denoted $\mathbf{v} + \mathbf{w}$.

The group $\mathrm{Sp}(2n,\mathbf{R})$ has a natural left action on $\mathcal{L}_n$ and $\mathcal{L}_n^{\mathrm{d}}$:

$$g \cdot L := \{g(x)\}_{x\in L},$$
$$g \cdot (L, (v_1, \ldots, v_n)) := \big(g \cdot L, (g(v_1), \ldots, g(v_n))\big).$$

These actions are transitive, hence the spaces $\mathcal{L}_n$ and $\mathcal{L}_n^{\mathrm{d}}$ are homogeneous spaces under the symplectic group. Furthermore, the map $\mathcal{L}_n^{\mathrm{d}} \to \mathcal{L}_n$ is $\mathrm{Sp}(2n,\mathbf{R})$-equivariant. The left action of $\mathrm{Sp}(2n,\mathbf{R})$ and the right action of $\mathrm{GL}(n,\mathbf{R})$ on $\mathcal{L}_n^{\mathrm{d}}$ commute.

Let $L_0 = \mathrm{Span}(\mathbf{e}_0)$ (where $(\mathbf{e}_0, \mathbf{f}_0)$ is the standard symplectic basis of $\mathbf{R}^{2n}$), and consider the stabilizers

$$P = \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbf{R})}(L_0), \quad U = \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbf{R})}((L_0, \mathbf{e}_0)).$$

The group P is a parabolic subgroup of $\mathrm{Sp}(2n,\mathbf{R})$, and $U \subset P$ is its unipotent radical. As homogeneous spaces, we have

$$\mathcal{L}_n = \mathrm{Sp}(2n,\mathbf{R})/P, \quad \mathcal{L}_n^{\mathrm{d}} = \mathrm{Sp}(2n,\mathbf{R})/U.$$

The action of $\mathrm{Sp}(2n,\mathbf{R})$ on $\mathcal{L}_n$ is not effective, its kernel is $\{\pm \mathrm{Id}\}$. The group of symmetries of $\mathcal{L}_n$ is the projective symplectic group $\mathrm{PSp}(2n,\mathbf{R})$. On the space $\mathcal{L}_n^{\mathrm{d}}$, the action is effective.

Definition 2.4. Two Lagrangians L_1, L_2 are called *transverse* if their intersection is trivial. Two decorated Lagrangians $\mathbf{v}_1, \mathbf{v}_2$ are called *transverse* if $\mathrm{Span}(\mathbf{v}_1)$ and $\mathrm{Span}(\mathbf{v}_2)$ are transverse.

2.2. **Configurations of Lagrangians.** For every integer $d \geqslant 2$, let us denote by $\mathrm{Conf}^d(\mathcal{L}_n)$ the moduli space of d-tuples of Lagrangians, i.e. the quotient of $(\mathcal{L}_n)^d$ by the diagonal action of $\mathrm{Sp}(2n,\mathbf{R})$. The natural action of the symmetric group $\mathfrak{S}_d$ on $(\mathcal{L}_n)^d$ descends to $\mathrm{Conf}^d(\mathcal{L}_n)$.

We will be particularly interested in $\mathrm{Conf}^3(\mathcal{L}_n), \mathrm{Conf}^4(\mathcal{L}_n)$ and $\mathrm{Conf}^5(\mathcal{L}_n)$ and certain of their subspaces. We will denote by $\mathrm{Conf}^{3*}(\mathcal{L}_n)$ the configuration space of triples of pairwise transverse Lagrangians. The subspace $\mathrm{Conf}^{3*}(\mathcal{L}_n) \subset \mathrm{Conf}^3(\mathcal{L}_n)$ is invariant by the action of $\mathfrak{S}_3$.

The subspace of $\mathrm{Conf}^4(\mathcal{L}_n)$ consisting of (orbits of) quadruples (L_1, M_1, L_2, M_2) of Lagrangians such that (L_1, M_1, L_2) and (L_1, M_2, L_2) belong to $\mathrm{Conf}^{3*}(\mathcal{L}_n)$ will be denoted by $\mathrm{Conf}^{4\diamondsuit}(\mathcal{L}_n)$. It is not invariant by the full symmetry group $\mathfrak{S}_4$ but is invariant by the Klein subgroup (generated by the double transpositions) and also the transpositions $(1,3)$ and $(2,4)$.

The map $(L_1, M_1, L_2, M_2) \mapsto (L_2, M_2, L_1, M_1)$ induces an automorphism of $\mathrm{Conf}^4(\mathcal{L}_n)$ denoted by κ.

2.3. **Maslov index.** In this section we review properties of the Maslov index, for a more general discussion we refer the reader to [18, Part I, Appendix A].

Let L_1, M, L_2 be three pairwise transverse Lagrangians. There is a unique linear map $M_{L_1\to L_2}$ from L_1 to L_2 such that $M = \{v \in \mathbf{R}^{2n} \mid \exists e \in L_1,\, v = e + M_{L_1\to L_2}(e)\}$. When this does not cause confusion, it will be denoted just by M.

Using the symplectic form ω, we can define a bilinear form β on L_1 in the following way: for $v_1, v_2 \in L_1$

$$\beta(v_1, v_2) := \omega(v_1, M(v_2)).$$

Definition 2.5. The bilinear form β is called the Maslov form and is denoted by $[L_1, M, L_2]$.

The following is well known, see [24]:

Proposition 2.6. *The Maslov form $[L_1, M, L_2]$ is symmetric and nondegenerate.*

Remark 2.7. Let $\mathbf{e}$ be a basis of L_1 and let $\mathbf{f}$ be the basis of L_2 such that $(\mathbf{e}, \mathbf{f})$ is a symplectic basis. Then the matrix $[M]_{\mathbf{e},\mathbf{f}}$ of M in these bases and the matrix $[\beta]_{\mathbf{e}}$ are equal: $[M]_{\mathbf{e},\mathbf{f}} = [\beta]_{\mathbf{e}}$.

We will denote the signature of β by

$$\mathrm{sgn}(\beta) = p - q,$$

where p is the dimension of a maximal subspace of L_1 on which β is positive definite and q is the dimension of a maximal subspace of L_1 on which β is negative definite. They satisfy $p + q = n$ so that $\mathrm{sgn}(\beta) = n \mod 2$.

Definition 2.8. The *Maslov index* of the triple of Lagrangians (L_1, M, L_2) is the signature $\mathrm{sgn}([L_1, M, L_2])$ and is denoted by $\mu_n(L_1, M, L_2)$.

For $n = 1$, the three Lagrangians (L_1, M, L_2) are pairwise distinct points in the circle $\mathbf{R}\mathbb{P}^1$. The Maslov index is 1 if the three points are cyclically ordered, and -1 otherwise.

There is a slightly more general definition of the Maslov index that works for any triple of Lagrangians, not just for the pairwise transverse triples. It can be defined as the signature $\mathrm{sgn}(\gamma)$ of the quadratic form

$$\begin{aligned}\gamma\colon L_1 \oplus M \oplus L_2 &\longrightarrow \mathbf{R}\\ (v, w, x) &\longmapsto \omega(v, w) + \omega(w, x) + \omega(x, v).\end{aligned}$$

When the triple is pairwise transverse, the two definitions agree.

Proposition 2.9 (Properties of Maslov index)**.** *The Maslov index*

- *has range $\{-n, -n+1, \ldots, n\}$;*
- *is invariant under the action of $\mathrm{Sp}(2n, \mathbf{R})$ on $\mathcal{L}_n^3$;*
- *is antisymmetric and, as a result, is cyclically invariant;*
- *satisfies the cocycle relation, namely for all $L_1, L_2, L_3, L_4 \in \mathcal{L}_n$*

$$\mu_n(L_1, L_2, L_3) - \mu_n(L_1, L_2, L_4) + \mu_n(L_1, L_3, L_4) - \mu_n(L_2, L_3, L_4) = 0$$

- *the group $\mathrm{Sp}(2n, \mathbf{R})$ acts transitively on the set of triples of pairwise transverse Lagrangians with the same Maslov index, i.e. the map $\mathrm{Conf}^{3*}(\mathcal{L}_n) \to \{-n, -n+2, \ldots, n-2, n\}$ induced by μ_n is a bijection.*

Note also that every decomposition of $n=p+q$ induces an injection $\mathcal{L}_p \times \mathcal{L}_q \to \mathcal{L}_n$ that is equivariant with respect to the homomorphism $\mathrm{Sp}(2p,\mathbf{R}) \times \mathrm{Sp}(2q,\mathbf{R}) \to \mathrm{Sp}(2n,\mathbf{R})$ and for which $\mu_n = \mu_p + \mu_q$.

2.4. **Universal coverings.** Let us also equip $\mathbf{R}^{2n}$ with its standard Euclidean structure. Then a maximal compact subgroup of $\mathrm{Sp}(2n,\mathbf{R})$ is the intersection $\mathrm{Sp}(2n,\mathbf{R}) \cap \mathrm{O}(2n)$ that identifies with $\mathrm{U}(n)$ via $A+iB \mapsto \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$.

The homomorphism $\det\colon \mathrm{U}(n) \to \mathbf{C}^*$ can be used to construct the universal cover of $\mathrm{U}(n)$:

$$\widetilde{\mathrm{U}}(n) := \{(u,t) \in \mathrm{U}(n) \times \mathbf{R} \mid \det(u) = e^{it}\}.$$

Of course $\widetilde{\mathrm{U}}(n)$ is a Lie subgroup of the universal cover $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$ of $\mathrm{Sp}(2n,\mathbf{R})$. For any decomposition $n=p+q$ there is a homomorphism $\widetilde{\mathrm{Sp}}(2p,\mathbf{R}) \times \widetilde{\mathrm{Sp}}(2q,\mathbf{R}) \to \widetilde{\mathrm{Sp}}(2n,\mathbf{R})$. (Similar statements hold for any decomposition of n.)

Remark 2.10. Let $(\mathbf{e}_0,\mathbf{f}_0)$ be the standard symplectic basis of $\mathbf{R}^{2n}$; it is also an orthonormal basis. The space $L_0 := \mathrm{Span}(\mathbf{e_0})$ is Lagrangian and its orthogonal complement is $L_0^\perp = \mathrm{Span}(\mathbf{f}_0)$. If, for a symplectic basis $(\mathbf{e},\mathbf{f})$, one has $\mathrm{Span}(\mathbf{e}) = L_0$ and $\mathrm{Span}(\mathbf{f}) = L_0^\perp$, then $(\mathbf{e},\mathbf{f})$ is an orthonormal basis of $\mathbf{R}^{2n}$ if and only if $\mathbf{f}$ is an orthonormal family (this holds in fact as soon as that $\mathrm{Span}(\mathbf{e})$ and $\mathrm{Span}(\mathbf{f})$ are orthogonal).

Lemma 2.11. *For every M in $\mathcal{L}_n$ that is transverse to L_0, there is a unique n-uple of real numbers $(\varphi_1,\varphi_2,\dots,\varphi_n)$ such that:*

- $0 < \varphi_1 \leqslant \varphi_2 \leqslant \cdots \leqslant \varphi_n < \pi$, *and*
- *there exist symplectic orthonormal bases* $(\mathbf{e},\mathbf{f})$ *with*

$$L_0 = \mathrm{Span}(\mathbf{e}),\ L_0^\perp = \mathrm{Span}(\mathbf{f}),\ \textit{and } M = \mathrm{Span}\{\cos(\varphi_i)e_i + \sin(\varphi_i)f_i\}_{1\leqslant i\leqslant n}.$$

Remark 2.12. The bases constructed in this lemma consist of one orbit under tha action of the group $\mathrm{Stab}_{\mathrm{O}(n)}(M)$.

Proof. Another way to state the conclusion is $M = \mathrm{Span}\{\cot(\varphi_i)e_i + f_i\}_{1\leqslant i\leqslant n}$.

Since M is transverse to L_0, it is the graph of a map $L_0^\perp \to L_0$ whose matrix in the bases $\mathbf{f}_0$, $\mathbf{e}_0$ is symmetric. The ortho-diagonalization together with Remark 2.10 furnishes a (unique) nonincreasing sequence $\lambda_1,\dots,\lambda_n$ and a symplectic orthonormal basis $(\mathbf{e},\mathbf{f})$ such that $L_0 = \mathrm{Span}(\mathbf{e})$, $L_0^\perp = \mathrm{Span}(\mathbf{f})$ and $M = \mathrm{Span}\{\lambda_i e_i + f_i\}_{1\leqslant i\leqslant n}$. Setting $\varphi_i = \cot^{-1}(\lambda_i)$ gives the result. □

For M transverse to L_0, we will denote by $(\varphi_1(M),\dots,\varphi_n(M))$ the n-uple provided by the previous lemma. The φ_i are continuous functions on the open and dense subset

$$\mathcal{U} := \{M \in \mathcal{L}_n \mid M \text{ is transverse to } L_0\}.$$

They admit (noncontinuous) extensions to $\mathcal{L}_n$:

Lemma 2.13. *For every M in $\mathcal{L}_n$, there is a unique n-tuple $0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \cdots \leqslant \varphi_n < \pi$ for which there exist symplectic orthonormal bases $(\mathbf{e},\mathbf{f})$ such that $L_0 = \mathrm{Span}(\mathbf{e})$, $L_0^\perp = \mathrm{Span}(\mathbf{f})$ and $M = \mathrm{Span}\{\cos(\varphi_i)e_i + \sin(\varphi_i)f_i\}_{1\leqslant i\leqslant n}$.*

Proof. Choose first $(e_1,\dots,e_k)$ an orthonormal basis of $N = M \cap L_0$ and apply the previous lemma to $M \cap N^\perp = (M^\perp \oplus N)^\perp$ which is a Lagrangian subspace of the $(2n-2k)$-dimensional symplectic vector space $N^\perp \cap N^{\perp_\omega}$. □

The space:

$$\widetilde{\mathcal{L}}_n := \left\{(M,\theta) \in \mathcal{L}_n \times \mathbf{R} \mid \sum_{i=1}^n \varphi_i(M) = \theta \mod \pi\right\}$$

is a submanifold of $\mathcal{L}_n \times \mathbf{R}$ and the natural map $\widetilde{\mathcal{L}}_n \to \mathcal{L}_n$ is a covering. The action of $\mathrm{Sp}(2n,\mathbf{R})$ on $\mathcal{L}_n$ lifts to an action of $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$ on $\widetilde{\mathcal{L}}_n$; the restriction of this action to $\widetilde{\mathrm{U}}(n)$ has an explicit expression:

$$(u,t)\cdot(M,\theta) = (u(M), t+\theta), \quad ((u,t)\in \widetilde{\mathrm{U}}(n),\ (M,\theta)\in\widetilde{\mathcal{L}}_n).$$

From this expression, it follows easily that the action of $\widetilde{\mathrm{U}}(n)$ is transitive and that the stabilizer of $(L_0,0)$ is the subgroup $\mathrm{SO}(n)\times\{0\}\subset\widetilde{\mathrm{U}}(n)$. This implies that $\widetilde{\mathcal{L}}_n$ is a connected and simply connected manifold so that $\widetilde{\mathcal{L}}_n \to \mathcal{L}_n$ is the universal covering.

2.5. **Souriau index.** Two elements of $\widetilde{\mathcal{L}}_n$ will be called *transverse* if their projections to $\mathcal{L}_n$ are transverse. For two transverse elements $L_1^\sim$, $L_2^\sim$ of $\widetilde{\mathcal{L}}_n$, there is $\tilde{u}$ in $\widetilde{\mathrm{U}}(n)$, M in $\mathcal{U}$, and θ in $\mathbf{R}$ such that

$$\tilde{u}\cdot L_1^\sim = (L_0,0), \quad \text{and,} \quad \tilde{u}\cdot L_2^\sim = (M,\theta).$$

Definition 2.14. The *Souriau index* (cf. [24, Section 4]) of $(L_1^\sim, L_2^\sim)$ is

$$m_n(L_1^\sim, L_2^\sim) = n + \frac{1}{\pi}\Big(\theta - \sum_{i=1}^n \varphi_i(M)\Big).$$

We will call also Souriau index the function defined on the space of transverse pairs of $\widetilde{\mathcal{L}}_n$. The reader may refer to [8] (in particular Sections 5 and 6) for a more general discussion. For the reader's convenience, we provide below short proofs of the main properties of the Souriau index.

Proposition 2.15. *The Souriau index is well defined, antisymmetric, and* $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$*-invariant.*

Proof. We note that $\varphi_j(g\cdot M) = \varphi_j(M)$ for every M in $\mathcal{U}$ and every g in $\mathrm{O}(n)$ so that the above formula (Definiton 2.14) does not depend on the choices since the stabilizer in $\widetilde{\mathrm{U}}(n)$ of $(L_0,0)$ is $\mathrm{SO}(n)$.

To prove the antisymmetry, one can assume that $L_1^\sim = (L_0,0)$ and $L_2^\sim = (M,\theta)$ with $M = \mathrm{Span}\{\cos(\varphi_i)e_{0,i} + \sin(\varphi_i)f_{0,i}\}$ (as above $(\mathbf{e}_0,\mathbf{f}_0)$ is the standard basis of $\mathbf{R}^{2n}$). Let $u\in\mathrm{U}(n)\subset\mathrm{GL}(n,\mathbf{C})$ the diagonal element whose coefficients are $\pm e^{-i\varphi_1}, e^{-i\varphi_2},\dots,e^{-i\varphi_n}$ where the sign is fixed so that $\tilde{u} := (u,-\theta)$ belongs to $\widetilde{\mathrm{U}}(n)$ (i.e. it is $+e^{-i\varphi_1}$ in the case $\sum_{i=1}^n \varphi_i = \theta \mod 2\pi$ and $-e^{-i\varphi_1}$ in the case $\sum_{i=1}^n\varphi_i = \theta+\pi \mod 2\pi$). Furthermore, $\tilde{u}\cdot L_2^\sim = (L_0,0)$, and $\tilde{u}\cdot L_1^\sim = (N,-\theta)$ with N a Lagrangian such that $\varphi_j(N) = \pi - \varphi_{n+1-j}(M)$ (for all j). A small calculation then gives $m_n(L_2^\sim, L_1^\sim) = -m_n(L_1^\sim, L_2^\sim)$.

The Souriau index is continuous on the space of transverse pairs and thus locally constant which implies that it is constant on $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$-orbits hence $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$-invariant. □

Let T be the element $(\mathrm{Id},2\pi)$ of $\widetilde{\mathrm{U}}(n)$. This element generates the kernel of $\widetilde{\mathrm{U}}(n)\to\mathrm{U}(n)$. The following equality is a direct consequence of the definitions:

$$m_n(L_1^\sim, T\cdot L_2^\sim) = m_n(L_1^\sim, L_2^\sim) + 2 \quad (L_1^\sim,\ L_2^\sim \in \widetilde{\mathcal{L}}_n), \tag{2.2}$$

more generally, for every element $\tilde{u} = (\pm\mathrm{Id}, k\pi)$ (i.e. $\tilde{u}$ belongs to the center of $\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$), one has

$$m_n(L_1^\sim, \tilde{u}\cdot L_2^\sim) = m_n(L_1^\sim, L_2^\sim) + k \quad (L_1^\sim,\ L_2^\sim \in \widetilde{\mathcal{L}}_n). \tag{2.3}$$

Furthermore, for every decomposition $n = p+q$, there is a natural map $\widetilde{\mathcal{L}}_p\times\widetilde{\mathcal{L}}_q \to \widetilde{\mathcal{L}}_n$ that is equivariant with respect to the homomorphism $\widetilde{\mathrm{Sp}}(2p,\mathbf{R})\times\widetilde{\mathrm{Sp}}(2q,\mathbf{R})\to\widetilde{\mathrm{Sp}}(2n,\mathbf{R})$ mentioned earlier. The Souriau indices behave naturally with respect to this map:

$$m_n = m_p + m_q. \tag{2.4}$$

Similar statements hold for any decomposition of n.

The Souriau index is strongly related to the Maslov index:

Lemma 2.16. *Let L_1, L_2, and L_3 be three pairwise transverse Lagrangians. Let $L_1^\sim$, $L_2^\sim$, and $L_3^\sim$ be lifts to $\widetilde{\mathcal{L}}_n$ of L_1, L_2, and L_3 respectively. Then*

$$\mu_n(L_1, L_2, L_3) = m_n(L_1^\sim, L_2^\sim) + m_n(L_2^\sim, L_3^\sim) + m_n(L_3^\sim, L_1^\sim).$$

Proof. Consider d_n the difference of the two terms above seen as a function on the space of triples of pairwise transverse elements of $\widetilde{\mathcal{L}}_n$. Since $\mathcal{L}_n$ is the quotient of $\widetilde{\mathcal{L}}_n$ by the center of $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$, Equation (2.3) implies that this function descends to a function on the space of pairwise transverse triples of Lagrangians. This last function is $\mathrm{Sp}(2n, \mathbf{R})$-invariant so that the equality $d_n = 0$ needs to be checked only on representatives of the finitely many $\mathrm{Sp}(2n, \mathbf{R})$-orbits. Since one can choose representatives coming from the embedding $(\mathcal{L}_1)^n \to \mathcal{L}_n$, thanks to Equation (2.4), we can further assume that $n = 1$. However, in the case $n = 1$, $\mathcal{L}_1$ can be identified with S^1 with μ_1 being the orientation cocycle, and $\widetilde{\mathcal{L}}_1$ is identified with $\mathbf{R}$ with the Souriau index being the sign function; the equality $d_1 \equiv 0$ follows then from a direct calculation. □

Finally, we are able to extend the Souriau index to any pair in $\widetilde{\mathcal{L}}_n$:

Proposition 2.17. *For all $L_1^\sim$ and $L_2^\sim$ in $\widetilde{\mathcal{L}}_n$, the following integer, defined for any $L_3^\sim$ that is transverse to both $L_1^\sim$ and $L_2^\sim$,*

$$\mu_n(L_1, L_2, L_3) - m_n(L_2^\sim, L_3^\sim) - m_n(L_3^\sim, L_1^\sim)$$

does not depend on $L_3^\sim$.

The resulting integer will be denoted by $m_n(L_1^\sim, L_2^\sim)$ and called the *Souriau index* of $(L_1^\sim, L_2^\sim)$. The Souriau index is then a $\mathbf{Z}$-valued, antisymmetric, and $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$-invariant function on the space $\widetilde{\mathcal{L}}_n \times \widetilde{\mathcal{L}}_n$; it satisfies Equations (2.2) and (2.3) for all $L_1^\sim$ and $L_2^\sim$ in $\widetilde{\mathcal{L}}_n$.

Proof. Call $\delta(L_1^\sim, L_2^\sim, L_3^\sim)$ the integer in the statement.

We need to prove that for any other $L_3''^\sim$, $\delta(L_1^\sim, L_2^\sim, L_3^\sim) = \delta(L_1^\sim, L_2^\sim, L_3''^\sim)$. Choosing an element $L_3'^\sim$ that is transverse to $L_1^\sim$, $L_2^\sim$, $L_3^\sim$, and $L_3''^\sim$, we will prove that $\delta(L_1^\sim, L_2^\sim, L_3^\sim) = \delta(L_1^\sim, L_2^\sim, L_3'^\sim)$ and $\delta(L_1^\sim, L_2^\sim, L_3'^\sim) = \delta(L_1^\sim, L_2^\sim, L_3''^\sim)$. In fact, only the first equality needs a proof, the second being a consequence of the first applied to the quadruple $(L_1^\sim, L_2^\sim, L_3'^\sim, L_3''^\sim)$. The equality $\delta(L_1^\sim, L_2^\sim, L_3^\sim) - \delta(L_1^\sim, L_2^\sim, L_3'^\sim) = 0$ is the result of a direct calculation using the definitions, the cocycle property of the Maslov index, Lemma 2.16, and the antisymmetry of the Souriau index. □

The relation between the Maslov index and the Souriau index is valid without assuming the transversality of the Lagrangians.

Proposition 2.18. *Let L_1, L_2, and L_3 be Lagrangians and let $L_1^\sim$, $L_2^\sim$, and $L_3^\sim$ be lifts of L_1, L_2, L_3 in $\widetilde{\mathcal{L}}_n$. Then*

$$\mu_n(L_1, L_2, L_3) = m_n(L_1^\sim, L_2^\sim) + m_n(L_2^\sim, L_3^\sim) + m_n(L_3^\sim, L_1^\sim).$$

Proof. By construction of the extension of the Souriau index, the identity holds true as soon as L_3 is transverse to both L_1 and L_2. Let then $M^\sim = (M, \theta)$ be an element of $\widetilde{\mathcal{L}}_n$ with M transverse to L_1, L_2, and L_3. Then, by the cocycle property of the Maslov index and its antisymmetry, one has $\mu_n(L_1, L_2, L_3) = \mu_n(L_1, L_2, M) + \mu_n(L_2, L_3, M) + \mu_n(L_3, L_1, M)$ and the result follows from a successive application of Proposition 2.17 and the antisymmetry of the Souriau index. □

By an easy induction one gets:

Lemma 2.19. *Let $L_1, L_2, \dots, L_r$ be Lagrangians and let $L_1^\sim, L_2^\sim, \dots, L_r^\sim$ be lifts of $L_1, L_2, \dots, L_r$ to $\widetilde{\mathcal{L}}_n$. Then*

$$\sum_{j=2}^{r-1} \mu_n(L_1, L_j, L_{j+1}) = m_n(L_r^\sim, L_1^\sim) + \sum_{j=1}^{r-1} m_n(L_j^\sim, L_{j+1}^\sim).$$

A consequence of this last lemma, the invariance of m_n and its antisymmetry is:

Lemma 2.20. *Let M, $L_1, \dots, L_{r-1}$ be Lagrangians. Let g be an element of $\mathrm{Sp}(2n, \mathbf{R})$ fixing M, let $M^\sim$ be a lift of M and let $\tilde{g} \in \widetilde{\mathrm{Sp}}(2n, \mathbf{R})$ be the lift of g fixing $M^\sim$. Let $L_1^\sim, \dots, L_{r-1}^\sim$ be lifts of $L_1, \dots, L_{r-1}$ in $\widetilde{\mathcal{L}}_n$ and set $L_r := g \cdot L_1$ so that $L_r^\sim := \tilde{g} \cdot L_1^\sim$ is a lift of L_r. Then*

$$\sum_{j=1}^{r-1} \mu_n(M, L_j, L_{j+1}) = \sum_{j=1}^{r-1} m_n(L_j^\sim, L_{j+1}^\sim).$$

2.6. **Translation number.** The translation number $\widetilde{\mathrm{Rot}} \colon \widetilde{\mathrm{Sp}}(2n, \mathbf{R}) \to \mathbf{R}$ is a conjugation invariant function defined in [7] using bounded cohomology. We will need the following properties:

Lemma 2.21 ([25])**.** *Let h be in $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$, $L^\sim$ be in $\widetilde{\mathcal{L}}_n$ such that the projection of $L^\sim$ in $\mathcal{L}_n$ is fixed by the projection of h in $\mathrm{Sp}(2n, \mathbf{R})$. Then*

$$\widetilde{\mathrm{Rot}}(h) = \frac{1}{2} m_n(h \cdot L^\sim, L^\sim).$$

Remark 2.22. For any h in $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$ and any $L^\sim$ in $\widetilde{\mathcal{L}}_n$ one has

$$\widetilde{\mathrm{Rot}}(h) = \lim_{k \to \infty} \frac{1}{2k} m_n(h^k \cdot L^\sim, L^\sim),$$

and this equality could serve as a definition for the translation number (cf. [8, Section 10]).

3. Invariants of Lagrangian subspaces

The action of $\mathrm{Sp}(2n, \mathbf{R})$ on pairs of transverse Lagrangians is transitive. However the action of $\mathrm{Sp}(2n, \mathbf{R})$ on triples of pairwise transverse Lagrangians is not transitive and all the more for the actions on quadruples and quintuples.In this section we describe invariants of such tuples of Lagrangians.

Equally the action of $\mathrm{Sp}(2n, \mathbf{R})$ on pairs of decorated Lagrangians is not transitive. We introduce below the symplectic Λ-length and investigate its properties. This will help to understand the orbits of $\mathrm{Sp}(2n, \mathbf{R})$.

3.1. **Cross ratio.** Let L_1, M_1, L_2, M_2 be four Lagrangians such that M_1 is transverse to L_2, M_2 is transverse to L_1, and L_1 is transverse to L_2. We use the linear isomorphisms ${M_1}_{L_1 \to L_2}$ and ${M_2}_{L_2 \to L_1}$, defined in Section 2.3, to introduce the map

$$[L_1, M_1, L_2, M_2] := -{M_2}_{L_2 \to L_1} \circ {M_1}_{L_1 \to L_2} \colon L_1 \to L_1$$

which is a linear endomorphism of L_1.

Definition 3.1. The map $[L_1, M_1, L_2, M_2] \colon L_1 \to L_1$ is called the *cross ratio* of the quadruple of Lagrangians (L_1, M_1, L_2, M_2).

For related invariants of four Lagrangians, see [3, 4, 15, 23]. Cross ratios for quadruples of matrices have been defined by Hua [16, Section 5] and later extended to operator by Zelikin [27]; these can be used to describe a cross ratio for quadruples of n-planes in $\mathbf{R}^{2n}$ (cf. Section 5.7 in [19, Chapter 2]). Noncommutative cross ratios have been defined by Retakh [21], and later related in [22] to the work of Berenstein and Retakh [2] (cf. also Section 11.10 below).

For $n = 1$, the cross ratio is a linear map from a line to itself. This is just the multiplication by a scalar, which is exactly the classical cross ratio of four lines in $\mathbf{R}^2$.

Proposition 3.2 (Properties of cross ratio)**.**

(1) The cross ratio is equivariant under the action of $\mathrm{Sp}(2n, \mathbf{R})$ *on* $\mathcal{L}_n$*, that is, for any* $g \in \mathrm{Sp}(2n, \mathbf{R})$, $[gL_1, gM_1, gL_2, gM_2] = h[L_1, M_1, L_2, M_2]h^{-1}$ *where* $h\colon L_1 \to g(L_1)$ *is the restriction of* g *to* L_1.

(2) If furthermore M_1 *is transverse to* L_1 *and* M_2 *is transverse to* L_2*, then the maps* ${M_1}_{L_1 \to L_2}$ *and* ${M_2}_{L_2 \to L_1}$ *are bijective and one has:* $[L_1, M_1, L_2, M_2] = [L_1, M_2, L_2, M_1]^{-1}$*;*

(3) Under the same hypothesis, $[L_1, M_1, L_2, M_2] = {M_1}_{L_1 \to L_2}^{-1} \circ [L_2, M_2, L_1, M_1] \circ {M_1}_{L_1 \to L_2} = {M_2}_{L_2 \to L_1} \circ [L_2, M_2, L_1, M_1] \circ {M_2}_{L_2 \to L_1}^{-1}$.

(4) Denote $\psi_1\colon L_1 \to M_1 \mid v \mapsto v + M_1(v)$*, then, when* M_1 *and* M_2 *are transverse,* $[M_1, L_2, M_2, L_1] = \psi_1 [L_1, M_1, L_2, M_2]^{-1} \psi_1^{-1}$.

Proposition 3.3. *The cross ratio* $B := [L_1, M_1, L_2, M_2]$ *is selfadjoint with respect to the Maslov forms* $[L_1, M_1, L_2]$ *and* $[L_1, M_2, L_2]$.

Proof. Let $\beta_1 = [L_1, M_1, L_2]$ and $\beta_2 = [L_2, M_2, L_1]$. Thus β_2 is a symmetric bilinear form on L_2. Let $v, w \in L_1$. Then:

$$\beta_1(Bv, w) = \omega(-M_2 M_1 v, M_1 w) = \omega(M_1 w, M_2 M_1 v) = \beta_2(M_1 w, M_1 v), \tag{3.1}$$

and, exchanging v and w, $\beta_1(Bw, v) = \beta_2(M_1 v, M_1 w)$. Since β_1 and β_2 are symmetric, B is selfadjoint with respect to β_1. Exchanging the Lagrangians, we obtain that $[L_2, M_2, L_1, M_1]$ is selfadjoint with respect to $[L_2, M_2, L_1]$. By the second equality in (3) of Proposition 3.2, $B = M_2 \circ [L_2, M_2, L_1, M_1] \circ M_2^{-1}$ and since $M_2\colon L_2 \to L_1$ is isometric with respect to $[L_2, M_2, L_1]$ (on L_2) and $[L_1, M_2, L_2]$ (on L_1), we get that B is selfadjoint with respect to $[L_1, M_2, L_2]$. □

Corollary 3.4. *If the Maslov form* $[L_1, M_1, L_2]$ *is positive definite, then* $[L_1, M_1, L_2, M_2]$ *is diagonalizable. If the Maslov form* $[L_2, M_2, L_1]$ *is also positive definite, then* $[L_1, M_1, L_2, M_2]$ *has positive eigenvalues.*

Proof. We set as before $\beta_1 = [L_1, M_1, L_2]$ and $\beta_2 = [L_2, M_2, L_1]$. Since $B = [L_1, M_1, L_2, M_2]$ is selfadjoint with respect to β_1, there is a β_1-orthonormal basis $\mathbf{e}$ of L_1 diagonalizing B, i.e. such that $[\beta_1]_{\mathbf{e}} = \mathrm{Id}$ and $[B]_{\mathbf{e}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Let $\mathbf{f}$ be the unique basis of L_2 such that $\omega(\mathbf{e}, \mathbf{f}) = \mathrm{Id}$. By Remark 2.7, $[M_1]_{\mathbf{e}, \mathbf{f}} = \mathrm{Id}$, i.e. for all i, $f_i = M_1 e_i$. By Equation (3.1), for all i, one has

$$\lambda_i = \beta_1(Be_i, e_i) = \beta_2(f_i, f_i) > 0,$$

since β_2 is positive definite. □

Definition 3.5. A quadruple of Lagrangians (L_1, M_1, L_2, M_2) is said to be *positive* if $[L_1, M_1, L_2]$ and $[L_2, M_2, L_1]$ are positive definite.

A symplectic basis $(\mathbf{e}, \mathbf{f})$ will be said to be in *standard position* with respect to a positive quadruple of Lagrangians (L_1, M_1, L_2, M_2) if there exists a diagonal matrix Λ with positive nondecreasing coefficients such that

$$L_1 = \mathrm{Span}(\mathbf{e}),\ L_2 = \mathrm{Span}(\mathbf{f}),\ M_1 = \mathrm{Span}(\mathbf{e} + \mathbf{f}),\ M_2 = \mathrm{Span}(\mathbf{e} - \mathbf{f} \cdot \Lambda).$$

The matrix Λ is uniquely determined by the quadruple (L_1, M_1, L_2, M_2) since Λ^{-1} is equal to the cross ratio $[L_1, M_1, L_2, M_2]$ (in the basis $\mathbf{e}$) and is in fact the unique diagonal matrix with nondecreasing coefficients representing this endomorphism.

The configuration space of positive quadruples will be denoted by $\mathrm{Conf}^{4+}(\mathcal{L}_n) \subset \mathrm{Conf}^4(\mathcal{L}_n)$; it is contained in the space of pairwise transverse quadruples, and in particular in $\mathrm{Conf}^{4\diamondsuit}(\mathcal{L}_n)$.

Note that (L_2, M_2, L_1, M_1) is positive as soon as (L_1, M_1, L_2, M_2) is positive. In fact the cocycle property of the Maslov index (Proposition 2.9) implies that (L_1, M_1, L_2, M_2) is positive if and only if (M_1, L_2, M_2, L_1) is positive (see also Lemma 4.32).

The previous corollary implies the existence of symplectic bases in standard position (actually its proof constructs one). The following proposition complements the conclusion of the corollary (see also Proposition 9.3 for a more general statement without the positivity assumption):

Proposition 3.6. *Let (L_1, M_1, L_2, M_2) be a positive quadruple of Lagrangians.*

(1) There exists a symplectic basis $(\mathbf{e}, \mathbf{f})$ in standard position.

(2) Any other symplectic basis in standard position with respect to (L_1, M_1, L_2, M_2) is of the form $(\mathbf{e} \cdot h, \mathbf{f} \cdot h)$ for h in $\mathrm{O}(n)$ commuting with Λ.

(3) Let $u = \Lambda^{1/2}$. Then

$$(\mathbf{e}', \mathbf{f}') := (\mathbf{e}, \mathbf{f}) \cdot \begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix} = (\mathbf{f} \cdot u, -\mathbf{e} \cdot u^{-1})$$

is a symplectic basis that is in standard position with respect to (L_2, M_2, L_1, M_1).

Proof. Point (1) was already proved.

Let us make first two observations: for any n-tuple $\mathbf{v}$ and any g in $\mathrm{GL}(n, \mathbf{R})$, $\mathrm{Span}(\mathbf{v} \cdot g) = \mathrm{Span}(\mathbf{v})$; for any pairs $(\mathbf{v}, \mathbf{w})$ of n-tuples such that the family $(\mathbf{v}, \mathbf{w})$ is free and for any m, m' in $M_n(\mathbf{R})$, $\mathrm{Span}(\mathbf{v} + \mathbf{w} \cdot m) = \mathrm{Span}(\mathbf{v} + \mathbf{w} \cdot m')$ if and only if $m = m'$.

Let h be as in (2), and let $(\mathbf{e}_1, \mathbf{f}_1) = (\mathbf{e} \cdot h, \mathbf{f} \cdot h)$. Then $\mathrm{Span}(\mathbf{e}_1) = \mathrm{Span}(\mathbf{e} \cdot h) = \mathrm{Span}(\mathbf{e}) = L_1$, similarly $\mathrm{Span}(\mathbf{f}_1) = L_2$, and $\mathrm{Span}(\mathbf{e}_1 + \mathbf{f}_1) = \mathrm{Span}\big((\mathbf{e} + \mathbf{f}) \cdot h\big) = M_1$; and finally, since h commutes with Λ, $\mathrm{Span}(\mathbf{e}_1 - \mathbf{f}_1 \cdot \Lambda) = \mathrm{Span}\big((\mathbf{e} - \mathbf{f} \cdot \Lambda) \cdot h\big) = M_2$.

Conversely, let $(\mathbf{e}_1, \mathbf{f}_1)$ be a symplectic basis in standard position with respect to the quadruple (L_1, M_1, L_2, M_2). There is a unique symplectic matrix g such that $(\mathbf{e}_1, \mathbf{f}_1) = (\mathbf{e}, \mathbf{f}) \cdot g$. The equalities $\mathrm{Span}(\mathbf{e}_1) = \mathrm{Span}(\mathbf{e})$ and $\mathrm{Span}(\mathbf{f}_1) = \mathrm{Span}(\mathbf{f})$ imply that the matrix g is block diagonal: $g = \left(\begin{smallmatrix} h & 0 \\ 0 & {}^T h^{-1} \end{smallmatrix}\right)$ with h in $\mathrm{GL}(n, \mathbf{R})$. Since

$$\mathrm{Span}(\mathbf{e}_1 + \mathbf{f}_1) = \mathrm{Span}(\mathbf{e} \cdot h + \mathbf{f} \cdot {}^T h^{-1}) = \mathrm{Span}(\mathbf{e} \cdot h\, {}^T h + \mathbf{f}) = \mathrm{Span}(\mathbf{e} + \mathbf{f}),$$

one has $h\,{}^T h = \mathrm{Id}$, i.e. $h \in \mathrm{O}(n)$ and $g = \left(\begin{smallmatrix} h & 0 \\ 0 & h \end{smallmatrix}\right)$. Similarly, one obtains $\mathrm{Span}(\mathbf{e} - \mathbf{f} \cdot h \Lambda h^{-1}) = \mathrm{Span}(\mathbf{e} - \mathbf{f} \cdot \Lambda)$ so that h and Λ commute.

Point (3) follows from similar considerations. □

Let Δ_n be the space of diagonal matrices with positive nondecreasing coefficients. Another way to phrase the conclusion of the proposition is:

Corollary 3.7. *The map that associates to a quadruple (L_1, M_1, L_2, M_2) its invariant Λ induces an homeomorphism between $\mathrm{Conf}^{4+}(\mathcal{L}_n)$ and Δ_n.*

3.2. **Maximal triples of decorated Lagrangians.** The following lemma will be used when giving the parameterization of maximal decorated representations.

Lemma 3.8. *Let L_a, L_b, and L_c be Lagrangians. Let $(\mathbf{e}_a, \mathbf{f}_a)$, $(\mathbf{e}_b, \mathbf{f}_b)$, and $(\mathbf{e}_c, \mathbf{f}_c)$ be symplectic bases such that*

$$\begin{aligned} L_a &= \mathrm{Span}(\mathbf{e}_c) = \mathrm{Span}(\mathbf{f}_b) = \mathrm{Span}(\mathbf{e}_a + \mathbf{f}_a) \\ L_b &= \mathrm{Span}(\mathbf{e}_a) = \mathrm{Span}(\mathbf{f}_c) = \mathrm{Span}(\mathbf{e}_b + \mathbf{f}_b) \\ L_c &= \mathrm{Span}(\mathbf{e}_b) = \mathrm{Span}(\mathbf{f}_a) = \mathrm{Span}(\mathbf{e}_c + \mathbf{f}_c). \end{aligned}$$

Let the matrices A, B, and C be defined uniquely by the equalities

$$\mathbf{f}_b = -\mathbf{e}_c \cdot A, \;\; \mathbf{f}_c = -\mathbf{e}_a \cdot B, \;\; \mathbf{f}_a = -\mathbf{e}_b \cdot C.$$

Then they are orthogonal and satisfy the equation $CBA = -\operatorname{Id}$.

Furthermore,

$$\mathbf{e}_b = (\mathbf{e}_c + \mathbf{f}_c) \cdot A, \;\; \mathbf{e}_c = (\mathbf{e}_a + \mathbf{f}_a) \cdot B, \;\; \mathbf{e}_a = (\mathbf{e}_b + \mathbf{f}_b) \cdot C.$$

Remark 3.9. This result will be used in the presence of three other Lagrangians (M_a, M_b, M_c) such that the quadruple (L_a, L_b, L_c, M_b), (L_b, L_c, L_a, M_c), and (L_c, L_a, L_b, M_a) are positive and applied to symplectic bases in standard position with respect to these quadruples.

We will in particular use the conclusion in the situation where $(\mathbf{e}_b, \mathbf{f}_b) = (\mathbf{e}_c, \mathbf{f}_c) \cdot \left(\begin{smallmatrix} A & -A \\ A & 0 \end{smallmatrix}\right)$.

Proof. The pair $(\mathbf{f}_b, -\mathbf{e}_b - \mathbf{f}_b)$ is a symplectic basis. One has $\operatorname{Span}(\mathbf{f}_b) = L_a = \operatorname{Span}(\mathbf{e}_c)$ and $\operatorname{Span}(-\mathbf{e}_b - \mathbf{f}_b) = L_b = \operatorname{Span}(\mathbf{f}_c)$. Thus, for the uniquely defined matrix A above,

$$(\mathbf{f}_b, -\mathbf{e}_b - \mathbf{f}_b) = (\mathbf{e}_c, \mathbf{f}_c) \cdot \begin{pmatrix} -A & 0 \\ 0 & -{}^T\!A^{-1} \end{pmatrix}.$$

Thus $\mathbf{e}_b = \mathbf{e}_c \cdot A + \mathbf{f}_c \cdot {}^T\!A^{-1}$. In particular, $\operatorname{Span}(\mathbf{e}_c \cdot A\,{}^T\!A + \mathbf{f}_c) = \operatorname{Span}(\mathbf{e}_b) = L_c = \operatorname{Span}(\mathbf{e}_c + \mathbf{f}_c)$ which implies $A\,{}^T\!A = \operatorname{Id}$, i.e. A is orthogonal. The same property holds for B and C. To conclude, the composition of the three changes of symplectic bases must be the identity, that is

$$\begin{pmatrix} C & -C \\ C & 0 \end{pmatrix} \begin{pmatrix} B & -B \\ B & 0 \end{pmatrix} \begin{pmatrix} A & -A \\ A & 0 \end{pmatrix} = \operatorname{Id},$$

and the identity $CBA = -\operatorname{Id}$ follows. □

Remark 3.10. We will often use the fact that if, for two symplectic bases $(\mathbf{e}, \mathbf{f})$ and $(\mathbf{e}', \mathbf{f}')$, one has $\operatorname{Span}(\mathbf{e}) = \operatorname{Span}(\mathbf{e}')$, $\operatorname{Span}(\mathbf{f}) = \operatorname{Span}(\mathbf{f}')$, $\operatorname{Span}(\mathbf{e} + \mathbf{f}) = \operatorname{Span}(\mathbf{e}' + \mathbf{f}')$, then $\mathbf{e}' = \mathbf{e}A$ and $\mathbf{f}' = \mathbf{f}A$ for A in $\mathrm{O}(n)$.

The converse statement is:

Lemma 3.11. *Let A, B, and C be orthogonal matrices such that $CBA = -\operatorname{Id}$. Let $(\mathbf{e}_c, \mathbf{f}_c)$ be a symplectic basis and set*

$$(\mathbf{e}_b, \mathbf{f}_b) := (\mathbf{e}_c, \mathbf{f}_c) \cdot \begin{pmatrix} A & -A \\ A & 0 \end{pmatrix} \text{ and } (\mathbf{e}_a, \mathbf{f}_a) := (\mathbf{e}_b, \mathbf{f}_b) \cdot \begin{pmatrix} B & -B \\ B & 0 \end{pmatrix}.$$

Then $(\mathbf{e}_c, \mathbf{f}_c) = (\mathbf{e}_a, \mathbf{f}_a) \cdot \begin{pmatrix} C & -C \\ C & 0 \end{pmatrix}$ *and the three Lagrangians*

$$\begin{aligned} L_a &:= \operatorname{Span}(\mathbf{e}_c) = \operatorname{Span}(\mathbf{f}_b) = \operatorname{Span}(\mathbf{e}_a + \mathbf{f}_a) \\ L_b &:= \operatorname{Span}(\mathbf{e}_a) = \operatorname{Span}(\mathbf{f}_c) = \operatorname{Span}(\mathbf{e}_b + \mathbf{f}_b) \\ L_c &:= \operatorname{Span}(\mathbf{e}_b) = \operatorname{Span}(\mathbf{f}_a) = \operatorname{Span}(\mathbf{e}_c + \mathbf{f}_c) \end{aligned}$$

are pairwise transverse.

3.3. **Positive quintuples.** A quintuple $(L_a, M_c, L_b, L_c, M_b)$ of Lagrangians is said *positive* if the quadruples (L_a, L_b, L_c, M_b) and (L_a, M_c, L_b, L_c) are positive or equivalently if the triples (L_a, L_b, L_c), (L_a, L_c, M_b), and (L_a, M_c, L_b) are positive.

Using Proposition 3.6, we get $(\mathbf{e}_b, \mathbf{f}_b)$ a symplectic basis in standard position with respect to (L_a, L_b, L_c, M_b) and $(\mathbf{e}_c, \mathbf{f}_c)$ a symplectic basis in standard position with respect to (L_b, L_c, L_a, M_c). The matrix A in $\mathrm{GL}(n, \mathbf{R})$ such that $\mathbf{f}_c = \mathbf{e}_b \cdot A$ belongs to $\mathrm{O}(n)$. It is called the *angle invariant* of the positive quintuple $(L_a, M_c, L_b, L_c, M_b)$ (we refer to Sections 4.11 and 5.8 for the choice of this terminology); only its class modulo right multiplication by $\operatorname{Stab}_{\mathrm{O}(n)}(\Lambda_b)$ and modulo left multiplication by $\operatorname{Stab}_{\mathrm{O}(n)}(\Lambda_c)$ is well defined (where Λ_b and Λ_c are the elements of Δ_n associated with the positive quadruples (L_a, L_b, L_c, M_b) and (L_b, L_c, L_a, M_c)).

3.4. **Symplectic Λ-lengths.** In this subsection we introduce an invariant of two transverse decorated Lagrangians. Since this invariant is closely related to Penner's λ-lengths in the case when $n = 1$, we call it the symplectic Λ-length.

Definition 3.12. The *symplectic Λ-length* of two decorated Lagrangians $\mathbf{v}, \mathbf{w}$ is the $n \times n$-matrix

$$\Lambda_{\mathbf{v},\mathbf{w}} := \omega(\mathbf{v}, \mathbf{w}) = (\omega(v_i, w_j))_{i,j \in \{1,\dots,n\}}.$$

To shorten the notation, when $\{\mathbf{v}_i\}_{i \in I}$ is a family of decorated Lagrangians, we will write

$$\Lambda_{ij} := \Lambda_{\mathbf{v}_i,\mathbf{v}_j}, \text{ for } i, j \in I.$$

Lemma 3.13. *For all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{L}_n^{\mathrm{d}}$, we have $\Lambda_{12} = -{}^T\Lambda_{21}$; $\mathbf{v}_1$ and $\mathbf{v}_2$ are transverse if and only if Λ_{12} is not singular, in which case*

$$[\omega]_{\mathbf{v}_1,\mathbf{v}_2} = \begin{pmatrix} 0 & \Lambda_{12} \\ \Lambda_{21} & 0 \end{pmatrix}.$$

Remark 3.14. The symplectic Λ-lengths generalize Penner's λ-lengths for the decorated Teichmüller space ([20]), and one can check when $n = 1$, the symplectic Λ-length is a square root of Penner's λ-length.

3.5. **Ptolemy equation, exchange and triangle relations.** Penner's λ-lengths satisfy the famous Ptolemy equation. Given four b-isotropic vectors w_1, w_2, w_3, w_4 in $\mathbf{R}^3$ (b is a form of signature $(1,2)$ on $\mathbf{R}^3$), contained in the same component of the isotropic cone and cyclically order (see Figure 3.1), we have the relation

$$\sqrt{b(w_2,w_4)b(w_1,w_3)} = \sqrt{b(w_2,w_3)b(w_1,w_4)} - \sqrt{b(w_1,w_2)b(w_3,w_4)},$$

where the terms $b(w_i, w_j)$ are Penner's λ-lenghts. Our symplectic Λ-lengths satisfy a noncommutative version of the Ptolemy equation. We also call this identity the exchange relation. Moreover, they satisfy a triangle relation, which is trivial in the case of $\mathrm{SL}(2, \mathbf{R})$.

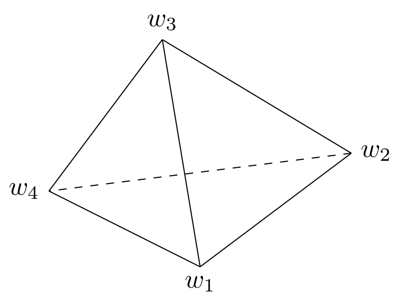

FIGURE 3.1. The tetrahedron illustrating the exchange relation

Lemma 3.15. *Let $\mathbf{v}_1$, $\mathbf{v}_3$ be two transverse decorated Lagrangians. Consider a third decorated Lagrangian $\mathbf{v}_2$, so that there exists a unique pair (A, B) of $n \times n$-matrices such that*

$$\mathbf{v}_2 = \mathbf{v}_1 A + \mathbf{v}_3 B.$$

Then

$$A = \Lambda_{31}^{-1}\Lambda_{32}, \quad B = \Lambda_{13}^{-1}\Lambda_{12},$$

and the matrix $\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12}$ is symmetric.

Proof. One has $\Lambda_{12} = \omega(\mathbf{v}_1, \mathbf{v}_2) = \omega(\mathbf{v}_1, \mathbf{v}_1 A + \mathbf{v}_3 B) = \omega(\mathbf{v}_1, \mathbf{v}_3 B) = \Lambda_{13} B$, so $\Lambda_{13}^{-1}\Lambda_{12} = B$. Similarly $A = \Lambda_{31}^{-1}\Lambda_{32}$.

Also the equality $\omega(\mathbf{v}_2, \mathbf{v}_2) = 0$ gives ${}^T A \Lambda_{13} B + {}^T B \Lambda_{31} A = 0$. Since $\Lambda_{13} = -{}^T\Lambda_{31}$, we get that ${}^T A \Lambda_{13} B = {}^T\Lambda_{32}\, {}^T\Lambda_{31}^{-1}\Lambda_{12}$ is symmetric and the last result follows from the equalities ${}^T\Lambda_{32} = -\Lambda_{23}$ and ${}^T\Lambda_{31} = -\Lambda_{13}$. □

The symmetry of the matrix $\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12}$ can be restated as follow (thanks to the relations ${}^T\Lambda_{ij} = -\Lambda_{ji}$):

Corollary 3.16 (Triangle relation)**.** *One has $\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12} + \Lambda_{12}\Lambda_{31}^{-1}\Lambda_{32} = 0$, and when $\mathbf{v}_2$ is transverse to $\mathbf{v}_1$ and to $\mathbf{v}_3$, $\Lambda_{32}^{-1}\Lambda_{13}\Lambda_{21}^{-1}\Lambda_{23}\Lambda_{13}^{-1}\Lambda_{12} = -\operatorname{Id}$.*

Proposition 3.17 (Ptolemy relation)**.** *Let $\mathbf{v}_i$, $i \in \{1,2,3,4\}$, be decorated Lagrangians such that $\mathbf{v}_1$ and $\mathbf{v}_3$ are transverse. Then*

$$\Lambda_{24} = \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.$$

Proof. Using Lemma 3.15, we have

$$\begin{aligned}\Lambda_{24} = \omega(\mathbf{v}_2, \mathbf{v}_4) &= \omega(\mathbf{v}_1\Lambda_{31}^{-1}\Lambda_{32}, \mathbf{v}_3\Lambda_{13}^{-1}\Lambda_{14}) + \omega(\mathbf{v}_3\Lambda_{13}^{-1}\Lambda_{12}, \mathbf{v}_1\Lambda_{31}^{-1}\Lambda_{34})\\ &= {}^T(\Lambda_{31}^{-1}\Lambda_{32})\omega(\mathbf{v}_1, \mathbf{v}_3)\Lambda_{13}^{-1}\Lambda_{14} + {}^T(\Lambda_{13}^{-1}\Lambda_{12})\omega(\mathbf{v}_3, \mathbf{v}_1)\Lambda_{31}^{-1}\Lambda_{34}\\ &= \Lambda_{23}\Lambda_{13}^{-1}\Lambda_{14} + \Lambda_{21}\Lambda_{31}^{-1}\Lambda_{34}.\end{aligned}$$

□

3.6. **Symplectic Λ-lengths, Maslov index and cross ratios.** If we choose bases for all the Lagrangian subspaces, the Maslov index and the cross ratio can be expressed in terms of the symplectic Λ-lengths.

Lemma 3.18. *Let $(L_i, \mathbf{v}_i) \in \mathcal{L}_n^{\mathrm{d}}$, for $i \in \{1,2,3\}$, be three pairwise transverse decorated Lagrangians. Then the matrix $\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31}$ is symmetric and the Maslov index is given by its signature:*

$$\mu_n(L_1, L_2, L_3) = \operatorname{sgn}(\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31}).$$

Proof. The symmetry of the matrix was established in Lemma 3.15. Changing the bases of L_1, L_2, and L_3 transconjugates the matrix $\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31}$ and in particular it does not change its signature. Thus we can assume that $(\mathbf{v}_1, \mathbf{v}_3)$ is a symplectic basis and that there is $p \in \{0, \ldots, n\}$ such that $v_{2,j} = v_{1,j} + v_{3,j}$ if $j \leqslant p$ and $v_{2,j} = v_{1,j} - v_{3,j}$ if $j > p$ so that $\mu_n(L_1, L_2, L_3) = p - (n-p) = 2p - n$. In this situation $\Lambda_{31} = \Lambda_{32} = -\operatorname{Id}$ and $\Lambda_{12} = \left(\begin{smallmatrix}\operatorname{Id}_p & 0\\ 0 & -\operatorname{Id}_{n-p}\end{smallmatrix}\right)$. □

Lemma 3.19 (Cross ratio in terms of symplectic Λ-lengths)**.** *Let $(L_i, \mathbf{v}_i)$, for $i \in \{1,2,3,4\}$, be four pairwise transverse decorated Lagrangians. Then*

$$[L_1, L_2, L_3, L_4]_{\mathbf{v}_1} = -\Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31} = -\Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21},$$

where $[L_1, L_2, L_3, L_4]_{\mathbf{v}_1}$ denotes the matrix of the cross ratio in the basis $\mathbf{v}_1$.

Proof. By Lemma 3.15, if A and B are the matrices such that $\mathbf{v}_2 = \mathbf{v}_1 A + \mathbf{v}_3 B$, then $A = \Lambda_{31}^{-1}\Lambda_{32}$ and $B = \Lambda_{13}^{-1}\Lambda_{12}$. Since $L_2 = \operatorname{Span}(\mathbf{v}_1 A + \mathbf{v}_3 B) = \operatorname{Span}(\mathbf{v}_1 + \mathbf{v}_3 B A^{-1})$, the matrix of the linear map $L_2 \colon L_1 \to L_3$ is BA^{-1}, that is

$$[L_2]_{\mathbf{v}_1, \mathbf{v}_3} = \Lambda_{13}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31}$$

and using the triangle relation:

$$= -\Lambda_{13}^{-1}\Lambda_{13}\Lambda_{23}^{-1}\Lambda_{21} = -\Lambda_{23}^{-1}\Lambda_{21}.$$

Similarly

$$[L_4]_{\mathbf{v}_3,\mathbf{v}_1} = \Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{13} = -\Lambda_{41}^{-1}\Lambda_{43}.$$

Therefore, on one hand

$$[L_1, L_2, L_3, L_4]_{\mathbf{v}_1} = -\Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{13}\Lambda_{13}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31} = -\Lambda_{31}^{-1}\Lambda_{34}\Lambda_{14}^{-1}\Lambda_{12}\Lambda_{32}^{-1}\Lambda_{31},$$

and on the other hand

$$[L_1, L_2, L_3, L_4]_{\mathbf{v}_1} = -\Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21}.$$

□

4. Moduli spaces of framed and decorated local systems

This section describes the framed and decorated local systems whose moduli spaces are later parameterized. The interpretation of these spaces in term of "genuine" representations are also given. We introduce Maslov indices for triples associated with framed local systems and explain how these can be used to calculate the Toledo number. The characterization of maximal representations in term of these Maslov indices will lead to a definition of maximal framed local system.

4.1. Topological data.

From now on, S will denote an oriented, connected, finite-type surface with nonempty boundary and without punctures. There is an essentially unique connected oriented compact surface $\bar{S}$ with nonempty boundary $\partial\bar{S}$ and a finite set $R \subset \partial\overline{S}$ so that S is the complement $\bar{S} \smallsetminus R$. The boundary of S can be noncompact.

We denote by g the genus of $\bar{S}$, by k the number of connected components of $\partial\bar{S}$, and by r the cardinality of R. We will also denote by $p \in \{0, \dots, k\}$ the number of compact connected components of ∂S, i.e. the number of connected components of $\partial\bar{S}$ not intersecting R.

We will always assume that

$$4g - 4 + 2k + r > 0.$$

This precisely means that the double of S along its boundary $\partial S = \partial\bar{S} \smallsetminus R$ has negative Euler characteristic.

Remark 4.1. The conditions we assume on S can be equivalently expressed saying that S admits a complete hyperbolic structure of finite volume with nonempty, possibly noncompact, geodesic boundary and without cusps.

As examples, let's see some special and "extreme" cases. On the one hand, r can be zero. This is the case when S is compact, with negative Euler characteristic. On the other hand, the surface $\bar{S}$ can be a disk ($g = 0, k = 1$), or an annulus ($g = 0, k = 2$). In these cases, our hypothesis implies that $r \geqslant 3$ for the disc, and $r \geqslant 1$ for the annulus.

4.2. Fundamental group.

The fundamental group of S is a free group of rank $2g + k - 1$.

For definiteness, we denote by b_0 the base point of the surface S, i.e. $\pi_1(S) = \pi_1(S, b_0)$. Thus elements of $\pi_1(S)$ are classes of loops based at b_0 and, for α and γ in $\pi_1(S)$, represented by loops a and g respectively, their product $\alpha\gamma$ or $\alpha * \gamma$ is represented by the loop $a * g$ obtained juxtaposing g then a, in that order. More generally, for two paths $a\colon [0,1] \to S$, $g\colon [0,1] \to S$ such that $g(1) = a(0)$, we will denote by $a * g$ the juxtaposition of g and a. With this convention, the fundamental group $\pi_1(S)$ acts on the *right* on the universal cover $\widetilde{S}$.

For later purpose, we *fix* a path $\alpha_C\colon [0,1] \to S$ between b_0 and $C \cap S$ ($= C \smallsetminus R$) for every component C of $\partial\bar{S}$. This enables us in particular to give p elements of $\pi_1(S)$, $c_1, \dots, c_p$, representing the compact components of ∂S. Obviously, only the conjugacy class of c_j is intrinsically defined. The other components of $\partial\bar{S}$ will be numbered $C_1, \dots, C_{k-p}$; they contain respectively $r_1, \dots, r_{k-p}$ elements of R; r_ℓ is a positive integer and $\sum_\ell r_\ell = r$. The arc α_{C_ℓ} joining b_0 and C_ℓ will also be denoted by α_ℓ.

4.3. **Triangulations.** An *ideal triangulation* $\mathcal{T}$ of S is a triangulation of $\bar{S}$ such that the vertex set of $\mathcal{T}$ is contained in ∂S and that every connected component of ∂S contains exactly one vertex of $\mathcal{T}$. It will later prove useful to consider a triangulation as a maximal set of arcs in S, that do not intersect. An *arc* is (the homotopy class of) a map of pairs $([0,1],\{0,1\}) \to (S, \partial S))$. Hence, if an arc α belongs to $\mathcal{T}$, its reverse $\bar{\alpha}$ belongs also to $\mathcal{T}$. Nevertheless, in many situations, α and $\bar{\alpha}$ will be identified and we will speak of nonoriented edges.

It is well known that triangulations can be obtained from one another by a series of elementary moves called *flips*. Two triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ are related by a flip if there exist (oriented) edges e_0 in $\mathcal{T}_0$ and e_1 in $\mathcal{T}_1$ such that $\mathcal{T}_0 \smallsetminus \{e_0, \bar{e}_0\} = \mathcal{T}_1 \smallsetminus \{e_1, \bar{e}_1\}$ (we refer to the later figure 11.1 for an illustration of this classical situation).

The number of vertices of $\mathcal{T}$ is $p+r$, the number of pairs of edges $\{\alpha, \bar{\alpha}\}$ is $3|\chi(\bar{S})|+2r$, among those there are r pairs of *external* edges, i.e. contained in exactly one face of the triangulation, and there are $3|\chi(\bar{S})|+r$ pairs of *internal* edges (contained in two faces), finally the number of faces (or triangles) is $2|\chi(\bar{S})|+r$.

The r_ℓ (pairs of) arcs connecting cyclically the components of $C_\ell \smallsetminus R$ (for every ℓ in $\{1,\dots,k-p\}$) always belongs to the triangulations $\mathcal{T}$ of S. These are the arcs that account for the r external edges.

4.4. **Framed symplectic local systems.** Let $\mathcal{F}$ be a $\mathrm{Sp}(2n,\mathbf{R})$-local system on the surface S, i.e. $\mathcal{F}$ is a right principal $\mathrm{Sp}(2n,\mathbf{R})_d$-bundle where $\mathrm{Sp}(2n,\mathbf{R})_d$ is the topological group $\mathrm{Sp}(2n,\mathbf{R})$ equipped with the discrete topology (or, what amounts to the same, the change of trivializations are locally constant maps into $\mathrm{Sp}(2n,\mathbf{R})$). Using the parallel transport, we find a homomorphism $\rho\colon \pi_1(S) \to \mathrm{Sp}(2n,\mathbf{R})$ so that $\mathcal{F}$ is the quotient $\pi_1(S)\backslash(\widetilde{S}\times\mathrm{Sp}(2n,\mathbf{R}))$ by the left diagonal action on $\widetilde{S}\times\mathrm{Sp}(2n,\mathbf{R})$ determined by ρ; namely $\gamma\cdot(p,g) = (p\cdot\gamma^{-1}, \rho(\gamma)g)$ for all γ in $\pi_1(S)$ and all $(p,g) \in \widetilde{S}\times\mathrm{Sp}(2n,\mathbf{R})$. Given $\mathcal{F}$ there is an associated bundle with fiber $\mathcal{L}_n$, the Lagrangian Grassmannian, which we denote by $\mathcal{F}_{\mathcal{L}_n}$. It is the quotient $\pi_1(S)\backslash(\widetilde{S}\times\mathcal{L}_n)$.

Definition 4.2. A *framing* of a symplectic local system $\mathcal{F}$ is a flat section of $\mathcal{F}_{\mathcal{L}_n}|_{\partial S}$. A *framed symplectic local system* is a pair $(\mathcal{F},\sigma)$ where σ is a framing of the symplectic local system $\mathcal{F}$.

Remark 4.3. The framing σ is equivalently given as a locally constant and ρ-equivariant map $\tilde{\sigma}\colon \partial\tilde{S}$; this means that, for all γ in $\pi_1(S)$ and all p in $\partial\tilde{S}$, $\gamma\cdot(p,\tilde{\sigma}(p)) = (p\cdot\gamma^{-1}, \tilde{\sigma}(p\cdot\gamma^{-1}))$. This last equality can be rewritten $\tilde{\sigma}(p\cdot\gamma^{-1}) = \rho(\gamma)\cdot\tilde{\sigma}(p)$.

The moduli space of framed symplectic local systems is denoted by $\mathrm{Loc}^{\mathrm{f}}(S,\mathrm{Sp}(2n,\mathbf{R}))$.

A framed local system gives rise to the following data (with the notation introduced in Section 4.2)

- a representation $\rho\colon \pi_1(S)\to\mathrm{Sp}(2n,\mathbf{R})$,
- for every $j=1,\dots,p$, a $\rho(c_j)$-invariant Lagrangian L_j. It is obtained using the parallel transport along the arc α_C where C is the compact boundary component homotopic to c_j.
- for every $\ell = 1,\dots,k-p$, a r_ℓ-tuple $(L_{1,\ell},\dots,L_{r_\ell,\ell})$ of Lagrangians, obtained again using the parallel transport along the arc α_ℓ and along arcs connecting cyclically the r_ℓ components of $C_\ell \smallsetminus R$.

The tuple $(\rho, \{L_j\}_{j=1}^p, \{(L_{j,\ell})_{j=1}^{r_\ell}\}_{\ell=1}^{k-p})$ will be called the *holonomy* of the framed local system $(\mathcal{F},\sigma)$; it is only defined up to the action of the symplectic group and it determines completely the isomorphism class of the framed symplectic local system $(\mathcal{F},\sigma)$. We are thus led to consider the following definition.

Definition 4.4. A *framed symplectic representation* is a tuple

$$\left(\rho, \{L_j\}_{j=1}^{p}, \{(L_{j,\ell})_{j=1}^{r_\ell}\}_{\ell=1}^{k-p}\right) \in \operatorname{Hom}(\pi_1(S), \operatorname{Sp}(2n, \mathbf{R})) \times \mathcal{L}_n^p \times \prod_{\ell=1}^{k-p} \mathcal{L}_n^{r_\ell}$$

satisfying the condition $\rho(c_j) \cdot L_j = L_j$ for all $j = 1, \dots, p$.

We denote by $\operatorname{Hom}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$ the space of all framed symplectic representations. This space carries a natural action of $\operatorname{Sp}(2n, \mathbf{R})$, and we will denote the quotient by

$$\operatorname{Rep}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R})) = \operatorname{Hom}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R})) / \operatorname{Sp}(2n, \mathbf{R}).$$

Remark 4.5. When $\operatorname{Rep}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$ is endowed with the quotient topology, it is in general not a Hausdorff space. In this work, we will be led to consider *noncontinuous* ($\operatorname{Sp}(2n, \mathbf{R})$-invariant) functions on $\operatorname{Hom}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$, and it is therefore more relevant not to consider any Hausdorffication of the moduli space.

Remark 4.6. One can give a more intrinsic way to describe the moduli space $\operatorname{Rep}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$ by considering at the same time all the possible arcs α_ℓ and all the possible representatives c_j of the compact boundary components. We will not write this definition here explicitly.

We can identify $\operatorname{Loc}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$ and $\operatorname{Rep}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$:

Lemma 4.7. *The holonomy map is a bijection between the moduli space* $\operatorname{Loc}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$ *and the moduli space* $\operatorname{Rep}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$*.*

We use this identification to endow the moduli space $\operatorname{Loc}^{\mathrm{f}}(S, \operatorname{Sp}(2n, \mathbf{R}))$ with a (non-Hausdorff) topology.

4.5. **Twisted local systems.** We denote by $T'S$ the *punctured* tangent bundle of S, i.e. the tangent bundle TS with the zero section removed. Its fundamental group is a central extension of $\pi_1(S)$

$$\mathbf{Z} \longrightarrow \pi_1(T'S) \longrightarrow \pi_1(S)$$

that necessarily splits as $\pi_1(S)$ is a free group. The generating element δ of the kernel of the extension is the class in $\pi_1(T'S)$ of any loop representing the homotopy of a fiber of $T'S \to S$.

Let G be a group and let δ_G be a central element in G. For the case when $G = \operatorname{Sp}(2n, \mathbf{R})$ we will always take $\delta_G = -\operatorname{Id}$. For the case when G is a connected Lie group locally isomorphic to $\operatorname{Sp}(2n, \mathbf{R})$, the relevant element δ_G is described in Section 10.3.

Definition 4.8. A G-local system $\mathcal{F}$ on $T'S$ is said to be *twisted* (or δ-twisted) if its holonomy in restriction to any fiber of $T'S \to S$ is equal to δ_G. The moduli space of δ-twisted G-local systems is denoted by $\operatorname{Loc}_\delta(S, G)$.

A representation $\rho \colon \pi_1(T'S) \to G$ is said δ-*twisted* if $\rho(\delta) = \delta_G$.

The space of δ-twisted representations is denotes by $\operatorname{Hom}_\delta(S, G)$, the space of their G-conjugacy classes by $\operatorname{Rep}_\delta(S, G)$.

The holonomy map gives an identification between $\operatorname{Loc}_\delta(S, G)$ and $\operatorname{Rep}_\delta(S, G)$.

Remark 4.9. Using an isomorphism $\pi_1(T'S) \simeq \mathbf{Z} \times \pi_1(S)$, the space $\operatorname{Hom}_\delta(S, G)$ is in one-to-one correspondence with $\operatorname{Hom}(\pi_1(S), G)$. This identification is mapping class group equivariant but is not canonical since it relies on the choice of isomorphism. We will rely on this correspondence and describe in Section 6 and sqq. moduli spaces of (decorated) local systems.

4.6. **Vector fields.** Since S as nonempty boundary there exists a nonvanishing vector field on S. Such a vector field gives rise to an isomorphism $\pi_1(T'S) \simeq \mathbf{Z} \times \pi_1(S)$. We describe now a slightly different identification, which relies on the existence of a vector field with isolated zeros. This is a bit more involved but useful for later considerations.

Let $\vec{x}$ be a vector field with isolated zeros and without any zero on the compact components of ∂S. We denote by $Z(\vec{x})$ the set of its zeroes in the *interior* of S. For every u in $Z(\vec{x})$, the index $\operatorname{ind}(u, \vec{x})$ (later simply denoted $\operatorname{ind}(u)$) is the number of "turns" that $\vec{x}$ does around u (e.g. it is $+1$ if $\vec{x}$ is tangent to a small circle around u, it is -1 if $\vec{x}$ presents a saddle singularity). For every $u \in Z(\vec{x})$, let us fix γ_u an element in $\pi_1(S \smallsetminus Z(\vec{x}))$ that is freely homotopic to a small (directly oriented) circle around u.

The vector field $\vec{x}$ provides a trivialization of $T'(S \smallsetminus Z(\vec{x}))$, that we can use to obtain:

Lemma 4.10. *The homomorphism $\pi_1(T'(S \smallsetminus Z(\vec{x}))) \to \pi_1(T'S)$ induces an isomorphism between $\pi_1(T'S)$ and the quotient of $\mathbf{Z} \times \pi_1(S \smallsetminus Z(\vec{x})) \simeq \pi_1(T'(S \smallsetminus Z(\vec{x})))$ by the normal subgroup generated by the elements $(-\operatorname{ind}(u), \gamma_u)$ for u in $Z(\vec{x})$.*

As a consequence:

Corollary 4.11. *The isomorphism of Lemma 4.10 gives a one-to-one G-equivariant correspondence between the space $\operatorname{Hom}_\delta(S, G)$ and the space*

$$\operatorname{Hom}_\delta(S \smallsetminus Z(\vec{x}), G) := \{\rho \in \operatorname{Hom}(\pi_1(S \smallsetminus Z(\vec{x})), G) \mid \forall u \in Z(\vec{x}),\ \rho(\gamma_u) = \delta_G^{\operatorname{ind(u)}}\}.$$

We call the representations in $\operatorname{Hom}_\delta(S \smallsetminus Z(\vec{x}), G)$ also twisted representations.

Remark 4.12. The surface $S \smallsetminus Z(\vec{x})$ is (almost) of the same type as the surface S so that the discussion of Sections 4.1 and 4.2 applies. In particular we will also denote by $c_1, \dots, c_p$ representatives in $\pi_1(S \smallsetminus Z(\vec{x}))$ of the compact boundary components of S. These elements depend now on choices of arcs in $S \smallsetminus Z(\vec{x})$. Similarly, paths α_ℓ connecting in $S \smallsetminus Z(\vec{x})$ the base point to $C_\ell \smallsetminus R$ will be chosen.

Remark 4.13. One can go a little further in the conclusion of Corollary 4.11 and remove only the zeroes of $\vec{x}$ whose index is not a multiple of the order of δ_G. When $G = \operatorname{Sp}(2n, \mathbf{R})$, so that δ_G is of order 2, we will sometimes be able to use vector fields whose indices are all even. In this case, the space $\operatorname{Hom}_\delta(S, \operatorname{Sp}(2n, \mathbf{R}))$ is in fact a space of representations of $\pi_1(S)$.

4.7. **Triangulations and vector fields.** Given a triangulation $\mathcal{T}$ of S we can construct a vector field $\vec{x}_\mathcal{T}$ on S that satisfies the condition required in Section 4.6. It is constructed from a fixed model on each triangle of $\mathcal{T}$ that we now describe:

- The zeroes of the vector field are
  - the vertices of the triangles, and, in a neighborhood of these vertices, the vector field is tangent to the clockwise oriented circles centered at the vertices;
  - and the center of the triangle, and it is tangent to the counter-clockwise oriented circles in a neighborhood of this center (index 1).
  - the midpoints of its edges, where it is "half" of a saddle connection.
- there are 3 complete flow lines emanating and ending at the midpoints of the edges; these flow lines cut the triangle into 4 regions, 3 of them contain exactly one vertex and the last one contains the center. Furthermore, in each of the 3 regions containing a vertex, the flow lines foliate the region into a family of homotopic curves going from one half edge to another half edge, the last region contains a zero of the vector field and the other flow lines are counter-clockwise circles around this zero.

This basically determines uniquely the vector field, it is illustrated in Figure 4.1.

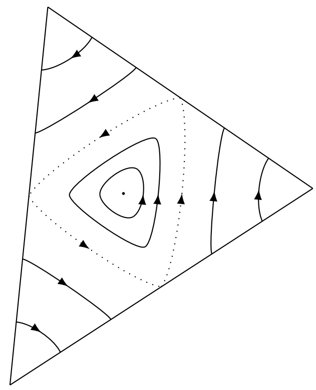

FIGURE 4.1. The model vector field. Dotted lines are complete flow lines.

To be complete, we remove from this local model a small neighborhood of every vertex of the triangle if this vertex belongs to a compact component of ∂S.

We use this vector field in Section 5 to describe the spaces of twisted local systems introduced here in terms of twisted local systems on a quiver.

4.8. **Decorated twisted local systems.** In this section we introduce decorated twisted symplectic systems. Given a symplectic local system $\mathcal{F}$ on $T'S$ we denote by $\mathcal{F}_{\mathcal{L}_n^{\mathrm{d}}}$ the associated bundle over $T'S$ with fiber $\mathcal{L}_n^{\mathrm{d}}$, the space of decorated Lagrangians.

We use the orientation of S and an auxiliary Riemannian metric to define a section of $T'S|_{\partial S}$. We denote the image of ∂S under this section by $\vec{\partial} S \subset T'S|_{\partial S}$ and normalize the section so that tangent vectors forming an angle of $\pi/2$ with $\vec{\partial} S$ are pointing inward.

Definition 4.14. A *decoration* of a twisted symplectic local system $\mathcal{F}$ on $T'S$ is a flat section β of the restriction $\mathcal{F}_{\mathcal{L}_n^{\mathrm{d}}}|_{\vec{\partial} S}$. The pair $(\mathcal{F}, \beta)$ is called a *decorated twisted symplectic local system.*

The moduli space of decorated twisted symplectic local systems is denoted $\mathrm{Loc}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$.

In order to translate this notion in terms of representations, it is convenient to choose vector fields $\vec{x}$ that have some compatibility with $\vec{\partial} S$. The vector field $\vec{x}$ will be said to be *tangent to the compact boundary of* S if, for every b belonging to a compact component of ∂S, $\vec{x}(b)$ is tangent to ∂S and the vector forming an angle of $\pi/2$ with it points inward the surface. In particular, the vector field $\vec{x}_{\mathcal{T}}$ constructed in the previous paragraph has this property.

Using the elements $c_1, \dots, c_p$ in $\pi_1(S \smallsetminus Z(\vec{x}))$ (cf. Remark 4.12) as well as arcs (in $S \smallsetminus Z(\vec{x})$) connecting the base point to the components of $\partial \bar{S}$ containing elements of R, one gets the following

Proposition 4.15. *Let $\vec{x}$ be a vector field on S with isolated zeros and no zeros on the compact components of ∂S, which furthermore is tangent to the compact boundary of S. The parallel transport gives rise to a one-to-one correspondence between the space* $\mathrm{Loc}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ *and the space of conjugacy classes of tuples*

$$(\rho, \{\mathbf{v}_j\}_{j=1}^p, \{(v_{j,\ell})_{j=1}^{r_\ell}\}_{\ell=1}^{k-p}) \in \mathrm{Hom}_\delta(S \smallsetminus Z(\vec{x}), \mathrm{Sp}(2n, \mathbf{R})) \times (\mathcal{L}_n^{\mathrm{d}})^p \times \prod_{\ell=1}^{k-p} (\mathcal{L}_n^{\mathrm{d}})^{r_\ell}$$

such that, for all $j = 1, \dots, p$, $\rho(c_j) \cdot \mathbf{v}_j = \mathbf{v}_j$.

Remark 4.16. Note that when $r > 0$ (i.e. when $R \neq \varnothing$), there are always nonvanishing vector fields that are tangent to the compact boundary.

Under some parity conditions, there are vector fields tangent to the compact boundary and whose indices are all even (cf. Remark 4.13) so that the space can be interpreted as representations of $\pi_1(S)$:

Corollary 4.17. *Suppose that* $p = k$ *(i.e.* $R = \varnothing$*) and that* k *is even. Then there is a one-to-one correspondence between* $\mathrm{Loc}^{\mathrm{d}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$ *and conjugacy classes of tuples*

$$(\rho, \{\mathbf{v}_j\}_{j=1}^{k}) \in \mathrm{Hom}(S, \mathrm{Sp}(2n, \mathbf{R})) \times (\mathcal{L}^{\mathrm{d}}_n)^k$$

such that, for all $j = 1, \dots, k$*,* $\rho(c_j) \cdot \mathbf{v}_j = \mathbf{v}_j$*.*

Without parity condition, we can use a vector field that has index 0 at the boundary components and whose all other indices are even to get the following.

Corollary 4.18. *Suppose that* $p = k$ *(i.e.* $R = \varnothing$*). Then there is a one-to-one correspondence between* $\mathrm{Loc}^{\mathrm{d}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$ *and conjugacy classes of tuples*

$$(\rho, \{\mathbf{v}_j\}_{j=1}^{k}) \in \mathrm{Hom}(S, \mathrm{Sp}(2n, \mathbf{R})) \times (\mathcal{L}^{\mathrm{d}}_n)^k$$

such that, for all $j = 1, \dots, k$*,* $\rho(c_j) \cdot \mathbf{v}_j = -\mathbf{v}_j$*.*

4.9. Framed twisted local system.

Definition 4.19. A *framing* of a twisted local system $\mathcal{F}$ is a flat section σ of the restriction of $\mathcal{F}_{\mathcal{L}_n}$ to $T'S|_{\partial S}$. The pair $(\mathcal{F}, \sigma)$ is called a *framed twisted local system.*

The moduli space of framed twisted local systems is denoted by $\mathrm{Loc}^{\mathrm{f}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$. Since the element $-\mathrm{Id}$ of $\mathrm{Sp}(2n, \mathbf{R})$ acts trivially on $\mathcal{L}_n$, the restriction of the $\mathcal{L}_n$-local system $\mathcal{F}_{\mathcal{L}_n}$ to a fiber of $T'S \to S$ is the trivial bundle and the section σ is constant in restriction to such a fiber of $T'S \to S$ (with base point in ∂S). Thus the section is entirely determined by its restriction to $\vec{\partial} S \subset T'S|_{\partial S}$. Therefore we also call a flat section σ of the restriction of $\mathcal{F}_{\mathcal{L}_n}$ to $\vec{\partial} S$ a *framing.* With this point of view, using the natural projection $\mathcal{L}^{\mathrm{d}}_n \to \mathcal{L}_n$ and the induced map $\mathcal{F}_{\mathcal{L}^{\mathrm{d}}_n} \to \mathcal{F}_{\mathcal{L}_n}$, there is a natural map

$$\mathrm{Loc}^{\mathrm{d}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathrm{Loc}^{\mathrm{f}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$$

and thus, relying on the above identification $\mathrm{Loc}^{\mathrm{f}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R})) \simeq \mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ (depending on the choice of a nonvanishing vector field), a map $\mathrm{Loc}^{\mathrm{d}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R})) \to \mathrm{Loc}^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$.

4.10. Transverse local systems.

Let $\alpha \colon ([0,1], \{0,1\}) \to (S, \partial S)$ be an arc and let $(\mathcal{F}, \sigma)$ be a framed symplectic local system on S. The restriction of $\mathcal{F}$ to α (more precisely, its pull-back by α) is the trivial local system $[0,1] \times \mathrm{Sp}(2n, \mathbf{R})$ and the decoration gives a pair of Lagrangians (L^t, L^b) (L^t coming from the fiber above $\alpha(0)$ and L^b from $\alpha(1)$). Only the $\mathrm{Sp}(2n, \mathbf{R})$-orbit of the pair (L^t, L^b) is well defined. However it makes sense to say when this pair is transverse in which case we will say that the framed local system $(\mathcal{F}, \sigma)$ is *α-transverse.*

As the pair associated with $\bar{\alpha}$ is (L^b, L^t), a framed local system is $\bar{\alpha}$-transverse if and only if it is α-transverse.

Definition 4.20. Let $\mathcal{T}$ be an ideal triangulation of S. A framed symplectic local system is said to be *$\mathcal{T}$-transverse* if it is α-transverse for every α in $\mathcal{T}$.

We will denote by $\mathrm{Loc}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ the space of $\mathcal{T}$-transverse decorated symplectic local systems.

Let $(\mathcal{F}, \sigma)$ be a framed *twisted* local system. If $(\mathcal{F}', \sigma')$ is the corresponding framed local system (obtained via the choice of a nonvanishing vector field), we will say that $(\mathcal{F}, \sigma)$ is *α-transverse* or *$\mathcal{T}$-transverse* if $(\mathcal{F}', \sigma')$ is so. As different nonvanishing vector fields $\vec{x}_1$, $\vec{x}_2$ differ by "twists" in the fiber of $T'S \to S$ (more precisely, for any arc α, one can, up to homotopy, assume that $\vec{x}_1$

and $\vec{x}_2$ coincide at the extremities of α, and the loop $\vec{x}_1(\alpha) \sqcup \vec{x}_2(\bar{\alpha})$ is homotopic to a power of δ), and again as $-\mathrm{Id}$ acts trivially on $\mathcal{L}_n$, this condition does not depend on the choice of the nonvanishing vector field. Hence the space $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ of $\mathcal{T}$-transverse framed twisted symplectic local system is well defined and isomorphic to $\mathrm{Loc}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{Sp}(2n, \mathbf{R}))$.

Similarly, a decorated δ-twisted symplectic local system $(\mathcal{F}, \beta)$ is said to be *α-transverse* or *$\mathcal{T}$-transverse* if the associated framed local system is so. Their moduli space is denoted $\mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}}(S, \mathrm{Sp}(2n, \mathbf{R}))$.

4.11. **Configurations associated with framed local systems.** For $\ell \geqslant 2$, an *ℓ-gon* is (the homotopy class of) a map $(\mathbb{D}, \mu_\ell) \to (S, \partial S)$ where $\mathbb{D}$ is the closed unit disk in $\mathbf{C}$, $\mathbb{D} = \{z \in \mathbf{C} \mid |z| \leqslant 1\}$ and μ_ℓ is the set of ℓ-roots of unity, $\mu_\ell = \{z \in \mathbf{C} \mid z^\ell = 1\}$. A 3-gon will be sometime called a *triangle* (although this may cause confusion with the triangles of a triangulation) and a 4-gon will be called a *quadrilateral.* A 2-gon is the same thing (up to homotopy) than an arc in S.

The pull back of a framed local system by an ℓ-gon is the trivial local system on $\mathbb{D}$ together with the data of a Lagrangian for each element in μ_ℓ, i.e. it gives a well defined element in $\mathrm{Conf}^\ell(\mathcal{L}_n)$. We have thus, associated with any ℓ-gon τ, a map

$$f_\tau \colon \mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathrm{Conf}^\ell(\mathcal{L}_n).$$

For $\ell = 2$, the maps f_α for α an edge of a triangulation are precisely the maps we used above to define transverse local systems.

In a similar way, an ℓ-gon provides a map from the space of decorated symplectic local systems to the configuration space of decorated Lagrangians.

Let $\mathcal{T}$ be a triangulation of S. Any face T (i.e. triangle) of $\mathcal{T}$ gives rise to three 3-gons, τ_1, τ_2, τ_3 obtained one from the other via precomposition by the rotation of angle $2\pi/3$. In other words, for every framed symplectic local system $(\mathcal{F}, \sigma)$, the three configurations of triples of Lagrangians $f_{\tau_i}(\mathcal{F}, \sigma)$ $(i = 1, 2, 3)$ are obtained one from the other by cyclic permutation.

Definition 4.21. The common value of the Maslov index $\mu_n(f_{\tau_i}(\mathcal{F}, \sigma))$ depends only on T and $(\mathcal{F}, \sigma)$ (cf. Proposition 2.9) and will be called *Maslov index of the triangle* T for $(\mathcal{F}, \sigma)$ and denoted by $\mu^T(\mathcal{F}, \sigma)$.

In a similar vein, any internal (oriented) edge α of T is the diagonal of a quadrilateral in T and gives rise to a 4-gon τ. In this situation the map f_τ will be denoted by q_α:

$$q_\alpha \colon \mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathrm{Conf}^4(\mathcal{L}_n).$$

As the 4-gon associated with $\bar{\alpha}$ is obtained from τ by the precomposition with the rotation of angle π, the map $q_{\bar{\alpha}}$ is equal to $\kappa \circ q_\alpha$ (κ is the automorphism of $\mathrm{Conf}^4(\mathcal{L}_n)$ induced by the permutation $(13)(24)$, see Section 2.2).

An *angle* θ of $\mathcal{T}$ is a pair of edges $\{\alpha, \alpha'\}$ of $\mathcal{T}$ contained in the same face and having the same endpoint. It is called *internal* if both edges α, α' are internal. If the angle θ is internal, it gives rise to a well defined 5-gon τ (composed with the three faces of $\mathcal{T}$ containing $\{\alpha, \alpha'\}$) and hence a map

$$c_\theta := f_\tau \colon \mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathrm{Conf}^5(\mathcal{L}_n).$$

4.12. **Toledo number and maximal representations.** In this section, we assume that $r = 0$ (i.e. $R = \varnothing$), so that $p = k$.

An important invariant for a symplectic local system $\mathcal{F}$ on S (or for the associated holonomy representation $\rho \colon \pi_1(S) \to \mathrm{Sp}(2n, \mathbf{R})$) is the Toledo number, here denoted by $T_{\mathcal{F}}$ (or T_ρ), which was defined in [7] using bounded cohomology. Note that the Toledo number depends on the topological surface S and not only on its fundamental group. It is a real number which satisfies the Milnor–Wood inequality:

$$-n|\chi(S)| \leqslant T_{\mathcal{F}} \leqslant n|\chi(S)|.$$

The representations where this invariant achieves its maximum have particularly nice geometric properties, see [7].

Definition 4.22. If $R = \varnothing$, a symplectic local system $\mathcal{F}$ is called *maximal* if $T_{\mathcal{F}} = n\chi(S) = -n|\chi(S)|$.

Remark 4.23. The choice of the sign in the definition is not really relevant (pulling back with an orientation reversing diffeomorphism changes the Toledo number to its opposite). The chosen sign here makes Corollary 4.27 below looks more natural; equally uniformizations are maximal for this choice.

We denote by $\mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$ the subspace of $\mathrm{Loc}(S, \mathrm{Sp}(2n, \mathbf{R}))$ consisting of maximal local systems. In a similar fashion, we denote by $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ the subspace of $\mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ of framed maximal local systems. The following facts are proven in [7].

Proposition 4.24. *Suppose that R is empty.*

(a) The natural map

$$\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$$

is surjective. In other words, every maximal local system admits a framing.

(b) Framed maximal local systems are transverse with respect to any ideal triangulation $\mathcal{T}$:

$$\mathcal{M}^{f}(S, \mathrm{Sp}(2n, \mathbf{R})) \subset \mathrm{Loc}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{Sp}(2n, \mathbf{R})).$$

(c) Maximal representations are reductive; the spaces $\mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$ and $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ are hence Hausdorff (cf. Remark 4.5).

Remark 4.25. The argument given in [7] implies that not only maximal representations are reductive, but also representations that are *almost maximal*, i.e. where $T_\rho < -(n-1)|\chi(S)|$.

The next result shows that the Toledo number of a framed local system can be computed easily using an ideal triangulation. In the special case of a pair of pants this has been proven in [25].

Theorem 4.26. *Suppose that R is empty. Let $\mathcal{T}$ be an ideal triangulation of S and $(\mathcal{F}, \sigma) \in \mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$. The Toledo number $T_{\mathcal{F}}$ can be computed via the following formula:*

$$T_{\mathcal{F}} = -\frac{1}{2}\sum_{T \in \mathcal{T}} \mu^T(\mathcal{F}, \sigma).$$

This proposition implies the Milnor–Wood inequality and the integrality property of the Toledo invariant for framed representations. Another consequence is that framed maximal representations can be detected using a triangulation:

Corollary 4.27. *Given a local system $\mathcal{F}$ admitting framings, and an ideal triangulation $\mathcal{T}$ of S, for any framing σ of $\mathcal{F}$, we have that $\mathcal{F}$ is maximal if and only if, for every triangle T in $\mathcal{T}$, the Maslov index $\mu^T(\mathcal{F}, \sigma)$ is n.*

Note also that this corollary implies point (b) of Proposition 4.24.

The proof of Theorem 4.26 will take the rest of this subsection. It will use the Souriau index (see Section 2.5) and the translation number (see Section 2.6).

Lemma 4.28. *Let $(\mathcal{F}, \sigma)$ be a framed local system. Then the sum $\sum_{T \in \mathcal{T}} \mu^T(\mathcal{F}, \sigma)$ does not depend on the triangulation $\mathcal{T}$.*

Remark 4.29. Of course, a consequence of Theorem 4.26 is that this sum does not depend on σ either. This can be proved a priori thanks to Lemma 2.20.

Proof. The result of the lemma follows from

(1) Any two ideal triangulations are related by a sequence of flips.
(2) The invariance of the sum under a flip is equivalent to the cocycle condition satisfied by the Maslov index (see Proposition 2.9). ☐

The following result will be our starting point in the proof of Theorem 4.26.

Lemma 4.30 ([7, Thm. 12])**.** *Let $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{Sp}(2n, \mathbf{R}))$ and let $a_1, \dots, a_g$, $b_1, \dots, b_g$, $c_1, \dots, c_k$ be based loops in S such that the complement $S \smallsetminus \big(a_1 \cup \cdots \cup c_k\big)$ is the disjoint union of k punctured disks bounded by $c_1, \dots, c_k$ and a disk bounded by $c_1^{-1} \cdots c_k^{-1}[b_g, a_g] \cdots [b_1, a_1]$.*

Since $\pi_1(S)$ is a free group, the representation ρ can be lifted to $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$, and let $\tilde{\rho}$ be such a lift. The Toledo number of ρ can be computed as:

$$T_\rho = -\sum_{i=1}^{k} \widetilde{\mathrm{Rot}}(\tilde{\rho}(c_i)).$$

Proof of Theorem 4.26. Via the choice of a finite volume, complete, hyperbolic structure on the interior $S \smallsetminus \partial S$ of S, we can identify the interior of $\widetilde{S}$ with the hyperbolic plane $\mathbb{H}^2$. The fundamental group $\pi_1(S)$ acts on the right. The boundary at infinity of the interior of $\widetilde{S}$, which we denote by $\partial_\infty \mathbb{H}^2$, is also endowed with a right $\pi_1(S)$-action. It contains points whose stabilizers are generated by one element that is a conjugate of one of the boundary representatives $c_1, \dots, c_k$ (cf. Section 4.2, recall that we are assuming here that $p = k$). Note that, if the stabilizer of p is generated by γ, then, for every $g \in \pi_1(S)$, $g^{-1}\gamma g$ generates the stabilizer of $p \cdot g$.

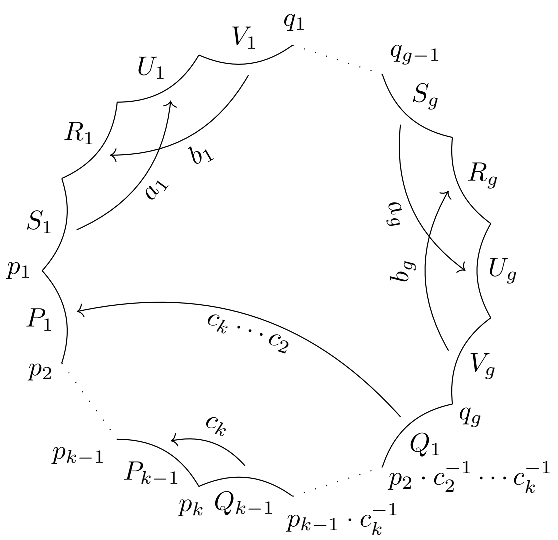

FIGURE 4.2. The fundamental ideal polygon and its sides identifications

A fundamental domain for the right action of $\pi_1(S)$ on $\mathbb{H}^2$ is an ideal $(4g + 2k - 2)$-gon (see Figure 4.2), which we now describe in some details. The sides of this polygon are geodesic rays denoted (in counter-clockwise order and with their "counter-clockwise" orientation):

$$P_1, \dots, P_{k-1}, Q_{k-1}, \dots, Q_1, V_g, U_g, R_g, S_g, \dots, V_1, U_1, R_1, S_1,$$

and are subject to the following identifications (where a bar means the opposite orientation):

> For all $i = 1, \dots, g$, $S_i \cdot a_i = \overline{U}_i$, $V_i \cdot b_i = \overline{R}_i$ and for all $j = 1, \dots, k-1$, $Q_j \cdot (c_k \cdots c_{j+1}) = \overline{P}_j$.

It means that, on the interior of the surface S, the rays P_j and Q_j have the same image, as well as the rays S_i and U_i, and the rays V_i and R_i. The images of the rays $P_1, \dots, P_{k-1}$ connect the boundary components and the complement in the interior of S of the images of all the rays is a topological disk.

Let $p_1, \dots, p_k$ be the elements of $\partial_\infty \mathbb{H}^2$ that are the extremities of the sides $P_1, \dots, P_{k-1}$: for all $j = 1, \dots, k-1$, P_j goes form p_j to p_{j+1}. The stabilizer of p_j in $\pi_1(S)$ is the subgroup generated by c_j.

The ideal extremities of the fundamental polygon are then, in the counter-clockwise order (cf. Figure 4.2):

$$p_1, \dots, p_k, p_{k-1} \cdot c_k^{-1}, \dots, p_2 \cdot (c_k \cdots c_3)^{-1}, q_g := p_1 \cdot (c_k \cdots c_2)^{-1}, q_g \cdot b_g a_g b_g^{-1}, q_g \cdot b_g a_g, q_g \cdot b_g,$$
$$q_{g-1} := q_g \cdot b_g a_g b_g^{-1} a_g^{-1}, \dots, q_1 := q_2 \cdot b_2 a_2 b_2^{-1} a_2^{-1}, q_1 \cdot b_1 a_1 b_1^{-1}, q_1 \cdot b_1 a_1, q_1 \cdot b_1.$$

The equality $p_1 = q_1 \cdot b_1 a_1 b_1^{-1} a_1^{-1}$ follows from the relation satisfied by $a_1, \dots, a_g$, $b_1, \dots, b_g$, $c_1, \dots, c_k$.

Generators of the corresponding stabilizers are respectively (using the notation $x^y := y^{-1} x y$ and $[x, y] = x y x^{-1} x^{-1}$):

$$c_1, \dots, c_k,\ c_k c_{k-1} c_k^{-1} = c_{k-1}^{c_k^{-1}}, \dots, c_2^{(c_k \cdots c_2)^{-1}},\ c_1^{(c_k \cdots c_1)^{-1}} = c_1^{[a_1,b_1]\cdots[a_g,b_g]},\ c_1^{[a_1,b_1]\cdots[a_g,b_g]a_g},$$
$$c_1^{[a_1,b_1]\cdots[a_g,b_g]a_g b_g},\ c_1^{[a_1,b_1]\cdots[a_g,b_g]a_g b_g a_g^{-1}},\ c_1^{[a_1,b_1]\cdots[a_{g-1},b_{g-1}]}, \dots, c_1^{[a_1,b_1]}, c_1^{a_1}, c_1^{a_1 b_1}, c_1^{a_1 b_1 a_1^{-1}}.$$

Let now $(\mathcal{F}, \sigma)$ be a framed local system and call $(\rho, \{L_1, \dots, L_k\})$ its holonomy (cf. Lemma 4.7). For all $j = 1, \dots, k$, $\rho(c_j)$ fixes the Lagrangian L_j. For $j = 1, \dots, k$, let $z_j \in \widetilde{\mathcal{L}}_n$ be a lift of L_j. Since $\pi_1(S)$ is freely generated by $\{a_i, b_i\}_{i=1,\dots,g} \cup \{c_j\}_{j=2,\dots,k}$, one can choose a lift $\tilde{\rho} \colon \pi_1(S) \to \widetilde{\mathrm{Sp}}(2n, \mathbf{R})$ such that, for all $j = 2, \dots, k$, $\tilde{\rho}(c_j)$ fixes z_j. Thus, for all $j = 2, \dots, k$, $\widetilde{\mathrm{Rot}}(\tilde{\rho}(c_j)) = 0$ and $\widetilde{\mathrm{Rot}}(\tilde{\rho}(c_1)) = 1/2\, m_n(\tilde{\rho}(c_1) \cdot z_1, z_1)$ (see Lemma 2.21). Therefore, by Lemma 4.30, $2T_\rho = -m_n(\tilde{\rho}(c_1) \cdot z_1, z_1)$.

For all $i = 1, \dots, g$, define

$$M_i = \rho([b_i, a_i] \cdots [b_1, a_1]) \cdot L_1 \text{ and } y_i = \tilde{\rho}([b_i, a_i] \cdots [b_1, a_1]) \cdot z_1.$$

The Lagrangians associated (ρ-equivariantly) to the ideal vertices of the fundamental polygon are thus (cf. Remark 4.3):

> $L_1, \dots, L_k$, $\rho(c_k) \cdot L_{k-1}, \dots, \rho(c_k \cdots c_1) L_1 = M_g$, $\rho(b_g a_g^{-1} b_g^{-1}) \cdot M_g$, $\rho(a_g^{-1} b_g^{-1}) \cdot M_g$, $\rho(b_g^{-1}) \cdot M_g$, $M_{g-1}, \dots, M_1$, $\rho(b_1 a_1^{-1} b_1^{-1}) \cdot M_1$, $\rho(a_1^{-1} b_1^{-1}) \cdot M_1$, $\rho(b_1^{-1}) \cdot M_1$.

Lifts of those to $\widetilde{\mathcal{L}}_n$ are hence:

> $z_1, \dots, z_k = \tilde{\rho}(c_k) \cdot z_k$, $\tilde{\rho}(c_k c_{k-1}) \cdot z_{k-1}, \dots, \tilde{\rho}(c_k \cdots c_1) \cdot z_1 = y_g$, $\tilde{\rho}(b_g a_g^{-1} b_g^{-1}) \cdot y_g$, $\tilde{\rho}(a_g^{-1} b_g^{-1}) \cdot y_g$, $\tilde{\rho}(b_g^{-1}) \cdot y_g$, $y_{g-1}, \dots, y_1$, $\tilde{\rho}(b_1 a_1^{-1} b_1^{-1}) \cdot y_1$, $\tilde{\rho}(a_1^{-1} b_1^{-1}) \cdot y_1$, $\tilde{\rho}(b_1^{-1}) \cdot y_1$.

Let now $\mathcal{T}$ be the ideal triangulation of S induced by the ideal triangulation of the fundamental polygon obtained by adding edges between p_1 and every other vertex.

Thanks to Lemma 2.19, one has

$$\begin{aligned}\sum_{T\in\mathcal{T}}\mu^T(\mathcal{F},\sigma) =& \sum_{j=1}^{k-1} m_n(z_j,z_{j+1}) + \sum_{j=k-1}^{1} m_n\big(\tilde\rho(c_k\cdots c_{j+1})\cdot z_{j+1},\tilde\rho(c_k\cdots c_j)\cdot z_j\big)\\ &+\sum_{i=g}^{1}\Big(m_n\big(y_i,\tilde\rho(b_g a_g^{-1}b_g^{-1})\cdot y_i\big)+m_n\big(\tilde\rho(b_g a_g^{-1}b_g^{-1})\cdot y_i,\tilde\rho(a_g^{-1}b_g^{-1})\cdot y_i\big)\\ &\qquad +m_n\big(\tilde\rho(a_g^{-1}b_g^{-1})\cdot y_i,\tilde\rho(b_g^{-1})\cdot y_i\big)+m_n\big(\tilde\rho(b_g^{-1})\cdot y_i,\tilde\rho(a_g^{-1}b_g^{-1}a_g^{-1}b_g^{-1})\cdot y_i\big)\Big).\end{aligned}$$

By the equivariance and the antisymmetry of the Souriau index, for all $i=1,\dots,g$, one has

$$m_n\big(\tilde\rho(a_g^{-1}b_g^{-1})\cdot y_i,\tilde\rho(b_g^{-1})\cdot y_i\big) = m_n\big(\tilde\rho(b_g a_g^{-1}b_g^{-1})\cdot y_i,y_i\big) = -m_n\big(y_i,\tilde\rho(b_g a_g^{-1}b_g^{-1})\cdot y_i\big)$$

and

$$\begin{aligned}m_n\big(\tilde\rho(b_g^{-1})\cdot y_i,\tilde\rho(a_g^{-1}b_g^{-1}a_g^{-1}b_g^{-1})\cdot y_i\big) &= m_n\big(\tilde\rho(a_g b_g^{-1})\cdot y_i,\tilde\rho(b_g^{-1}a_g^{-1}b_g^{-1})\cdot y_i\big)\\ &= -m_n\big(\tilde\rho(b_g^{-1}a_g^{-1}b_g^{-1})\cdot y_i,\tilde\rho(a_g b_g^{-1})\cdot y_i\big).\end{aligned}$$

Thus the last sum cancels out. For similar reasons

$$\sum_{j=2}^{k-1}\big(m_n(z_j,z_{j+1})+m_n(\tilde\rho(c_k\cdots c_{j+1})\cdot z_{j+1},\tilde\rho(c_k\cdots c_j)\cdot z_j)\big)=0;$$

it remains

$$\sum_{T\in\mathcal{T}}\mu^T(\rho,D) = m_n(z_1,z_2)+m_n(z_2,\tilde\rho(c_1)\cdot z_1)$$

applying Proposition 2.18:

$$\begin{aligned}&= m_n(\tilde\rho(c_1)\cdot z_1,z_1)+\mu_n(D(c_1),D(c_2),D(c_1))\\ &= -2T_\rho+0.\end{aligned}$$

□

4.13. **Maximal framed local system.** We return now to the general situation (i.e. R can be empty or not).

Based on Corollary 4.27, we introduce the following definition.

Definition 4.31. A framed local system $(\mathcal{F},\sigma)$ is said to be *maximal* if there exists a triangulation $\mathcal{T}$ of S for which $\mu^T(\mathcal{F},\sigma)=n$ for every triangle T of $\mathcal{T}$.

The space of maximal framed symplectic local systems is denoted by $\mathcal{M}^{\mathrm{f}}(S,\mathrm{Sp}(2n,\mathbf{R}))$.

Reasoning similarly to the proof of Lemma 4.28 and with the help of finite coverings of S, one has

Lemma 4.32. *A framed symplectic local system is maximal*

- *if and only if, for every triangulation $\mathcal{T}$ and every triangle T of $\mathcal{T}$, $\mu^T(\mathcal{F},\sigma)=n$;*
- *if and only if, for every 3-gon τ, the Maslov index of $f_\tau(\mathcal{F},\sigma)$ is maximal.*

Since every arc is an edge of at least one 3-gon, one gets:

Corollary 4.33. *A maximal framed local system is α-tranverse for every arc α.*

Remark 4.34. When $r>0$, the notion of maximality really depends on the pair $(\mathcal{F},\sigma)$ and not only on $\mathcal{F}$; this is flagrant when S is a disk. Summing the Maslov indices for all the triangles of $\mathcal{T}$, one could define a notion of Toledo number in this wider setting; the interest of this notion seems nevertheless limited.

Remark 4.35. When $\bar{S}$ is a disk with $r \geqslant 3$ points in its boundary, the space $\mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ is isomorphic to the space $\mathrm{Conf}^r(\mathcal{L}_n)$. The space $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ of maximal framed local system is sometimes denoted $\mathrm{Conf}^{r+}(\mathcal{L}_n)$ and its elements are called *positive* configurations of Lagrangians. When r is 3, $\mathrm{Conf}^{3+}(\mathcal{L}_n)$ contains one element: the orbit of triples with Maslov index equal to n. When r is 4, $\mathrm{Conf}^{4+}(\mathcal{L}_n)$ is the space of positive quadruples introduced above (cf. Definition 3.5).

4.14. **Maximal framed twisted local systems.** Any nonvanishing vector field $\vec{x}$ gives rise to an identification $\mathrm{Loc}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \simeq \mathrm{Loc}^{\mathrm{f}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$. By the same arguments as in Section 4.10, the image of $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ by this isomorphism does not depend on $\vec{x}$. It will be denoted by $\mathcal{M}^{\mathrm{f}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$ and its elements are call *maximal framed twisted* local systems.

5. Local systems on quivers and their framings

In order to control and parametrize the spaces introduced in the previous section we relate them to local systems on graphs embedded in the surface S. For every ideal triangulation we define a quiver (i.e. an oriented graph), which will be used systematically in the rest of the paper. We recall the correspondence between local systems on S and local systems on this quiver. We introduce twisted local systems and framed local systems on this quiver which then provide the counterparts of the moduli spaces introduced in the previous section.

5.1. **The quiver $\Gamma_{\mathcal{T}}$.** Let $\mathcal{T}$ be an ideal triangulation of S (cf. Section 4.3).

We consider the following quiver (aka. oriented graph whose oriented edges are called arrows) $\Gamma_{\mathcal{T}}$ embedded in the surface S (see Figure 5.1).

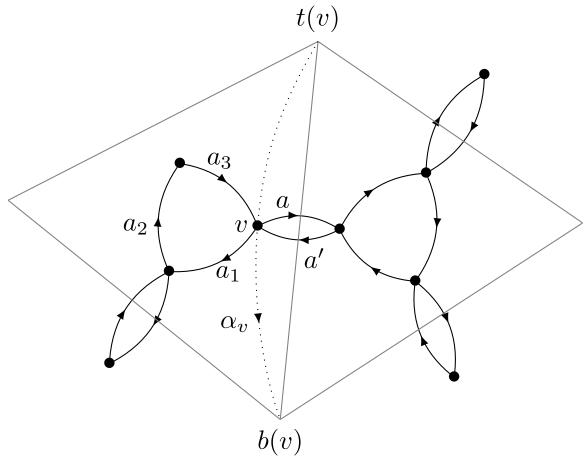

FIGURE 5.1. The quiver $\Gamma_{\mathcal{T}}$; one external edge is drawn (on the upper left), the other 4 edges are internal.

The vertex set V of the quiver $\Gamma_{\mathcal{T}}$ is constructed as follow. For each face f of $\mathcal{T}$ and each (nonoriented) edge of f we add a point inside f near the midpoint of the edge. The r external edges contribute to r vertices of $\Gamma_{\mathcal{T}}$, we will call these vertices *external* as well; the $3|\chi(\bar{S})| + r$ internal edges contribute to $6|\chi(\bar{S})| + 2r$ vertices that we call *internal*.

The arrows of $\Gamma_{\mathcal{T}}$ have two origins:

(E) for every internal (nonoriented) edge e of $\mathcal{T}$, we have two vertices v, v' of $\Gamma_{\mathcal{T}}$, belonging respectively to triangles T, T'. We add the cycle $C_e = \{a, a'\}$ made up of two arrows a, a'

connecting v to v' and v' to v within $T \cup T'$. The arrows a and a' both intersect the edge e exactly once.

(F) for every face f of $\mathcal{T}$ we have the three vertices of $\Gamma_{\mathcal{T}}$ that are contained in f. We add the cycle $C_f = \{a_1, a_2, a_3\}$ made up of three arrows a_1, a_2, a_3 connecting these three vertices within the face f and clockwise oriented.

We denote by A_2 the arrows that are part of such a 2-cycle and by A_3 the arrows that are part of a 3-cycle. The set A of arrows of $\Gamma_{\mathcal{T}}$ is then the disjoint union $A_2 \sqcup A_3$.

As the quiver $\Gamma_{\mathcal{T}}$ is embedded in S, it has a natural ribbon structure. We consider the boundary components of the ribbon graph $\Gamma_{\mathcal{T}}$ *not* containing external vertices. There are three possible types, which can be indexed by:

(1) the compact boundary components of S. There are k such cycles.
(2) the edges of $\mathcal{T}$; for each (nonoriented) edge e, we have the 2-cycle C_e. There are $3|\chi(\bar{S})|+r$ such cycles.
(3) the faces of $\mathcal{T}$; for each face f, one has the 3-cycle C_f. There are $2|\chi(\bar{S})| + r$ such cycles.

Each vertex v of $\Gamma_{\mathcal{T}}$ has, by construction, a closest (nonoriented) edge e in $\mathcal{T}$ and the endpoints of this edge are then contained in a component of ∂S. The orientation of S can be used to distinguish between them:

If v is to the left of e, then $t(v) \in \pi_0(\partial S)$ is the component up and $b(v)$ is down (see Figure 5.1).

For every internal vertex v in V, we denote by α_v the (oriented) edge of $\mathcal{T}$ such that $r(\alpha_v)$ "goes through" v; precisely α_v is the edge next to v going from the component $t(v)$ to the component $b(v)$.

Remark 5.1. Using the vector field $\vec{x}_{\mathcal{T}}$ (cf. Section 4.7), the quiver $\Gamma_{\mathcal{T}}$ is also embedded in $T'S$.

5.2. **Local systems on a quiver.** Local systems on quivers can be described concretely.

Let Γ be a quiver with vertex set V and arrow set A. Any arrow a in A has a start point $v^-(a)$ and an endpoint $v^+(a)$.

Definition 5.2. A *rank-d local system* on Γ is the data of

(1) for each $v \in V$, a vector space F_v of dimension d;
(2) for each $a \in A$, a linear isomorphism $g_a \colon F_{v^-(a)} \to F_{v^+(a)}$.

Such a local system will be denoted simply as a pair $(\{F_v\}_{v\in V}, \{g_a\}_{a\in A})$ or (F_v, g_a).

Remark 5.3. Representations of quivers are a more general notion: the dimensions of the vector spaces can vary and the linear morphisms are not necessarily invertible.

Two local systems (F_v, g_a) and (F'_v, g'_a) are *equivalent* if there is a family of linear isomorphisms $\psi_v \colon F_v \to F'_v$ $(v \in V)$ such that, for every arrow a of Γ, the following diagram commutes:

$$\begin{array}{ccc} F_{v^-(a)} & \xrightarrow{g_a} & F_{v^+(a)} \\ {\scriptstyle \Psi_{v^-(a)}}\downarrow & & \downarrow{\scriptstyle \Psi_{v^+(a)}} \\ F'_{v^-(a)} & \xrightarrow{g'_a} & F_{v^+(a)} \end{array} .$$

We will denote by $\mathrm{Loc}(\Gamma, \mathrm{GL}(d, \mathbf{R}))$ the moduli space of rank-d local systems on Γ.

From time to time, the elements g_a will be called *transition maps.*

A *symplectic local system* (of rank $d = 2n$) is a local system (F_v, g_a) and the data of a symplectic form ω_v on F_v (for each v in V) such that the maps g_a are symplectic isomorphisms. Equivalence of symplectic local systems is defined likewise. Local systems with respect to a general Lie group are discussed in Section 10.2.

5.3. **Local systems and basis.** Given a local system (F_v, g_a) on Γ it is often more convenient to give the elements g_a in matrix coordinates, i.e. to fix a basis $\mathbf{b}_v$ of F_v for each v in V. Hence a rank-d local system can be thought of as the data $((F_v, \mathbf{b}_v), G_a)$, where for each a in A, G_a belongs to $\mathrm{GL}(d, \mathbf{R})$; G_a is the matrix of the linear transformation $g_a \colon F_{v^-(a)} \to F_{v^+(a)}$ in the bases $\mathbf{b}_{v^-(a)}$ and $\mathbf{b}_{v^+(a)}$.

We sometimes call G_a *transition matrices.*

We will often say that the family of bases $(\mathbf{b}_v)_{v\in V}$ is a *basis* of the local system (F_v, g_a).

One can even retain only the data $\{G_a\}_{a\in A}$ for a local system on Γ. However, it will be useful to keep the complete data $(F_v, \mathbf{b}_v, G_a)$ (or even $(F_v, \mathbf{b}_v, g_a, G_a)$) in order to later give the framing or a decoration and to have a precise understanding of equivalent local systems. We call these tuples also a local system on Γ.

Lemma 5.4. *Let (F_v, g_a) and (F'_v, g'_a) be two rank-d local systems on Γ. Then the following statements are equivalent:*

(i) (F_v, g_a) and (F'_v, g'_a) are equivalent local systems.
(ii) For every basis $(\mathbf{b}_v)$ of (F_v, g_a), there is a basis $(\mathbf{b}'_v)$ of (F'_v, g'_a) such that, for every arrow a in Γ, the matrices G_a and G'_a of g_a and g'_a respectively are equal.
(iii) There are a basis $(\mathbf{b}_v)$ of (F_v, g_a) and a basis $(\mathbf{b}'_v)$ of (F'_v, g'_a) such that, for all arrow a, the matrices G_a and G'_a are equal.

The proofs of (i) ⇒ (ii) ⇒ (iii) ⇒ (i) are straightforward.

The lemma will be used in the following (equivalent) form:

Lemma 5.5. *Two local systems $(F_v, \mathbf{b}_v, g_a, G_a)$ and $(F'_v, \mathbf{b}'_v, g'_a, G'_a)$ are equivalent if and only if there is a basis $(\mathbf{c}_v)_{v\in V}$ of (F_v, g_a) such that, for all arrow a of Γ, G'_a is the matrix of the map $g_a \colon F_{v^-(a)} \to F_{v^+(a)}$ in the bases $\mathbf{c}_{v^-(a)}$ and $\mathbf{c}_{v^+(a)}$.*

Of course, our main concern here is symplectic local systems, for those we work with symplectic bases of F_v given as pairs $(\mathbf{e}_v, \mathbf{f}_v)$ and the matrices G_a will be elements of the symplectic group $\mathrm{Sp}(2n, \mathbf{R})$.

5.4. **Local systems on $\Gamma_{\mathcal{T}}$ and local systems on S.** A local systems on S can be restricted to $\Gamma_{\mathcal{T}}$ and thus gives rise to a local system on $\Gamma_{\mathcal{T}}$. As the embedding $\Gamma_{\mathcal{T}} \hookrightarrow S$ induces an epimorphism between the fundamental groups, the restriction map

$$\mathrm{Loc}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathrm{Loc}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$$

is injective. In order to describe its image, we need to keep track which cycles in $\Gamma_{\mathcal{T}}$ become trivial in S. This leads to the following definition:

Definition 5.6. A local system (F_v, g_a) on $\Gamma_{\mathcal{T}}$ is said S-*compatible* if

(1) for all $\{a, a'\}$ as in (E) (Section 5.1, p. 28), $g_a g_{a'} = \mathrm{Id}$.
(2) for all $\{a_1, a_2, a_3\}$ as in (F) (Section 5.1, p. 29), $g_{a_3} g_{a_2} g_{a_1} = \mathrm{Id}$.

The moduli space of S-compatible local systems is denoted $\mathrm{Loc}_S(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$.

Proposition 5.7. *The restriction map gives an isomorphism between $\mathrm{Loc}(S, \mathrm{Sp}(2n, \mathbf{R}))$ and $\mathrm{Loc}_S(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$.*

5.5. **Twisted local systems on $\Gamma_{\mathcal{T}}$.** As with local systems on S, one can use local sysems on $\Gamma_{\mathcal{T}}$ to understand δ-twisted local systems on $T'S$. Since $\Gamma_{\mathcal{T}}$ can be embedded in $T'S$ (Remark 5.1) we can defined δ-twisted local systems on $\Gamma_{\mathcal{T}}$ as follows.

Definition 5.8. A symplectic local system (F_v, g_a) on $\Gamma_{\mathcal{T}}$ is said to be δ-*twisted* (or *twisted*) if

(1) for all $\{a, a'\}$ as in (E), p. 28, $g_a g_{a'} = -\mathrm{Id}$.

(2) for all $\{a_1, a_2, a_3\}$ as in (F), p. 29, $g_{a_3} g_{a_2} g_{a_1} = -\mathrm{Id}$.

Then, along the same line than the results above, denoting by $\mathrm{Loc}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ the moduli space of δ-twisted local systems, we get:

Proposition 5.9. *The restriction map induces an isomorphism between* $\mathrm{Loc}_\delta(S, \mathrm{Sp}(2n, \mathbf{R}))$ *and* $\mathrm{Loc}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$.

5.6. **Framed twisted local systems on $\Gamma_{\mathcal{T}}$.** In the above correspondence (Theorem 5.9), we would like to keep track of framed twisted local systems. For this we will use the ribbon structure on $\Gamma_{\mathcal{T}}$, in particular the two maps $t \colon V \to \pi_0(\partial S)$ and $b \colon V \to \pi_0(\partial S)$ defined in Section 5.1; recall that V is the set of vertices of $\Gamma_{\mathcal{T}}$.

A framing of a twisted local system on $\Gamma_{\mathcal{T}}$ is an equivariant data of Lagrangians, more precisely:

Definition 5.10. Let (F_v, g_a) be a symplectic δ-twisted local system. A *framing* of (F_v, g_a) is the data of two families of Lagrangians L_v^t and L_v^b of F_v (v in V), such that, for every arrow a of $\Gamma_{\mathcal{T}}$, one has:

- if a is in A_2, i.e. if a crosses an edge of $\mathcal{T}$, $g_a(L^t_{v^-(a)}) = L^b_{v^+(a)}$ and $g_a(L^b_{v^-(a)}) = L^t_{v^+(a)}$.
- if a is in A_3, i.e. if the arrow a is contained in a face of $\mathcal{T}$, $g_a(L^b_{v^-(a)}) = L^t_{v^+(a)}$.

The tuple (F_v, g_a, L_v^t, L_v^b) is called a *framed twisted local system.*

Equivalence of framed twisted local systems has to take care of the framings:

Definition 5.11. We define two framed symplectic twisted local systems (F_v, g_a, L_v^t, L_v^b) and $(F'_v, g'_a, L'^t_v, L'^b_v)$ to be *equivalent* if there is a family $\{\psi_v \colon F_v \to F'_v\}_{v \in V}$ that is an isomorphism between the symplectic local systems (F_v, g_a) and (F'_v, g'_a), and such that, for all v in V, $\psi_v(L_v^t) = L'^t_v$ and $\psi_v(L_v^b) = L'^b_v$.

The restriction of a framed twisted local system on S gives a framed twisted local system on $\Gamma_{\mathcal{T}}$: starting from a framed twisted symplectic local system $(\mathcal{F}, \sigma)$, the restriction gives first a local system (F_v, g_a) on $\Gamma_{\mathcal{T}}$, then, for every vertex v in V, using parallel transport along the arc α_v described in Section 5.1, and the section σ defined at the extremities of α_v, we obtain two Lagrangians L_v^t and L_v^b in F_v. One has

Proposition 5.12. *The restriction maps induces an isomorphism between* $\mathrm{Loc}^{\mathrm{d}}_\delta(S, \mathrm{Sp}(2n, \mathbf{R}))$ *and the moduli space* $\mathrm{Loc}^{\mathrm{d}}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ *of framed twisted symplectic local systems on* $\Gamma_{\mathcal{T}}$.

We will denote by $\mathcal{M}^{\mathrm{f}}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ the framed local systems on $\Gamma_{\mathcal{T}}$ corresponding to maximal ones.

5.7. **Symplectic bases of framed local systems.** Local systems on $\Gamma_{\mathcal{T}}$ are easier to tackle when given as a family $\{G_a\}_{a \in A}$ of elements of $\mathrm{Sp}(2n, \mathbf{R})$, i.e. the linear isomorphism g_a associated to an arrow of the quiver is given by an explicit transition matrix. As explained in Section 5.3, this is equivalent to equipping each of the spaces F_v with a symplectic basis.

In the presence of a framing, it is desirable that the symplectic bases are adapted to the Lagrangians. This motivates the following definition:

Definition 5.13. A framed δ-twisted symplectic local system (F_v, g_a, L_v^t, L_v^b) is called *transverse* if, for all $v \in V$, the two Lagrangians L_v^t and L_v^b are transverse: $F_v = L_v^t \oplus L_v^b$.

A symplectic basis $\{(\mathbf{e}_v, \mathbf{f}_v)\}_{v \in V}$ is said to *generate* (or *generating*) the framing if, for all v, $L_v^t = \mathrm{Span}(\mathbf{e}_v)$ and $L_v^b = \mathrm{Span}(\mathbf{f}_v)$; in this case (L_v^t, L_v^b) will be called the *generated framing.*

Lemma 5.14. *(1) A framed δ-twisted symplectic local system on $\Gamma_{\mathcal{T}}$ is transverse if and only if the corresponding framed twisted local system on S is transverse with respect to $\mathcal{T}$ (Definition 4.20).*

(2) A framed δ-twisted symplectic local system on $\Gamma_{\mathcal{T}}$ is transverse if and only if it admits a generating basis.

In other words, the isomorphism of Proposition 5.12 restricts to an isomorphism between the space $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R}))$ of transverse framed δ-twisted symplectic local system and the space $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(S, \mathrm{Sp}(2n,\mathbf{R}))$ of transverse framed twisted local systems on S.

For a framed twisted local system with a generating symplectic basis, the matrices $\{G_a\}_{a\in A}$ have a special form:

Lemma 5.15. *Let (F_v, g_a, L^t_v, L^b_v) be a transverse framed symplectic twisted local system on $\Gamma_{\mathcal{T}}$. Let $\{(\mathbf{e}_v, \mathbf{f}_v)\}_v$ be a generating symplectic basis and, for each $a \in A$, G_a the matrix of the transformation g_a in these bases.*

Then, if the arrow a belongs to A_2, the matrix G_a has the form $\left(\begin{smallmatrix} 0 & -{}^T\!B^{-1} \\ B & 0 \end{smallmatrix}\right)$ for some B in $\mathrm{GL}(n,\mathbf{R})$; and if the arrow a belongs to A_3, the matrix G_a has the form $\left(\begin{smallmatrix} MC & -{}^T\!C^{-1} \\ C & 0 \end{smallmatrix}\right)$ for some C in $\mathrm{GL}(n,\mathbf{R})$ and M in $\mathrm{Sym}(n,\mathbf{R})$.

Proof. Let v_1 and v_2 be the start point and endpoint of a. In the case when $a \in A_2$ the conditions

$$g_a(\mathrm{Span}(\mathbf{e}_{v_1})) = g_a(L^t_{v_1}) = L^b_{v_2} = \mathrm{Span}(\mathbf{f}_{v_2}) \text{ and } g_a(\mathrm{Span}(\mathbf{f}_{v_1})) = \mathrm{Span}(\mathbf{e}_{v_2})$$

translate into the following conditions on G_a, with $(\mathbf{e}_0, \mathbf{f}_0)$ the standard symplectic basis of $\mathbf{R}^{2n}$:

$$\mathrm{Span}(\mathbf{e}_0 \cdot G_a) = \mathrm{Span}(\mathbf{f}_0) \text{ and } \mathrm{Span}(\mathbf{f}_0 \cdot G_a) = \mathrm{Span}(\mathbf{e}_0)$$

which precisely means that the matrix G_a has the announced off diagonal form.

In the second case ($a \in A_3$), the condition on G_a is $\mathrm{Span}(\mathbf{f}_0 \cdot G_a) = \mathrm{Span}(\mathbf{e}_0)$ and this implies the result. □

Conversely, if a twisted local system is given by a family $\{G_a\}_{a\in A}$ of symplectic matrices satisfying the special form given in the conclusions of the preceding lemma then it comes from a unique generating symplectic basis on a framed local system.

Proposition 5.16. *Let $\{G_a\}_{a\in A}$ be a family of elements of $\mathrm{Sp}(2n,\mathbf{R})$ such that*

(1) $\{G_a\}_a$ is a δ-twisted local system (i.e. for each edge e of $\mathcal{T}$, the product of the G_as along the cycle C_e is $-\mathrm{Id}$ and for each triangle f of $\mathcal{T}$, the product of the G_as along C_f is $-\mathrm{Id}$ — see Section 5.5.)

(2) for all arrow a of $\Gamma_{\mathcal{T}}$ in A_2, the matrix G_a has the form $\left(\begin{smallmatrix} 0 & -{}^T\!B^{-1} \\ B & 0 \end{smallmatrix}\right)$ for some $B \in \mathrm{GL}(n,\mathbf{R})$.

(3) for all arrow a in A_3, the matrix G_a has the form $\left(\begin{smallmatrix} MC & -{}^T\!C^{-1} \\ C & 0 \end{smallmatrix}\right)$ for some $C \in \mathrm{GL}(n,\mathbf{R})$ and $M \in \mathrm{Sym}(n,\mathbf{R})$.

Then there is a unique up to equivalence framed δ-twisted symplectic local system on $\Gamma_{\mathcal{T}}$ with generating symplectic basis such that the transition matrices are precisely the G_a ($a \in A$).

Proof. Uniqueness is clear since the bases on two local systems will entirely determine the isomorphism between them.

Let us prove existence. For each v in V, set $(F_v, (\mathbf{e}_v, \mathbf{f}_v))$ to be the space $\mathbf{R}^{2n}$ with its canonical symplectic basis. Let (L^t_v, L^b_v) the Lagrangians generated by these bases. Finally let $g_a \colon F_{v^-(a)} \to F_{v^+(a)}$ the symplectic isomorphism whose matrix is G_a (again in the given bases). The hypothesis on G_a precisely means that the family (L^t_v, L^b_v) is a framing of the δ-twisted local system (F_v, g_a). □

As a conclusion, a framed transverse local system (symplectic and δ-twisted) is entirely determined by a family $\{G_a\}_{a\in A}$ satisfying the hypothesis of the proposition. By a little abuse of terminology, such a family $\{G_a\}_{a\in A}$ will be called a transverse framed twisted local system.

5.8. **Configurations associated with framed local systems on $\Gamma_{\mathcal{T}}$.** The maps from local systems to configurations of Section 4.11 can be seen as defined on the space of framed twisted local systems on $\Gamma_{\mathcal{T}}$.

When v is a vertex of $\Gamma_{\mathcal{T}}$, hence contained in some triangle f of $\mathcal{T}$, we can define a 3-gon $\tau\colon (\mathbb{D},\mu_3)\to(S,\partial S)$ (cf. Section 4.11) such that $\tau(1)=t(v)$. We will rather denote

$$t_v := f_\tau\colon \operatorname{Loc}^{\mathrm{d}}_\delta(\Gamma_{\mathcal{T}},\mathrm{Sp}(2n,\mathbf{R}))\longrightarrow \operatorname{Conf}^3(\mathcal{L}_n),$$

the map defined previously.

When v is an internal vertex, the map q_{α_v} will be denoted by q_v (cf. Section 5.1 for the definition of α_v). When θ is an internal angle, we obtain as well a map $\operatorname{Loc}^{\mathrm{d}}_\delta(\Gamma_{\mathcal{T}},\mathrm{Sp}(2n,\mathbf{R}))\to\operatorname{Conf}^5(\mathcal{L}_n)$ that comes from the map defined in Section 4.11 and will be denoted as well c_θ.

For a triangle T of $\mathcal{T}$, we will denote by the same letter μ^T the Maslov index of the triangle T.

5.9. **Maslov indices, cross ratios, and Toledo number for framed local systems.** We end this section by explaining how to calculate the mentioned quantities for a framed δ-twisted local system $x=(F_v,g_a,L^t_v,L^b_v)$ given explicitely as a family of matrices $\{G_a\}_{a\in A}$ thanks to $\{(\mathbf{e}_v,\mathbf{f}_v)\}_v$ a generating symplectic basis.

Explicitely, the given data is:

- for every arrow a in A_2, $G_a=\begin{pmatrix}0 & -{}^T\!B_a^{-1}\\ B_a & 0\end{pmatrix}$ for an element $B_a\in\mathrm{GL}(n,\mathbf{R})$. These elements satisfy $B_a=B_{a'}$ for every cycle $\{a,a'\}$ in A_2.
- for every arrow a in A_3, $G_a=\begin{pmatrix}M_aC_a & -{}^T\!C_a^{-1}\\ C_a & 0\end{pmatrix}$ for a symmetric matrix M_a and an element C_a in $\mathrm{GL}(n,\mathbf{R})$. These satisfy the identities given in Remark 5.18 below.

Lemma 5.17. *Let T be a triangle of $\mathcal{T}$ and $a\in A_3$ an arrow of $\Gamma_{\mathcal{T}}$ contained in T, set $v=v^+(a)$. Then the configuration $t_v(x)\in\operatorname{Conf}^3(\mathcal{L}_n)$ is the class of $\big(\operatorname{Span}(\mathbf{e}_0),\operatorname{Span}(\mathbf{f}_0),\operatorname{Span}(\mathbf{e}_0+\mathbf{f}_0\cdot M_a)\big)$, where $(\mathbf{e}_0,\mathbf{f}_0)$ is the standard basis of $\mathbf{R}^{2n}$. The Maslov index $\mu^T(x)$ is equal to the signature of the symmetric matrix M_a.*

Proof. The triple of Lagrangians is

$$L_1=\operatorname{Span}(\mathbf{e}_v),\ L_2=\operatorname{Span}(\mathbf{f}_v),\ \text{and } M=g_a(\operatorname{Span}(\mathbf{e}_{v^-(a)})).$$

Since $g_a(\mathbf{e}_{v^-(a)})=\mathbf{e}_v\cdot M_aC_a+\mathbf{f}_v\cdot C_a=\big(\mathbf{e}_v\cdot M_a+\mathbf{f}_v\big)\cdot C_a$, one has $M=\operatorname{Span}(\mathbf{e}_v\cdot M_a+\mathbf{f}_v)$ and the result follows identifying $(F_v,\mathbf{e}_v,\mathbf{f}_v)$ with $(\mathbf{R}^{2n},\mathbf{e}_0,\mathbf{f}_0)$ (cf. Section 2.9). ☐

Remark 5.18. As a consequence, the signature of the M_a's have to coincide along the arrows of a 3-cycle in A_3. This property can be also obtained from the equality

$$\begin{pmatrix}M_cC_c & -{}^T\!C_c^{-1}\\ C_a & 0\end{pmatrix}\begin{pmatrix}M_bC_b & -{}^T\!C_b^{-1}\\ C_b & 0\end{pmatrix}\begin{pmatrix}M_aC_a & -{}^T\!C_a^{-1}\\ C_c & 0\end{pmatrix}=-\operatorname{Id}$$

which holds if and only if $M_cC_c{}^T\!C_b^{-1}C_a=\operatorname{Id}$, $M_c^{-1}=C_cM_b\,{}^T\!C_c$, and $M_b^{-1}=C_bM_a\,{}^T\!C_b$. Another consequence is that the symmetric matrices M_a, M_b, and M_c must be invertible.

Corollary 5.19. *Let $A'\subset A_3$ be a subset containing exactly one arrow of every 3-cycle in A_3. Then*

(1) if $R=\varnothing$ (so that the Toledo number is defined on the space $\operatorname{Loc}^{\mathrm{d}}_\delta(\Gamma_{\mathcal{T}},\mathrm{Sp}(2n,\mathbf{R}))$), $T_x=-\frac{1}{2}\sum_{a\in A'}\operatorname{sgn}(M_a)=-\frac{1}{6}\sum_{a\in A_3}\operatorname{sgn}(M_a)$;

(2) x is maximal if and only if for all a in A', M_a is positive definite, and if and only if, for all $a\in A_3$, M_a is positive definite.

Proof. Using the preceeding lemma, this results from Theorem 4.26 and its corollary 4.27 or the lemma 4.32. □

We also have an understanding of the cross ratios which we state under the additional condition that $M_a = \mathrm{Id}$ for all $a \in A_3$.

Lemma 5.20. *Suppose furthermore that $M_a = \mathrm{Id}$ for all $a \in A_3$, then for all internal vertex v of $\Gamma_{\mathcal{T}}$, the cross ratio of the quadruple $q_v(x)$ is the class of the symmetric matrix $B_a\,{}^TB_a$, where a is one of the arrow in A_2 whose endpoint is the vertex v.*

Proof. Let w be the vertex of $\Gamma_{\mathcal{T}}$ connected to v by a. The two symplectic vector spaces F_v and F_w are equipped with symplectic basis $(\mathbf{e}_v, \mathbf{f}_v)$ and $(\mathbf{e}_w, \mathbf{f}_w)$ respectively.

By Lemma 5.17. The triple of Lagrangians in F_v is $\mathrm{Span}(\mathbf{e}_v)$, $\mathrm{Span}(\mathbf{f}_v)$, $\mathrm{Span}(\mathbf{e}_v + \mathbf{f}_v)$, and similarly in F_w.

The matrix of $g_a \colon F_w \to F_v$ in the symplectic basis is $\left(\begin{smallmatrix} 0 & -{}^TB_a^{-1} \\ B_a & 0 \end{smallmatrix}\right)$, and $g_a(\mathrm{Span}(\mathbf{e}_w)) = \mathrm{Span}(\mathbf{f}_v \cdot B_a) = \mathrm{Span}(\mathbf{f}_v)$, $g_a(\mathrm{Span}(\mathbf{f}_w)) = \mathrm{Span}(\mathbf{e}_v)$, and $g_a(\mathrm{Span}(\mathbf{e}_w + \mathbf{f}_w)) = \mathrm{Span}(-\mathbf{e}_v \cdot {}^TB_a^{-1} + \mathbf{f}_v \cdot B_a) = \mathrm{Span}(\mathbf{e}_v - \mathbf{f}_v \cdot B_a\,{}^TB_a)$, hence the result. □

The angle invariant (cf. Section 3.3) can also be determined easily in the case when the matrices B_a are diagonal.

Lemma 5.21. *Suppose that $M_a = \mathrm{Id}$ for all a in A_3 and that $B_a \in \Delta_n$ for all a in A_2. Let $\theta = (\alpha, \alpha')$ be an internal angle (α and α' are internal edges of the triangulation and are sides of the same triangle). Let b be the arrow in A_3 that is "next" to θ, i.e. b joins α and α'.*

Then the matrix C_b is orthogonal and represents the angle invariant of the quintuple $c_\theta(x)$.

Proof. The fact that C_b is orthogonal follows directly from the hypothesis on the M_as (cf. Remark 5.18). Let $v = v^-(b)$ and $v' = v^+(b)$ so that v is next to α and v' is next to α'. The hypothesis on the matrices B_as implies that the symplectic basis $(\mathbf{e}_v, \mathbf{f}_v)$ (resp. $(\mathbf{e}_v, \mathbf{f}_v)$) is in standard position with respect to the quadruple $q_v(x)$ (resp. $q_{v'}(x)$). Since $\mathbf{e}_{v'} = -\mathbf{f}_v \cdot C_b$, the result follows from the definition of the angle invariant (Section 3.3). □

6. $\mathcal{X}$-coordinates for maximal representations

In this section we introduce positive $\mathcal{X}$-coordinates. They give a parameterization of the space of maximal representations: we restrict our attention here to this special case because the definition is significantly simpler than in the general case. We refer the reader to Section 9 for the definition of general $\mathcal{X}$-coordinates.

6.1. A space of coordinates.

Let $\mathcal{T}$ be an ideal triangulation of S and let $\Gamma_{\mathcal{T}}$ the quiver constructed in Section 5.1. The set of arrows (oriented edges) of $\Gamma_{\mathcal{T}}$ is denoted by A. Recall that arrows of $\Gamma_{\mathcal{T}}$ are of two types: $A = A_2 \sqcup A_3$ where A_2 is the set of arrows crossing an edge of $\mathcal{T}$ and A_3 is the set of arrows that are contained in one triangle of $\mathcal{T}$.

We denote by $\mathcal{X}^+(\mathcal{T}, n)$ the space of tuples

$$x = \left(\{x(a)\}_{a \in A_2}, \{x(a)\}_{a \in A_3}\right) \in \left(\mathrm{Sym}^+(n, \mathbf{R})\right)^{A_2} \times \mathrm{O}(n)^{A_3}$$

such that

- for every 3-cycle (a_1, a_2, a_3) in A_3, $x(a_3)x(a_2)x(a_1) = \mathrm{Id}$.
- for every 2-cycle (a, a') in A_2, $x(a) = x(a')$.

To every $x \in \mathcal{X}^+(\mathcal{T}, n)$, we associate a framed δ-twisted symplectic local system $\mathrm{hol}_{\mathcal{T}}^{\mathcal{X},+}(x)$ as follows: the local system is given by the matrices $\{G_a(x)\}_{a \in A}$, where

- for every a in A_2 (using the square root function on the set of positive definite symmetric matrices)
$$G_a(x) = \begin{pmatrix} 0 & -x(a)^{-1/2} \\ x(a)^{1/2} & 0 \end{pmatrix};$$
- and for every a in A_3, $G_a(x) = \begin{pmatrix} x(a) & -x(a) \\ x(a) & 0 \end{pmatrix}$.

This local system $\{G_a(x)\}$ is δ-twisted: this follows from the conditions on the $\{z(a)\}_{a\in A_3}$ and the fact that $\left(\begin{smallmatrix} 0 & -\Lambda^{-1} \\ \Lambda & 0 \end{smallmatrix}\right)^2 = -\operatorname{Id}$ for every matrix Λ. Therefore, by Proposition 5.16, the family $\{G_a(x)\}_{a\in A}$ defines a framed δ-twisted symplectic local system on $\Gamma_{\mathcal{T}}$, which is $\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T}}(x)$.

Lemma 6.1. *For every $x \in \mathcal{X}^+(\mathcal{T},n)$, $\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T}}(x)$ is a maximal framed δ-compatible local system on $\Gamma_{\mathcal{T}}$.*

Proof. This follows from Corollary 5.19 since, in the notation of the corollary, $M_a = \operatorname{Id}$ for every $a \in A_3$. □

In fact, only the equivalence class of the framed local system $\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T}}(x)$ is well defined, so we get an element in $\mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, \operatorname{Sp}(2n,\mathbf{R}))$. Thus we have a well defined map

$$\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T}} \colon \mathcal{X}^+(\mathcal{T},n) \longrightarrow \mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, \operatorname{Sp}(2n,\mathbf{R})). \tag{6.1}$$

We will see shortly (Proposition 6.5) that this map is surjective. This map is instead not injective, and the rest of this section will be an investigation of this lack of injectivity. We will restrict the map $\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T}}$ to some suitable subsets of $\mathcal{X}^+(\mathcal{T},n)$, and we will describe the fibers of the restricted maps.

6.2. **Positive $\mathcal{X}$-coordinates.** Let Δ_n be the set of diagonal matrices with positive nondecreasing entries.

Definition 6.2 (Positive $\mathcal{X}$-coordinates)**.** We will call *positive $\mathcal{X}$-coordinates* and denote by $\mathcal{X}^+_{\Delta}(\mathcal{T},n)$ the space of tuples
$$x = \left(\{x(a)\}_{a\in A_2}, \{x(a)\}_{a\in A_3}\right) \in \mathcal{X}^+(\mathcal{T},n)$$
such that for every a in A_2, $x(a)$ belongs to Δ_n.

Sometimes we will say that x is a *system of positive $\mathcal{X}$-coordinates* of rank n on $(S,\mathcal{T})$.

Let A' be a subset of A_3 such that, for each 3-cycle C in A_3, $C \cap A'$ has 2 elements and let E be a subset of A_2 containing exactly one element in every 2-cycle in A_2. Then the map
$$\begin{aligned} \mathcal{X}^+_{\Delta}(\mathcal{T},n) &\longrightarrow \Delta_n^E \times \mathrm{O}(n)^{A'} \\ x &\longmapsto (\{x(a)\}_{a\in E}, \{x(a)\}_{a\in A'}) \end{aligned}$$
is a diffeomorphism. Note that $\sharp E = 3|\chi(\bar{S})| + r$, and $\sharp A' = 2\sharp\mathcal{T} = 4|\chi(\bar{S})| + 2r$.

Since $\mathcal{X}^+_{\Delta}(\mathcal{T},n) \subset \mathcal{X}^+(\mathcal{T},n)$, by restriction of the map $\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T}}$ we have a map
$$\operatorname{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta} \colon \mathcal{X}^+_{\Delta}(\mathcal{T},n) \longrightarrow \mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, \operatorname{Sp}(2n,\mathbf{R})).$$

The geometric significance of the positive $\mathcal{X}$-coordinates is the following statement:

Lemma 6.3. *Let x be in $\mathcal{X}^+_{\Delta}(\mathcal{T},n)$ and let $(\mathcal{F},\sigma) = \operatorname{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta}(x)$. Then, for every a in A_2, the cross ratio of the quadruple $q_{v^+(a)}(\mathcal{F},\sigma)$ is $x(a)^{-1}$ and, for every internal angle $\theta = \{\alpha,\alpha'\}$, the angle invariant of the quintuple $c_\theta(\mathcal{F},\sigma)$ is $x(a)$ where a is the arrow in A_3 going from α to α'.*

Proof. The statement concerning the cross ratio results from Lemma 5.20, the one concerning the angle invariant results from Lemma 5.21. □

6.3. **Maximal framed symplectic local systems.** In this subsection, we will prove the following result:

Theorem 6.4. *The map* $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta}$

(1) is onto the space of maximal twisted framed representations $\mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R}))$.

(2) For x *and* x' *in* $\mathcal{X}^+_\Delta(\mathcal{T},n)$*, one has* $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta}(x) = \mathrm{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta}(x')$ *if and only if*
- *For every* a *in* A_2, $x(a) = x'(a)$
- *There is a family* $(r_v)_{v\in V}$ *of orthogonal matrices such that:*
  - *For all* $a \in A_2$*, one has* $r_{v^-(a)} = r_{v^+(a)}$*, and* $r_{v^-(a)}$ *commutes with* $x(a)$.
  - *for every* a *in* A_3, $x'(a) = r_{v^+(a)} x(a) r^{-1}_{v^-(a)}$.

The surjectivity of the map $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta}$ is the following proposition.

Proposition 6.5. *Let* $x = (F_v, g_a, L^t_v, L^b_v)$ *be a maximal framed* δ*-twisted symplectic local system on the quiver* $\Gamma_{\mathcal{T}}$.

Then there exists a symplectic basis $(\mathbf{e}_v, \mathbf{f}_v)_{v\in V}$ *generating the framing and, denoting* $G_a \in \mathrm{Sp}(2n,\mathbf{R})$ *the matrices of* g_a*, one has:*

(1) For every a *in* A_2*, then* $G_a = \begin{pmatrix} 0 & -\Lambda_a^{-1} \\ \Lambda_a & 0 \end{pmatrix}$ *for some* $\Lambda_a \in \Delta_n$.

(2) For each $a \in A_3$, $G_a = \begin{pmatrix} u_a & -u_a \\ u_a & 0 \end{pmatrix}$ *for some* $u_a \in \mathrm{O}(n)$.

We will say that a symplectic basis of $\{F_v\}_{v\in V}$ is in *standard* $\mathcal{X}$-position if it satisfies the properties of the above conclusion. The tuples of matrices $((\Lambda_a^2)_{a\in A_2}, (u_a)_{a\in A_3})$ is then an element of $\mathcal{X}^+_\Delta(\mathcal{T},n)$ (see Lemma 3.8) and will be called *the system of positive coordinates* associated with the basis $(\mathbf{e}_v, \mathbf{f}_v)$.

Proof. Let first v be an external vertex of $\Gamma_{\mathcal{T}}$. Let a be the arrow in A_3 such that $v = v^+(a)$. As the local system x is maximal, the Maslov index of the triple $(L^t_v, g_a(L^t_{v^-(a)}), L^b_v)$ is maximal and there is thus a symplectic basis $(\mathbf{e}_v, \mathbf{f}_v)$ such that $L^t_v = \mathrm{Span}(\mathbf{e}_v)$, $g_a(L^t_{v^-(a)}) = \mathrm{Span}(\mathbf{e}_v + \mathbf{f}_v)$, and $L^b_v = \mathrm{Span}(\mathbf{f}_v)$

Let $E \subset A_2$ be a subset containing exactly one of the two arrows for every cycle in A_2. The set V of vertices of $\Gamma_{\mathcal{T}}$ is the disjoint union of the doubletons $\{v^-(a), v^+(a)\}$ for $a \in E$.

Let $a \in E$ and call $v = v^+(a)$, $v' = v^-(a)$. The quadruple $(L_1, M_1, L_2, M_2) = q_v(x)$ is then positive since the two triples $t_v(x)$ and $t_{v'}(x)$ are maximal. By Proposition 3.6.(1), there is a symplectic basis $(\mathbf{e}_v, \mathbf{f}_v)$ in standard position with respect to that quadruple: there is a matrix Λ in Δ_n so that $L_1 = \mathrm{Span}(\mathbf{e}_v)$, $L_2 = \mathrm{Span}(\mathbf{f}_v)$, $M_1 = \mathrm{Span}(\mathbf{e}_v + \mathbf{f}_v)$ and $M_2 = \mathrm{Span}(\mathbf{e}_v - \mathbf{f}_v \cdot \Lambda)$.

By construction of the bases, the matrices of the g_a, $a \in A'$, have the desired form. The property $g_{a_1} g_{a_2} = -\mathrm{Id}$ for all 2-cycle (a_1, a_2) in A_2 implies that the same holds for every $a \in A_2$. The fact that the matrices of the g_a ($a \in A_3$) have the requested form is a consequence of Lemma 3.8. □

Using Lemma 5.5, the description of the fibers of $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T},\Delta}$ is a consequence of the following statement.

Proposition 6.6. *Let* $x = (F_v, g_a, L^t_v, L^b_v)$ *be a maximal framed* δ*-twisted symplectic local system on the quiver* $\Gamma_{\mathcal{T}}$.

Let $(\mathbf{e}_v, \mathbf{f}_v)$ *and* $(\mathbf{e}'_v, \mathbf{f}'_v)$ *be two symplectic bases in standard* $\mathcal{X}$*-position. And let* x *and* x' *be the two associated systems of positive coordinates.*

Let $\{\psi_v\}_{v\in V}$ *be the family of elements of* $\mathrm{Sp}(2n,\mathbf{R})$ *defined by the equalities:* $(\mathbf{e}_v, \mathbf{f}_v) = (\mathbf{e}'_v, \mathbf{f}'_v) \cdot \psi_v$ $(v \in V)$.

Then

- *For all a in A_2, $x'(a) = x(a)$.*
- *For all v in V, there is an orthogonal matrix u_v such that $\psi_v = \begin{pmatrix} u_v & 0 \\ 0 & u_v \end{pmatrix}$;*
- *For every arrow a in A_2, then $u_{v^+(a)} = u_{v^-(a)}$ and $u_{v^+(a)}$ commutes with $x(a)$.*
- *For every a in A_3, $x'(a) = u_{v^+(a)} x(a) u_{v^-(a)}^{-1}$.*

Proof. By Lemma 5.20, the cross ratio of the quadruple $q_{v^+(a)}(x)$ is given by the matrices $x(a)$ and $x'(a)$, hence the equality.

Let v be a vertex of $\Gamma_{\mathcal{T}}$.

The symplectic bases $(\mathbf{e}_v, \mathbf{f}_v)$ and $(\mathbf{e}'_v, \mathbf{f}'_v)$ are in standard position. Applying Proposition 3.6, there is an element u_v in $\mathrm{O}(n)$ such that $(\mathbf{e}_v, \mathbf{f}_v) = (\mathbf{e}'_v \cdot u_v, \mathbf{f}'_v \cdot u_v)$.

Using the fact that the transition matrices with respect to the basis $(\mathbf{e}_v, \mathbf{f}_v)$ are deduced from the other by multiplying by the matrices ψ_v, we get the other statements. □

6.4. **Reparameterization of the $\mathcal{X}$-coordinates.** When $R = \varnothing$, the space $\mathcal{X}^+_\Delta(\mathcal{T}, n)$ gives a generically finite-to-one parameterization of the space of maximal framed local systems; this is not anymore the case when $R \neq \varnothing$. In the present subsection, we introduce first another parameterization of the space of maximal framed local systems that is always finite-to-one.

In order to describe another parameterization for maximal representations we make use of a spanning tree $\mathcal{S}$ of the graph $\Gamma_{\mathcal{T}}/A_2$, i.e. the graph where all the arrows in A_2 have been collapsed. Thus $\mathcal{S}$ can be thought of as a subset of A_3, though the orientation of the arrows is not relevant here. Geometrically, the graph $\Gamma_{\mathcal{T}}/A_2$ is obtained in the following way: for every (nonoriented) edge e of $\mathcal{T}$ mark a midpoint in e and, for every triangle f of $\mathcal{T}$, connect the three midpoints of the three sides of f. Thus the number of vertices of $\Gamma_{\mathcal{T}}/A_2$ is $3|\chi(\bar{S})| + 2r$ and its number of edges is $3\sharp\mathcal{T} = 6|\chi(\bar{S})| + 3r$.

The spanning tree has to connect all the vertices, hence the cardinality of the set of edges of $\mathcal{S}$ is equal the number of vertices minus one, i.e. $\sharp\mathcal{S} = 3|\chi(\bar{S})| + 2r - 1$ (in a tree the number of edges is the number of vertices minus 1).

We fix an arrow a_0 in A_2 and denote by $\mathcal{X}^+_{\mathcal{S},a_0}$ the subset of $\mathcal{X}^+(\mathcal{T}, n)$ consisting of the tuples

$$y = (\{y(a)\}_{a \in A_2}, \{y(a)\}_{a \in A_3}) \in \mathrm{Sym}^+(n, \mathbf{R})^{A_2} \times \mathrm{O}(n)^{A_3}$$

such that

- $y(a_0)$ belongs to Δ_n;
- for all a in $\mathcal{S}$, $y(a) = \mathrm{Id}$.

Of course, the equations $y(a) = y(a')$ ($\{a, a'\}$ 2-cycle in A_2) and $y(a_3)y(a_2)y(a_1) = \mathrm{Id}$ ($\{a_1, a_2, a_3\}$ cycle in A_3) are satisfied.

We choose another set $\mathcal{R} \subset A_3$, disjoint from $\mathcal{S}$ and such that $\mathcal{S} \sqcup \mathcal{R}$ contains two of the arrows of every 3-cycle in A_3 and subset $E \subset A_2$ containing a_0 and containing exactly one of the arrows of every 2-cycle in A_2. With this we have

$$\begin{aligned} \mathcal{X}^+_{\mathcal{S},a_0} &\longrightarrow \Delta_n \times \mathrm{Sym}^+(n, \mathbf{R})^{E \smallsetminus \{a_0\}} \times \mathrm{O}(n)^{\mathcal{R}} \\ y &\longmapsto \big(y(a_0), \{y(a)\}_{a \in E \smallsetminus \{a_0\}}, \{y(a)\}_{a \in \mathcal{R}}\big) \end{aligned}$$

is a diffeomorphism. Recall that $\sharp(E \smallsetminus \{a_0\}) = 3|\chi(\bar{S})| + r - 1$ and $\sharp\mathcal{R} = |\chi(\bar{S})| + 1$ (since $\sharp\mathcal{S} = 3|\chi(\bar{S})| + 2r - 1$ and $\sharp(\mathcal{S} \cup \mathcal{R}) = 2\sharp\mathcal{T} = 4|\chi(\bar{S})| + 2r$).

We have $\mathcal{X}^+_{\mathcal{S},a_0} \subset \mathcal{X}^+(\mathcal{T}, n)$, hence by restriction of the map $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T}}$ we have a map

$$\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S},a_0} \colon \mathcal{X}^+_{\mathcal{S},a_0} \longrightarrow \mathcal{M}^{\mathrm{f}}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R})).$$

Theorem 6.7. *The map* $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S},a_0}$ *is onto the space of maximal framed* δ*-twisted local system* $\mathcal{M}^{\mathrm{f}}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R})) \subset \mathrm{Loc}^{\mathrm{f}}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R}))$. *Two elements* y *and* y' *in* $\mathcal{X}^+_{\mathcal{S},a_0}$ *have the same holonomy if and only if,* $y(a_0) = y'(a_0)$ *and there is an element* $u \in \mathrm{O}(n)$ *commuting with* $y(a_0)$ *such that, for every* a *in* A, $y'(a) = uy(a)u^{-1}$.

6.5. **Image and fibers of the holonomy map.** This section is devoted to the proof of Theorem 6.7.

Proposition 6.8. *For all* x *in* $\mathcal{M}^{\mathrm{f}}_\delta(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R}))$ *there is* y *in* $\mathcal{X}^+_{\mathcal{S},a_0}$ *such that* $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S},a_0}(y) = x$.

Proof. By Theorem 6.4, there is $z \in \mathcal{X}^+(\mathcal{T}, n)$ such that $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T}}(z) = x$ and $z(a_0) \in \Delta_n$. Let $(F_v, g_a, L^t_v, L^b_v, (\mathbf{e}_v, \mathbf{f}_v))$ be the local system with generating symplectic basis defined by z.

By induction on $\sharp\mathcal{S}'$ we prove:

> For every subtree $\mathcal{S}'$ of $\mathcal{S}$ (connected to a_0), there is z' in $\mathcal{X}^+(\mathcal{T}, n)$ and a generating basis $(\mathbf{e}'_v, \mathbf{f}'_v)$ of (F_v, g_a, L^t_v, L^b_v) such that $z'(a_0)$ belongs to Δ_n, the matrices of g_a in these bases are $\{G_a(z')\}$ and z' satisfies the following: for all $a \in \mathcal{S}'$, $z'(a) = \mathrm{Id}$.

For $\mathcal{S}' = \varnothing$, we set $z' = z$.

Now let $\mathcal{S}' = \mathcal{S}'' \cup \{b\}$ for some $b \in A_3 \smallsetminus \mathcal{S}''$ (and with $\mathcal{S}''$ a tree, i.e. b is a leaf of $\mathcal{S}'$). Let z'' and $(\mathbf{e}''_v, \mathbf{f}''_v)$ be given by the induction step for $\mathcal{S}''$. We denote by v_0 and v_1 the extremities of b. For definiteness, suppose that v_0 is connected to $\mathcal{S}''$ and $v_0 = v^-(b)$, hence $v^+(b) = v_1$. In the case where v_1 is an internal vertex, let v_2 be the vertex of $\Gamma_{\mathcal{T}}$ connected to it by a cycle $\{c, c'\}$ of A_2. Since b is a leaf of $\mathcal{S}'$, the vertex v_2 is not the extremity of any of the arrows in $\mathcal{S}'$, so that we can change the basis in F_{v_2} without affecting the transitions matrices for $a \in \mathcal{S}'$.

We now define the symplectic basis $(\mathbf{e}'_v, \mathbf{f}'_v)$. For $v \neq v_1, v_2$, set $(\mathbf{e}'_v, \mathbf{f}'_v) = (\mathbf{e}''_v, \mathbf{f}''_v)$. For v_1, we set:

$$\begin{aligned}(\mathbf{e}'_{v_1}, \mathbf{f}'_{v_1}) &= g_a(\mathbf{e}'_{v_0}, \mathbf{f}'_{v_0}) \cdot \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}^{-1} \\ &= g_a(\mathbf{e}''_{v_0}, \mathbf{f}''_{v_0}) \cdot \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} = (\mathbf{e}''_{v_1}, \mathbf{f}''_{v_1}) \cdot \begin{pmatrix} z''(b) & -z''(b) \\ z''(b) & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} \\ &= (\mathbf{e}''_{v_1}, \mathbf{f}''_{v_1}) \cdot \begin{pmatrix} z''(b) & 0 \\ 0 & z''(b) \end{pmatrix} = (\mathbf{e}''_{v_1} \cdot z''(b), \mathbf{f}''_{v_1} \cdot z''(b)).\end{aligned}$$

Finally, when v_1 is internal, let $(\mathbf{e}'_{v_2}, \mathbf{f}'_{v_2}) = (\mathbf{e}''_{v_2} \cdot z''(b), \mathbf{f}''_{v_2} \cdot z''(b))$. It is easily checked that the transition matrices are $\{G_a(z')\}$ with $z' \in \mathcal{X}^+(\mathcal{T}, n)$ such that $z'(a) = z''(a)$ for all arrows a distinct from b (and from c or c') and $z'(b) = \mathrm{Id}$ (and, in the case when v_1 is internal, $z'(c') = z'(c) = z''(b)z''(c)z''(b)^{-1}$ again symmetric). □

The uniqueness in Theorem 6.7 follows from:

Proposition 6.9. *Let* $x = (F_v, g_a, L^t_v, L^b_v)$ *be a maximal framed local system. Let* $(\mathbf{e}_v, \mathbf{f}_v)$ *be a symplectic basis generating the framing and such that the transition matrices are* $\{G_a(y)\}$ *for some* y *in* $\mathcal{X}^+_{\mathcal{S},a_0}$.

(1) *Then, for all* $u \in \mathrm{O}(n)$ *commuting with* $y(a_0)$*, the symplectic basis* $(\mathbf{e}_v \cdot u, \mathbf{f}_v \cdot u)$ *generates the framing, the tuple* y' *defined by* $y'(a) = uy(a)u^{-1}$ $(a \in A)$ *is in* $\mathcal{X}^+_{\mathcal{S},a_0}$*, and the transition matrices with respect to this symplectic basis are* $\{G_a(y')\}$.

(2) *If* $(\mathbf{e}'_v, \mathbf{f}'_v)$ *is another generating basis and if there is an element* y' *in* $\mathcal{X}^+_{\mathcal{S},a_0}$ *such that the transition matrices are* $\{G_a(y')\}$*, then there is an element* $u \in \mathrm{O}(n)$*, commuting with* $y(a_0)$ *and such that, for every vertex* v *in* $\Gamma_{\mathcal{T}}$*,* $(\mathbf{e}'_v, \mathbf{f}'_v) = (\mathbf{e}_v \cdot u, \mathbf{f}_v \cdot u)$.

Proof. The first part of the statement is clear. Let us address the second part. Applying Proposition 3.6, up to conjugating by an element commuting with $y(a_0)$ we can assume that the generating bases $(\mathbf{e}_v, \mathbf{f}_v)$ and $(\mathbf{e}'_v, \mathbf{f}'_v)$ coincide at one of the endpoint v of a_0. We show now that they coincide at every vertex by traveling in the spanning tree $\mathcal{S}$. For this it is enough to note that

- If the basis coincide at one of the extremity of an arrow a in $\mathcal{S}$ then they coincide at the other extremity (since the transition matrices coincide).
- Let $a \in A_2$ and suppose that the basis coincide at $v = v^-(a)$. At $w = v^+(a)$, since the basis are generating, we get, from the analysis in Lemma 5.17,
$$\mathrm{Span}(\mathbf{e}_v) = \mathrm{Span}(\mathbf{e}'_v),\ \mathrm{Span}(\mathbf{f}_v) = \mathrm{Span}(\mathbf{f}'_v),\ \mathrm{Span}(\mathbf{e}_v + \mathbf{f}_v) = \mathrm{Span}(\mathbf{e}'_v + \mathbf{f}'_v).$$
This implies that there is $u \in \mathrm{O}(n)$ such that $(\mathbf{e}'_v, \mathbf{f}'_v) = (\mathbf{e}_v \cdot u, \mathbf{f}_v \cdot u)$. Thus $y'(a) = y(a)u$ and the matrices $y(a)$, $y'(a)$ are symmetric and positive definite. Uniqueness in the polar decomposition implies that $u = \mathrm{Id}$ and the bases coincide at w. □

6.6. **Over-parameterizations.** In the preceding subsection, the space of parameters has the same dimension as the space of framed representations and the holonomy maps were generically finite-to-one. In this section we give an (over)-parametrization which has too many parameters, but becomes injective after taking the quotient by the action of a group.

Let us denote by $\mathcal{X}^+_{\mathcal{S}}(\mathcal{T}, n)$ the subset of $\mathcal{X}^+(\mathcal{T}, n)$ consisting of tuples z with $z(a) = \mathrm{Id}$ for all a in the spanning tree $\mathcal{S}$. We denote by $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S}}$ the map to framed local systems; it is the restriction of $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{T}}$. It has already been observed that $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S}}(z)$ is maximal (Lemma 6.1).

Theorem 6.10. *The map* $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S}} \colon \mathcal{X}^+_{\mathcal{S}}(\mathcal{T}, n) \to \mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ *is onto. Two tuples* z *and* z' *have the same image by* $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S}}$ *if and only there is* u *in* $\mathrm{O}(n)$ *such that* $z'(c) = uz(c)u^{-1}$ *for all* c *in* A.

Proof. Surjectivity is assured by Theorem 6.7 since $\mathcal{X}^+_{\mathcal{S},a_0} \subset \mathcal{X}^+_{\mathcal{S}}(\mathcal{T}, n)$. It is clear that if two elements of $\mathcal{X}^+_{\mathcal{S}}(\mathcal{T}, n)$ are conjugate by an element in $\mathrm{O}(n)$ then they will give rise to the equivalent framed local system. Conversely, one can always use such a conjugation to bring the elements back to $\mathcal{X}^+_{\mathcal{S},a_0}$. Theorem 6.7 then gives the result. □

7. Topology of the space of maximal framed representations

From the parameterizations introduced in the previous section, we construct explicit retractions of the space of maximal framed representations to subspaces which have a simpler parametrization and whose topological properties are easier to describe. This enables us to give the homotopy type of the space of maximal framed local systems as well as some topological properties of the space of maximal representations.

This section relies on the previous sections for the parameterizations of maximal framed twisted local systems and on Section 4.9 for their correspondence with nontwisted local systems. Hence all results here will be expressed for the space $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$.

7.1. **Subspace of "singular" local systems.** We consider here subsets of maximal local systems where the holonomy elements have as few different eigenvalues as possible.

For any framed $\mathrm{SL}(2, \mathbf{R})$-local system $(\mathcal{F}, \sigma)$, there is an associated framed $\mathrm{Sp}(2n, \mathbf{R})$-local system $(\breve{\mathcal{F}}, \breve{\sigma})$ defined as follows. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to $\mathrm{SL}(2, \mathbf{R})$ and $\ell = \mathrm{Span}(e \cdot x + f \cdot y)$ belongs to $\mathbf{R}\mathbb{P}^1$, with (e, f) being the canonical basis of $\mathbf{R}^2$ and $x, y \in \mathbf{R}$, then we get an embedding $\mathrm{SL}(2, \mathbf{R}) \to \mathrm{Sp}(2n, \mathbf{R})$ (the diagonal embedding) and a $\mathrm{SL}(2, \mathbf{R})$-equivariant application $\mathbf{R}\mathbb{P}^1 \to \mathcal{L}_n$ by setting
$$\breve{g} := \begin{pmatrix} a\,\mathrm{Id} & b\,\mathrm{Id} \\ c\,\mathrm{Id} & d\,\mathrm{Id} \end{pmatrix}, \text{ and } \breve{\ell} := \mathrm{Span}(\mathbf{e} \cdot x + \mathbf{f} \cdot y).$$

The local system $\breve{\mathcal{F}}$ is obtained from $\mathcal{F}$ by "composing" with the above homomorphism and the section $\breve{\sigma}$ is the composition of σ with the natural map $\mathcal{F}_{\mathbf{R}\mathbb{P}^1} \to \breve{\mathcal{F}}_{\mathcal{L}_n}$ induced by the map $\mathbf{R}\mathbb{P}^1 \to \mathcal{L}_n$.

Moreover the formula

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, u\right) \longmapsto \begin{pmatrix} au & bu \\ cu & du \end{pmatrix}$$

defines an homomorphism $\mathrm{SL}_2(\mathbf{R}) \times \mathrm{O}(n) \to \mathrm{Sp}(2n, \mathbf{R})$ and the above embedding $\mathbf{R}\mathbb{P}^1 \to \mathcal{L}_n$ is also equivariant with respect to this homomorphism. This implies that if $\mathcal{E}$ is a $\mathrm{O}(n)$-local system on S, then $\mathcal{F} \otimes \mathcal{E}$ is a twisted $\mathrm{Sp}(2n, \mathbf{R})$-local system, and $\breve{\sigma}$ is a framing of this local system.

The following is easily checked:

Lemma 7.1. *The framed local system $(\mathcal{F} \otimes \mathcal{E}, \breve{\sigma})$ is transverse with respect to a triangulation $\mathcal{T}$ if and only if $(\mathcal{F}, \sigma)$ is tranverse with respect to $\mathcal{T}$.*

The local system $(\mathcal{F} \otimes \mathcal{E}, \breve{\sigma})$ is maximal if and only $(\mathcal{F}, \sigma)$ is maximal.

As a consequence, there is a well defined application

$$\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R})) \times \mathrm{Loc}(S, \mathrm{O}(n)) \to \mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$$

whose image is denoted by $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$. Equally there is an application

$$\mathcal{M}(S, \mathrm{SL}(2, \mathbf{R})) \times \mathrm{Loc}(S, \mathrm{O}(n)) \to \mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$$

whose image is denoted by $\mathcal{M}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$

Last let us introduce a "base point" in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R}))$. For a triangulation $\mathcal{T}$, let $(\mathcal{F}_{\mathcal{T}}, \sigma_{\mathcal{T}})$ be the framed twisted local system associated with the element x in $\mathcal{X}^+_{\Delta_n}(\mathcal{T}, 1)$ such that, for all $c \in A$, $x(c) = 1$. We will denote by

$$\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)})$$

the subset of $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$ which is the image of $\{(\mathcal{F}_{\mathcal{T}}, \sigma_{\mathcal{T}})\} \times \mathrm{Loc}(S, \mathrm{O}(n))$. Also we denote by

$$\mathcal{D}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)}) \tag{7.1}$$

the subset of $\mathcal{M}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$ which is the image of $\{(\mathcal{F}_{\mathcal{T}})\} \times \mathrm{Loc}(S, \mathrm{O}(n))$.

Theorem 7.2. *(1) The subspaces $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)})$ and $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$ are strong deformation retracts of $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))$.*

(2) For every spanning tree $\mathcal{S}$, $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)})$ (respectively $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$) is the image by $\mathrm{hol}^{\mathcal{X},+}_{\mathcal{S}}$ of the subset of $\mathcal{X}^+_{\mathcal{S}}(\mathcal{T}, n)$ consisting of the elements z such that, for every arrow a in A_2, $z(a) = \mathrm{Id}$ (respectively $z(a) \in \mathbf{R}_{>0} \mathrm{Id}$).

(3) The spaces $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)})$ and $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O}(n))$ are respectively diffeomorphic to $\mathrm{O}(n)^{|\chi(\bar{S})|+1} / \mathrm{O}(n)$ (quotient by simultaneous conjugation) and to $\mathbf{R}^{3|\chi(\bar{S})|+r} \times (\mathrm{O}(n)^{|\chi(\bar{S})|+1} / \mathrm{O}(n))$.

7.2. **Coordinates for the subspaces.** Using an analysis similar to the one in Section 5, the space $\mathrm{Loc}(S, \mathrm{O}(n))$ also admits parameterizations via local systems on the quiver $\Gamma_{\mathcal{T}}$. We will use the following result: the holonomy map gives a surjective map

$$\mathrm{hol}\colon \mathcal{U} \longrightarrow \mathrm{Loc}(S, \mathrm{O}(n))$$

where $\mathcal{U}$ is the subset of u in $\mathrm{O}(n)^A$ with $u(a) = \mathrm{Id}$ for all $a \in A_2 \sqcup \mathcal{S}$ and $u(a_3)u(a_2)u(a_1) = \mathrm{Id}$ for every 3-cycle (a_1, a_2, a_3) in A_3.

The maps between the moduli spaces of framed local systems can be promoted to maps between the parameters spaces:

$$\begin{aligned} \mathcal{X}_{\mathcal{S}}^+(\mathcal{T},1) &\longrightarrow \mathcal{X}_{\mathcal{S}}^+(\mathcal{T},n) \\ (z,u) &\longmapsto (z(a)u(a))_{a\in A} \end{aligned}$$

and

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathcal{X}_{\mathcal{S}}^+(\mathcal{T},n) \\ u &\longmapsto (u(a))_{a\in A} \end{aligned}$$

The image of the first map will be denoted by $\mathcal{X}_{\mathcal{S}}^+(\mathrm{SL}(2,\mathbf{R})\otimes \mathrm{O}(n))$. It is the set of tuples z in $\mathcal{X}_{\mathcal{S}}^+(\mathcal{T},n)$ such that $z(a)\in \mathbf{R}_{>0}\,\mathrm{Id}$ for all $a\in A_2$; it is isomorphic to $\mathbf{R}_{>0}^{3|\chi(\bar{S})|+r}\times \mathrm{O}(n)^{|\chi(\bar{S})|+1}$.

The image of the second map is denoted by $\mathcal{X}_{\mathcal{S}}^+(\mathcal{F}_{\mathcal{T}}\otimes \mathrm{O}(n))$. It is the set of tuples z such that $z(a)=\mathrm{Id}$ for all $a\in A_2$; it is isomorphic to $\mathrm{O}(n)^{|\chi(\bar{S})|+1}$.

The above maps are also compatible with the holonomy maps to the moduli spaces. This proves the assertion in Theorem 7.2 concerning the parameterizations of $\mathcal{M}^{\mathrm{f}}(S,\mathrm{SL}(2,\mathbf{R})\otimes \mathrm{O}(n))$ and of $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes \mathrm{O(n)})$ and their topologies.

7.3. **Retractions.** Recall that a subspace A of a topological space X is called a *strong deformation retract* if there exists a strong deformation retraction $H\colon X\times[0,1]\to X$, i.e. a continuous map H such that $H(a,t)=a$ for all a in A and $t\in[0,1]$, $H(x,1)=x$ for all $x\in X$, and $H(x,0)\in A$ for all $x\in X$.

To prove Theorem 7.2, it is thus enough to find $\mathrm{O}(n)$-equivariant retractions of $\mathcal{X}_{\mathcal{S}}^+(\mathcal{T},n)$ on $\mathcal{X}_{\mathcal{S}}^+(\mathcal{F}_{\mathcal{T}}\otimes \mathrm{O}(n))$ and on $\mathcal{X}_{\mathcal{S}}^+(\mathrm{SL}(2,\mathbf{R})\otimes \mathrm{O}(n))$ respectively. Or, more concretely, to find $\mathrm{O}(n)$-equivariant retractions of $\mathrm{Sym}^+(n,\mathbf{R})^{3|\chi(\bar{S})|+r}\times \mathrm{O}(n)^{|\chi(\bar{S})|+1}$ on $\mathrm{O}(n)^{|\chi(\bar{S})|+1}$ and on $\mathbf{R}_{>0}^{3|\chi(\bar{S})|+r}\times \mathrm{O}(n)^{|\chi(\bar{S})|+1}$.

Since the action of $\mathrm{O}(n)$ respects the decompositions into products, it all boils down to finding $\mathrm{O}(n)$-equivariant retractions of $\mathrm{Sym}^+(n,\mathbf{R})$ on $\{\mathrm{Id}\}$ and on $\mathbf{R}_{>0}\,\mathrm{Id}$. As the exponential $\exp:\mathrm{Sym}(n,\mathbf{R})\to \mathrm{Sym}^+(n,\mathbf{R})$ is an $\mathrm{O}(n)$-equivariant diffeomorphism, the question translates now to finding equivariant retractions of $\mathrm{Sym}(n,\mathbf{R})$ on $\{0\}$ and on $\mathbf{R}\,\mathrm{Id}$. The first one is given by the family of linear maps $\mathrm{Sym}(n,\mathbf{R})\to\mathrm{Sym}(n,\mathbf{R})$: $\{t\,\mathrm{Id}\}_{t\in[0,1]}$. The second one is obtained similarly using first that $\mathrm{Sym}(n,\mathbf{R})$ is the direct sum of two $\mathrm{O}(n)$-invariant subspaces: $\mathbf{R}\,\mathrm{Id}$ and $\mathrm{Sym}_0(n,\mathbf{R})$, the subspace of traceless matrices.

7.4. **Connected components.** An immediate consequence of Theorem 7.2 is:

Corollary 7.3.
- *The space $\mathcal{M}^{\mathrm{f}}(S,\mathrm{Sp}(2n,\mathbf{R}))$ has $2^{|\chi(\bar{S})|+1}$ connected components.*
- *The space $\mathcal{M}^{\mathrm{f}}(S,\mathrm{PSp}(2n,\mathbf{R}))$ has $2^{|\chi(\bar{S})|+1}$ connected components if n is even. If n is odd, it is connected.*

A more general statement can be found in Corollary 10.20; parameterizations of the space $\mathcal{M}^{\mathrm{f}}(S,\mathrm{PSp}(2n,\mathbf{R}))$ are also described in Section 10.

In order to get some information about the space of maximal representations we prove:

Theorem 7.4. *If $r=0$ (i.e. $R=\varnothing$ and $S=\bar{S}$), the natural map $\mathcal{M}^{\mathrm{f}}(S,\mathrm{Sp}(2n,\mathbf{R}))\to \mathcal{M}(S,\mathrm{Sp}(2n,\mathbf{R}))$ induces a bijection at the level of connected components.*

Theorem 7.4 will be proved in Section 7.6. In this way we obtain a new proof of the following statement:

Corollary 7.5 ([25, Theorem 4]). *If $r=0$, the number of components of $\mathcal{M}(S,\mathrm{Sp}(2n,\mathbf{R}))$ is $2^{|\chi(S)|+1}$.*

7.5. **Path lifting property.** A continuous map $f\colon X \to Y$ is said to have *the path-lifting property* if for every continuous $\sigma\colon [0,1] \to Y$ and every $x \in f^{-1}(\sigma(0))$ there exists, up to reparameterization, a lift $\tilde{\sigma}$ of σ starting at x, namely $\tilde{\sigma}\colon [0,1] \to X$ is continuous, $\tilde{\sigma}(0) = x$ and there is a continuous, increasing, surjective function $\psi\colon [0,1] \to [0,1]$ such that $f \circ \tilde{\sigma} = \sigma \circ \psi$. This is the case, for example, when f is a covering; this is also the case when, for any σ as above, the space $\sigma^*(f) = \{(t,x) \in [0,1] \times X \mid \sigma(t) = f(x)\}$ is path-connected. A piecewise linear map between simplicial spaces that is surjective and with connected fibers has the path lifting property.

Proposition 7.6. *If $r = 0$, the map $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \to \mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$ has the path-lifting property.*

We will work in this section with spaces of representations instead of local systems (cf. Section 4.4).

Let $\mathrm{Hom}_{\max}(S, \mathrm{Sp}(2n, \mathbf{R}))$ and $\mathrm{Hom}^{\mathrm{f}}_{\max}(S, \mathrm{Sp}(2n, \mathbf{R}))$ the spaces of maximal representations and of framed maximal representations (cf. Lemma 4.7). The proposition is a direct consequence of the following lemmas.

Lemma 7.7. *The map $p\colon \mathrm{Hom}_{\max}(S, \mathrm{Sp}(2n, \mathbf{R})) \to \mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$ has the path lifting property.*

Proof. As these space are real algebraic, connectedness and path-connectedness are equivalent notions here. In fact, the same is true for the spaces $\sigma^*(p)$ for any continuous path $\sigma\colon [0,1] \to \mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$. Actually the fibers of $\sigma^*(p) \to [0,1]$ are connected, since they are orbits for the action of the connected group $\mathrm{Sp}(2n, \mathbf{R})$, and thus the total space is indeed connected. □

Lemma 7.8. *The map $\mathrm{Hom}^{\mathrm{f}}_{\max}(S, \mathrm{Sp}(2n, \mathbf{R})) \to \mathrm{Hom}_{\max}(S, \mathrm{Sp}(2n, \mathbf{R}))$ has the path lifting property.*

Proof. Since we can choose independently the framing at the punctures (see Lemma 4.7), it is enough to answer positively the following problem:

> Let $\mathcal{P} \subset \mathrm{Sp}(2n, \mathbf{R})$ be the set of elements g having at least one invariant Lagrangian and $\mathcal{Q} = \{(g, L) \in \mathcal{P} \times \mathcal{L}_n \mid g \cdot L = L\}$. Then the natural map $\pi\colon \mathcal{Q} \to \mathcal{P}$ has the path lifting property.

Let L_0 be the base point of $\mathcal{L}_n$ and P its stabilizer (see Section 2.1). We then have a surjective map $\mathrm{Sp}(2n, \mathbf{R}) \times P \to \mathcal{Q} \mid (h, p) \mapsto (hph^{-1}, h \cdot L_0)$ and it is enough to prove that its composition with π has the path lifting property, i.e. that $\mathrm{Sp}(2n, \mathbf{R}) \times P \to \mathcal{P} \mid (h,p) \mapsto hph^{-1}$ has the path lifting property. This last map is algebraic so that there exists simplicial structures on $\mathrm{Sp}(2n, \mathbf{R}) \times P$ and on $\mathcal{P}$ such that the map is piecewise linear; since it is as well surjective, it has the path lifting property. □

7.6. **Framing of singular representations.** Any singular representation has a unique framing:

Proposition 7.9. *The map $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)}) \to \mathcal{D}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)})$ (see Equation (7.1)) is an isomorphism.*

Let us first conclude the theorem 7.4:

Proof of Theorem 7.4. The map

$$\pi_0\big(\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))\big) \longrightarrow \pi_0\big(\mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))\big)$$

is surjective since $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R})) \to \mathcal{M}(S, \mathrm{Sp}(2n, \mathbf{R}))$ is surjective. Let us prove its injectivity. By Theorem 7.2, the map

$$\pi_0\big(\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S, \mathrm{SL}(2, \mathbf{R}) \otimes \mathrm{O(n)})\big) \longrightarrow \pi_0\big(\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(2n, \mathbf{R}))\big)$$

is an isomorphism and by Proposition 7.9

$$\pi_0\big(\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})\big)\longrightarrow\pi_0\big(\mathcal{D}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})\big)$$

is also bijective. Hence we are reduced to show that

$$\pi_0\big(\mathcal{D}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})\big)\longrightarrow\pi_0\big(\mathcal{M}(S,\mathrm{Sp}(2n,\mathbf{R}))\big)$$

is injective.

Let $\rho_0,\rho_1\in\mathcal{D}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})$ be connected by a path $\sigma\colon[0,1]\to\mathcal{M}(S,\mathrm{Sp}(2n,\mathbf{R}))$. By Proposition 7.9, there is a lift $\tilde{\sigma}$ of (a reparameterization of) σ to $\mathcal{M}^{\mathrm{f}}(S,\mathrm{Sp}(2n,\mathbf{R}))$. The elements $\tilde{\sigma}(0)$ and $\tilde{\sigma}(1)$ are the unique framings of ρ_0 and ρ_1. Applying Theorem 7.2 again, we get a continuous path in $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})$ between $\tilde{\sigma}(0)$ and $\tilde{\sigma}(1)$, as a result ρ_0 and ρ_1 belongs to the same connected component of $\mathcal{D}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})$. This concludes the injectivity and the proof. □

Proof of Proposition 7.9. Let ρ be a representation in $\mathcal{D}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})$. We have to prove that, for each loop c representing a boundary component, $\rho(c)$ has a unique invariant Lagrangian. By construction (see Section 6.4) and up to conjugation, $\rho(c)$ is a finite product of elements of the form

$$\begin{pmatrix}0&-\mathrm{Id}\\ \mathrm{Id}&0\end{pmatrix}\begin{pmatrix}-C&C\\ -C&0\end{pmatrix}=\begin{pmatrix}C&0\\ -C&C\end{pmatrix}=\begin{pmatrix}C&0\\ 0&C\end{pmatrix}\begin{pmatrix}\mathrm{Id}&0\\ -\mathrm{Id}&\mathrm{Id}\end{pmatrix}=\begin{pmatrix}\mathrm{Id}&0\\ -\mathrm{Id}&\mathrm{Id}\end{pmatrix}\begin{pmatrix}C&0\\ 0&C\end{pmatrix}$$

with $C\in\mathrm{O}(n)$. Hence it is equal to

$$\begin{pmatrix}\mathrm{Id}&0\\ m\,\mathrm{Id}&\mathrm{Id}\end{pmatrix}\begin{pmatrix}B&0\\ 0&B\end{pmatrix}$$

for some $m\neq 0$ and $B\in\mathrm{O}(n)$. Up to taking the inverse and conjugate by a block diagonal matrix, we can assume that $m=1$. The powers of the above matrix M are then

$$M^k=\begin{pmatrix}\mathrm{Id}&0\\ k\,\mathrm{Id}&\mathrm{Id}\end{pmatrix}\begin{pmatrix}B^k&0\\ 0&B^k\end{pmatrix}=\begin{pmatrix}B^k&0\\ kB^k&B^k\end{pmatrix}.$$

Let us write elements in $\mathbf{R}^{2n}$ as pairs (x,y) with x, y in $\mathbf{R}^n$ so that the symplectic form has the expression $\omega((x,y),(x',y'))={}^Txy'-{}^Tyx'$.

Suppose that L is an M-invariant Lagrangian. Suppose that there is x in $\mathbf{R}^n$ with $(x,0)$ in L. For every $k\geqslant 0$, $M^k(x,0)=(B^kx,kB^kx)$ belongs to the Lagrangian L and the relation $\omega(M^k(x,0),M^\ell(x,0))=0$ (k, ℓ in $\mathbf{N}$) says $(\ell-k)^TxB^{\ell-k}x=0$. It follows that, denoting V the B-invariant subspace of $\mathbf{R}^n$ generated by B^kx ($k\geqslant 0$), $B(V)$ is orthogonal to V. This is possible only if $V=0$ and $x=0$.

Let $(\mathbf{e},\mathbf{f})$ denote the standard symplectic basis of $\mathbf{R}^{2n}$. The Lagrangian L is thus transverse to $\mathrm{Span}(\mathbf{e})$ and there exists a unique symmetric matrix P such that $L=\mathrm{Span}(\mathbf{e}P+\mathbf{f})$. The equation $M(L)=L$ translate into $\mathrm{Span}(\mathbf{e}BP+\mathbf{f}B(P+\mathrm{Id}))=\mathrm{Span}(\mathbf{e}P+\mathbf{f})$ which implies that $P+\mathrm{Id}$ is invertible (hence -1 does not belong to the spectrum of P) and the equality $PBP+B=BP$. From this last equality, if x is an eigenvector of P for the eigenvalue λ, then Bx is an eigenvector of P for the eigenvalue $\lambda/(\lambda+1)$. Thus the spectrum of P is invariant by the Möbius transformation $\lambda\mapsto\lambda/(\lambda+1)$. However the only finite invariant set for this transformation is $\{0\}$ and we obtain that $P=0$. It turns out that $L=\mathrm{Span}(\mathbf{f})$, establishing the uniqueness. □

8. Singularities of the space of framed maximal representations into $\mathrm{Sp}(4,\mathbf{R})$

In this section we analyse the singularities of the space of framed maximal $\mathrm{Sp}(4,\mathbf{R})$-local systems. We show that the singular locus corresponds exactly to local systems into subgroups that we describe first.

8.1. **Space of block diagonal local systems.** If (e_1, e_2, f_1, f_2) denotes the canonical basis of $\mathbf{R}^4$, the two subspaces $V = \operatorname{Span}(e_1, f_1)$, $W = \operatorname{Span}(e_2, f_2)$ are in direct sum and this decomposition induces a homomorphism

$$\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}) \longrightarrow \mathrm{Sp}(4,\mathbf{R})$$

as wells as an equivariant map $\mathbf{R}\mathbb{P}^1 \times \mathbf{R}\mathbb{P}^1 \to \mathcal{L}_2$. In particular we get an induced map

$$\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R})) \times \mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R})) \to \mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4,\mathbf{R}))$$

which is a two-to-one ramified covering. We will denote the image of this map by $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}))$.

In Section 7.1, we introduced the subspaces $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}))$ and $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{O}(2))$, and we can as well define $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{SO}(2))$.

8.2. **Singular points.** We will prove the following statement:

Theorem 8.1. *A point x of the space $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4,\mathbf{R}))$ is*

- *a smooth point if and only if x does not belong to the union*
$$\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})) \cup \mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{SO}(2)).$$
- *an orbifold point with isotropy group $\mathbf{Z}/2\mathbf{Z}$ if and only if x belongs to*
$$\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})) \smallsetminus \mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{SO}(2));$$
in which case a neighborhood of x is isomorphic to a neighborhood of 0 in $\mathbf{R}^{6|\chi(\bar{S})|+2r} \times (\mathbf{R}^{4|\chi(\bar{S})|+r}/\{\pm \mathrm{Id}\})$;
- *a non-orbifold singular point if and only if the point x belongs to the connected subspace $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{SO}(2))$; furthermore*
  - *if x belongs to $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{SO}(2)) \smallsetminus \mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}))$, a neighborhood of x is isomorphic to a neighborhood of 0 in the space $\mathbf{R}^{4|\chi(\bar{S})|+r+1} \times (\mathbf{C}^{3|\chi(\bar{S})|+r}/\,\mathrm{U}(1))$.*
  - *if x belongs to $\mathcal{M}^{\mathrm{f}}(S, \mathrm{SL}(2,\mathbf{R}))$, a neighborhood of x is isomorphic to a neighborhood of 0 in the space $\mathbf{R}^{3|\chi(\bar{S})|+r} \times \big(\mathbf{R}^{|\chi(\bar{S})|+1} \times (\mathbf{C}^{3|\chi(\bar{S})|+r}/\,\mathrm{U}(1))\big)/\sigma$ where σ is the involution on the space $\mathbf{R}^{|\chi(\bar{S})|+1} \times (\mathbf{C}^{3|\chi(\bar{S})|+r}/\,\mathrm{U}(1))$ induced by $-\mathrm{Id}$ on the factor $\mathbf{R}^{|\chi(\bar{S})|+1}$ and the complex conjugation on the factor $\mathbf{C}^{3|\chi(\bar{S})|+r}/\,\mathrm{U}(1)$.*

Remark 8.2. The claims about the singularity types in the above theorem are consequences of the following considerations:

(1) The *join* $X \curlyvee Y$ of two topological spaces is the quotient of $X \times [0,1] \times Y$ by the equivalence relation whose classes are $\{x\} \times \{0\} \times Y$, $X \times \{1\} \times \{y\}$, and $\{(x,t,y)\}$ ($x \in X$, $t \in (0,1)$, $y \in Y$). If a neighborhood of a point a in a space A is the cone over X (i.e. the quotient of $[0,1] \times X$ by the equivalence relation whose only nontrivial class is $\{0\} \times X$), and a neighborhood of $b \in B$ is the cone over Y, then a neighborhood of (a,b) in $A \times B$ is the join $X \curlyvee Y$. Joins of spheres are spheres.

(2) The join is also the disjoint union of U, the image in $X \curlyvee Y$ of $X \times [0,1) \times Y$, and of V, the image of $X \times (0,1] \times Y$. The space X is a deformation retract of U, Y is a deformation retract of V and $X \times Y$ is a deformation retract of $U \cap V$. From this and the Mayer–Vietoris long exact sequence, we get the following formula, relating the Euler characteristics χ_F over a field F of these spaces:
$$\chi_F\big(X \curlyvee Y\big) = -\chi_F(X)\chi_F(Y) + \chi_F(X) + \chi_F(Y).$$

(3) A neighborhood of 0 in $\mathbf{R}^{4|\chi(S)|}/\{\pm \mathrm{Id}\}$ is the cone over $\mathbf{R}\mathbb{P}^{4|\chi(S)|-1}$ so that a neighborhood of 0 in $\mathbf{R}^{6|\chi(S)|} \times (\mathbf{R}^{4|\chi(S)|}/\{\pm \mathrm{Id}\})$ is the join $S^{6|\chi(S)|-1} \curlyvee \mathbf{R}\mathbb{P}^{4|\chi(S)|-1}$. The above formula (and the knowledge of the cohomology of real projective spaces) can be used to show that

the Euler characteristic of this join varies with the field F and therefore it cannot be homeomorphic to a sphere. Thus no neighborhood of the points in the second item ot the theorem are homeomorphic to a manifold.

(4) By similar arguments, for the last case in the theorem, the neighborhood of the singularity is either the join of a sphere and the projective space $\mathbf{C}\mathbb{P}^{3|\chi(S)|-1}$ or a quotient of this join by $\mathbf{Z}/2\mathbf{Z}$. From this, again by cohomological argument, it can be seen that the singularity is not homeomorphic to an orbifold singularity.

8.3. **Parameters.** Analogous to the map between moduli spaces mentioned in Section 8.1, there is a map between the parameter spaces of Theorem 6.10:

$$\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},1)\times\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},1)\to\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},2).$$

Writing $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},1)\simeq\mathbf{R}_{>0}^{3|\chi(\bar{S})|+r}\times\mathrm{O}(1)^{|\chi(\bar{S})|+1}$ and similarly for $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},2)$, the map is explicitly given by

$$\begin{aligned}\left(\mathbf{R}_{>0}^{3|\chi(\bar{S})|+r}\times\mathrm{O}(1)^{|\chi(\bar{S})|+1}\right)^2&\longrightarrow\mathrm{Sym}^{+}(2,\mathbf{R})^{3|\chi(\bar{S})|+r}\times\mathrm{O}(2)^{|\chi(\bar{S})|+1}\\(\{\lambda_{j,i}\}_i,\{\epsilon_{j,\ell}\}_\ell)_{j=1,2}&\longmapsto\left(\{\mathrm{diag}(\lambda_{1,i},\lambda_{2,i})\}_i,\{\mathrm{diag}(\epsilon_{1,\ell},\epsilon_{2,\ell})\}_\ell\right).\end{aligned}$$

From this, an element $z=(\{s_i\}_i,\{r_\ell\}_\ell)$ in $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},2)$ represents a framed representation in $\mathcal{M}^{\mathrm{f}}(S,\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R}))$ if, up to replacing z by $r\cdot z$ with r in $\mathrm{O}(2)$, all the coordinates s_i, and r_ℓ of z are diagonal.

In approaching the above theorem, it is useful to consider different coordinates for the space of symmetric positive matrices. The map

$$\begin{aligned}\mathbf{R}\times\mathbf{C}&\longrightarrow\mathrm{Sym}^{+}(2,\mathbf{R})\\(t,a+ib)&\longmapsto\exp\left(t\,\mathrm{Id}+\begin{pmatrix}a&b\\b&-a\end{pmatrix}\right)\end{aligned}$$

is a diffeomorphism. Via this isomorphism, the group $\mathrm{O}(2)$ acts on $\mathbf{R}\times\mathbf{C}$. The action is trivial on the factor $\mathbf{R}$ whereas on the factor $\mathbf{C}$

- the element $\left(\begin{smallmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{smallmatrix}\right)$ acts by multiplication by $e^{2i\theta}$,
- the element $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ acts by $z\mapsto\bar{z}$.

The space $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},2)$ is then $\mathrm{O}(2)$-isomorphic to $\mathbf{R}^{3|\chi(\bar{S})|+r}\times\mathbf{C}^{3|\chi(\bar{S})|+r}\times\mathrm{O}(2)^{|\chi(\bar{S})|+1}$.

Recapping the above and Section 7.1:

Proposition 8.3. *Let z be an element*

$$(\{t_i\}_i,\{z_j\}_j,\{r_\ell\}_\ell)\in\mathbf{R}^{3|\chi(\bar{S})|+r}\times\mathbf{C}^{3|\chi(\bar{S})|+r}\times\mathrm{O}(2)^{|\chi(\bar{S})|+1}\simeq\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T},2),$$

and let x be the corresponding framed maximal representation: $x\in\mathcal{M}^{\mathrm{f}}(S,\mathrm{Sp}(4,\mathbf{R}))$. Then

(1) x belongs to $\mathcal{M}^{\mathrm{f}}(\mathrm{SL}(2,\mathbf{R}))$ if and only if, for all j, $z_j=0$, and, for all ℓ, $r_\ell=\pm\mathrm{Id}$;

(2) x belongs to $\mathcal{M}^{\mathrm{f}}(\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{SO}(2))\smallsetminus\mathcal{M}^{\mathrm{f}}(\mathrm{SL}(2,\mathbf{R}))$ if and only if, for all j, $z_j=0$, and $\{r_\ell\}_\ell\in\mathrm{SO}(2)^{|\chi(\bar{S})|+1}\smallsetminus\{\pm\mathrm{Id}\}^{|\chi(\bar{S})|+1}$;

(3) x belongs to $\mathcal{M}^{\mathrm{f}}(\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R}))\smallsetminus\mathcal{M}^{\mathrm{f}}(\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{SO}(2))$ if and only if, there is $\theta\in\mathbf{R}$ such that, for all j, $z_j\in\mathbf{R}e^{i\theta}$, for all ℓ, $r_\ell\in\left\{\pm\mathrm{Id},\pm\left(\begin{smallmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{smallmatrix}\right)\right\}$, and either $\{z_j\}_j\neq0$ or $\{r_\ell\}_\ell\notin\{\pm\mathrm{Id}\}^{|\chi(\bar{S})|+1}$.

(4) otherwise x does not belong to the union $\mathcal{M}^{\mathrm{f}}(S,\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R}))\cup\mathcal{M}^{\mathrm{f}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{SO}(2))$.

In particular, $\mathcal{M}^{\mathrm{f}}(\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{SO}(2))$ is the image of the connected set $\mathbf{R}^{3|\chi(\bar{S})|+r}\times\{0\}\times\mathrm{SO}(2)^{|\chi(\bar{S})|+1}$ and is thus itself connected. This proves the connectedness claim in the third item of Theorem 8.1.

Remark 8.4. Note that, in the last case, the element x is the holonomy of the element $z' = \left(\begin{smallmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{smallmatrix}\right) \cdot z = (\{t'_i\}, \{z'_j\}, \{r'_\ell\})$ with $z'_j \in \mathbf{R}$, and $r'_\ell \in \left\{\pm \mathrm{Id}, \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\}$.

8.4. **Stabilizers.** We are going to calculate stabilizers of elements in $\mathbf{R}^{3|\chi(\bar{S})|+r} \times \mathbf{C}^{3|\chi(\bar{S})|+r} \times \mathrm{O}(2)^{|\chi(\bar{S})|+1} \simeq \mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)$. In the calculation we make use of the following observations:

- the stabilizer (here the centralizer) of an element $r \in \mathrm{O}(2)$ is equal to $\mathrm{O}(2)$ if and only if $r = \pm \mathrm{Id}$;
- the stabilizer of an element $r \in \mathrm{O}(2)$ is equal to $\mathrm{SO}(2)$ if and only if $r \in \mathrm{SO}(2) \smallsetminus \pm\{\mathrm{Id}\}$;
- the stabilizer of an element $r \in \mathrm{O}(2)$ is finite if and only if $r \in \mathrm{O}(2) \smallsetminus \mathrm{SO}(2)$; in this case the stabilizer is equal to $\{\pm \mathrm{Id}, \pm r\}$;
- The stabilizer of an element $z \in \mathbf{C}$ is:
  - equal to $\mathrm{O}(2)$ if $z = 0$;
  - equal to $\left\{\pm \mathrm{Id}, \pm\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)\right\}$ if $z \in \mathbf{R}^* e^{i\theta}$.

As a consequence:

Proposition 8.5. *Let $z = (\{t_i\}_i, \{z_j\}_j, \{r_\ell\}_\ell) \in \mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)$. Then the stabilizer of z in $\mathrm{O}(2)$ is equal to*

(1) $\mathrm{O}(2)$ *if and only if, for all j, $z_j = 0$, and, for all ℓ, $r_\ell = \pm \mathrm{Id}$.*
(2) $\mathrm{SO}(2)$ *if and only if, for all j, $z_j = 0$, and $\{r_\ell\}_\ell \in \mathrm{SO}(2)^{|\chi(\bar{S})|+1} \smallsetminus \{\pm \mathrm{Id}\}^{|\chi(\bar{S})|+1}$.*
(3) the finite group $G_\theta = \left\{\pm \mathrm{Id}, \pm\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)\right\}$ (for some $\theta \in \mathbf{R}$) if and only if, for all j, $z_j \in \mathbf{R}e^{i\theta}$, and, for all ℓ, $r_\ell \in G_\theta$, and if either at least one of the z_j is non-zero or at least one of the r_ℓ is not $\pm \mathrm{Id}$.
(4) $\{\pm \mathrm{Id}\}$ *otherwise.*

Note that $-\mathrm{Id}$ acts trivially on $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)$, so the above stabilizers are more meaningful in the quotient group $\mathrm{O}(2)/\{\pm \mathrm{Id}\}$.

Observe also that the group $\mathrm{O}(2)$ acts diagonally on the product $\mathbf{R}^{3|\chi(\bar{S})|+r} \times \mathbf{C}^{3|\chi(\bar{S})|+r} \times \mathrm{O}(2)^{|\chi(\bar{S})|+1}$ and that the action is trivial on the first factor. Hence

$$\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)/\mathrm{O}(2) = \mathbf{R}^{3|\chi(\bar{S})|+r} \times \left(\mathbf{C}^{3|\chi(\bar{S})|+r} \times \mathrm{O}(2)^{|\chi(\bar{S})|+1}\right)/\mathrm{O}(2).$$

8.5. **Singularity types.** To conclude the proof of Theorem 8.1, it now remains to analyze the quotient singularities in each case of Proposition 8.3. Let $z = (\{t_i\}_i, \{z_j\}_j, \{r_\ell\}_\ell)$ be in $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)$ and let x be the corresponding framed representation in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4, \mathbf{R}))$.

8.5.1. *Case (4) of Proposition 8.3.* By Proposition 8.5.(4), the stabilizer of z in $\mathrm{O}(2)/\{\pm \mathrm{Id}\}$ is trivial. Therefore a neighborhood of x in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4, \mathbf{R}))$ is isomorphic to a neighborhood of z in $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)$ and is thus a smooth point.

8.5.2. *Case (3) of Proposition 8.3.* One can assume that $\theta = 0$ (see the remark 8.4). We treat this case under the assumption that $\{r_\ell\}_\ell \notin \{\pm \mathrm{Id}\}^{|\chi(\bar{S})|+1}$; under the assumption that $\{z_j\}_j \neq 0$, the reasoning is similar.

By Proposition 8.5.(3), the stabilizer of z in $\mathrm{O}(2)/\{\pm \mathrm{Id}\}$ is isomorphic to the group generated by the involution $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Therefore a neighborhood of x in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4, \mathbf{R}))$ is isomorphic to a neighborhood (of the class) of z in $\mathcal{X}_{\mathcal{S}}^{+}(\mathcal{T}, 2)/\{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$.

Applying as many times as needed on the factor $\mathrm{O}(2)^{|\chi(\bar{S})|+1}$, $\mathrm{O}(2)$-equivariant transformations of the form

$$\begin{aligned} \mathrm{O}(2) \times \mathrm{O}(2) &\longrightarrow \mathrm{O}(2) \times \mathrm{O}(2) \\ (a, b) &\longmapsto (ab, b) \end{aligned}$$

or

$$(a,b) \longmapsto (b,a)$$

we can assume that $r_1 = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $r_\ell = \pm \mathrm{Id}$ for all $\ell > 1$. We can parameterize a neighborhood of z in $\mathcal{X}_{\mathcal{S}}^+(\mathcal{T}, 2)$ by

$$\begin{aligned} \Psi\colon \mathbf{R}^{3|\chi(\bar{S})|+r} \times \mathbf{C}^{3|\chi(\bar{S})|+r} \times \mathbf{R}^{|\chi(\bar{S})|+1} &\longrightarrow \mathcal{X}_{\mathcal{S}}^+(\mathcal{T},2) \\ (\{\tau_i\}_i, \{\zeta_j\}_j, \{\theta_\ell\}_\ell) &\longmapsto \Big(\{t_i+\tau_i\}_i, \{z_j+\zeta_j\}_j, \big\{\left(\begin{smallmatrix} \cos\theta_\ell & -\sin\theta_\ell \\ \sin\theta_\ell & \cos\theta_\ell \end{smallmatrix}\right) r_\ell\big\}_\ell\Big) \end{aligned}$$

In these coordinates, the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is trivial on the first factor $\mathbf{R}^{3|\chi(\bar{S})|+r}$ and is given by the complex conjugation on $\mathbf{C}^{3|\chi(\bar{S})|+r}$ and by $-\mathrm{Id}$ on $\mathbf{R}^{|\chi(\bar{S})|+1}$. The locally free action of $\mathrm{SO}(2)$ on a neighborhood of z in $\mathcal{X}_{\mathcal{S}}^+(\mathcal{T}, 2)$ comes from a free action of $\mathbf{R}$ on these coordinates: for $\theta \in \mathbf{R}$, the action of θ is given by

$$(\{\tau_i\}_i, \{\zeta_j\}_j, \{\theta_\ell\}_\ell) \longmapsto (\{\tau_i\}_i, \{\zeta_j + 2i\theta\}_j, \{\theta_1+\theta, \theta_\ell\}_{\ell>1}).$$

We can then work on the hyperplane $\theta_1 = 0$ and grouping furthermore the real and imaginary parts of the parameters in $\mathbf{C}$, we conclude that a neighborhood of x in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4,\mathbf{R}))$ is diffeomorphic to a neighborhood of 0 in

$$\mathbf{R}^{6|\chi(\bar{S})|+2r} \times \big(\mathbf{R}^{4|\chi(\bar{S})|+r}/\{\pm \mathrm{Id}\}\big).$$

8.5.3. *Case (2) of Proposition 8.3.* In that case, since the stabilizer of z is $\mathrm{SO}(2)$, a neighborhood of x in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4,\mathbf{R}))$ is isomorphic to a neighborhood of z in $\mathcal{X}_{\mathcal{S}}^+(\mathcal{T}, 2)/\mathrm{SO}(2)$. Since, for all ℓ, r_ℓ is in $\mathrm{SO}(2)$, we can restrict to the subspace $\mathbf{R}^{3|\chi(\bar{S})|+r} \times \mathbf{C}^{3|\chi(\bar{S})|+r} \times \mathrm{SO}(2)^{|\chi(\bar{S})|+1}$. As $\mathrm{SO}(2)$ is abelian, we can conclude that a neighborhood of x in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4,\mathbf{R}))$ is isomorphic to a neighborhood of $(0,\dots,0,\mathrm{Id},\dots,\mathrm{Id})$ in the space

$$\mathbf{R}^{3|\chi(\bar{S})|+r} \times \big(\mathbf{C}^{3|\chi(\bar{S})|+r}/\mathrm{SO}(2)\big) \times \mathrm{SO}(2)^{|\chi(\bar{S})|+1},$$

and this is what we wanted to prove.

8.5.4. *Case (1) of Proposition 8.3.* The situation is similar to the previous case, but we have to take care of the remaining action by $\mathrm{O}(2)/\mathrm{SO}(2) \simeq \mathbf{Z}/2\mathbf{Z}$. Precisely, we conclude that a neighborhood of x in $\mathcal{M}^{\mathrm{f}}(S, \mathrm{Sp}(4,\mathbf{R}))$ is isomorphic to a neighborhood of 0 in the quotient of the space

$$\mathbf{R}^{3|\chi(\bar{S})|+r} \times \big(\mathbf{C}^{3|\chi(\bar{S})|+r}/\mathrm{SO}(2)\big) \times \mathbf{R}^{|\chi(\bar{S})|+1},$$

by the involution whose action is trivial on the factor $\mathbf{R}^{3|\chi(\bar{S})|+r}$, is the action induced by the complex conjugation on the factor $\mathbf{C}^{3|\chi(\bar{S})|+r}/\mathrm{SO}(2)$, and is $-\mathrm{Id}$ on $\mathbf{R}^{|\chi(\bar{S})|+1}$.

9. General $\mathcal{X}$-coordinates

In this section we introduce general (i.e. not necessarily positive) $\mathcal{X}$-coordinates with respect to a chosen ideal triangulation $\mathcal{T}$ of S. This relies on the analysis of pairs of nondegenerate quadratic forms which is given in the appendix. We obtain a generically finite-to-one parameterization of the space of transverse framed local systems (Section 9.6) as well as a kind of cellular decomposition of that space (Section 9.9). Relaxing the condition on the parameter space (Section 9.10), we deduce topological results for the space of transverse framed local systems (Section 9.11).

9.1. **Pairs of real symmetric forms.** In Appendix A are established a number of results about pairs of quadratic forms that we recap here. The reader might want to read the appendix first where the statements are obtained more progressively.

9.1.1. *Parameter space.* We denote by $\mathcal{D}(n)$ the set of

- quintuple $\mathbf{n} = (\{\underline{n}_x\}_{x\in\{\pm1\}^2}, 2\underline{m})$ of sequences of integers $\underline{n}_x = (n_{x,j})_{j=1,\dots,k_x} \in (\mathbf{Z}_{>0})^{k_x}$ (for $x \in \{\pm1\}^2$) and $2\underline{m} = (2m_j)_{j=1,\dots,k_0} \in (2\mathbf{Z}_{>0})^{k_0}$ of lengths k_x ($x \in \{\pm1\}^2$) and k_0 in $\mathbf{Z}_{\geqslant 0}$, such that
- the sum of all the integers in the 5 sequences in $\mathbf{n}$ is equal to n.

For every $\mathbf{n} = (\{\underline{n}_{(\varepsilon,\eta)}\}_{(\varepsilon,\eta)\in\{\pm1\}^2}, 2\underline{m}) \in \mathcal{D}(n)$, the quintuple $(\{\underline{n}_{(\eta,\varepsilon)}\}_{(\varepsilon,\eta)\in\{\pm1\}^2}, 2\underline{m})$ is also in $\mathcal{D}(n)$ and will be denoted by $\iota(\mathbf{n})$. Thus ι is an involution of $\mathcal{D}(n)$.

We will denote $\mathcal{E}(n)$ the space of pairs $(\mathbf{n}, \boldsymbol{\lambda})$ with

- $\mathbf{n} \in \mathcal{D}(n)$, and
- $\boldsymbol{\lambda}$ a quintuple $(\{\underline{\lambda}_x\}, \underline{\lambda})$ of decreasing sequences $\underline{\lambda}_x = (\lambda_{x,j})_{j=1,\dots,k_x} \in \mathbf{R}^{k_x}$ ($x \in \{\pm1\}^2$) of real numbers and a sequence $\underline{\lambda} = (\lambda_j)_{j=1,\dots,k_0} \in \mathbb{H}^{k_0}$ of decreasing (for the lexicographic order on $\mathbf{C}$ and where $\mathbb{H} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) > 0\}$) complex numbers with positive imaginary part and of the same lengths as the sequences in $\mathbf{n}$;
- For any $x = (\varepsilon, \eta) \in \{\pm1\}^2$ and any $1 \leqslant \ell \leqslant k_x$, the product $\varepsilon\eta\lambda_{x,\ell}$ is positive, i.e. $\lambda_{1,1,\ell}$, $\lambda_{-1,-1,\ell}$ are positive and $\lambda_{-1,1,\ell}$, $\lambda_{1,-1,\ell}$ negative;
- where the sequences in $\boldsymbol{\lambda}$ are constant, the corresponding sequences in $\mathbf{n}$ are decreasing: if $r < s$, $\lambda_{x,r} = \lambda_{x,s}$ (resp. $\lambda_r = \lambda_s$), then $n_{x,r} \geqslant n_{x,s}$ (resp. $m_r \geqslant m_s$). Equivalently, if $r < s$ and $n_{x,r} < n_{x,s}$ (resp. $m_r < m_s$), then $\lambda_{x,r} > \lambda_{x,s}$ (resp. $\lambda_r > \lambda_s$) (where a pair in one of the sequences in $\mathbf{n}$ is strictly increasing, the corresponding elements in $\boldsymbol{\lambda}$ are strictly decreasing).

The space $\mathcal{E}(n)$ has also an involution denoted by ι and defined by

$$(\mathbf{n}, \boldsymbol{\lambda}) \longmapsto \big(\iota(\mathbf{n}), (\{\underline{\lambda}_{(\eta,\varepsilon)}\}_{(\varepsilon,\eta)\in\{\pm1\}^2}, \underline{\lambda})\big).$$

There is a natural projection $\pi\colon \mathcal{E}(n) \to \mathcal{D}(n)$; it is ι-equivariant and its fibers are contractible:

Lemma 9.1. *For every $\mathbf{n} = (\{\underline{n}_x\}_{x\in\{\pm1\}^2}, 2\underline{m})$ in $\mathcal{D}(n)$, the fiber $V(\mathbf{n}) := \pi^{-1}(\mathbf{n})$ is a convex cone of dimension $d(\mathbf{n}) := \sum_{x\in\{\pm1\}^2} k_x + 2k_0$ where k_x is the length of $\underline{n}_x$ ($x \in \{\pm1\}^2$) and k_0 is the length of $2\underline{m}$.*

Of course, the precise part of the closure of the cone $V(\mathbf{n})$ that must be included is described by the conditions on the $\boldsymbol{\lambda}$s.

9.1.2. *Matrices.* For each n in $\mathbf{Z}_{>0}$, and $\lambda \in \mathbf{C}$, let C_n and $J_n(\lambda)$ be the following $n \times n$-matrices

$$C_n := \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad J_n(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & \dots \\ 0 & \lambda & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots & \lambda \end{pmatrix}. \tag{9.1}$$

When λ is real and non-zero, let us define the (symmetric) $n \times n$-matrix

$$\Phi_n(\lambda) := |\lambda|^{1/2} C_n \sum_{\ell=0}^{n-1} a_\ell \lambda^{-\ell} J_n(0)^\ell \tag{9.2}$$

where $\sum_{\ell=0}^{\infty} a_\ell t^\ell$ is the Taylor series of $t \mapsto (1+t)^{1/2}$ (i.e. for all ℓ, $a_\ell = \frac{\prod_{j=0}^{\ell-1}(1/2-j)}{\ell!}$), cf. Section A.4.

For sequences $\underline{n} = (n_1, \dots, n_\ell)$ in $(\mathbf{Z}_{>0})^\ell$ and $\underline{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ in $\mathbf{C}^\ell$ we define $C(\underline{n})$ to be the block diagonal matrix whose blocks are $C_{n_1}, \dots, C_{n_\ell}$; and we define $J(\underline{n}, \underline{\lambda})$ to be the block diagonal matrix whose blocks are $J_{n_1}(\lambda_1), \dots, J_{n_\ell}(\lambda_\ell)$. When $\underline{\lambda} \in (\mathbf{R}^*)^\ell$, the matrix $\Phi(\underline{n}, \underline{\lambda})$ is the block diagonal matrix whose blocks are $\Phi_{n_1}(\lambda_1), \dots, \Phi_{n_\ell}(\lambda_\ell)$.

Let us denote, for an even number $2m$ and $\lambda = a + ib \in \mathbf{C}$, the $2m \times 2m$-matrices

$$C'_{2m} := \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \\ \vdots & \vdots & \iddots & \iddots & \iddots \\ 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \end{pmatrix}, \quad \text{and} \quad J'_{2m}(\lambda) := \begin{pmatrix} a & -b & 1 & 0 & \dots & 0 & 0 \\ b & a & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & a & -b & \ddots & 0 & 0 \\ 0 & 0 & b & a & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a & -b \\ 0 & 0 & 0 & 0 & \dots & b & a \end{pmatrix} \tag{9.3}$$

If $\lambda \neq 0$, let $c + id$ the biggest (for the lexicographic order) square root of λ. We define (again with $\sum a_\ell t^\ell = (1+t)^{1/2}$)

$$\Psi_{2m}(\lambda) := \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & c & -d \\ 0 & 0 & \dots & 0 & 0 & -d & -c \\ 0 & 0 & \iddots & c & -d & \vdots & \vdots \\ 0 & 0 & \iddots & -d & -c & \vdots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ c & -d & 0 & 0 & \dots & 0 & 0 \\ -d & -c & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \sum_{\ell=0}^{m-1} \frac{a_\ell}{(a^2+b^2)^\ell} \begin{pmatrix} 0 & 0 & a & b & 0 & 0 & \dots & \dots \\ 0 & 0 & -b & a & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 & a & b \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 & -b & a \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}^\ell . \tag{9.4}$$

When $2\underline{m} = (2m_1, \dots, 2m_\ell)$ is a sequence of even integers and $\underline{\lambda} = (\lambda_1, \dots, \lambda_\ell) \in \mathbf{C}^\ell$, let $C'(2\underline{m})$ be the matrix with diagonal blocks $C'_{2m_1}, \dots, C'_{2m_k}$ and $J'(2\underline{m}, \underline{\lambda})$ the matrix with diagonal blocks $J'_{2m_1}(\lambda_1), \dots, J'_{2m_k}(\lambda_k)$. When $\underline{\lambda} \in (\mathbf{C}^*)^\ell$, the matrix $\Psi(2\underline{m}, \underline{\lambda})$ has the diagonal blocks $\Psi_{2m_1}(\lambda_1), \dots, \Psi_{2m_\ell}(\lambda_\ell)$.

We now introduce the matrices that will serve as normal forms for pairs of quadratic forms.

Let $(\mathbf{n}, \boldsymbol{\lambda})$ be in $\mathcal{E}(n)$, thus $\mathbf{n}$ is quintuple $(\underline{n}_{1,1}, \underline{n}_{1,-1}, \underline{n}_{-1,1}, \underline{n}_{-1,-1}, 2\underline{m})$ and similarly for $\boldsymbol{\lambda}$. We define $C(\mathbf{n})$ to be the block diagonal matrix

$$C(\mathbf{n}) := \begin{pmatrix} C(\underline{n}_{1,1}) & & & & \\ & C(\underline{n}_{1,-1}) & & & \\ & & -C(\underline{n}_{-1,1}) & & \\ & & & -C(\underline{n}_{-1,-1}) & \\ & & & & C'(2\underline{m}) \end{pmatrix},$$

We define $J(\mathbf{n}, \boldsymbol{\lambda})$ to be the block diagonal matrix

$$J(\mathbf{n}, \boldsymbol{\lambda}) := \begin{pmatrix} J(\underline{n}_{1,1}, \underline{\lambda}_{1,1}) & & & & \\ & J(\underline{n}_{1,-1}, \underline{\lambda}_{1,-1}) & & & \\ & & J(\underline{n}_{-1,1}, \underline{\lambda}_{-1,1}) & & \\ & & & J(\underline{n}_{-1,-1}, \underline{\lambda}_{-1,-1}) & \\ & & & & J'(2\underline{m}, \underline{\lambda}) \end{pmatrix}.$$

Also we define $D(\mathbf{n}, \boldsymbol{\lambda}) := C(\mathbf{n}) J(\mathbf{n}, \boldsymbol{\lambda})$.

Finally, since none of the elements in $\boldsymbol{\lambda}$ are zero, let $\Phi(\mathbf{n}, \boldsymbol{\lambda})$ be

$$\begin{pmatrix} \Phi(\underline{n}_{1,1}, \underline{\lambda}_{1,1}) & & & & \\ & 0 & \Phi(\underline{n}_{1,-1}, \underline{\lambda}_{1,-1}) & & \\ & \Phi(\underline{n}_{-1,1}, \underline{\lambda}_{-1,1}) & 0 & & \\ & & & \Phi(\underline{n}_{-1,-1}, \underline{\lambda}_{-1,-1}) & \\ & & & & \Psi(2\underline{m}, \underline{\lambda}) \end{pmatrix}.$$

9.1.3. *Normal forms.* Theorem A.20 and Proposition A.21 of the appendix say

Theorem 9.2. *Let V be a n-dimensional real vector space, let b_0 and b_1 be two symmetric forms on V with b_0 and b_1 nondegenerate.*

Then there is a unique element $(\mathbf{n}, \boldsymbol{\lambda}) = (\mathbf{n}(b_0, b_1), \boldsymbol{\lambda}(b_0, b_1))$ in $\mathcal{E}(n)$ such that there is a basis $\mathbf{e}$ of V for which the matrices of the symmetric forms b_0 and b_1 are $C(\mathbf{n})$ and $D(\mathbf{n}, \boldsymbol{\lambda})$ respectively.

For the pair (b_1^*, b_0^*) of quadratic forms on the dual space V^*, one has

$$(\mathbf{n}(b_1^*, b_0^*), \boldsymbol{\lambda}(b_1^*, b_0^*)) = \iota(\mathbf{n}(b_0, b_1), \boldsymbol{\lambda}(b_0, b_1)).$$

If $\mathbf{v}^*$ is the basis of V^* dual to the basis $\mathbf{v} = \mathbf{e}\,\Phi(\mathbf{n}, \boldsymbol{\lambda})$, then the matrices of the symmetric forms b_1^* and b_0^* in the basis $\mathbf{v}^*$ are $C(\iota(\mathbf{n}))$ and $D(\iota(\mathbf{n}, \boldsymbol{\lambda}))$ respectively.

There is also a uniqueness statement for the standard bases: two such bases are conjugate under the action of the subgroup of $\mathrm{GL}(n, \mathbf{R})$ which is the intersection of the orthogonal group of $C(\mathbf{n})$ and the orthogonal group of $D(\mathbf{n}, \boldsymbol{\lambda})$; it is also the centralizer of $\Phi(\mathbf{n}, \boldsymbol{\lambda})$ in the orthogonal group of $C(\mathbf{n})$. This group naturally identifies with $\mathrm{O}(b_0) \cap \mathrm{O}(b_1)$ and is described in Section A.9.

9.2. **Quadruples of transverse Lagrangians.**

Based on the normal forms of the above theorem, one deduces normal forms for quadruples of transverse Lagrangians, generalizing Proposition 3.6.

Proposition 9.3. *Let $x = (L_1, M_1, L_2, M_2)$ be a quadruple of Lagrangians in $\mathbf{R}^{2n}$ such that (L_1, M_1, L_2), and (L_1, M_2, L_2) are transverse triples. Then there is unique element $(\mathbf{n}, \boldsymbol{\lambda}) = (\mathbf{n}(x), \boldsymbol{\lambda}(x))$ such that there is a symplectic basis $(\mathbf{e}, \mathbf{f})$ with*

$$L_1 = \mathrm{Span}(\mathbf{e}),\ L_2 = \mathrm{Span}(\mathbf{f}),\ M_1 = \mathrm{Span}(\mathbf{e} + \mathbf{f}C(\mathbf{n})),\ M_2 = \mathrm{Span}(\mathbf{e} - \mathbf{f}D(\mathbf{n}, \boldsymbol{\lambda})).$$

Let $y = \kappa(x)$ (cf. Section 2.2) be the quadruple (L_2, M_2, L_1, M_1). Then $(\mathbf{n}(y), \boldsymbol{\lambda}(y)) = \iota(\mathbf{n}, \boldsymbol{\lambda})$ and, setting $\mathbf{e}' = \mathbf{f}\,\Phi(\mathbf{n}, \boldsymbol{\lambda})$, and $\mathbf{f}' = -\mathbf{e}\,{}^T\Phi(\mathbf{n}, \boldsymbol{\lambda})^{-1}$ one has

$$L_2 = \mathrm{Span}(\mathbf{e}'),\ L_1 = \mathrm{Span}(\mathbf{f}'),\ M_2 = \mathrm{Span}(\mathbf{e}' + \mathbf{f}'C(\iota(\mathbf{n}))),\ M_1 = \mathrm{Span}(\mathbf{e}' - \mathbf{f}'D(\iota(\mathbf{n}, \boldsymbol{\lambda}))),$$

and $(\mathbf{e}', \mathbf{f}')$ is a symplectic basis.

A basis $(\mathbf{e}, \mathbf{f})$ as in the proposition is said in *standard position* with respect to x.

Remark 9.4. The cross ratio of the quadruple of Lagrangians is then $D(\mathbf{n}, \boldsymbol{\lambda})^{-1}C(\mathbf{n}) = J(\mathbf{n}, \boldsymbol{\lambda})^{-1}$ (cf. the definition of the cross-ratio in Section 3.2).

As a consequence (cf. Section 2.2 for the notation):

Corollary 9.5. *The map $x \mapsto (\mathbf{n}(x), \boldsymbol{\lambda}(x))$ induces an isomorphism between $\mathrm{Conf}^{4\Diamond}(\mathcal{L}_n)$ and the space $\mathcal{E}(n)$.*

The uniqueness statement for standard basis takes the following form:

Proposition 9.6. *Let $(\mathbf{e}_1, \mathbf{f}_1)$ and $(\mathbf{e}_2, \mathbf{f}_2)$ be bases in standard position with respect to a quadruple $x = (L_1, M_1, L_2, M_2)$. Then*

1. *(1) there is a (unique) element r of $\mathrm{GL}(n, \mathbf{R})$ that is orthogonal with respect to the symmetric matrices $C(\mathbf{n}(x))$ and $D(\mathbf{n}(x), \boldsymbol{\lambda}(x))$ and such that $(\mathbf{e}_2, \mathbf{f}_2) = (\mathbf{e}_1 \cdot r, \mathbf{f}_1 \cdot r)$;*
2. *(2) for every s in $\mathrm{GL}(n, \mathbf{R})$ orthogonal with respect to $C(\mathbf{n}(x))$ and $D(\mathbf{n}(x), \boldsymbol{\lambda}(x))$, the basis $(\mathbf{e}_1 \cdot s, \mathbf{f}_1 \cdot s)$ is in standard position with respect to x.*

9.3. **Triple of decorated Lagrangians.** We give now a generalization of Lemma 3.8 and of Lemma 3.11 established in Section 3.2 dropping now the maximality assumption.

We adopt here a point of view closer to the framed local systems introduced in Section 5.6.

Lemma 9.7. *Let F_a, F_b, and F_c be three symplectic vector spaces dimension $2n$ and let $(\mathbf{e}_a, \mathbf{f}_a)$, $(\mathbf{e}_b, \mathbf{f}_b)$, and $(\mathbf{e}_c, \mathbf{f}_c)$ be symplectic bases of F_a, F_b, and F_c. For $x = a, b, c$ set $L_x^t = \mathrm{Span}(\mathbf{e}_x)$ and $L_x^b = \mathrm{Span}(\mathbf{f}_x)$ (L_x^t and L_x^b are thus Lagrangians in F_x). Let S_a, S_b, and S_c be symmetric $n \times n$-matrices and let $A\colon F_b \to F_c$, $B\colon F_c \to F_a$, and $C\colon F_a \to F_b$ be symplectic isomorphisms such that*

(1) $CBA = -\mathrm{Id}$,
(2) $A(L_b^b) = L_c^t$, $B(L_c^b) = L_a^t$, *and* $C(L_a^b) = L_b^t$,
(3) $A(L_b^t) = \mathrm{Span}(\mathbf{e}_c + \mathbf{f}_c \cdot S_c)$, $B(L_c^t) = \mathrm{Span}(\mathbf{e}_a + \mathbf{f}_a \cdot S_a)$, $C(L_a^t) = \mathrm{Span}(\mathbf{e}_b + \mathbf{f}_b \cdot S_b)$.

Then the matrices S_a, S_b, and S_c are invertible; there are Y_a, Y_b, and Y_c in $\mathrm{GL}(n, \mathbf{R})$ such that, with respect to the symplectic bases, the matrices of A, B, and C are respectively given by

$$\begin{pmatrix} S_c^{-1} Y_a & -{}^T Y_a^{-1} \\ Y_a & 0 \end{pmatrix}, \quad \begin{pmatrix} S_a^{-1} Y_b & -{}^T Y_b^{-1} \\ Y_b & 0 \end{pmatrix}, \quad \begin{pmatrix} S_b^{-1} Y_c & -{}^T Y_c^{-1} \\ Y_c & 0 \end{pmatrix}, \tag{9.5}$$

and the following relations hold

$$Y_c\, {}^T Y_b^{-1} Y_a = S_b, \quad Y_b\, {}^T Y_a^{-1} Y_c = S_a, \quad Y_a\, {}^T Y_c^{-1} Y_b = S_c.$$

Proof. From the hypothesis (2), we get that the matrices of A, B, and C have the following form

$$\begin{pmatrix} M_a Y_a & -{}^T Y_a^{-1} \\ Y_a & 0 \end{pmatrix}, \quad \begin{pmatrix} M_b Y_b & -{}^T Y_b^{-1} \\ Y_b & 0 \end{pmatrix}, \quad \begin{pmatrix} M_c Y_c & -{}^T Y_c^{-1} \\ Y_c & 0 \end{pmatrix},$$

for some (uniquely defined) Y_a, Y_b, and Y_c in $\mathrm{GL}(n, \mathbf{R})$ and symmetric matrices M_a, M_b, and M_c. Since $CBA = -\mathrm{Id}$, a small calculation (see Remark 5.18) implies that M_a, M_b, and M_c are nonsingular and the relations $Y_c\, {}^T Y_b^{-1} Y_a = M_c^{-1}$, $Y_b\, {}^T Y_a^{-1} Y_c = M_b^{-1}$, $Y_a\, {}^T Y_c^{-1} Y_b = M_a^{-1}$ are satisfied. The relations between the symmetric matrices come from the equality $\mathrm{Span}(\mathbf{e}_c + \mathbf{f}_c \cdot S_c) = A(\mathrm{Span}(\mathbf{e_b})) = \mathrm{Span}(\mathbf{e}_c \cdot M_a Y_a + \mathbf{f}_c \cdot Y_a) = \mathrm{Span}(\mathbf{e}_c \cdot M_a + \mathbf{f}_c)$ which gives $S_c = M_a^{-1}$ and similarly, $S_a = M_b^{-1}$, and $S_b = M_c^{-1}$. ☐

Remark 9.8. The matrices Y_x ($x \in \{a, b, c\}$) are isometries in the following sense: $Y_c S_a^{-1}\, {}^T Y_c = S_b$, $Y_a S_b^{-1}\, {}^T Y_a = S_c$, $Y_b S_c^{-1}\, {}^T Y_b = S_a$.

Conversely:

Lemma 9.9. *Let S_b be a nonsingular symmetric matrix and let Y_a, and Y_b be in $\mathrm{GL}(n, \mathbf{R})$. Define $Y_c := S_b Y_a^{-1}\, {}^T Y_b$, $S_c := Y_a S_b^{-1}\, {}^T Y_a$, and $S_a := Y_b S_c^{-1}\, {}^T Y_b$.*

Then S_a and S_c are symmetric and nonsingular and one has

$$Y_b\, {}^T Y_a^{-1} Y_c = S_a, \; Y_a\, {}^T Y_c^{-1} Y_b = S_c, \; \text{and} \; S_b = Y_c S_a^{-1}\, {}^T Y_c.$$

The matrices A, B, and C defined by Equation (9.5) satisfy $CBA = -\mathrm{Id}$.

Remark 9.10. Lemmas 10.5 and 10.7 below generalize these statements.

9.4. **Space of $\mathcal{X}$-coordinates.** We denote by $\mathcal{X}_{\mathcal{E}}(\mathcal{T}, n)$ the set of tuples

$$\left(\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}_{a \in A_2}, \{S_v\}_{v \in V}, \{Y_a\}_{a \in A_3}\right)$$

such that

- for all a in A_2, $(\mathbf{n}_a, \boldsymbol{\lambda}_a)$ belongs to $\mathcal{E}(n)$;
- for all cycles $\{a, a'\}$ in A_2, $(\mathbf{n}_a, \boldsymbol{\lambda}_a) = \iota(\mathbf{n}_{a'}, \boldsymbol{\lambda}_{a'})$;

- for all v in V, S_v is a symmetric matrix. If $v = v^+(a)$ for some $a \in A_2$, then $S_v = C(\mathbf{n}_a)$, and if not (i.e. when v is an external edge), S_v is a diagonal matrix $I_{p,q} = \left(\begin{smallmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{smallmatrix}\right)$ $(p+q=n)$;
- for all a in A_3, Y_a belongs to $\mathrm{GL}(n,\mathbf{R})$; and
- for all cycle (a,b,c) in A_3, the equality $Y_c\,{}^T Y_b^{-1} Y_a = S_v$ holds where $v = v^+(c) = v^-(b)$.

Of course the family $\{S_{v^+(a)}\}_{a\in A_2}$ is completely determined by $\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}$ but it is helpful to keep it. Likewise, the matrices S_v, for v an external vertex are completely determined by the signature associated with the triangle containing v. Unless $A_2 = \varnothing$ (which happens only in the case of the disk with $\sharp R = 3$), these signatures are equally determined by the family $\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}$.

9.5. **Positive locus.** The subset of elements

$$u = \big(\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}_{a\in A_2}, \{S_v\}_{v\in V}, \{Y_a\}_{a\in A_3}\big)$$

of $\mathcal{X}_{\mathcal{E}}(\mathcal{T}, n)$ for which, for all v in V, S_v is positive definite is called the *positive locus* of the parameter space. In this case, for every a in A_2 the elements $\mathbf{n}_a = (\{\underline{n}_x\}_{x\in\{\pm1\}^2}, 2\underline{m}), \boldsymbol{\lambda}_a = (\{\underline{\lambda}_x\}_{x\in\{\pm1\}^2}, 2\underline{\lambda})$ have a simpler form: only the sequence $\underline{n}_{1,1} = (n_1, \dots, n_k)$ is nontrivial and all its entries are equal to 1 (so $k = n$) and the sequence $\underline{\lambda}_{1,1} = (\lambda_1, \dots, \lambda_n)$ is decreasing. Hence, this data can be encoded by the diagonal matrix $x(a)$ with entries $(\lambda_1, \dots, \lambda_n)$. Also, all the matrices Y_a $(a \in A_3)$ belong to $\mathrm{O}(n)$ and we set $x(a) = Y_a$. The tuple $f(u) := (x(a))_{a\in A}$ belongs then to $\mathcal{X}_\Delta^+(\mathcal{T}, n)$ (cf. Section 6.2) and the map f is an isomorphism between the positive locus and $\mathcal{X}_\Delta^+(\mathcal{T}, n)$.

In the sequel, we will rather consider $\mathcal{X}_\Delta^+(\mathcal{T}, n)$ as a subspace of $\mathcal{X}_{\mathcal{E}}(\mathcal{T}, n)$ (i.e. the positive locus) without reference to f.

Our last observation si that, in this case, the matrix $\Phi(\mathbf{n}_a, \boldsymbol{\lambda}_a)$ $(a \in A_2)$ constructed above is equal to $x(a)^{1/2}$.

9.6. **From coordinates to representations.** Similarly to Section 6.3 and based again on Section 5, we associate a framed δ-twisted symplectic local system on $\Gamma_{\mathcal{T}}$ to every $x = \big(\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}_{a\in A_2}, \{S_v\}_{v\in V}, \{Y_a\}_{a\in A_3}\big)$ by specifying the transition matrices:

(1) for every a in A_2,

$$G_a = \begin{pmatrix} 0 & -{}^T\Phi(\mathbf{n}_a, \boldsymbol{\lambda}_a)^{-1} \\ \Phi(\mathbf{n}_a, \boldsymbol{\lambda}_a) & 0 \end{pmatrix};$$

(2) for every a in A_3,

$$G_a = \begin{pmatrix} S_{v^+(a)}^{-1} Y_a & -{}^T Y_a^{-1} \\ Y_a & 0 \end{pmatrix}.$$

The fact that this procedure defines indeed a δ-twisted local system follows from the equalities, for every $(\mathbf{n}, \boldsymbol{\lambda}) \in \mathcal{E}(n)$, ${}^T\Phi(\iota(\mathbf{n}, \boldsymbol{\lambda})) = \Phi(\mathbf{n}, \boldsymbol{\lambda})$ (in turn a consequence of the the fact that the matrices $\Phi_n(\lambda)$ and $\Psi_{2m}(\lambda)$ are symmetric) and the relations between the Y_as and the S_vs.

We will denote by $\mathrm{hol}_{\mathcal{T}}^{\mathcal{X}}(x)$ the element of $\mathrm{Loc}_\delta^{\mathrm{f}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ constructed above.

Remark 9.11. In restriction to the positive locus, the map $\mathrm{hol}_{\mathcal{T}}^{\mathcal{X}}$ is exactly the map $\mathrm{hol}_{\mathcal{T}}^{\mathcal{X},+}$ defined in Section 6.2.

Our main result is the following

Theorem 9.12. *The map*

$$\mathrm{hol}_{\mathcal{T}}^{\mathcal{X}} \colon \mathcal{X}_{\mathcal{E}}(\mathcal{T}, n) \longrightarrow \mathrm{Loc}_\delta^{\mathrm{f}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$$

is onto the subspace $\mathrm{Loc}_{\delta,\mathcal{T}}^{\mathrm{f}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ *of transverse framed local systems.*

Two elements $x = \big(\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}, \{S_v\}, \{Y_a\}\big)$ *and* $x' = \big(\{(\mathbf{n}'_a, \boldsymbol{\lambda}'_a)\}, \{S'_v\}, \{Y'_a\}\big)$ *have the same image under* $\mathrm{hol}_{\mathcal{T}}^{\mathcal{X}}$ *if and only if*

(1) *for all* a *in* A_2, $(\mathbf{n}_a, \boldsymbol{\lambda}_a) = (\mathbf{n}'_a, \boldsymbol{\lambda}'_a)$ *(thus* $S_{v^+(a)} = S'_{v^+(a)}$*);*
(2) *for every external vertex* v, $S_v = S'_v$*;*
(3) *and there is a family of matrices* $\{r_v\}_{v\in V}$*, such that*
  - *for all* v *in* V*,* r_v *is orthogonal with respect to* S_v*;*
  - *for all arrow* a *in* A_2*,* $r_{v^-(a)} = r_{v^+(a)}$ *and* $r_{v^+(a)}$ *commutes with* $\Phi(\mathbf{n}_a, \boldsymbol{\lambda}_a)$*;*
  - *for all* a *in* A_3*,* $Y'_a = r_{v^+(a)} Y_a r_{v^-(a)}^{-1}$.

9.7. **Standard bases for framed local systems.** Proposition 9.3 gives a notion of standard basis for a quadruple of Lagrangians. Here we generalize this notion to framed local systems.

Let (F_v, g_a, L_v^t, L_v^b) be a framed δ-twisted symplectic local system on the quiver $\Gamma_{\mathcal{T}}$.

Definition 9.13. A generating symplectic basis $\{(\mathbf{e}_v, \mathbf{f}_v)\}_{v\in V}$ (see Definition 5.13) is said to be *in standard position* with respect to the framing if, for every *internal* vertex v in V,

(1) if $x = q_v(F_v, g_a, L_v^t, L_v^b)$ is the associated quadruple of Lagrangians in F_v (see Section 5.8), then the basis $(\mathbf{e}_v, \mathbf{f}_v)$ is in standard position with respect to x (cf. Proposition 9.3);
(2) if a is the arrow in A_2 such that $v = v^+(a)$, the matrix of g_a is
$$G_a = \begin{pmatrix} 0 & -{}^T\Phi(\mathbf{n}(x), \boldsymbol{\lambda}(x))^{-1} \\ \Phi(\mathbf{n}(x), \boldsymbol{\lambda}(x)) & 0 \end{pmatrix},$$
where $(\mathbf{n}(x), \boldsymbol{\lambda}(x))$ is given by Proposition 9.3;

and if, for every *external* vertex v, there exists $(p, q) \in \mathbf{N}^2$ with $p + q = n$ such that the triple $t_v(F_v, g_a, L_v^t, L_v^b)$ of Lagrangians in F_v is $(\mathrm{Span}(\mathbf{e}_v), \mathrm{Span}(\mathbf{e}_v + \mathbf{f}_v \cdot I_{p,q}), \mathrm{Span}(\mathbf{f}_v))$. This means in particular that, if a is the element of A_3 such that $v = v^+(a)$, then $g_a(L^t_{v^-(a)}) = \mathrm{Span}(\mathbf{e}_v + \mathbf{f}_v \cdot I_{p,q})$ (the other equalities $L_v^t = \mathrm{Span}(\mathbf{e}_v)$ and $L_v^b = \mathrm{Span}(\mathbf{f}_v)$ are already satisfied by the assumption that the basis is generating).

The moduli space of framed local systems equipped with a standard basis will be denoted by $\mathrm{Loc}_\delta^{\mathrm{f,st}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$. The holonomy construction performed above (Section 9.6) defines in fact a map
$$\mathrm{hol}_{\mathcal{T}}^{\mathcal{X},\mathrm{st}} \colon \mathcal{X}_{\mathcal{E}}(\mathcal{T}, n) \longrightarrow \mathrm{Loc}_\delta^{\mathrm{f,st}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R})),$$
(cf. Section 5.3). Since the transition matrices completely determine the parameters, and since the bases entirely determine the transition matrices, the map $\mathrm{hol}_{\mathcal{T}}^{\mathcal{X},\mathrm{st}}$ is a one-to-one correspondence.

The following proposition is a direct consequence of the existence and uniqueness of standard bases for quadruples of Lagrangians (Proposition 9.3 and Proposition 9.6) and of the fact that triples of pairwise transverse Lagrangians are classified by their Maslov index; it readily implies Theorem 9.12.

Proposition 9.14.

(1) *Every transverse framed* δ*-twisted symplectic local system* (F_v, g_a, L_v^t, L_v^b) *admits a standard basis* $(\mathbf{e}_v, \mathbf{f}_v)$*.*
(2) *For every family* $(r_v)_{v\in V}$ *such that*
  - *for every external vertex* v *in* V*,* r_v *is orthogonal with respect to* $I_{p,q}$ *where* $p = (n + s_T)/2$*,* $q = (n - s_T)/2$ *and* $s_T = \mu^T(F_v, g_a, L_v^t, L_v^b)$ *the Maslov index for the triangle* T *containing* v*;*
  - *for every internal vertex* v *in* V*,* r_v *is orthogonal with respect to* $C(\mathbf{n}(x))$ *and with respect to* $D(\mathbf{n}(x), \boldsymbol{\lambda}(x))$ *where* x *is the quadruple* $q_v(F_v, g_a, L_v^t, L_v^b)$*;*
  - *for every* a *in* A_2*,* $r_{v^-(a)} = r_{v^+(a)}$*;*

  the family $\{(\mathbf{e}_v \cdot r_v, \mathbf{f}_v \cdot {}^T r_v^{-1})\}$ *is in standard position.*

(3) For every basis $\{(\mathbf{e}'_v, \mathbf{f}'_v)\}$ in standard position, there is a (unique) family $(r_v)_{v\in V}$ as in (2) and such that $\{(\mathbf{e}'_v, \mathbf{f}'_v)\} = \{(\mathbf{e}_v \cdot r_v, \mathbf{f}_v \cdot {}^T r_v^{-1})\}$.

9.8. **Maslov indices.** A framed transverse local system gives a family of integers $\{s_T\}_{T\in\mathcal{T}}$ where s_T is the Maslov index of the configuration of three Lagrangians associated with the triangle T. The integer s_T belongs to $\{-n, -n+2, \dots, n\}$.

When the local system is the holonomy of an element $(\{(\mathbf{n}_a, \boldsymbol{\lambda}_a)\}, \{S_v\}, \{Y_a\})$, then, for every triangle T and every vertex v contained in T, the integer s_T is the signature of the symmetric matrix S_v, which is equal to $C(\mathbf{n}_a)$ if $v = v^+(a)$ for some $a \in A_2$. It can be easily calculated noting that, for every integer m, the signature of C_{2m} and of C'_{2m} are 0 and the signature of C_{2m+1} is 1.

To state a precise result, for every $\mathbf{n} = (\{\underline{n}_x\}_{x\in\{\pm1\}^2}, 2\underline{m})$ in $\mathcal{D}(n)$, set, for $x \in \{\pm1\}^2$,

$$p_x(\mathbf{n}) := \sharp\big\{\ell \in \{1,\dots,k_x\} \mid n_{x,\ell} = 1 \mod 2\big\}.$$

Lemma 9.15. *For every $\mathbf{n}$ in $\mathcal{D}(n)$, the signature of $C(\mathbf{n})$ is equal to*

$$p_{1,1}(\mathbf{n}) + p_{1,-1}(\mathbf{n}) - p_{-1,1}(\mathbf{n}) - p_{-1,-1}(\mathbf{n}),$$

and the signature of $C(\iota(\mathbf{n}))$ is equal to

$$p_{1,1}(\mathbf{n}) - p_{1,-1}(\mathbf{n}) + p_{-1,1}(\mathbf{n}) - p_{-1,-1}(\mathbf{n}).$$

Let us introduce $\mathcal{D}(\mathcal{T}, n)$ the subspace of $\{-n, -n+2, \dots, n-2, n\}^{\mathcal{T}} \times \mathcal{D}(n)^{A_2}$ consisting of tuples $(\{s_T\}_{T\in\mathcal{T}}, \{\mathbf{n}_a\}_{a\in A_2})$ such that

- for all cycle $\{a, a'\}$ in A_2, $\mathbf{n}_a = \iota(\mathbf{n}_{a'})$;
- for all a in A_2, if $v^+(a)$ belongs to the triangle T of $\mathcal{T}$, the signature of $C(\mathbf{n}_a)$ is equal to s_T.

We note that for every possible choice of the indices $\{s_T\}_{T\in\mathcal{T}}$ there exist elements in $\mathcal{D}(\mathcal{T}, n)$ realizing this choice:

Lemma 9.16. *For every $\{s_T\}_{T\in\mathcal{T}}$ in $\{-n, -n+2, \dots, n\}^{\mathcal{T}}$, there exists $\{\mathbf{n}_a\}_{a\in A_2}$ in $\mathcal{D}(n)^{A_2}$ such that the tuple $(\{s_T\}_{T\in\mathcal{T}}, \{\mathbf{n}_a\}_{a\in A_2})$ is in $\mathcal{D}(\mathcal{T}, n)$.*

Proof. Let $E \subset A_2$ be a subset containing exactly one of the elements in every cycle in A_2. By definition of $\mathcal{D}(\mathcal{T}, n)$ it is enough to specify $\{\mathbf{n}_a\}_{a\in E}$.

Let thus a be in E. Let T be the triangle containing $v^+(a)$ and let T' be the triangle containing $v^-(a)$. By Lemma 9.15, we have

$$p_{1,1}(\mathbf{n}_a) + p_{1,-1}(\mathbf{n}_a) - p_{-1,1}(\mathbf{n}_a) - p_{-1,-1}(\mathbf{n}_a) = s_T$$
$$p_{1,1}(\mathbf{n}_a) - p_{1,-1}(\mathbf{n}_a) + p_{-1,1}(\mathbf{n}_a) - p_{-1,-1}(\mathbf{n}_a) = s_{T'}.$$

Therefore, we have to prove that there exists $\mathbf{n}_a$ in $\mathcal{D}(n)$ such that

$$p_{1,1}(\mathbf{n}_a) - p_{-1,-1}(\mathbf{n}_a) = \frac{s_T + s_{T'}}{2}$$
$$p_{1,-1}(\mathbf{n}_a) - p_{-1,1}(\mathbf{n}_a) = \frac{s_T - s_{T'}}{2}.$$

Define the element $\mathbf{n}_a = (\{\underline{n}_x\}_{x\in\{\pm1\}^2}, 2\underline{m})$ by

- $\underline{n}_{1,1}$ is a sequence of 1 whose length is $\max(0, (s_T + s_{T'})/2)$,
- $\underline{n}_{-1,-1}$ is a sequence of 1 whose length is $\max(0, -(s_T + s_{T'})/2)$,
- $\underline{n}_{1,-1}$ is a sequence of 1 whose length is $\max(0, (s_T - s_{T'})/2)$,
- $\underline{n}_{-1,1}$ is a sequence of 1 whose length is $\max(0, (-s_T + s_{T'})/2)$,
- $2\underline{m}$ is a sequence whose length is 0 or 1 chosen so that $\mathbf{n}_v$ belongs to $\mathcal{D}(n)$.

The bounds on s_T and $s_{T'}$ and their parity properties imply that the above construction is legitimate. □

9.9. **Pieces.** The natural projection $\pi\colon \mathcal{X}_{\mathcal{E}}(\mathcal{T},n) \to \{-n,-n+2,\dots,n-2,n\}^{\mathcal{T}} \times \mathcal{D}(n)^{A_2}$ associates to $x = (\{(\mathbf{n}_a,\boldsymbol{\lambda}_a)\},\{S_v\},\{Y_a\})$ the family $(\{s_T\},\{\mathbf{n}_a\})$ where, for each triangle T of $\mathcal{T}$, s_T is the common signature of the matrices S_v for v in T; it takes values in $\mathcal{D}(\mathcal{T},n)$. For every $\mathbf{x} = (\{s_T\},\{\mathbf{n}_a\})$ in $\mathcal{D}(\mathcal{T},n)$, we denote by $\mathcal{X}_{\mathcal{E},\mathbf{x}}(\mathcal{T},n)$ the fiber $\pi^{-1}(\mathbf{x})$. This subspace will be call a *piece* of $\mathcal{X}_{\mathcal{E}}(\mathcal{T},n)$.

Let $E \subset A_2$ be a subset containing exactly one of the elements in every cycle in A_2. Using the notation of Lemma 9.1, we define

$$V(\mathbf{x}) := \prod_{a\in E} V(\mathbf{n}_a).$$

Then $V(\mathbf{x})$ is a convex cone of dimension $d(\mathbf{x}) := \sum_{a\in E} d(\mathbf{n}_a)$.

Building on Lemma 9.9, for every 3-cycle $\{a,b,c\}$ in A_3, contained in a triangle T of $\mathcal{T}$, the space of triples of matrices (Y_a,Y_b,Y_c) satisfying the hypothesis of the lemma (or the conditions in Section 9.4) is isomorphic to

$$G_{s_T} := \mathrm{O}\Big(\frac{n+s_T}{2},\frac{n-s_T}{2}\Big)^2.$$

The product of these groups, for T in $\mathcal{T}$, is

$$G(\mathbf{x}) := \prod_{T\in\mathcal{T}} G_{s_T}.$$

This Lie group is of dimension $(2|\chi(\bar{S})|+r)n(n-1)$ as all the orthogonal groups involved have the same dimension $n(n-1)/2$ and as $\sharp\mathcal{T} = 2|\chi(\bar{S})|+r$.

The following result is a consequence of the fact that, in the description of the space of $\mathcal{X}$-coordinates (Section 9.4), the conditions on the family $\{\boldsymbol{\lambda}_a\}_{a\in A_2}$ depend only on $\{\mathbf{n}_a\}_{a\in A_2}$ and the conditions on the family $\{Y_a\}_{a\in A_3}$ depend only on $\{S_v\}_{v\in V}$.

Proposition 9.17. *(1) The piece $\mathcal{X}_{\mathcal{E},\mathbf{x}}(\mathcal{T},n)$ is isomorphic to $V(\mathbf{x})\times G(\mathbf{x})$ and is of dimension $d(\mathbf{x}) + (2|\chi(\bar{S})|+r)n(n-1)$.*
(2) The restriction of $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$ to $\mathcal{X}_{\mathcal{E},\mathbf{x}}(\mathcal{T},n)$ is continuous.
(3) Furthermore, if two elements of $\mathcal{X}_{\mathcal{E}}(\mathcal{T},n)$ have the same image under $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$, then they belong to the same piece.

Proof. Only the continuity needs a comment. It results from the continuity of the maps $\boldsymbol{\lambda} \mapsto \Phi(\mathbf{n},\boldsymbol{\lambda})$. □

We also note

Corollary 9.18. *For $\mathbf{x} = (\{s_T\}_{T\in\mathcal{T}},\{\mathbf{n}_a\}_{a\in A_2})$ in $\mathcal{D}(\mathcal{T},n)$, the piece $\mathcal{X}_{\mathcal{E},\mathbf{x}}(\mathcal{T},n)$ has nonempty interior if and only if, for all v in V, with $\mathbf{n}_v = (\{\underline{n}_x\}_{x\in\{\pm1\}^2}, 2\underline{m})$ all the integers in the sequences $\underline{n}_x$ $(x\in\{\pm1\}^2)$ and $\underline{m}$ are equal to 1.*

In this situation, the restriction of $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}$ to $\mathcal{X}_{\mathcal{E},\mathbf{x}}(\mathcal{T},n)$ is generically finite-to-one.

The decomposition of $\mathcal{X}_{\mathcal{E}}(\mathcal{T},n)$ into pieces induces a decomposition of the space of transverse local systems $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}},\mathrm{Sp}(2n,\mathbf{R}))$. However, it seems difficult to obtain topological information from this decomposition.

9.10. **Over-parameterization.** The formulas used in Section 9.6 are valid on a wider set; let

$$\mathcal{X}(\mathcal{T},n) \subset \mathrm{Sym}(n,\mathbf{R})^V \times \mathrm{GL}(n,\mathbf{R})^A$$

be the subset of tuples $z = (\{S_v\}_{v\in V},\{Y_a\}_{a\in A})$ such that

- for all cycle (a,a') in A_2, $Y_{a'} = {}^T Y_a$;
- for all v in V, S_v is nonsingular;
- for all cycle (a,b,c) in A_3, the equality $Y_c\,{}^T Y_b^{-1} Y_a = S_v$ holds where $v = v^+(c) = v^-(b)$.

For such a z, we will denote again by $\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}}(z)$ the framed δ-twisted symplectic local system arising from the following family $\{G_a(z)\}_{a\in A}$ of transition matrices:

- for a in A_2, $G_a(z) = \begin{pmatrix} 0 & -{}^T Y_a^{-1} \\ Y_a & 0 \end{pmatrix}$;
- for a in A_3, $G_a(z) = \begin{pmatrix} S_{v^+(a)}^{-1} Y_a & -{}^T Y_a^{-1} \\ Y_a & 0 \end{pmatrix}$.

The group

$$G_{\mathcal{X}} := \mathrm{GL}(n,\mathbf{R})^V$$

acts on $\mathcal{X}(\mathcal{T},n)$ via the following formula: if $z = (\{S_v\},\{Y_a\})$ belongs to $\mathcal{X}(\mathcal{T},n)$ and $g = \{g_v\}_{v\in V}$ belong to $G_{\mathcal{X}}$, then

$$g\cdot z := \big(\{\{g_v S_v\, {}^T g_v\}_{v\in V}, \{g_{v^+(a)} Y_a\, {}^T g_{v^-(a)}\}_{a\in A}\big).$$

Collecting the information about equivalent local systems (Section 5.3) and using Theorem 9.12, we get:

Theorem 9.19. *The map*

$$\mathrm{hol}^{\mathcal{X}}_{\mathcal{T}} \colon \mathcal{X}(\mathcal{T},n) \longrightarrow \mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R}))$$

is continuous, onto, and its fibers are the orbits of the action of $G_{\mathcal{X}}$.

9.11. **Connected components.** The previous theorem 9.19 can be used to determined the number of connected components of the moduli space of transverse framed local systems.

For this, let $H := \{\pm 1\} \simeq \pi_0(\mathrm{GL}(n,\mathbf{R}))$ be the group of connected components of $\mathrm{GL}(n,\mathbf{R})$, the quotient map $\mathrm{GL}(n,\mathbf{R}) \to H$ will be denoted by π_0. Let

$$Z(\mathcal{T},n) \subset \{-n,-n+2,\dots,n\}^V \times H^A$$

be the set of tuples $(\{s_v\}_{v\in V}, \{h_a\}_{a\in A_3})$ such that

- for all a in A_3, $s_{v^+(a)} = s_{v^-(a)}$;
- for all cycle (a,a') in A_2, $h_{a'} = h_a$;
- for all cycle (a,b,c) in A_3, $h_c h_b h_a = (-1)^{(n-s_{v^+(a)})/2}$.

The group $F_Z := H^V$ acts on $Z(\mathcal{T},n)$: if $h = \{h_v\}$ is in F_Z and $z = (\{s_v\},\{h_a\})$ is in $Z(\mathcal{T},n)$, then

$$h\cdot z := \big(\{s_v\}_{v\in V}, \{h_a h_{v^+(a)} h_{v^-(a)}\}_{a\in A}\big).$$

Using the fact that the space of nonsingular symmetric matrices of a given signature is connected (it is an orbit under the action of the connected group $\mathrm{GL}^+(n,\mathbf{R})$), the explicit description of $\mathcal{X}(\mathcal{T},n)$ gives:

Proposition 9.20. *The map*

$$\begin{aligned} \pi_{\mathcal{X}} \colon \mathcal{X}(\mathcal{T},n) &\longrightarrow Z(\mathcal{T},n) \\ (\{\Phi_a\},\{S_v\},\{Y_a\}) &\longmapsto (\{\pi_0(\Phi_a)\},\{\mathrm{sgn}(S_v)\},\{\pi_0(Y_a)\}) \end{aligned}$$

induces a bijection between the space of connected components of $\mathcal{X}(\mathcal{T},n)$ *and the set* $Z(\mathcal{T},n)$*. This map is equivariant with respect to the morphism*

$$\begin{aligned} \pi_G \colon G_{\mathcal{X}} &\longrightarrow F_Z \\ \{g_v\}_{v\in V} &\longmapsto \{\pi_0(g_v)\}_{v\in V}. \end{aligned}$$

In turn, there is a well defined map

$$\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R})) \simeq G_{\mathcal{X}}\backslash \mathcal{X}(\mathcal{T},n) \longrightarrow F_Z\backslash Z(\mathcal{T},n)$$

that induces a bijection between the set of connected components of $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n,\mathbf{R}))$ *and* $F_Z\backslash Z(\mathcal{T},n)$.

Proof. Let $A' \subset A_2$ be a subset containing exactly one of the arrows of every 2-cycle, let $T \subset V$ be a subset containing, for every triangle f, exactly one of the three vertices of $\Gamma_{\mathcal{T}}$ that are in f. Let B be a subset of A_3 containing exactly two of the arrows of every 3-cycle.

Thanks to Lemma 9.9, the map

$$\begin{aligned} \mathcal{X}(\mathcal{T}, n) &\longrightarrow \mathrm{Sym}^*(n, \mathbf{R})^T \times \mathrm{GL}(n, \mathbf{R})^{A' \sqcup B} \\ (\{S_v\}_{v\in V}, \{Y_a\}_{a\in A}) &\longmapsto (\{S_v\}_{v\in T}, \{Y_a\}_{a\in A'\sqcup B}) \end{aligned}$$

is a diffeomorphism where $\mathrm{Sym}^*(n, \mathbf{R})$ is the space of nonsingular symmetric matrices. Similarly, the map

$$\begin{aligned} Z(\mathcal{T}, n) &\longrightarrow \{-n, -n+2, \dots, n\}^T \times H^{A' \sqcup B} \\ (\{s_v\}_{v\in V}, \{h_a\}_{a\in A}) &\longmapsto (\{s_v\}_{v\in T}, \{h_a\}_{a\in A'\sqcup B}) \end{aligned}$$

is a bijection. Using these isomorphisms, the map $\pi_{\mathcal{X}}$ become

$$\begin{aligned} \mathrm{Sym}^*(n, \mathbf{R})^T \times \mathrm{GL}(n, \mathbf{R})^{A' \sqcup B} &\longrightarrow \{-n, -n+2, \dots, n\}^T \times H^{A' \sqcup B} \\ (\{S_v\}_{v\in T}, \{Y_a\}_{a\in A'\sqcup B}) &\longmapsto (\{\mathrm{sgn}(S_v)\}_{v\in T}, \{\pi_0(Y_a)\}_{a\in A'\sqcup B}). \end{aligned}$$

It induces a bijection at the level of connected components since $\pi_0 \colon \mathrm{GL}(n, \mathbf{R}) \to H$ and $\mathrm{sgn} \colon \mathrm{Sym}^*(n, \mathbf{R}) \to \{-n, -n+2, \dots, n\}$ do. The other statements follow from similar consideration. □

Finally, noting that the diagonal subgroup $H \subset F_Z$ acts trivially on $Z(\mathcal{T}, n)$ and that the quotient group F_Z/H acts freely on $Z(\mathcal{T}, n)$, one obtains:

Corollary 9.21. *The number of connected components of* $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ *is equal to* $2^{|\chi(\bar{S})|+1} \times (n+1)^{2|\chi(\bar{S})|+r}$.

10. $\mathcal{X}$-coordinates for representations into isogenic groups

In this section we investigate framed local systems for groups that are isogenic to $\mathrm{Sp}(2n, \mathbf{R})$ as well as coordinates on their moduli space. We first describe the groups in Section 10.1 and then the moduli spaces in Section 10.2. In order to introduce their twisted version, we need a few special elements of the Lie groups, which we introduce in Section 10.3). Finally we describe the corresponding local systems on the quiver $\Gamma_{\mathcal{T}}$ and "parametrize" them. From this we draw a few topological consequences.

10.1. **Groups.** Let G be a connected finite cover of $\mathrm{PSp}(2n, \mathbf{R})$. Thus G is a Lie group that is isomorphic to the quotient $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})/\Lambda_G$ for some uniquely determined subgroup Λ_G of $\pi_1(\mathrm{PSp}(2n, \mathbf{R})) \simeq Z(\widetilde{\mathrm{Sp}}(2n, \mathbf{R})) \subset Z(\widetilde{\mathrm{U}}(n))$. As a subgroup of $\widetilde{\mathrm{U}}(n)$ (see Section 2.4), the group $\pi_1(\mathrm{PSp}(2n, \mathbf{R}))$ is the subgroup generated by the elements

$$(\mathrm{Id}, 2\pi), \quad \text{and} \quad (-\mathrm{Id}, n\pi).$$

Depending on the parity of n, a generating system for that group is

(1) $(-\mathrm{Id}, \pi)$ if n is odd in which case $\pi_1(\mathrm{PSp}(2n, \mathbf{R}))$ is isomorphic to $\mathbf{Z}$;
(2) $(\mathrm{Id}, 2\pi)$ and $(-\mathrm{Id}, 0)$ if n is even in which case $\pi_1(\mathrm{PSp}(2n, \mathbf{R}))$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and its 2-torsion is generated by $(-\mathrm{Id}, 0)$.

Thus the subgroup Λ_G is either

(I) generated by $((-1)^{x_G}\,\mathrm{Id}, x_G\pi)$ for some uniquely determined $x_G \in \mathbf{Z}_{>0}$ when n is odd;
(II) generated by $(-\mathrm{Id}, 0)$ and $(\mathrm{Id}, x_G\pi)$ for some uniquely determined $x_G \in 2\mathbf{Z}_{>0}$ when n is even and Λ_G contains the torsion of $\pi_1(\mathrm{PSp}(2n, \mathbf{R}))$;

(III) generated by $(\mathrm{Id}, x_G\pi)$ or by $(-\mathrm{Id}, x_G\pi)$ for some uniquely determined $x_G \in 2\mathbf{Z}_{>0}$ when n is even and Λ_G does not contain the torsion of $\pi_1(\mathrm{PSp}(2n,\mathbf{R}))$.

We will designate by L and by K the subgroups of G that are the preimages of the subgroups $\mathrm{PGL}(n,\mathbf{R})$ and $\mathrm{PO}(n)$ by the homomorphism $\pi\colon G \to \mathrm{PSp}(2n,\mathbf{R})$. Thus K is a maximal compact subgroup in L and the polar decomposition induces a K-equivariant diffeomorphism between L and $K \times \mathrm{Sym}(n,\mathbf{R})$. Let also $\hat{L}$ and $\hat{K}$ be the preimages of $\mathrm{PGL}(n,\mathbf{R})$ and $\mathrm{PO}(n)$ in $\widetilde{\mathrm{PSp}}(2n,\mathbf{R})$. The group K is hence isomorphic to the quotient $\hat{K}/\Lambda_G$ (note that Λ_G is contained in the kernel of $\widetilde{\mathrm{Sp}}(2n,\mathbf{R}) \to \mathrm{PSp}(2n,\mathbf{R})$ and therefore in $\hat{K}$).

As a subgroup of $\widetilde{\mathrm{U}}(n)$ (see Section 2.4), $\hat{K}$ is

$$\hat{K} = \{(u,\theta) \in \widetilde{\mathrm{U}}(n) \mid u \in \mathrm{O}(n)\} = \mathrm{SO}(n) \times 2\pi\mathbf{Z} \bigsqcup \big(\mathrm{O}(n) \smallsetminus \mathrm{SO}(n)\big) \times (\pi + 2\pi\mathbf{Z}),$$

and its neutral component $\hat{K}_0$ is $\mathrm{SO}(n) \times \{0\}$. Let r be any element in $\mathrm{O}(n) \smallsetminus \mathrm{SO}(n)$. The homomorphism

$$\begin{aligned} \tau\colon \mathbf{Z} &\longrightarrow \hat{K} \\ n &\longmapsto (r^n, n\pi) \end{aligned}$$

induces an isomorphism $\mathbf{Z} \simeq \pi_0(\hat{K}) = \hat{K}/\hat{K}_0$ that does not depend on the choice of r. The natural map $\pi_0(\hat{K}) \to \pi_0(K)$ is onto and its kernel is $\Lambda_G\hat{K}_0$ (seen as a subgroup of $\pi_0(\hat{K})$). Using again the fact that $\mathrm{SO}(n)$ is connected, one get that in cases (I), (II) and (III) above:

- $\tau^{-1}(\Lambda_G\hat{K}_0)$ is the subgroup $x_G\mathbf{Z}$.

As a conclusion:

Lemma 10.1. *The group $\pi_0(K) \simeq \pi_0(L)$ is isomorphic to $\mathbf{Z}/x_G\mathbf{Z}$.*

10.2. **Local systems.** There is a natural action of G on $\mathcal{L}_n$ through the homomorphism $G \to \mathrm{PSp}(2n,\mathbf{R})$. A *framing* for a G-local system $\mathcal{F}$ on S is a section σ of the restriction of $\mathcal{F}_{\mathcal{L}_n}$ to ∂S. The pair $(\mathcal{F},\sigma)$ is called a *framed local system*. The moduli space of framed local system will be denoted by

$$\mathrm{Loc}^{\mathrm{f}}(S,G).$$

Any framed G-local system $(\mathcal{F},\sigma)$ induces a framed $\mathrm{PSp}(2n,\mathbf{R})$-local system $(\mathcal{F}',\sigma)$; the framed local system $(\mathcal{F},\sigma)$ is said *maximal* if $(\mathcal{F}',\sigma)$ is maximal. The subspace of maximal framed local systems will be denoted by $\mathcal{M}^{\mathrm{f}}(S,G)$. When an ideal triangulation $\mathcal{T}$ of S is given, the definition of a transverse framing with respect to $\mathcal{T}$ is as in Section 4.10 and the moduli space of transverse framed local systems will be denoted $\mathrm{Loc}^{\mathrm{f}}_{\mathcal{T}}(S,G)$.

Remark 10.2. When $R = \varnothing$, we also say that a G-local system is *maximal* if the associated $\mathrm{PSp}(2n,\mathbf{R})$-local system is maximal.

10.3. **Some elements in G.** The exponential map $\exp_G\colon \mathfrak{sp}(2n,\mathbf{R}) \to G$ enables us to define the following elements:

$$s_G := \exp_G\left(\frac{\pi}{2}\begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}\right), \quad \delta_G := s_G^2,$$
$$u_M := \exp_G\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad v_M := \exp_G\begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \quad (M \in \mathrm{Sym}(n,\mathbf{R})).$$

Remark 10.3. In the case when $G = \mathrm{Sp}(2n,\mathbf{R})$, one has $s_{\mathrm{Sp}(2n,\mathbf{R})} = \left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$, $\delta_{\mathrm{Sp}(2n,\mathbf{R})} = -\mathrm{Id}$, $u_M = \left(\begin{smallmatrix} \mathrm{Id} & M \\ 0 & \mathrm{Id} \end{smallmatrix}\right)$, and $v_M = \left(\begin{smallmatrix} \mathrm{Id} & 0 \\ M & \mathrm{Id} \end{smallmatrix}\right)$.

Lemma 10.4. *The element δ_G belongs to the center of G. As elements of $\widetilde{\mathrm{U}}(n)/\Lambda_G$, s_G is represented by $(i\,\mathrm{Id}, n\pi/2)$ and δ_G by $(-\mathrm{Id}, n\pi)$.*

Proof. The matrix $\left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ in $\mathfrak{sp}(2n,\mathbf{R})$ correspond to the matrix $i\,\mathrm{Id}$ in $\mathfrak{u}(n)$. Since, for every θ in $\mathbf{R}$, $\exp_{\widetilde{\mathrm{U}}(n)}(i\theta\,\mathrm{Id}) = (e^{i\theta}\,\mathrm{Id}, n\theta)$, the sought for equality follows applying $\theta = \pi/2$ and π. □

The automorphism of G

$$g \longmapsto g^* := s_G g s_G^{-1}$$

is therefore an involution that stabilizes the subgroup L. For the case $G = \mathrm{Sp}(2n,\mathbf{R})$, the restriction of the involution to $L = \mathrm{GL}(n,\mathbf{R})$ is $g \mapsto {}^T g^{-1}$. Using this last fact, for every S in $\mathrm{Sym}(n,\mathbf{R}) \subset \mathfrak{gl}(n,\mathbf{R}) = \mathrm{Lie}(L)$, one has $\exp_L(S)^* = \exp_L(-S)$. Using the representative of s_G in $\widetilde{\mathrm{U}}(n)$ (Lemma 10.4), we get that the restriction of $g \mapsto g^*$ to K is the trivial automorphism. Consequently, $g \mapsto g^*$ is a Cartan involution of L. (In fact it is already a Cartan involution of G.)

Furthermore, for every M in $\mathrm{Sym}(n,\mathbf{R})$

$$s_G u_M s_G^{-1} = v_{-M} \quad \text{and} \quad s_G v_M s_G^{-1} = u_{-M}.$$

The group L acts on $\mathrm{Sym}(n,\mathbf{R})$ via the homomorphism $\pi\colon L \to \mathrm{PGL}(n,\mathbf{R})$: for every g in L and every M in $\mathrm{Sym}(n,\mathbf{R})$,

$$g \cdot M = \pi(g) M\, {}^T\pi(g).$$

(This formula involves a lift of $\pi(g)$ to $\mathrm{GL}(n,\mathbf{R})$ but the result does not depend on the lift.) The stabilizer of $\mathrm{Id} \in \mathrm{Sym}(n,\mathbf{R})$ is the group K. This action is related to the adjoint action of L since, for g and M as above, $g u_M g^{-1} = u_{g\cdot M}$ and $g v_M g^{-1} = v_{g^*\cdot M}$. Furthermore, one can check that $g \cdot M$ is nonsingular if and only if M is, and in that case $(g\cdot M)^{-1} = g^{*-1}\cdot M^{-1}$.

The following lemmas are a direct generalization of Lemmas 9.7 and 9.9.

Lemma 10.5. *Let M_a, M_b, and M_c be in* $\mathrm{Sym}(n,\mathbf{R})$ *and let ℓ_a, ℓ_b, and ℓ_c be in L. Define the elements of G*

$$A := u_{M_a} s_G \ell_a,\ B := u_{M_b} s_G \ell_b,\ \textit{and } C := u_{M_c} s_G \ell_c.$$

Suppose that $CBA = \delta_G$. Then

- *the matrices M_a, M_b, and M_c are nonsingular;*
- *one has $M_c^{-1} = \ell_c \cdot M_b$, $M_b^{-1} = \ell_b \cdot M_a$, and $M_a^{-1} = \ell_a \cdot M_c$;*
- *the element $u_{M_c} v_{-\ell_c\cdot M_b} u_{\ell_c^*\ell_b\cdot M_a} s_G$ belongs to L and is equal to $(\ell_c\ell_b^*\ell_a)^{-1}$.*

Remark 10.6. When $G = \mathrm{Sp}(2n,\mathbf{R})$, the product $u_M s_{\mathrm{Sp}(2n,\mathbf{R})} g$ is equal to $\left(\begin{smallmatrix} MY & -{}^TY^{-1} \\ Y & 0 \end{smallmatrix}\right)$ where $g = \left(\begin{smallmatrix} Y & 0 \\ 0 & {}^TY^{-1} \end{smallmatrix}\right)$.

Proof. One has

$$\begin{aligned} CBA &= u_{M_c} s_G \ell_c\ u_{M_b} s_G \ell_b\ u_{M_a} s_G \ell_a \\ &= u_{M_c} s_G (u_{\ell_c\cdot M_b}\ell_c) s_G \ell_b u_{M_a} s_G \ell_a \quad (\text{as } \ell_c u_{M_b} \ell_c^{-1} = u_{\ell_c\cdot M_b}) \\ &= u_{M_c} s_G u_{\ell_c\cdot M_b} (s_G \ell_c^*) \ell_b u_{M_a} s_G \ell_a \quad (\text{using that } s_G^{-1}\ell_c s_G = \ell_c^*) \\ &= u_{M_c} (v_{-\ell_c\cdot M_b} (s_G)^2) \ell_c^* \ell_b u_{M_a} s_G \ell_a \quad (s_G u_{\ell_c\cdot M_b} s_G^{-1} = v_{-\ell_c\cdot M_b}) \\ &= u_{M_c} v_{-\ell_c\cdot M_b} \delta_G (u_{\ell_c^*\ell_b\cdot M_a} \ell_c^*\ell_b) s_G \ell_a \quad (\ell_c^*\ell_b u_{M_a} (\ell_c^*\ell_b)^{-1} = u_{\ell_c^*\ell_b\cdot M_a}) \\ &= u_{M_c} v_{-\ell_c\cdot M_b} \delta_G u_{\ell_c^*\ell_b\cdot M_a} s_G \ell_c \ell_b^* \ell_a, \end{aligned}$$

thus $u_{M_c} v_{-\ell_c\cdot M_b} u_{\ell_c^*\ell_b\cdot M_a} s_G$ is equal to $(\ell_c\ell_b^*\ell_a)^{-1}$ and hence must belongs to L. Denote by $N_b = \ell_c \cdot M_b$ and $N_a = \ell_c^*\ell_b \cdot M_a$, then the following product of symplectic matrices

$$\begin{pmatrix} \mathrm{Id} & M_c \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -N_b & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & N_a \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$$

projects to $\pi(u_{M_c}v_{-\ell_c\cdot M_b}u_{\ell_c^*\ell_b\cdot M_a}s_G)$ in $\mathrm{PSp}(2n,\mathbf{R})$ and must thus belong to $\mathrm{GL}(n,\mathbf{R})\subset\mathrm{Sp}(2n,\mathbf{R})$. Since the above product of matrices is equal to:

$$\begin{pmatrix}(\mathrm{Id}-M_cN_b)N_a+M_c & M_cN_b-\mathrm{Id}\\ -N_bN_a+\mathrm{Id} & N_b\end{pmatrix},$$

it turns that $M_cN_b=\mathrm{Id}$ and $N_bN_a=\mathrm{Id}$. This implies the first two conclusions of the lemma and the third one was already observed. ☐

Conversely:

Lemma 10.7. *Let M_a be in $\mathrm{Sym}^*(n,\mathbf{R})$ and let ℓ_b, and ℓ_c be in L. Define $M_b:=\ell_b^*\cdot M_a^{-1}$ and $M_c:=\ell_c^*\ell_b\cdot M_a$. Then the product $h:=u_{M_c}v_{-\ell_c\cdot M_b}u_{\ell_c^*\ell_b\cdot M_a}s_G$ belongs to L.*

Furthermore denoting $\ell_a:=\ell_b^{*-1}\ell_c^{-1}h^{-1}$, $A:=u_{M_a}s_G\ell_a$, $B:=u_{M_b}s_G\ell_b$, and $C:=u_{M_c}s_G\ell_c$, then $CBA=\delta_G$.

Remark 10.8. The element h is of course equal to $u_{M_c}v_{-M_c^{-1}}u_{M_c}s_G$ and varies continuously with M_c.

The element s_G could be expressed as a product of u_Ms and v_Ms, precisely:

Lemma 10.9. *The product $u_{\mathrm{Id}}v_{-\mathrm{Id}}u_{\mathrm{Id}}s_G$ is trivial in G. The product $u_{-\mathrm{Id}}v_{\mathrm{Id}}u_{-\mathrm{Id}}s_G$ is equal to δ_G.*

Proof. Let $u=\exp_{\widetilde{\mathrm{Sp}}(2n,\mathbf{R})}\left(\begin{smallmatrix}0&\mathrm{Id}\\0&0\end{smallmatrix}\right)$, $v=\exp_{\widetilde{\mathrm{Sp}}(2n,\mathbf{R})}\left(\begin{smallmatrix}0&0\\-\mathrm{Id}&0\end{smallmatrix}\right)$, and $s=\exp_{\widetilde{\mathrm{Sp}}(2n,\mathbf{R})}\left(\begin{smallmatrix}0&-\pi/2\,\mathrm{Id}\\ \pi/2\,\mathrm{Id}&0\end{smallmatrix}\right)$. It is then enough to prove the equality $uvus=e_{\widetilde{\mathrm{Sp}}(2n,\mathbf{R})}$. Using embeddings as in Section 2.4, one can assume furthermore that $n=1$, hence the equality has to be checked in $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ or even in $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$.

Elements of $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ will be represented by paths in $\mathrm{GL}^+(2,\mathbf{R})$. A path representing u is

$$\theta\in[0,\pi/2]\longmapsto\begin{pmatrix}1&\sin\theta\\0&1\end{pmatrix},$$

a path representing v is

$$\theta\in[0,\pi/2]\longmapsto\begin{pmatrix}1&0\\-\sin\theta&1\end{pmatrix},$$

and a path representing s is

$$\theta\in[0,\pi/2]\longmapsto\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}.$$

Thus the element $uvus$ of $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ is represented by the loop

$$\theta\in[0,\pi/2]\longmapsto\begin{pmatrix}\cos^3\theta+\cos^2\theta\sin^2\theta+\sin^2\theta & \cos\theta\sin\theta(\cos^2\theta-\cos\theta+1)\\ \sin\theta\cos\theta(\cos\theta-1) & \sin^2\theta+\cos^3\theta\end{pmatrix}.$$

This loop is contained in the subspace $T=\{A\in\mathrm{GL}^+(2,\mathbf{R})\mid \operatorname{tr}A>0\}$. Since, for every matrix A, and every t in $\mathbf{R}$, $\operatorname{tr}(tA+\mathrm{Id})=t\operatorname{tr}A+2$ and $\det(tA+\mathrm{Id})=t^2\det(A)+t\operatorname{tr}A+1$, the space T is contractible and the above loop is homotopically trivial in $\mathrm{GL}^+(2,\mathbf{R})$. This concludes the equality $uvus=e_{\widetilde{\mathrm{SL}}(2,\mathbf{R})}$.

Conjugating with s_G gives $s_Gu_{\mathrm{Id}}v_{-\mathrm{Id}}u_{\mathrm{Id}}=e_G$; taking inverses we get $u_{-\mathrm{Id}}v_{\mathrm{Id}}u_{-\mathrm{Id}}s_G^{-1}=e_G$, which implies the second equality. ☐

Remark 10.10. For any decomposition $n=p+q$, let $I_{p,q}$ be the symmetric matrix $\left(\begin{smallmatrix}\mathrm{Id}_p&0\\0&-\mathrm{Id}_q\end{smallmatrix}\right)$, then the element $u_{I_{p,q}}v_{-I_{p,q}}u_{I_{p,q}}s_G$ belongs to K and is represented by the element $(I_{p,q},q\pi)$ of $\hat{K}\subset\widetilde{\mathrm{U}}(n)$.

10.4. **Twisted local systems.** Twisted local systems were defined in Section 4.5, their moduli space is denoted $\mathrm{Loc}_\delta(S,G)$. Using trivializations $T'S \simeq S \times \mathbf{C}^*$, or, what amounts to the same, global sections of $T'S$, one gets bijective correspondence with local systems on S:

Proposition 10.11. *Any nonvanishing vector field $\vec{x}$ on S induces, via pull back, an isomorphism*

$$\mathrm{Loc}_\delta(S,G) \longrightarrow \mathrm{Loc}(S,G).$$

For a δ-twisted G-local system $\mathcal{F}$, a *framing* of $\mathcal{F}$ is a flat section of the restriction of $\mathcal{F}_{\mathcal{L}_n}$ to $T'S|_{\partial S}$. The pair $(\mathcal{F},\sigma)$ is called a *framed local system.* Equivalently (cf. Section 4.9) a section of the restriction of $\mathcal{F}_{\mathcal{L}_n}$ to $\vec{\partial} S$ can be called a *framing* of $\mathcal{F}$.

Similar to Proposition 10.11, one has

Proposition 10.12. *Pulling back by any nonvanishing vector field gives an isomorphism between the space $\mathrm{Loc}^{\mathrm{f}}_\delta(S,G)$ of framed twisted G-local systems and $\mathrm{Loc}^{\mathrm{f}}(S,G)$.*

The respective images of $\mathrm{Loc}^{\mathrm{f}}_{\mathcal{T}}(S,G)$ and $\mathcal{M}^{\mathrm{f}}(S,G)$ will be denoted $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(S,G)$ and $\mathcal{M}^{\mathrm{f}}_\delta(S,G)$; their elements are called as well *transverse* (respectively *maximal*).

10.5. **Local systems on the quiver $\Gamma_{\mathcal{T}}$.** The discussion of Section 5 can be adapted to representations into G. If $\Gamma=(V,A)$ is a quiver, a *G-local system* on Γ is the data $(\{H_v\}_{v\in V},\{g_a\}_{a\in A})$ where, for all v in V, H_v is a right G-space with simply transitive G-action (thus isomorphic to the space G with the right action coming from the multiplication), and, for all a in A, g_a is a G-morphism from $H_{v^-(a)}$ to $H_{v^+(a)}$ (hence an isomorphism).

Two local systems (H_v,g_a) and (H'_v,g'_a) are *equivalent* if there is a family $\{\psi_v\}_{v\in V}$ such that, for every v in V, $\psi_v\colon H_v\to H'_v$ is a G-morphism, and, for every a in A, $\psi_{v^+(a)}\circ g_a = g'_a\circ\psi_{v^-(a)}$. The moduli space of G-local systems is denoted $\mathrm{Loc}(\Gamma,G)$.

Fixing a base point b_v in H_v, for every v in V, one obtains, for every a in A, an element G_a of G via the equality: $g_a(b_{v^-(a)}) = b_{v^+(a)}\cdot G_a$. The tuples $(\{H_v\},\{b_v\},\{g_a\})$ and $(\{H_v\},\{b_v\},\{g_a\},\{G_a\})$ or even the family $\{G_a\}$ will be called G-local systems. In fact, there is a one-to-one correspondence between equivalence classes of *based* local systems and the space G^A.

For the quiver $\Gamma_{\mathcal{T}}$ associated with an ideal triangulation $\mathcal{T}$ on the surface S (cf. Sections 4.7 and 5.1), one set:

Definition 10.13. A local system $(\{H_v\},\{g_a\})$ on the quiver $\Gamma_{\mathcal{T}}$ is *δ-twisted* if

- for every 2-cycle (a,a') in A_2, $g_a\circ g_{a'}=\delta_G^{-1}$ (i.e. for every b in $H_{v^+(a)}$, $g_a\circ g_{a'}(b)=b\cdot\delta_G^{-1}$; this holds if and only if there exists b in $H_{v^+(a)}$ such that $g_a\circ g_{a'}(b)=b\cdot\delta_G^{-1}$);
- for every 3-cycle (a,b,c) in $\Gamma_{\mathcal{T}}$, $g_c\circ g_b\circ g_a=\delta_G$.

Via the restriction map, the moduli space $\mathrm{Loc}_\delta(\Gamma_{\mathcal{T}},G)$ of δ-twisted local systems is isomorphic to the space $\mathrm{Loc}_\delta(S,G)$.

10.6. **Framed local systems.** For a G-local system $(\{H_v\},\{g_a\})$, the space $\mathcal{L}_v$ of Lagrangians in H_v is the quotient $G\backslash(H_v\times\mathcal{L}_n)$ by the diagonal action of G: $g\cdot(b,L):=(b\cdot g^{-1},g\cdot L)$. For every a in A, the isomorphism $g_a\colon H_{v^-(a)}\to H_{v^+(a)}$ induces an isomorphism $\mathcal{L}_{v^-(a)}\to\mathcal{L}_{v^+(a)}$ denoted again by g_a.

When the local system $(\{H_v\},\{g_a\})$ is on the quiver $\Gamma_{\mathcal{T}}$ and is δ-twisted, a *framing* is a family $\{(L^t_v,L^b_v)\}_{v\in V}$ such that, for every v in V, L^t_v and L^b_v belong to $\mathcal{L}_v$, and, for every a in A, $g_a(L^b_{v^-(a)})=L^t_{v^+(a)}$.

For a in A_2, let a' be in A_2 the arrow such that (a,a') is a 2-cycle, then one has $g_a(L^t_{v^-(a)}) = g_a\circ g_{a'}(L^b_{v^+(a)}) = L^b_{v^+(a)}$ since the action of $\delta_G^{-1}=g_a\circ g_{a'}$ is trivial on the Lagrangian variety.

The notion of equivalence for framed local systems is adapted directly from Section 5.6. Via the restriction map, the moduli space $\mathrm{Loc}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, G)$ of framed δ-twisted local systems is isomorphic to $\mathrm{Loc}^{\mathrm{f}}_{\delta}(S, G)$.

A framed local system $(\{H_v\}, \{g_a\}, \{(L^t_v, L^b_v)\})$ is called *transverse* if, for every v in V, the Lagrangians L^t_v and L^b_v are transverse. The space $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}}, G)$ of transverse framed δ-twisted local systems is isomorphic to $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(S, G)$.

A framed local system $(\{H_v\}, \{g_a\}, \{(L^t_v, L^b_v)\})$ is called *maximal* if, for every a in A_3, the triple of Lagrangians $(L^t_{v^+(a)}, g_a(L^t_{v^-(a)}), L^b_{v^+(a)})$ is maximal. Such a local system is automatically transverse. The moduli space $\mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}}, G)$ of maximal framed δ-twisted local systems is isomorphic to $\mathcal{M}^{\mathrm{f}}_{\delta}(S, G)$.

10.7. **Parameters.** We will denote by

$$\mathcal{X}(\mathcal{T}, G) \subset L^{A_2} \times \mathrm{Sym}(n, \mathbf{R})^V \times L^{A_3}$$

the subspace of tuples $z = \big(\{\phi_a\}_{a\in A_2}, \{M_v\}_{v\in V}, \{\ell_a\}_{a\in A_3}\big)$ such that

- for all 2-cycle (a, b) in A_2, $\phi^*_a = \phi^{-1}_b$,
- for all v in V, M_v is nonsingular,
- for all a in A_3, $M^{-1}_{v^+(a)} = \ell_a \cdot M_{v^-(a)}$,
- for all 3-cycle (a, b, c), $u_{M_{v^+(c)}} v_{-\ell_c\cdot M_{v^+(b)}} u_{\ell^*_c\ell_b\cdot M_{v^+(a)}} s_G\, \ell_c\ell^*_b\ell_a = e_G$.

The subspace $\mathcal{X}^+(\mathcal{T}, G)$ consists of the tuples z in $\mathcal{X}(\mathcal{T}, G)$ such that

- for every a in A_2, ϕ_a belongs to $\exp_L(\mathrm{Sym}(n, \mathbf{R}))$, and
- for every v in V, $M_v = \mathrm{Id}$.

Let z be in $\mathcal{X}^+(\mathcal{T}, G)$. For all a in A_3 the element ℓ_a belongs to K (since K is the stabilizer of Id for the action of L on $\mathrm{Sym}(n, \mathbf{R})$) and, for all 3-cycle (a, b, c), $\ell_c\ell_b\ell_a = e_G$ since, by Lemma 10.9, $u_{\mathrm{Id}} v_{-\mathrm{Id}} u_{\mathrm{Id}} s_G = e_G$ and since $g^* = g$ for every g in K. Also for every a in A_2, there is a unique S_a in $\mathrm{Sym}(n, \mathbf{R})$, such that $\phi_a = \exp_L(S_a)$ and, for every 2-cycle (a, b) in A_2, one has $S_a = S_b$ since $\phi^{-1}_b = \phi^*_a$.

Accordingly the space $\mathcal{X}^+(\mathcal{T}, G)$ can be also described as the set of tuples $(\{S_a\}_{a\in A_2}, \{\ell_a\}_{a\in A_3})$ in $\mathrm{Sym}(n, \mathbf{R})^{A_2} \times K^{A_3}$ such that, for every cycle (a, b) in A_2, $S_a = S_b$, and, for every 3-cycle (a, b, c) in A_3, $\ell_c\ell_b\ell_a = e_G$.

The group $G_{\mathcal{X}} := L^V$ acts on $\mathcal{X}(\mathcal{T}, G)$ via the following rule: for all $m = \{m_v\}_{v\in V}$ and all $z = \big(\{\phi_a\}_{a\in A_2}, \{M_v\}_{v\in V}, \{\ell_a\}_{a\in A_3}\big)$,

$$m \cdot z := \big(\{m_{v^+(a)}\phi_a m^{*-1}_{v^-(a)}\}_{a\in A_2}, \{m_v \cdot M_v\}_{v\in V}, \{m_{v^+(a)}\ell_a m^{*-1}_{v^-(a)}\}_{a\in A_3}\big).$$

Remark 10.14. The meaningless difference with respect to the formula for the action in Section 9.10 involves just the precomposition with $m \mapsto m^*$.

The subgroup $K_{\mathcal{X}} \subset K^V \subset G_{\mathcal{X}}$ consisting on tuples $\{k_v\}_{v\in V}$ for which, for every a in A_2, $k_{v^+(a)} = k_{v^-(a)}$ stabilizes the subspace $\mathcal{X}^+(\mathcal{T}, G)$ and there is therefore an induced action of $K_{\mathcal{X}}$ on $\mathcal{X}^+(\mathcal{T}, G)$. An element $k = \{k_v\}_{v\in V}$ acts on $z = (\{S_a\}_{a\in A_2}, \{\ell_a\}_{a\in A_3})$ in $\mathcal{X}^+(\mathcal{T}, G) \subset \mathrm{Sym}(n, \mathbf{R})^{A_2} \times K^{A_3}$ by

$$k \cdot z = \big(\{k_{v^+(a)} S_a k^{-1}_{v^-(a)}\}_{a\in A_2}, \{k_{v^+(a)}\ell_a k^{-1}_{v^-(a)}\}_{a\in A_3}\big)$$

Let us fix a subset $E \subset A_2$ containing one element of every 2-cycle and a subset $W \subset V$ containing one of the three vertices in every triangle of $\mathcal{T}$. Let us denote by B the subset of A_3 of the arrows a having one of its endpoints in W. The group $G_{\mathcal{X}}$ acts on $L^E \times \mathrm{Sym}^*(n, \mathbf{R})^W \times L^B$: for all $m = \{m_v\}_{v\in V}$ and all $z = \big(\{\phi_a\}_{a\in E}, \{M_v\}_{v\in W}, \{\ell_a\}_{a\in B}\big)$,

$$m \cdot z := \big(\{m_{v^+(a)}\phi_a m^{*-1}_{v^-(a)}\}_{a\in E}, \{m_v \cdot M_v\}_{v\in W}, \{m_{v^+(a)}\ell_a m^{*-1}_{v^-(a)}\}_{a\in B}\big).$$

Using Lemmas 10.5 and 10.7, one get:

Lemma 10.15. *The map*

$$\mathcal{X}(\mathcal{T},G) \longrightarrow L^E \times \mathrm{Sym}^*(n,\mathbf{R})^W \times L^B$$
$$\big(\{\phi_a\}_{a\in A_2},\{M_v\}_{v\in V},\{\ell_a\}_{a\in A_3}\big) \longmapsto \big(\{\phi_a\}_{a\in E},\{M_v\}_{v\in W},\{\ell_a\}_{a\in B}\big)$$

is a $G_{\mathcal{X}}$-equivariant diffeomorphism.

Similarly, there is a natural action of $K_{\mathcal{X}}$ on $\mathrm{Sym}(n,\mathbf{R})^E \times K^B$ and:

Lemma 10.16. *The map*

$$\mathcal{X}^+(\mathcal{T},G) \longrightarrow \mathrm{Sym}(n,\mathbf{R})^E \times K^B$$
$$\big(\{S_a\}_{a\in A_2},\{\ell_a\}_{a\in A_3}\big) \longmapsto \big(\{S_a\}_{a\in E},\{\ell_a\}_{a\in B}\big)$$

is a $K_{\mathcal{X}}$-equivariant diffeomorphism.

10.8. **Holonomy.** Let $z = (\{\phi_a\}_{a\in A_2},\{M_v\}_{v\in V},\{\ell_a\}_{a\in A_3})$ be in $\mathcal{X}(\mathcal{T},G)$ (respectively $z = (\{S_a\}_{a\in A_2},\{\ell_a\}_{a\in A_3})$ in $\mathcal{X}^+(\mathcal{T},G)$). We associate to z the (based) framed δ-twisted G-local system $\mathrm{hol}^{\mathcal{X}}_{G,\mathcal{T}}(z) := (\{H_v\}_{v\in V},\{g_a\}_{a\in A},\{(L^t_v,L^b_v)\}_{v\in V})$ (respectively $\mathrm{hol}^{\mathcal{X},+}_{G,\mathcal{T}}$) where:

- for every v in V, $H_v = G$ (as a right G-space); thus $\mathcal{L}_v = \mathcal{L}_n$ is the space of Lagrangians in $\mathbf{R}^{2n}$;
- for every v in V, $L^t_v = \mathrm{Span}(\mathbf{e}_0)$ and $L^b_v = \mathrm{Span}(\mathbf{f}_0)$ where $(\mathbf{e}_0,\mathbf{f}_0)$ is the standard symplectic basis of $\mathbf{R}^{2n}$;
- for every a in A_2, $g_a\colon G\to G \mid h\mapsto hs_G^{-1}\phi_a$ (respectively $h\mapsto hs_G^{-1}\exp_L(S_a)$);
- for every a in A_3, g_a is the right multiplication by $u_{M_{v^+(a)}}s_G\ell_a$ (respectively by $u_{\mathrm{Id}}s_G\ell_a$).

The fact that this construction gives indeed a framed twisted local system follows directly from the conditions defining $\mathcal{X}(\mathcal{T},G)$ and the fact that the action of s_G on $\mathcal{L}_n$ permutes $\mathrm{Span}(\mathbf{e}_0)$ and $\mathrm{Span}(\mathbf{f}_0)$. This local system is clearly transverse (and maximal in the case $z\in\mathcal{X}^+(\mathcal{T},G)$).

The following results are direct generalizations of Theorem 9.19 and of Section 6.6.

Theorem 10.17. *The map*

$$\mathrm{hol}^{\mathcal{X}}_{G,\mathcal{T}}\colon \mathcal{X}(\mathcal{T},G) \longrightarrow \mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}},G)$$

is onto and its fibers are the orbits of $G_{\mathcal{X}}$, hence the quotient $G_{\mathcal{X}}\backslash\mathcal{X}(\mathcal{T},G)$ is isomorphic to $\mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}},G)$.

Theorem 10.18. *The map*

$$\mathrm{hol}^{\mathcal{X},+}_{G,\mathcal{T}}\colon \mathcal{X}^+(\mathcal{T},G) \longrightarrow \mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}},G)$$

is onto and its fibers are the orbits of $K_{\mathcal{X}}$, hence the quotient $K_{\mathcal{X}}\backslash XplusTG$ is isomorphic to $\mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}},G)$.

10.9. **Connected components.** Let H be the group $\pi_0(L)\simeq\pi_0(K)$; H is isomorphic to $\mathbf{Z}/x_G\mathbf{Z}$ (cf. Lemma 10.1, where $x_G\in\mathbf{Z}_{>0}$ is the integer that determines the group Λ_G).

For s in $\{-n,-n+2,\dots,n\}$, the class in H of the element $u_M v_{-M^{-1}} u_M s_G$ does not depend on the nonsingular symmetric matrix M of signature s (cf. Remark 10.8). We will denote $d(s)$ this element of H. In the isomorphism $H\simeq\mathbf{Z}/x_G\mathbf{Z}$ given by Lemma 10.1, $d(s)$ is represented by the integer $(n-s)/2$ modulo x_G (cf. Remark 10.10).

Similarly to Section 9.11, let $Z(\mathcal{T},G)$ be the subset of tuples $(\{h_a\}_{a\in A_2},\{s_v\}_{v\in V},\{h_a\}_{a\in A_3})$ in $H^{A_2}\times\{-n,-n+2,\dots,n\}^V\times H^{A_3}$ such that

- for all 2-cycle (a,b) in A_2, $h_ah_b = e_H$;
- for all v and v' in V that are in the same triangle of $\mathcal{T}$, $s_v = s_{v'}$;

- for all 3-cycle (a,b,c) in A_3, $h_c h_b h_a = d(s_{v^+(a)})$.

The group $F_Z = H^V$ acts in an "obvious" way on $Z(\mathcal{T},G)$. Generalizing Proposition 9.20 and Corollary 9.21, we have:

Proposition 10.19. *The natural map $\mathcal{X}(\mathcal{T},G) \to Z(\mathcal{T},G)$ is equivariant with respect to the natural homomorphism $G_{\mathcal{X}} \to F_Z$. This map and this homomorphism induce bijection at the level of connected components. As a result, the corresponding map between the set of connected components of $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}},G)$ and $F_Z\backslash Z(\mathcal{T},G)$ is a bijection.*

As the cardinality of $H = \pi_0(L)$ is equal to x_G, we get:

Corollary 10.20. *The number of connected components of $\mathrm{Loc}^{\mathrm{f}}_{\delta,\mathcal{T}}(\Gamma_{\mathcal{T}},G)$ is equal to*

$$x_G^{|\chi(\bar{S})|+1} \times (n+1)^{2|\chi(\bar{S})|+r}.$$

The number of connected components of $\mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}},G)$ is equal to $x_G^{|\chi(\bar{S})|+1}$.

10.10. **Homotopy type of the space of maximal framed local systems.** Theorem 10.17 and the fact that there exists a K-equivariant retraction of $\mathrm{Sym}(n,\mathbf{R})$ on $\{0\}$ imply

Corollary 10.21. *Let $\mathcal{X}_0^+(\mathcal{T},G)$ be the space of tuples $(\{S_a\}_{a\in A_2},\{\ell_a\}_{a\in A_3})$ such that $S_a = 0$ for all $a \in A_2$. Then $K_{\mathcal{X}}\backslash\mathcal{X}_0^+(\mathcal{T},G)$ is a strong deformation retract of $\mathcal{M}^{\mathrm{f}}_{\delta}(\Gamma_{\mathcal{T}},G)$.*

Furthermore the quotient $K_{\mathcal{X}}\backslash\mathcal{X}_0^+(\mathcal{T},G)$ is isomorphic to $K\backslash K^{|\chi(\bar{S})|+1}$ (quotient by the diagonal conjugation action).

In the notation of Section 7.1, when $G = \mathrm{Sp}(2n,\mathbf{R})$, the image of $\mathcal{X}_0^+(\mathcal{T},G)$ by the holonomy map is the subspace $\mathcal{D}^{\mathrm{f}}_{\mathcal{T}}(S,\mathrm{SL}(2,\mathbf{R})\otimes\mathrm{O(n)})$. Since, for any G, framed representations in the subspace $K_{\mathcal{X}}\backslash\mathcal{X}_0^+(\mathcal{T},G)$ projects in $\mathrm{PSp}(2n,\mathbf{R})$ to the subspace $\mathrm{PO}(n)_{\mathcal{Z}}\backslash\mathcal{X}_0^+(\mathcal{T},\mathrm{PSp}(2n,\mathbf{R}))$, one can apply, in the case $R=\varnothing$, Proposition 7.9 to deduce that these representations have a unique framing. From this, exactly as in Theorem 7.4, one deduces that, in that case, $\mathcal{M}^{\mathrm{f}}(S,G) \to \mathcal{M}(S,G)$ induces a bijection at the level of connected components.

Corollary 10.22. *Suppose that $R=\varnothing$. The space $\mathcal{M}(S,G)$ has $x_G^{|\chi(S)|+1}$ connected components. Here $x_G \in \mathbf{Z}_{>0}$ is the integer that determines the group Λ_G (Section 10.1).*

11. $\mathcal{A}$-coordinates for decorated local systems

In this section we investigate the coordinates on the space of decorated δ-twisted local systems given by the symplectic Λ-lengths. We prove that the Λ-lengths with respect to a triangulation give a one-to-one parameterization of the space of transverse (with respect to that triangulation) decorated representations. Furthermore we show that the Λ-lengths produce a geometric realization of the "noncommutative surfaces" introduced by Berenstein and Retakh [2]; those are noncommutative algebras associated with the surfaces S and that exhibit mapping class group invariant noncommutative cluster structure.

11.1. **Symplectic Λ-length.** For every arc α in $T'S$ —this means here that α is (the homotopy class of) a path $\alpha\colon [0,1] \to T'S$ with $\alpha(\{0,1\}) \subset \vec{\partial}S$— one associates a Λ-length Λ_α on the space $\mathrm{Loc}^{\mathrm{d}}_{\delta}(S,\mathrm{Sp}(2n,\mathbf{R}))$. Namely, to every decorated twisted local system $(\mathcal{F},\beta)$, its pull back by α gives a pair $(\mathbf{v}^t,\mathbf{v}^b)$ of decorated Lagrangians in $\mathbf{R}^{2n}$ well defined up to the action of $\mathrm{Sp}(2n,\mathbf{R})$ (compare with Section 4.11). Thus the matrix $\Lambda_\alpha(\mathcal{F},\beta) = \omega(\mathbf{v}^t,\mathbf{v}^b)$ is well defined.

11.2. **Field of rational functions on the space of decorated local systems.** The space $\mathrm{Loc}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ identifies with the quotient by the diagonal action of $\mathrm{Sp}(2n, \mathbf{R})$ of the space $\mathrm{Hom}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$, a subspace of

$$\mathrm{Hom}_\delta(S, \mathrm{Sp}(2n, \mathbf{R})) \times (\mathcal{L}_n^{\mathrm{d}})^p \times \prod_{\ell=1}^{k-p} (\mathcal{L}_n^{\mathrm{d}})^{r_\ell}$$

defined by algebraic equations (cf. Proposition 4.15). Hence $\mathrm{Hom}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ is an affine variety, since the decorated Lagrangian Grassmannian is an affine variety (it is a subvariety of $(\mathbf{R}^{2n})^n$). This variety is even defined over $\mathbf{Q}$ but this will play no role in the sequel. We will denote by $\mathcal{K}$ the field of rational functions on the affine variety $\mathrm{Hom}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$. For any arc α, the Λ-length Λ_α will be consider as an element of the algebra $M_n(\mathcal{K})$. Of course, it is even a matrix with entries in the subfield of $\mathrm{Sp}(2n, \mathbf{R})$-invariant elements of $\mathcal{K}$.

Proposition 11.1. *For every arc α, the element Λ_α belongs to $\mathrm{GL}(n, \mathcal{K})$.*

Proof. Since $\mathrm{Hom}_\delta^{\mathrm{d}}(S, \mathrm{Sp}(2n, \mathbf{R}))$ is an affine variety, it is enough to find one decorated twisted local system $(\mathcal{F}, \beta)$ for which $\Lambda_\alpha(\mathcal{F}, \beta)$ is an invertible matrix. This is clearly satisfied for every maximal local system since a maximal local system is α-transverse (Corollary 4.33), hence the proposition is proved. ☐

This proposition enables us to consider the element Λ_α^{-1} of $\mathrm{GL}(n, \mathcal{K})$.

11.3. **Decorated local systems on $\Gamma_{\mathcal{T}}$.** Let $\mathcal{T}$ be an ideal triangulation of S. Using the vector field $\vec{x}_{\mathcal{T}}$, the space of decorated local systems is isomorphic to the moduli space $\mathrm{Loc}_\delta^{\mathrm{d}}(\Gamma_{\mathcal{T}}, \mathrm{Sp}(2n, \mathbf{R}))$ of decorated δ-twisted symplectic local systems on the quiver $\Gamma_{\mathcal{T}}$, where we used the following definition (V denotes the vertex set of $\Gamma_{\mathcal{T}}$ and $A = A_2 \sqcup A_3$ its arrow set):

Definition 11.2. A tuple $(\{F_v, \mathbf{f}_v^t, \mathbf{f}_v^b\}_{v \in V}, \{g_a\}_{a \in A})$ is a *decorated* δ-twisted symplectic local system if

1. $(\{F_v\}_{v \in V}, \{g_a\}_{a \in A})$ is a δ-twisted symplectic local system (cf. Definition 5.8);
2. for all v in V, f_v^t and f_v^b are decorated Lagrangians in F_v;
3. for all a in A, $g_a(f_{v^-(a)}^b) = f_{v^+(a)}^t$.

The associated framed local system is $(\{F_v, L_v^t, L_v^b\}_{v \in V}, \{g_a\}_{a \in A})$ where, for all v, $L_v^t = \mathrm{Span}(\mathbf{f}_v^t)$ and $L_v^b = \mathrm{Span}(\mathbf{f}_v^b)$.

The decorated local system will be called transverse (or $\mathcal{T}$-transverse) if the associated framed local system is transverse, i.e. if and only if, for every v in V, the pair $(\mathbf{f}_v^t, \mathbf{f}_v^b)$ is a basis of F_v. This happens if and only if, for any arc α in $T'S$ that projects (in S) to an edge of $\mathcal{T}$, the symplectic Λ-length Λ_α is an invertible matrix.

11.4. **Lifting arcs.** Let $\alpha\colon ([0,1], \{0,1\}) \to (S, \partial S)$ be an arc in S. We construct a *lift*

$$r(\alpha)\colon ([0,1], \{0,1\}) \to (T'S, \vec{\partial} S)$$

via the following procedure: choose first a $\mathcal{C}^1$-representative of α having the minimal possible self-intersections (for example the geodesic representative for an hyperbolic structure) and then smooth this representative at its endpoints so that it becomes tangent at the boundary there; then $r(\alpha) : [0,1] \to T'S$ is the tangent curve of this last curve.

It is easy to observe that one can choose representatives of $r(\alpha)$ and of $r(\bar{\alpha})$ so that $r(\alpha) \sqcup r(\bar{\alpha})$ is homotopic to a fiber $T'S \to S$. In particular, the holonomy around this loop of a twisted local system is $-\mathrm{Id}$, therefore the equality $\Lambda_{\overline{r(\alpha)}} = -\Lambda_{r(\bar{\alpha})}$ follows.

11.5. **$\mathcal{A}$-space.** Note that the following properties hold:

(1) for every arc α in S, $\Lambda_{r(\bar\alpha)} = {}^T\Lambda_{r(\alpha)}$, simply since $\Lambda_{r(\bar\alpha)} = -\Lambda_{\overline{r(\alpha)}}$ (see above) and since $\Lambda_{\overline{r(\alpha)}} = -{}^T\Lambda_{r(\alpha)}$;

(2) for every triangle $(\alpha_1, \alpha_2, \alpha_3)$ in S, the matrix $\Lambda_{r(\alpha_1)}(\Lambda_{r(\bar\alpha_2)})^{-1}\Lambda_{r(\alpha_3)}$ is symmetric and is equal to $\Lambda_{r(\bar\alpha_3)}(\Lambda_{r(\alpha_2)})^{-1}\Lambda_{r(\bar\alpha_1)}$ (cf. Corollary 3.16).

Let $\mathcal{T}$ an ideal triangulation of S, here better thought as a maximal intersection-free subset of $\Gamma(S)$ that it is stable by the orientation reversion $\alpha \mapsto \bar\alpha$.

We define $\mathcal{A}(\mathcal{T}, n)$ to be the space of tuples $\{G_\alpha\}_{\alpha\in\mathcal{T}} \in \mathrm{GL}(n, \mathbf{R})^{\mathcal{T}}$ satisfying the above equations:

- for all α in $\mathcal{T}$, $G_{\bar\alpha} = {}^TG_\alpha$, and
- for every triangle $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathcal{T}$, $G_{\alpha_1}(G_{\bar\alpha_2})^{-1}G_{\alpha_3} = G_{\bar\alpha_3}(G_{\alpha_2})^{-1}G_{\bar\alpha_1}$.

We call $\mathcal{A}(\mathcal{T}, n)$ the space of $\mathcal{A}$-coordinates.

Our main result is

Theorem 11.3. *The map*

$$\Psi_{\mathcal{T}} = \{\Lambda_{r(\alpha)}\}_{\alpha\in\mathcal{T}} \colon \mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}}(S, \mathrm{Sp}(2n, \mathbf{R})) \longrightarrow \mathcal{A}(\mathcal{T}, n)$$

is a one-to-one correspondence.

Proof. We need first to prove that a transverse decorated δ-twisted symplectic local system $x = (F_v, f_v^t, f_v^b, g_a)$ is completely determined by the matrices $H_\alpha := \Lambda_{r(\alpha)}(x)$ $(\alpha \in \mathcal{T})$.

For every v in V, the arc α_v (cf. Section 5.1) is an edge of $\mathcal{T}$ and we will write H_v instead of H_{α_v}.

By definition of the symplectic Λ-length (and the correspondence between decorated local systems and decorated representations) one has $H_v = \omega(\mathbf{f}_v^t, \mathbf{f}_v^b)$ where ω is the symplectic form on the space F_v. Also the pair $(\mathbf{f}_v^t, \mathbf{f}_v^b)$ is a basis of F_v.

To prove injectivity of $\Psi_{\mathcal{T}}$, it is enough to show that the transition maps g_a are uniquely determined. This is obvious when a belongs to A_2 since, the local system being δ-twisted, one has $g_a(f^t_{v^-(a)}) = -f^b_{v^+(a)}$. Thus the matrix of g_a in the bases $(\mathbf{f}^t_{v^-(a)}, \mathbf{f}^b_{v^-(a)})$ and $(\mathbf{f}^t_{v^+(a)}, \mathbf{f}^b_{v^+(a)})$ is $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$.

Suppose now that a belongs to a 3-cycle (a, b, c) of A_3, so that $v^+(a) = v^-(b)$, $v^+(b) = v^-(c)$, $v^+(c) = v^-(a)$, and $g_cg_bg_a = -\mathrm{Id}$. Together with the condition $g_a(\mathbf{f}^b_{v^-(a)}) = \mathbf{f}^t_{v^+(a)}$, one has $g_a(\mathbf{f}^t_{v^-(a)}) = g_a \circ g_c(\mathbf{f}^b_{v^-(c)}) = -g_b^{-1}(\mathbf{f}^b_{v^-(c)})$. Also

$$\omega\big(g_a(\mathbf{f}^t_{v^-(a)}), \mathbf{f}^t_{v^-(b)}\big) = \omega\big(g_a(\mathbf{f}^t_{v^-(a)}), g_a(\mathbf{f}^b_{v^-(a)})\big) = \omega(\mathbf{f}^t_{v^-(a)}, \mathbf{f}^b_{v^-(a)}) = H_{v^-(a)};$$

similarly $\omega\big(\mathbf{f}^b_{v^-(b)}, g_a(\mathbf{f}^t_{v^-(a)})\big) = -\omega\big(\mathbf{f}^b_{v^-(b)}, g_b^{-1}(\mathbf{f}^b_{v^+(c)})\big) = -H_{v^-(c)}$. By Lemma 3.15 this implies that $g_a(f^t_{v^-(a)}) = \mathbf{f}^t_{v^-(b)} \cdot {}^TH^{-1}_{v^-(b)}H_{v^-(c)} - \mathbf{f}^b_{v^-(b)} \cdot H^{-1}_{v^-(b)}{}^TH_{v^-(a)}$ and the matrix of g_a is

$$\begin{pmatrix} {}^TH^{-1}_{v^-(b)}H_{v^-(c)} & \mathrm{Id} \\ -H^{-1}_{v^-(b)}{}^TH_{v^-(a)} & 0 \end{pmatrix}.$$

(Of course this matrix does not belong to the group $\mathrm{Sp}(2n, \mathbf{R})$, it is nevertheless the matrix of a symplectic isomorphism $F_{v^-(a)} \to F_{v^+(a)}$ in the given bases.)

Conversely given a family $\{G_\alpha\}_{\alpha\in\mathcal{T}}$ in $\mathcal{A}(\mathcal{T}, n)$, we can define

- for each v in V, a symplectic vector space F_v with a basis $(\mathbf{f}_v^t, \mathbf{f}_v^b)$ such that $\omega(\mathbf{f}_v^t, \mathbf{f}_v^t) = 0$, $\omega(\mathbf{f}_v^b, \mathbf{f}_v^b) = 0$, $\omega(\mathbf{f}_v^t, \mathbf{f}_v^b) = G_{\alpha_v}$. To simplify a little the notation, the latter matrix will be denoted G_v;
- for each a in A_2, g_a to be the linear map $F_{v^-(a)} \to F_{v^+(a)}$ whose matrix (in the given bases) is $\left(\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right)$;

- for each a in A_3, hence belonging to a 3-cycle (a,b,c), the matrix of $g_a\colon F_{v^-(a)} \to F_{v^+(a)}$ is $\begin{pmatrix} {}^TG_{v^-(b)}^{-1}G_{v^-(c)} & \mathrm{Id} \\ -G_{v^-(b)}^{-1}\,{}^TG_{v^-(a)} & 0 \end{pmatrix}$.

Then $(F_v, \mathbf{f}_v^t, \mathbf{f}_v^b, g_a)$ is a decorated δ-twisted symplectic local system on $\Gamma_{\mathcal{T}}$ and its image under the map $\Psi_{\mathcal{T}}$ is the family $\{G_\alpha\}$. This concludes the proof ot the theorem. □

11.6. **Change of coordinates: flips.** For every pair $(\mathcal{T}_0, \mathcal{T}_1)$ of triangulations, we would like to understand the change of coordinates from $\mathcal{A}(\mathcal{T}_0, n)$ to $\mathcal{A}(\mathcal{T}_1, n)$. It is enough to do this when $\mathcal{T}_0$ and $\mathcal{T}_1$ differ by a flip, i.e. when there are an edge e_0 in $\mathcal{T}_0$ and an edge e_1 in $\mathcal{T}_1$ such that $\mathcal{T}_0 \smallsetminus \{e_0, \bar{e}_0\} = \mathcal{T}_1 \smallsetminus \{e_1, \bar{e}_1\}$. We will furthermore assume that the starting point of e_0 is in the triangle of $\mathcal{T}_1$ that is to the left of e_1 (cf. Figure 11.1).

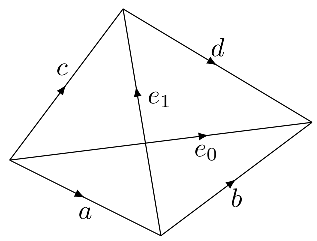

FIGURE 11.1. The edges involved in a flip.

The edge e_0 bounds two triangles in $\mathcal{T}_0$, one containing the starting point of e_1 and whose other edges are called a, b: precisely a, b, and $\bar{e}_0$ form a cycle bounding the corresponding triangle. The other triangle contains the endpoint of e_1, its edges c, d, and $\bar{e}_0$ forming as well a cycle in $\mathcal{T}_0$.

The following is an easy consequence of Proposition 3.17 (one has to take care of a few sign changes due to the fact that we are working with δ-twisted system, however it is easily verified that each term in Proposition 3.17 is changed with the same sign).

Proposition 11.4. *With the above notation, one has the identity (between functions defined on* $\mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}_1}(S, \mathrm{Sp}(2n, \mathbf{R}))$ *or in* $\mathrm{GL}(n, \mathcal{K})$*)*

$$\Lambda_{r(e_0)} = \Lambda_{r(a)}(\Lambda_{r(\bar{e}_1)})^{-1}\Lambda_{r(d)} + \Lambda_{r(c)}(\Lambda_{r(e_1)})^{-1}\Lambda_{r(b)}.$$

Remark 11.5. Of course, this result holds for any quadrilateral (cf. below Section 11.9) and not only for those coming from triangulations.

As a consequence:

Theorem 11.6. *The image by* $\Psi_{\mathcal{T}_1}$ *(cf. Theorem 11.3) of the space*

$$\mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}_0}(S, \mathrm{Sp}(2n, \mathbf{R})) \cap \mathrm{Loc}^{\mathrm{d}}_{\delta,\mathcal{T}_1}(S, \mathrm{Sp}(2n, \mathbf{R}))$$

is the set of tuples $\{G_e\}_{e \in \mathcal{T}_1}$ *in* $\mathcal{A}(\mathcal{T}_1, n)$ *such that* $G_a(G_{\bar{e}_1})^{-1}G_d + G_c(G_{e_1})^{-1}G_b$ *is invertible. The composition* $\Psi_{\mathcal{T}_0} \circ \Psi_{\mathcal{T}_1}^{-1}$ *is then* $\{G_e\}_{e \in \mathcal{T}_1} \mapsto \{H_e\}_{e \in \mathcal{T}_0}$ *where* $H_e = G_e$ *if* $e \notin \{e_0, \bar{e}_0\}$, $H_{e_0} = G_a(G_{\bar{e}_1})^{-1}G_d + G_c(G_{e_1})^{-1}G_b$*, and* $H_{\bar{e}_0} = {}^TH_{e_0}$.

As pointed out in Section 3.5 the formula for the flip in Proposition 11.4 can be seen as a noncommutative Ptolemy-relation. Thus the $\mathcal{A}$-space we introduce here is a noncommutative generalization of Penner's parametrization of the decorated Teichmüller space.

Remark 11.7. Since the Λ-lengths completely determine the decorated local system which in turn determine a framed local system, they also determine completely the possible $\mathcal{X}$-coordinates. However deriving the $\mathcal{X}$-coordinates from the Λ-lengths involve diagonalizing matrices, this nice formula for the flip does not descend to $\mathcal{X}$-coordinates directly.

11.7. **$\mathcal{A}$-coordinates for maximal decorated twisted local systems.** Using the results from Section 3.6 we can describe the subspace of $\mathcal{A}$-coordinates that parametrizes maximal decorated twisted representations.

Proposition 11.8. *Let $\mathcal{T}$ be an ideal triangulation of S.*

A decorated twisted representation in $\mathrm{Loc}^{\mathrm{d}}_{\delta}(S, \mathrm{Sp}(2n, \mathbf{R}))$ is maximal if and only if it is $\mathcal{T}$-transverse and, for every triangle (e_1, e_2, e_3) in $\mathcal{T}$, the symmetric matrix

$$\Lambda_{r(e_1)}\Lambda^{-1}_{r(\bar{e}_2)}\Lambda_{r(e_3)}$$

is positive definite.

Theorem 4.26 implies that these conditions are invariant under a flip, and thus independent of the triangulation (this can be also checked by a direct calculation using the formulas of Theorem 11.6). Therefore Proposition 11.8 gives a parametrization of the space of maximal decorated twisted representation.

11.8. **From $\mathcal{A}$-coordinates to $\mathcal{X}$-coordinates.** We now describe the relation between $\mathcal{X}$-coordinates and $\mathcal{A}$-coordinates, and derive explicit formulas for the flip.

In Section 3.6 we expressed the cross ratio of four pairwise transverse decorated Lagrangians in terms of the symplectic Λ-lengths, namely let $(L_i, \mathbf{v_i}) \in \mathcal{L}^{\mathrm{d}}_n$, with $i \in \{1, 2, 3, 4\}$, be four pairwise transverse framed Lagrangians. Then

$$[L_1, L_2, L_3, L_4]_{\mathbf{v_1}} = -\Lambda_{41}^{-1}\Lambda_{43}\Lambda_{23}^{-1}\Lambda_{21},$$

where $[L_1, L_2, L_3, L_4]_{\mathbf{v_1}}$ denotes the cross ratio expressed in the basis $\mathbf{v_1}$.

We think of the cross ratio $[L_1, L_2, L_3, L_4]_{\mathbf{v_1}}$ as being associated to an oriented edge/an arc α from the decorated Lagrangian $(L_1, \mathbf{v}_1)$ to the decorated Lagrangian $(L_3, \mathbf{v}_3)$. We write CR_α for this cross ratio, and call this the cross ratio of α.

This allows to define a map from the space of $\mathcal{A}$-coordinates to the space of $\mathcal{X}$-coordinates:

Given a decorated twisted local system $(\mathcal{F}, \beta)$ we get for every arc $\alpha \in \mathcal{T}$ a cross ratio CR_α, (or $\mathrm{CR}^{\mathcal{T}}_\alpha$ if we need to remember the triangulation). Lemma 3.19 implies that $\mathrm{CR}^{\mathcal{T}}_{\bar{\alpha}} = \Lambda_\alpha^{-1}\, {}^T\mathrm{CR}^{\mathcal{T}}_\alpha\, \Lambda_\alpha$.

The next proposition shows that the formulae for the change of the cross ratios CR_α under a flip have a nice form, that is just a noncommutative analog of the classical formula.

Arcs and triangles in the 8-gon are completely determined by their extremities and we will designate them by their extremities: e.g. 62 is the oriented edge from the vertex 6 to the vertex 2. Consider the following two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of the 8-gon: the triangles 234, 456, 678, and 812 belong to both $\mathcal{T}$ and $\mathcal{T}'$, the edge 62 belongs to $\mathcal{T}$ and the edge 84 belongs to $\mathcal{T}'$ (cf. Figure 11.2). Hence $\mathcal{T}$ and $\mathcal{T}'$ differ by a flip in the quadrilateral 2468.

Proposition 11.9. *Consider eight framed Lagrangians $(L_i, \mathbf{v_i})$, with $i \in \{1, \dots, 8\}$, thought as being associated with the vertices of a 8-gon. We have the following formulas for the flip along the edge 62:*

$$\begin{aligned}
\mathrm{CR}^{\mathcal{T}'}_{84} &= \Lambda_{68}^{-1}\, {}^T\mathrm{CR}^{\mathcal{T}\,-1}_{62}\, \Lambda_{68}\\
\mathrm{CR}^{\mathcal{T}'}_{64} &= \mathrm{CR}^{\mathcal{T}}_{64}(\mathrm{Id} + \mathrm{CR}^{\mathcal{T}\,-1}_{62})^{-1}, \quad \mathrm{CR}^{\mathcal{T}'}_{82} = (\mathrm{Id} + \Lambda_{68}^{-1}\, {}^T\mathrm{CR}^{\mathcal{T}\,-1}_{62}\, \Lambda_{68})\, \mathrm{CR}^{\mathcal{T}}_{82}\\
\mathrm{CR}^{\mathcal{T}'}_{68} &= (\mathrm{Id} + \mathrm{CR}^{\mathcal{T}}_{62})\, \mathrm{CR}^{\mathcal{T}}_{68}, \quad \mathrm{CR}^{\mathcal{T}'}_{42} = \mathrm{CR}^{\mathcal{T}}_{42}(\mathrm{Id} + \Lambda_{64}^{-1}\, \mathrm{CR}^{\mathcal{T}}_{62}\, \Lambda_{64}).
\end{aligned}$$

Note that in the case $n = 1$, these are precisely the formulas for the flip, see for example [9, Formula (1.30)], and here we have a noncommutative generalization of them.

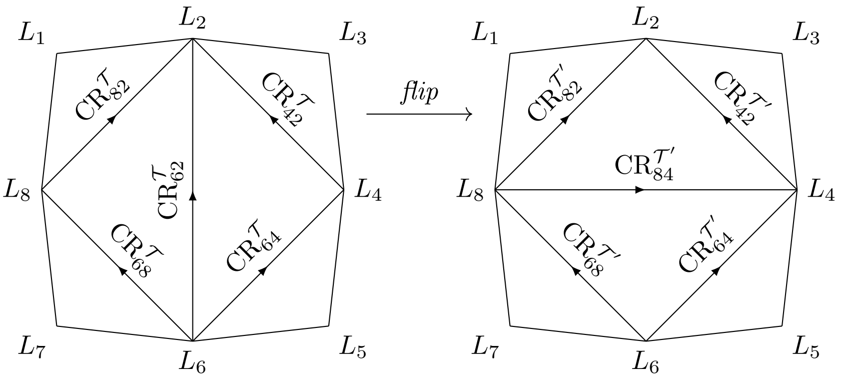

FIGURE 11.2. The flip in a 8-gon

Proof. One has $\mathrm{CR}^{\mathcal{T}}_{62} = -\Lambda_{46}^{-1}\Lambda_{42}\Lambda_{82}^{-1}\Lambda_{86}$ and $\mathrm{CR}^{\mathcal{T}'}_{84} = -\Lambda_{68}^{-1}\Lambda_{64}\Lambda_{24}^{-1}\Lambda_{28}$ so that, using the identities $\Lambda_{ji} = -{}^T\Lambda_{ij}$, ${}^T\mathrm{CR}^{\mathcal{T}-1}_{62} = -\Lambda_{64}\Lambda_{24}^{-1}\Lambda_{28}\Lambda_{68}^{-1}$. Thus $\mathrm{CR}^{\mathcal{T}'}_{84} = -\Lambda_{68}^{-1}(\Lambda_{64}\Lambda_{24}^{-1}\Lambda_{28}\Lambda_{68}^{-1})\Lambda_{68} = \Lambda_{68}^{-1}{}^T\mathrm{CR}^{\mathcal{T}-1}_{62}\Lambda_{68}$.

The flip relation for Λ-lengths implies

$$\Lambda_{84} = \Lambda_{86}\Lambda_{26}^{-1}\Lambda_{24} + \Lambda_{82}\Lambda_{62}^{-1}\Lambda_{64} = \Lambda_{86}(\mathrm{Id} + \Lambda_{86}^{-1}\Lambda_{82}\Lambda_{62}^{-1}\Lambda_{64}\Lambda_{24}^{-1}\Lambda_{26})\Lambda_{26}^{-1}\Lambda_{24}$$

by the triangle relation $\Lambda_{62}^{-1}\Lambda_{64}\Lambda_{24}^{-1}\Lambda_{26} = -\Lambda_{42}^{-1}\Lambda_{46}$:

$$= \Lambda_{86}(\mathrm{Id} - \Lambda_{86}^{-1}\Lambda_{82}\Lambda_{42}^{-1}\Lambda_{46})\Lambda_{26}^{-1}\Lambda_{24} = \Lambda_{86}(\mathrm{Id} + \mathrm{CR}^{\mathcal{T}-1}_{62})\Lambda_{26}^{-1}\Lambda_{24}.$$

Therefore $\mathrm{CR}^{\mathcal{T}'}_{64} = -\Lambda_{56}^{-1}\Lambda_{54}\Lambda_{84}^{-1}\Lambda_{86} = -\Lambda_{56}^{-1}\Lambda_{54}\Lambda_{24}^{-1}\Lambda_{26}(\mathrm{Id} + \mathrm{CR}^{\mathcal{T}-1}_{62})^{-1} = \mathrm{CR}^{\mathcal{T}}_{64}(\mathrm{Id} + \mathrm{CR}^{\mathcal{T}-1}_{62})^{-1}$.

The proof for other cross ratios is similar. □

11.9. **The algebra of Berenstein and Retakh.** In their paper [2], Berenstein and Retakh introduced the following noncommutative instance of cluster algebras, which we shortly recall. Let $\Gamma(S)$ be the set of homotopy classes (relative to the boundary) of arcs in S: elements of $\Gamma(S)$ are homotopy class of maps between pairs $\alpha\colon ([0,1],\{0,1\}) \to (S,\partial S)$. (Actually Berenstein and Retakh have a more refined version incorporating orbifolds points of order 2.)

Recall (Section 4.11) that, for every p in $\{2,3,\dots\}$, a p-gon is (the homotopy class of) a map $f\colon (\mathbb{D},\mu_p) \to (S,\partial S)$ where $\mu_p = \{z \in \mathbf{C} \mid z^p = 1\}$.

Every 3-gon t defines 6 elements in $\Gamma(S)$: there are the $\alpha_{k,\ell}(t)$ for $k \neq \ell$ in $\mathbf{Z}/3\mathbf{Z}$ where $\alpha_{k,\ell}(t)$ is the restriction of t to an arc going from $\mathbf{e}(k/3) = e^{2ik\pi/3}$ to $\mathbf{e}(\ell/3) = e^{2i\ell\pi/3}$ (one can choose the segment $[\mathbf{e}(k/3),\mathbf{e}(\ell/3)]$ in $\overline{\mathbb{D}}$). Of course $\overline{\alpha_{\ell,k}(t)} = \alpha_{k,\ell}(t)$.

A 4-gon q defines 12 elements $\alpha_{k,\ell}(q)$ in $\Gamma(S)$ for $k \neq \ell$ in $\mathbf{Z}/4\mathbf{Z}$.

Similarly, for every p-gon f, $\alpha_{k,\ell}(f)$ ($k \neq \ell$ in $\mathbf{Z}/p\mathbf{Z}$) will denote the class of the restriction of f to the segment $[\mathbf{e}(k/p),\mathbf{e}(\ell/p)]$ in $\mathbb{D}$.

Definition 11.10 (Noncommutative surface)**.** The *noncommutative surface* associated with S is the algebra $\mathcal{A}_S$ over $\mathbf{Q}$ generated by the elements x_α, x_α^{-1} ($\alpha \in \Gamma(S)$) subject to the relations:

(T) for every 3-gon $t\colon (\mathbb{D},\mu_3) \to (S,\partial S)$, abbreviating $x_{k\ell}^{\pm 1} = x_{\alpha_{k,\ell}(t)}^{\pm 1}$,

$$x_{12}x_{32}^{-1}x_{31} = x_{13}x_{23}^{-1}x_{21};$$

(Q) for every 4-gon $q\colon (\mathbb{D},\mu_4) \to (S,\partial S)$, abbreviating $x_{k\ell}^{\pm 1} = x_{\alpha_{k,\ell}(q)}^{\pm 1}$,

$$x_{13} = x_{12}x_{42}^{-1}x_{43} + x_{14}x_{24}^{-1}x_{23}.$$

11.10. **Geometric realization of the noncommutative surface.** The Λ-lengths over the space of decorated δ-twisted local system enables us to give a "geometric" realization of the algebra $\mathcal{A}_S$:

Theorem 11.11. *There is a (unique) algebra homomorphism*

$$\Psi\colon \mathcal{A}_S \longrightarrow M_n(\mathcal{K})$$

such that, for every arc α in S, $\Psi(x_\alpha) = \Lambda_{r(\alpha)}$.

Proof. Indeed, by the universal property of the algebra $\mathcal{A}_S$, the assignment $x_\alpha^{\pm 1} \mapsto \Lambda_{r(\alpha)}^{\pm 1}$ extends to an algebra homomorphism if and only if the defining relations between the x_α's (Definition 11.10) are satisfied by the family $\{\Lambda_{r(\alpha)}\}_{\alpha \in \Gamma(S)}$. This holds by from point (2) in Section 11.5 and Proposition 11.4. □

11.11. **Zig-zag path and expression of the Λ-lengths.** For any arc α in $\Gamma(S)$, there is a unique "minimal" triple (p, f, k) where p belongs to $\mathbf{N}$, $f\colon (\mathbb{D}, \mu_p) \to (S, \partial S)$ is a p-gon, and k is in $\mathbf{Z}/p\mathbf{Z}$ such that, for all i in $\mathbf{Z}/p\mathbf{Z}$, $\alpha_{i,i+1}(f)$ belongs to $\mathcal{T}$ and $\alpha = \alpha_{1k}(f)$. This can be proved, for example, using an hyperbolic structure on S with totally geodesic boundary and the geodesic realizations of the arcs.

Definition 11.12 (Zigzag sequence)**.** A *zigzag* sequence for α (or for $(\mathcal{T}, \alpha)$, sometimes called a α-zigzag sequence) is an odd length sequence $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{2m}, \delta_{2m+1})$ in $\mathcal{T}^{2m+1}$ such that:

- there is $(j_0, j_1, \ldots, j_{2m}, j_{2m+1}) \in (\mathbf{Z}/p\mathbf{Z})^{2m+2}$ with $j_0 = 1$, $j_{2m+2} = k$, and for all $\ell = 1, \ldots, 2m+1$, $\delta_\ell = \alpha_{j_{\ell-1} j_\ell}(f)$;
- for all odd ℓ, the segments (in $\mathbb{D}$) $[\mathbf{e}(j_{\ell-1}/p), \mathbf{e}(j_\ell/p)]$ and $[\mathbf{e}(1/p), \mathbf{e}(k/p)]$ do not intersect (besides endpoints for $\ell = 1$ of $\ell = 2m+1$);
- for all even ℓ, the segments $[\mathbf{e}(j_{\ell-1}/p), \mathbf{e}(j_\ell/p)]$ and $[\mathbf{e}(1/p), \mathbf{e}(k/p)]$ intersect at some point u_ℓ;
- the sequence $(u_2, u_4, \ldots, u_{2m})$ in $[\mathbf{e}(1/p), \mathbf{e}(k/p)]$ is increasing.

Note that our terminology differs from [2].

For such a $\boldsymbol{\delta}$ let us denote $x_{\boldsymbol{\delta}} = x_{\delta_1} x_{\bar{\delta}_2}^{-1} x_{\delta_3} \cdots x_{\bar{\delta}_{2m}}^{-1} x_{\delta_{2m+1}}$. This elements belongs thus to the subalgebra generated by the $x_\alpha^{\pm 1}$ for α in $\mathcal{T}$. Similarly, we denote by $\Lambda_{\boldsymbol{\delta}}$ the product $\Lambda_{\delta_1} \Lambda_{\bar{\delta}_2}^{-1} \Lambda_{\delta_3} \cdots \Lambda_{\bar{\delta}_{2m}}^{-1} \Lambda_{\delta_{2m+1}}$ of $\mathrm{GL}(n, \mathcal{K})$.

One spectacular result from [2] is the noncommutative Laurent phenomenon:

Theorem 11.13 ([2, Theorem 3.30])**.** *For every α in $\Gamma(S)$, one has*

$$x_\alpha = \sum_{\boldsymbol{\delta}} x_{\boldsymbol{\delta}}$$

where the sum runs over the α-zigzag sequences $\boldsymbol{\delta}$.

As an immediate corollary:

Corollary 11.14. *For every α in $\Gamma(S)$, one has, in $\mathrm{GL}(n, \mathcal{K})$,*

$$\Lambda_\alpha = \sum_{\boldsymbol{\delta}} \Lambda_{\boldsymbol{\delta}}$$

where the sum runs over the α-zigzag sequences $\boldsymbol{\delta}$.

Appendix A. Normal form for pair of quadratic forms

The well known spectral theorem says that for two symmetric bilinear forms b_0, b_1 on an n-dimensional real vector space V such that b_0 is positive definite, there exists a basis $\mathbf{e}$ such that $[b_0]_{\mathbf{e}} = \mathrm{Id}_n$, $[b_1]_{\mathbf{e}} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Therefore, the tuple $(\lambda_1, \dots, \lambda_n)$ defines the pair (b_0, b_1) up to change of basis of V. We can define the standard form of the pair of bilinear forms (b_0, b_1) to be the pair of matrices $(\mathrm{Id}_n, \mathrm{diag}(\lambda_1, \dots, \lambda_n))$ and say that the basis $\mathbf{e}$ puts (b_0, b_1) to the standard form. We used this standard form to define edge invariants for maximal representations in Section 6.

In this section, we define the standard form for a pair of bilinear forms (b_0, b_1) assuming only nondegeneracy of b_0. This standard form will be used to define edge invariants for general representations in Section 9.2.

A.1. **Bilinear forms and selfadjoint linear maps.** Let V be a n-dimensional vector space over a field K of characteristic 0, and let b_0, b_1 be symmetric bilinear forms on V. We assume that b_0 is not degenerate. We denote by $b_i^\dagger \colon V \to V^*$ the linear map corresponding to b_i, i.e. $b_i^\dagger(x)(y) = b_i(x, y)$ for all $x, y \in V$. Then we can consider $f = (b_0^\dagger)^{-1} \circ b_1^\dagger \colon V \to V$.

Lemma A.1. *For all $x, y \in V$, one has $b_1(x, y) = b_0(fx, y)$.*
The map f is selfadjoint with respect to b_0.

Proof. The thought for equality can be written: $\forall x \in V$, $b_1^\dagger(x) = b_0^\dagger(fx)$ and follows immediately from the definition of f. Since b_0 and b_1 are symmetric, this implies that f is selfadjoint. □

A.2. **More bilinear forms.** It will be useful to consider the bilinear forms, for all $k \in \mathbf{Z}_{\geqslant 0}$,

$$b_k(v, w) = b_0(f^k v, w) \quad (v, w \in V).$$

For all $p, q \geqslant 0$, one has $b_{p+q}(v, w) = b_0(f^p v, f^q w)$ and, for all k, f is selfadjoint with respect to b_k.

More generally, for every polynomial P in $K[X]$, $b_P(v, w) = b_0(P(f)v, w)$ is a symmetric form and f is selfadjoint with respect to b_P.

Lemma A.2. *The kernel of the symmetric form b_P is equal to the kernel of the endomorphism $P(f)$.*

Proof. Indeed, the kernel of b_P is

$$\begin{aligned} \ker(b_P) &= \{v \in V \mid \forall w \in V,\ b_P(v, w) = 0\} \\ &= \{v \in V \mid \forall w \in V,\ b_0(P(f)v, w) = 0\} \end{aligned}$$

and since b_0 is nondegenerate

$$= \{v \in V \mid P(f)v = 0\} = \ker P(f).$$ □

Let us equip the vector space V with its $K[X]$-module structure inherited from the endomorphism f, namely, for every $P \in K[X]$, $P \cdot v = P(f)v$.

Remark A.3.
(1) When f is nilpotent, V will be a module over the local ring $K_{[n]} = K[X]/(X^n)$.
(2) The homomorphism $\epsilon \colon K_{[n]} \to K[X]/(X) \simeq K$ is often called the augmentation. An element z in $K_{[n]}$ is invertible if and only if $\epsilon(z) \in K^*$ (one check that, if $\epsilon(z) \in K^*$, the finite sum $\epsilon(z)^{-1} \sum_{\ell=0}^{n-1} (1 - \epsilon(z)^{-1} z)^\ell$ is the inverse of z in $K_{[n]}$). Furthermore z is a square if and only if $\epsilon(z)$ is a square: one first gets back to the case $\epsilon(z) = 1$ and then checks that, if $\sum_\ell a_\ell t^\ell$ is the Taylor series of $t \mapsto (1+t)^{1/2}$ (the exact formula

being $a_\ell = \frac{\prod_{j=0}^{\ell-1}(1/2-j)}{\ell!}$), then $\sum_{\ell=0}^{n} a_\ell (z-1)^\ell$ is a square root of z. It follows that $K^*_{[n]}/K^{*2}_{[n]} \simeq K^*/K^{*2}$.

An encompassed way to put together all those bilinear forms is to consider forms with values in the following $K[X]$-module

$$K\langle\!\langle X^{-1}\rangle\!\rangle = K(\!(X^{-1})\!)/XK[X]$$

quotient of the field of Laurent series in X^{-1} modulo the polynomials in X without constant term. Every element in $K\langle\!\langle X^{-1}\rangle\!\rangle$ is represented uniquely as a series $\sum_{\ell\geqslant 0} a_\ell X^{-\ell}$. The relevance of $K\langle\!\langle X^{-1}\rangle\!\rangle$ is justified by the following lemma.

Lemma A.4. *With the above notation, the map*

$$\begin{aligned} b_{K\langle\!\langle X^{-1}\rangle\!\rangle} \colon V \times V &\longrightarrow K\langle\!\langle X^{-1}\rangle\!\rangle \\ (v,w) &\longmapsto \sum_{k\geqslant 0} b_k(v,w) X^{-k} \end{aligned}$$

is $K[X]$-bilinear and symmetric.

Proof. The symmetry and K-linearity follow at once from the same properties of the b_ks. The $K[X]$-linearity is a consequence of the fact that, for all $k \geqslant 0$, and all $v, w \in V$, $b_{k+1}(v,w) = b_k(fv,w)$. □

Remark A.5. Let Q be a polynomial such that $Q(f) = 0$. Then the above map $b_{K\langle\!\langle X^{-1}\rangle\!\rangle}$ takes values in the submodule $\mathrm{Tor}_Q = \{S \in K\langle\!\langle X^{-1}\rangle\!\rangle \mid QS = 0\}$ (i.e. if $\tilde{S} \in K(\!(X^{-1})\!)$ represents S, then $Q\tilde{S}$ is a polynomial without constant term). Furthermore the $K[X]$-module Tor_Q is isomorphic to the cyclic module $K[X]/(Q)$. We will exploit this additional fact mainly when f is nilpotent (i.e. $Q = X^n$), in which case we will shift the indices for $b_{K\langle\!\langle X^{-1}\rangle\!\rangle}$ in the positive range (see the proof of Lemma A.7).

A.3. **When f has one split Jordan block.** In this section, we start our investigation of normal forms under the additional assumption that the Jordan normal form of f has only one block. That is to say, there is a basis $\mathbf{e}$ of V and $\lambda \in K$, such that the matrix $[f]_{\mathbf{e}}$ of f in that basis is $J_n(\lambda) = \lambda \,\mathrm{Id} + N$, where $N = (\delta_{i+1,j})_{1\leqslant i,j\leqslant n}$ the regular nilpotent matrix and $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

Remark A.6.
(1) Such a basis is entirely determined by its last vector $v = e_n$ and the rule $e_i = f(e_{i+1}) - \lambda e_{i+1}$ ($i = n-1, n-2, \dots, 1$). The vector v must be chosen in $V \setminus \ker(f-\lambda)^{n-1}$ in order for the previous construction to lead to a basis.
(2) When $\lambda = 0$, the matrix of the bilinear form b_0 in that basis is then

$$[b_0]_{\mathbf{e}} = (b_{2n-i-j}(v,v))_{i,j=1,\dots,n}.$$

Let also C_n be the "antidiagonal" matrix $(\delta_{n+1,i+j})_{1\leqslant i,j\leqslant n}$ (see Equation (9.1) in Section 9.1.2).

To keep the discussion for a general field K, let us fix $\mathfrak{c} \subset K^*$ a set of representatives of K^*/K^{*2}; our main concerns are $K = \mathbf{R}$ ($\mathfrak{c} = \{\pm 1\}$) and $K = \mathbf{C}$ ($\mathfrak{c} = \{1\}$).

Lemma A.7. *Under the assumption that the Jordan normal form of f has only one block, there is a basis $\mathbf{e}$ of V and $\varepsilon \in \mathfrak{c}$, such has $[f]_{\mathbf{e}}$ is the Jordan block $J_n(\lambda)$ and $[b_0]_{\mathbf{e}}$ is εC_n.*

The basis $\mathbf{e}$ is unique up to multiplication by ± 1.

Remark A.8. The matrix of the symmetric bilinear form b_1 is then $[b_1]_{\mathbf{e}} = \varepsilon C_n J_n(\lambda) = \varepsilon^T J_n(\lambda) C_n$.

Proof. Up to changing f into $f-\lambda\,\mathrm{Id}$ (which amounts to changing b_1 into $b_1-\lambda b_0$), we can assume that $\lambda=0$. In this situation, $b_\ell=0$ for every $\ell\geqslant n$. We will consider V as a module over the algebra $K_{[n]}=K[X]/(X^n)$, and the $K_{[n]}$-bilinear symmetric form (compare with Section A.2):

$$\begin{aligned} b_{K_{[n]}}\colon V\times V &\longrightarrow K_{[n]} \\ (v,w) &\longmapsto \sum_{i=0}^{n-1} b_{n-1-i}(v,w)X^i. \end{aligned}$$

Following Remark A.6, we are searching for a vector v such that $b_{n-1}(v,v)\neq 0$ and $b_j(v,v)=0$ for all $j\neq n-1$, i.e. such that the element $b_{K_{[n]}}(v,v)$ belongs to $K^*\subset K_{[n]}$.

The kernel of the form b_{n-1} is equal to the kernel of f^{n-1} (Lemma A.2) and therefore is not zero by the assumption that the Jordan decomposition of f has only one block. Let thus w be in V with $b_{n-1}(w,w)\neq 0$. The element $b_{K_{[n]}}(w,w)$ is then in $K^*_{[n]}$ since $\epsilon(b_{K_{[n]}}(w,w))=b_{n-1}(w,w)\in K^*$, and there is an element $u\in K_{[n]}$ such that $\varepsilon:=u^2 b_{K_{[n]}}(w,w)$ belongs to $\mathfrak{c}$ (see Remark A.3). Define $v=u\cdot w\in V$. Then $b_{K_{[n]}}(v,v)=\varepsilon$ which is the sought for equality.

Let now v' be in V such that $b_{K_{[n]}}(v',v')\in\mathfrak{c}\subset K^*$. Since the $K_{[n]}$-module V is cyclic generated by v, there exists $t\in K_{[n]}$ such that $v'=t\cdot v$. Therefore $t^2\varepsilon=b_{K_{[n]}}(v',v')$. This equality implies that t belongs to K (assume that it is not the case, then $t=\nu+\alpha X^k$ and $t^2=\nu^2+2\nu\alpha X^k \bmod X^{k+1}$ cannot belong to K). Since $\mathfrak{c}$ is a set of representatives of K^*/K^{*2}, this implies that $t^2=1$ and $t=\pm 1$. The uniqueness (up to sign) of the basis follows. ☐

Remark A.9. Pursuing a little more the proof, one observes that V can be identified with $K_{[n]}$ and the bilinear for $b_{K_{[n]}}\colon K_{[n]}\times K_{[n]}\to K_{[n]}$ is simply ε times the multiplication in the algebra $K_{[n]}$. Shifting back to the point of view of Section A.2, the bilinear form $b_{K\langle\langle X^{-1}\rangle\rangle}$ is in that case:

$$\begin{aligned} b_{K\langle\langle X^{-1}\rangle\rangle}\colon K_{[n]}\times K_{[n]} &\longrightarrow K\langle\langle X^{-1}\rangle\rangle \\ (u,v) &\longmapsto \varepsilon X^{1-n}uv. \end{aligned}$$

A.4. **(Over the reals or the complex) Back transformation when f has one split Jordan block.** In this section we make the additional hypothesis that b_1 is nondegenerate and that f has one Jordan block and $K=\mathbf{R}$, or $\mathbf{C}$.

The dual vector space V^* has two bilinear forms b_0^* and b_1^* and, by definition of these forms, $b_0^{*\dagger}=(b_0^\dagger)^{-1}$ and $b_1^{*\dagger}=(b_1^\dagger)^{-1}$. In particular $(b_1^{*\dagger})^{-1}b_0^{*\dagger}=b_1^\dagger(b_0^\dagger)^{-1}$ is conjugate to $f=(b_0^\dagger)^{-1}b_1^\dagger$. Applying Lemma A.7 gives $\eta\in\{\pm1\}$, a basis of V^*, and hence a "predual" basis $\mathbf{v}$ of V, such that

$$[b_1^*]_{\mathbf{v}^*}=\eta C_n,\quad \text{and}\quad [b_0^*]_{\mathbf{v}^*}=\eta C_n J_n(\lambda).$$

Thus

$$[b_1]_{\mathbf{v}}={}^T(\eta C_n)^{-1}=\eta C_n,\quad \text{and}\quad [b_0]_{\mathbf{v}}=\eta C_n\,{}^T J_n(\lambda)^{-1}=\eta J_n(\lambda)^{-1}C_n.$$

Let Φ be the change-of-basis matrix from $\mathbf{e}$ to $\mathbf{v}$, i.e. $\mathbf{e}=\mathbf{v}\Phi$. The matrix Φ must satisfy the following two equalities:

$${}^T\Phi\eta C_n\Phi=\varepsilon C_n J_n(\lambda),\quad \text{and}\quad {}^T\Phi\eta J_n(\lambda)^{-1}C_n\Phi=\varepsilon C_n.$$

Denote again N the nilpotent matrix of maximal rank $(\delta_{i+1,j})_{1\leqslant i,j\leqslant n}$.

Lemma A.10. *With the above notation,*

(1) If $K=\mathbf{R}$, $\varepsilon\eta\lambda$ is positive (and in particular equal to $|\lambda|$).

(2) The matrix Φ is equal to, up to sign,

$$\sqrt{\varepsilon\eta\lambda}\sum_{\ell=0}^{n} a_\ell\lambda^{-\ell}C_nN^\ell, \tag{A.1}$$

where $\sum_{\ell=0}^{\infty} a_\ell t^\ell$ *is the Taylor series of* $t \mapsto (1+t)^{1/2}$ (*i.e. for all* ℓ, $a_\ell = \frac{\prod_{j=0}^{\ell-1}(1/2-j)}{\ell!}$) (*cf. also Equation* (9.2)).

Proof. Let $L = \varepsilon\eta J_n(\lambda) = \varepsilon\eta\lambda \operatorname{Id} + \varepsilon\eta N$. The two equations for Φ imply

$$ {}^T\Phi^{-1} = C_n \Phi L^{-1} C_n, \quad \text{and} \quad {}^T\Phi^{-1} = L^{-1} C_n \Phi C_n, $$

so that $C_n\Phi$ and L^{-1} commute and hence $C_n\Phi$ and N commute. This implies that $C_n\Phi$ is a polynomial in N: there are $c_0, \dots, c_{n-1}$ in K such that $C_n\Phi = c_0 \operatorname{Id} + \cdots + c_{n-1}N^{n-1}$. Thus $\Phi = c_0 C_n + c_1 C_n N + \cdots + c_{n-1} C_n N^{n-1}$ is a symmetric matrix (for all ℓ, $C_n N^\ell$ is symmetric). The first equation for Φ can now be written

$$ (C_n\Phi)^2 = \varepsilon\eta J_n(\lambda) $$

and as $(C_n\Phi)^2 = c_0^2 \operatorname{Id} + d_1 N + \cdots + d_{n-1}N^{n-1}$ (for some $d_1, \dots, d_{n-1}$ in K), we get that $\varepsilon\eta\lambda = c_0^2$ and is positive if $K = \mathbf{R}$. Dividing the last equation by $\varepsilon\eta\lambda$ give

$$ ((\varepsilon\eta\lambda)^{-1/2} C_n \Phi)^2 = \operatorname{Id} + \frac{1}{\lambda} N, $$

and this leads to the desired result. □

Definition A.11. The matrix given in Equation (A.1) is called the *back transformation* and denoted by $\Phi_n(\lambda)$. When $K = \mathbf{C}$, so that $\varepsilon = \eta = 1$, we always choose for $\sqrt{\lambda}$ the biggest square root of λ for the lexicographic order on $\mathbf{C} \simeq \mathbf{R}^2$.

A.5. **When the minimal polynomial of f has one eigenvalue.** In this section we assume that all the blocks of the Jordan decomposition of f are associated with the same eigenvalue λ in K or, what amounts to the same, that $(f-\lambda)^n = 0$ ($n = \dim V$). We denote again by $\mathfrak{c}$ a set of representatives of K^*/K^{*2}.

Proposition A.12. *There is a basis* $\mathbf{e}$ *of* V, *a sequence* $(n_1, \dots, n_p)$ *of integers, and a sequence* $(\varepsilon_1, \dots, \varepsilon_p)$ *in* $\mathfrak{c}$ *such that* $[f]_\mathbf{e}$ *and* $[b_0]_\mathbf{e}$ *are block diagonal with diagonal blocks being* $J_{n_1}(\lambda), \dots, J_{n_p}(\lambda)$, *and* $\varepsilon_1 C_{n_1}, \dots, \varepsilon_p C_{n_p}$ *respectively.*

Proof. We work by induction on n. Up to replacing f by $f - \lambda$, we may assume that $\lambda = 0$. Let $m \geqslant 1$ be the order of nilpotency of f, hence $f^m = 0$, $b_\ell = 0$ for all $\ell \geqslant m$, and $\ker f^{m-1} \neq V$. There is then a vector $w \in V$ such that $c = b_{m-1}(w, w) \neq 0$ (Lemma A.2). The $K[X]$-module W generated by w has for basis $w, fw, \dots, f^{m-1}w$ and the matrix of $b_0|_W$ in that basis is

$$ \begin{pmatrix} 0 & \dots & 0 & c \\ 0 & \dots & c & * \\ \vdots & \iddots & * & * \\ c & * & \dots & * \end{pmatrix} $$

(see Remark A.6) so that $b_0|_W$ is nondegenerate. Applying Lemma A.7 to W and the induction hypothesis to $W^{\perp_{b_0}}$ gives the result. □

A.6. **Orthogonality of generalized eigenspaces.** We return to the general setting: b_0 and b_1 are symmetric bilinear forms on a K-vector space V of dimension n, K is a field of characteristic 0, b_0 is nondegenerate, and $f = (b_0^\dagger)^{-1} b_1^\dagger$.

The generalized eigenspaces of f are the $V_\lambda = \ker(f-\lambda)^n$ for $\lambda \in K$.

Lemma A.13. *Generalized eigenspaces are orthogonal: if* $\lambda \neq \mu$, *then* $V_\mu \subset V_\lambda^{\perp_{b_0}}$.

Proof. The proof relies on the $K[X]$-module structure of V and on the bilinear form $b_{K\langle\langle X^{-1}\rangle\rangle}$ (Section A.2). Let P and Q be in $K[X]$ establishing a Bezout relation between $(X-\lambda)^n$ and $(X-\mu)^n$: $P(X-\lambda)^n + Q(X-\mu)^n = 1$.

Let v be in V_λ and w be in V_μ so that $(X-\lambda)^n v = (X-\mu)^n w = 0$. One has

$$\begin{aligned} b_{K\langle\langle X^{-1}\rangle\rangle}(v,w) &= P(X-\lambda)^n b_{K\langle\langle X^{-1}\rangle\rangle}(v,w) + Q(X-\mu)^n b_{K\langle\langle X^{-1}\rangle\rangle}(v,w) \\ &= P b_{K\langle\langle X^{-1}\rangle\rangle}((X-\lambda)^n v, w) + Q b_{K\langle\langle X^{-1}\rangle\rangle}(v,(X-\mu)^n w) \\ &= 0, \end{aligned}$$

and, for the least, $b_0(v,w)=0$. This is what had to be proved. □

Remark A.14. More generally, one can consider, for every irreducible polynomial $P \in K[X]$, the subspace $V_P = \ker P(f)^n$. The V_Ps are in direct orthogonal sum.

As a corollary, when the minimal polynomial of f is split over K, there is a basis $\mathbf{e}$ of V where the matrices of $[b_0]_{\mathbf{e}}$ and $[f]_{\mathbf{e}}$ are block diagonal with blocks as in Lemma A.7.

Elements commuting with f also stabilize the generalized eigenspaces. A more precise statement is the following:

Proposition A.15. *Let g be in $\mathrm{End}_K(V)$. The following are equivalent:*

(1) g belongs to the intersection of the orthogonal groups $\mathrm{O}(b_0)$ and $\mathrm{O}(b_1)$;
(2) g belongs to $\mathrm{O}(b_0)$ and commutes with f;
(3) g is $K[X]$-linear and is orthogonal with respect to the form $b_{K\langle\langle X^{-1}\rangle\rangle}$ (i.e. for all v,w in W, $b_{K\langle\langle X^{-1}\rangle\rangle}(gv,gw) = b_{K\langle\langle X^{-1}\rangle\rangle}(v,w)$);
(4) under the hypothesis that the minimal polynomial of f is split over K, g stabilizes every generalized eigenspace V_λ of f and the restriction $g|_{V_\lambda}$ is b_0-orthogonal.
(5) (with no assumption on f), for every irreducible polynomial $P \in K[X]$, g stabilizes $V_P = \ker P(f)^n$ and its restriction to V_P is b_0-orthogonal.

A.7. **(Over the reals) When f has two conjugate eigenvalues.** To have a complete understanding of the pair (b_0, f) over $\mathbf{R}$, we need to investigate the case when $V = V_P = \ker P(f)^n$ when $P = (X-\lambda)(X-\bar\lambda)$ for $\lambda = a+ib \notin \mathbf{R}$.

The complexification of V will be denoted $V_{\mathbf{C}} = V + iV$. It is a $\mathbf{C}$-vector space equipped with an antilinear involution (the complex conjugation) $v \mapsto \bar v$ whose fixed points set is precisely $V \subset V_{\mathbf{C}}$. Any $\mathbf{R}$-linear endomorphism $g\colon V \to V$ admits a complexification $g_{\mathbf{C}}\colon V_{\mathbf{C}} \to V_{\mathbf{C}}$. The $\mathbf{R}$-bilinear form $b_0\colon V\times V \to \mathbf{R}$ has also a $\mathbf{C}$-bilinear complexification $b_{0,\mathbf{C}}\colon V_{\mathbf{C}} \times V_{\mathbf{C}} \to \mathbf{C}$. In particular, $f_{\mathbf{C}}$ has two eigenvalues: λ and $\bar\lambda$ and the space $V_{\mathbf{C}}$ is the direct $b_{0,\mathbf{C}}$-orthogonal sum of $V_\lambda = \ker(f_{\mathbf{C}}-\lambda)^n$ and of $V_{\bar\lambda} = \ker(f_{\mathbf{C}}-\bar\lambda)^n$. The spaces V_λ and $V_{\bar\lambda}$ are exchanged by the complex conjugation $v\mapsto \bar v$, they intersect V trivially.

Lemma A.16. *The map $g \mapsto g_{\mathbf{C}}|V_\lambda$ induces an isomorphism between*

(1) the group of elements g in $\mathrm{O}(b_0)$ that commute with f, and
(2) the group of elements α in $\mathrm{O}(V_\lambda, b_{0,\mathbf{C}})$ that commute with $f_{\mathbf{C}}$.

Proof. If g commutes with f, $g_{\mathbf{C}}$ commutes with $f_{\mathbf{C}}$ and stabilizes V_λ. If furthermore g is b_0-orthogonal, then $g_{\mathbf{C}}$ is $b_{0,\mathbf{C}}$-orthogonal. Therefore the mentioned map from the centralizer $Z_{\mathrm{O}(b_0)}(f)$ to the centralizer $Z_{\mathrm{O}(V_\lambda,b_{0,\mathbf{C}})}(f_{\mathbf{C}})$ is well defined and easily seen to be a homomorphism.

Let us construct its inverse. For $\alpha \in Z_{\mathrm{O}(V_\lambda,b_{0,\mathbf{C}})}(f_{\mathbf{C}})$, let $\beta\colon V_{\bar\lambda} \to V_{\bar\lambda} \mid v \mapsto \overline{\alpha(\bar v)}$. Then β is $\mathbf{C}$-linear, $b_{0,\mathbf{C}}$-orthogonal, and commutes with $f_{\mathbf{C}}$. The pair (α,β) combines therefore into a $\mathbf{C}$-linear and $b_{0,\mathbf{C}}$-orthogonal map $\gamma\colon V_{\mathbf{C}} = V_\lambda \oplus V_{\bar\lambda} \to V_{\mathbf{C}}$ that commutes with $f_{\mathbf{C}}$. The map γ commutes with the complex conjugation and hence comes from a unique $\mathbf{R}$-linear map $g\colon V\to V$ (g is simply the restriction of γ to V). The map g is furthermore b_0-orthogonal and commutes with f since γ is $b_{0,\mathbf{C}}$-orthogonal and commutes with $f_{\mathbf{C}}$. □

Applying this lemma to $g = f$ and the results of Section A.3, we can give a normal form for the pair (b_0, f) when V is cyclic, i.e. when $f_{\mathbf{C}}|_{V_\lambda}$ has one Jordan block. For this, let us denote, for every matrix M in $M_m(\mathbf{C})$, $r(M)$ (respectively $s(M)$) the matrix in $M_{2m}(\mathbf{R})$ where each coefficient $x + iy$ of M is replaced by the bloc $\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$ (respectively by $\left(\begin{smallmatrix} x & -y \\ -y & -x \end{smallmatrix}\right)$). The following identities are easily verified: $r(MN) = r(M)r(N)$, $s(M) = Br(M) = r(\overline{M})B$ where B is the diagonal matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix},$$

and also: $s({}^T M) = {}^T s(M)$, $s(M^{-1}) = s(\overline{M})^{-1}$.

Lemma A.17. *Let (W, b) be a complex vector space equipped with a bilinear symmetric form. Denote by $W_{\mathbf{R}}$ the underlying real vector space and $b_{\mathbf{R}} := \Re(b)$ the real part of b. If S is the matrix of b in a basis $\mathbf{e} = (e_1, \dots, e_m)$ of W, then $s(M)$ is the matrix of $b_{\mathbf{R}}$ in the basis $\mathbf{f} = (e_1, ie_1, \dots, e_m, ie_m)$ of $W_{\mathbf{R}}$.*

Let f be an endomorphism of W. Then if F is the matrix of f in the basis $\mathbf{e}$, then matrix in the basis $\mathbf{f}$ of f considered as an endomorphism of $W_{\mathbf{R}}$ is $r(F)$.

We denote $C'_{2m} = s(C_m)$ and $J'_{2m}(\lambda) = r(J_m(\lambda))$ (see Equation (9.3) in Section 9.1.2). Then, using the basis of the complex vector space V_λ furnished by Lemma A.7 for $b_{0,\mathbf{C}}$ and $f_{\mathbf{C}}$, we get

Lemma A.18. *With the above hypothesis, there is a basis $\mathbf{e}$ of the real vector space V, such that $[b_0]_{\mathbf{e}} = C'_{2m}$ and $[f]_{\mathbf{e}} = J'_{2m}(\lambda)$. The basis $\mathbf{e}$ is unique up to multiplication by ± 1.*

For any two complex $m \times m$ matrices M and N, the following equalities are direct consequences of the identities recalled above:

$$s({}^T NMN) = {}^T r(N)s(M)r(N) = {}^T s(N)s(\overline{M})s(N)$$
$$s({}^T N {}^T M^{-1} N) = {}^T s(N) \, {}^T s(M)^{-1} s(N).$$

Let us denote $\Psi_{2m}(\lambda) := s(\Phi_m(\lambda))$ (cf. Equation (9.4)). Then the matrix $\Psi_{2m}(\lambda)$ is symmetric and satisfies ${}^T\Psi_{2m}(\lambda) \, {}^T C'^{-1}_{2m} \Psi_{2m}(\lambda) = C'_{2m} J'_{2m}(\lambda)$ and ${}^T\Psi_{2m}(\lambda) \, {}^T(C'_{2m} J'_{2m}(\lambda))^{-1} \Psi_{2m}(\lambda) = C'_{2m}$. Up to sign, this is the unique matrix satisfying these two equations.

Definition A.19. We call also the matrix $\Psi_{2m}(\lambda)$ the *back transformation*.

A.8. **Normal forms.** We now collect the results from the previous sections to establish normal forms for pairs of quadratic forms over $\mathbf{R}$.

For a finite sequence $\underline{n} = (n_1, \dots, n_k)$ of positive integers, we denote by $C(\underline{n})$ the square matrix of size $n_1 + \cdots + n_k$, diagonal by blocks, and whose blocks are $C_{n_1}, \dots, C_{n_k}$. If $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ is a further sequence of real numbers, we will denote by $J(\underline{n}, \underline{\lambda})$ the block diagonal matrix whose blocks are $J_{n_1}(\lambda_1), \dots, J_{n_k}(\lambda_k)$. Under the hypothesis that no λ_i is 0, similar notation $\Phi(\underline{n}, \underline{\lambda})$ is adopted for the matrix whose blocks are the back transformations $\Phi_{n_i}(\lambda_i)$ (see Definition A.11). When $2\underline{m} = (2m_1, \dots, 2m_k)$ is a sequence of even integers and $\underline{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbf{C}^k$, we introduce also, via the same procedure, the matrices $C'(2\underline{m})$, $J'(2\underline{m}, \underline{\lambda})$, and if $\underline{\lambda} \in (\mathbf{C}^*)^k$ $\Psi(2\underline{m}, \underline{\lambda})$.

Theorem A.20. *Let V be a real vector space of dimension n. Let b_0 and b_1 be two symmetric bilinear forms on V with b_0 nondegenerate. Let $b_i^\dagger$ $(i = 0, 1)$ be the induced morphisms $V \to V^*$ and let $f = (b_0^\dagger)^{-1} \circ b_1^\dagger$.*

(1) There are uniquely determined sequences $(\underline{n}_x, \underline{\lambda}_x) \in \mathbf{Z}_{>0}^{k_x} \times \mathbf{R}^{k_x}$ $(x \in \{\pm 1\}^2)$, and $(2\underline{m}, \underline{\lambda}) \in (2\mathbf{Z}_{>0})^{k_0} \times \mathbb{H}^{k_0}$ satisfying the following normalization:

- *for all* $x=(\varepsilon,\eta)\in\{\pm1\}^2$, *and all* $1\leqslant \ell\leqslant k_x$, $\varepsilon\eta\lambda_{x,\ell}\geqslant 0$;
- *the sequences* $\underline{\lambda}_x$ ($x\in\{\pm1\}^2$), *and* $\underline{\lambda}$ *are decreasing (for the last one, the lexicographic order is understood on* $\mathbf{C}\simeq\mathbf{R}^2$*)*;
- *for all* $x\in\{\pm1\}^2$ *and all* $1\leqslant k<\ell\leqslant k_x$, *if* $\lambda_{x,k}=\lambda_{x,\ell}$, *then* $n_{x,k}\geqslant n_{x,\ell}$;
- *for all* $1\leqslant k<\ell\leqslant k_0$, *if* $\lambda_k=\lambda_\ell$, *then* $m_k\geqslant m_\ell$;
- *the sum of the sequences* $\underline{n}_x$ ($x\in\{\pm1\}^2$), *and* $2\underline{m}$ *is equal to* n;

and such that

there exists a basis $\mathbf{e}$ *of* V *such that the matrix* $[b_0]_{\mathbf{e}}$ *is the block diagonal matrix whose blocks are* $C(\underline{n}_{1,1})$, $C(\underline{n}_{1,-1})$, $-C(\underline{n}_{-1,1})$, $-C(\underline{n}_{-1,-1})$, *and* $C'(2\underline{m})$ *and the matrix* $[f]_{\mathbf{e}}$ *is the block diagonal matrix with blocks* $J(\underline{n}_{1,1},\underline{\lambda}_{1,1})$, $J(\underline{n}_{1,-1},\underline{\lambda}_{1,-1})$, $J(\underline{n}_{-1,1},\underline{\lambda}_{-1,1})$, $J(\underline{n}_{-1,-1},\underline{\lambda}_{-1,-1})$, *and* $J'(2\underline{m},\underline{\lambda})$.

(2) *The form* b_1 *is nondegenerate if and only if no zero appears in the sequences* $\underline{\lambda}_x$ ($x\in\{\pm1\}^2$).

(3) *If the form* b_1 *is nondegenerate, the sequences associated to the pair* (b_1^*,b_0^*) *on* V^* *are, for* $x=(\varepsilon,\eta)$, $(\underline{n}_{\eta,\varepsilon},\underline{\lambda}_{\eta,\varepsilon})$, *and* $(2\underline{m},\underline{\lambda})$.

A basis $\mathbf{e}$ as in (1) will be called *in standard position* with respect to (b_0,b_1).

Proposition A.21. *Under the same assumption than Theorem A.20, let* Φ *the block matrix*

$$\Phi:=\begin{pmatrix}\Phi(\underline{n}_{1,1},\underline{\lambda}_{1,1}) &&&& \\ & & \Phi(\underline{n}_{1,-1},\underline{\lambda}_{1,-1}) && \\ & \Phi(\underline{n}_{-1,1},\underline{\lambda}_{-1,1}) &&& \\ &&& \Phi(\underline{n}_{-1,-1},\underline{\lambda}_{-1,-1}) & \\ &&&& \Psi(2\underline{m},\underline{\lambda})\end{pmatrix}.$$

If $\mathbf{e}$ *is a basis in standard position with respect to* (b_0,b_1), *let* $\mathbf{v}=\mathbf{e}\Phi$ *and* $\mathbf{v}^*$ *be the basis of* V^* *dual to* $\mathbf{v}$. *Then* $\mathbf{v}^*$ *is in standard position with respect to* (b_1^*,b_0^*).

The work done so far implies the existence of such basis and sequences. The uniqueness will follow from our analysis of the automorphism groups.

A.9. **Automorphism groups.** It is desirable to have an understanding on how much the basis in the above theorem can vary, or, what amounts to the same, to have an understanding of the centralizer of f in the orthogonal group $\mathrm{O}(b_0)$. By Proposition A.15, Lemma A.16, and up to replacing f by $f-\lambda\,\mathrm{Id}$, we are reduced to the following situation:

V is a K-vector space ($K=\mathbf{R}$ or $\mathbf{C}$) and f is nilpotent.

From the analysis done in Section A.5, we can assume that there is a decreasing sequence $(m_1,\dots,m_r)$ of integers and sequences $(p_1,\dots,p_r)$, $(q_1,\dots,q_r)$ such that, the $K[X]$-module V is the direct $b_{K\langle\langle X^{-1}\rangle\rangle}$-orthogonal sum

$$V=K_{[m_1]}^{p_1+q_1}\oplus\cdots\oplus K_{[m_r]}^{p_r+q_r}$$

and, for each $i=1,\dots,r$, the bilinear form on the factor $K_{[m_i]}^{p_i+q_i}$ is

$$\begin{aligned} K_{[m_i]}^{p_i+q_i}\times K_{[m_i]}^{p_i+q_i} &\longrightarrow K\langle\langle X^{-1}\rangle\rangle \\ ((x_1,\dots,x_{p_i+q_i}),(y_1,\dots,y_{p_i+q_i})) &\longmapsto X^{1-m_i}\,{}^T\!x I_{p_i,q_i} y \\ &= X^{1-m_i}(x_1y_1+\cdots+x_{p_i}y_{p_i}-x_{p_i+1}y_{p_i+1}-\cdots-x_{p_i+q_i}y_{p_i+q_i}), \end{aligned}$$

where $I_{p,q}=\begin{pmatrix}\mathrm{Id}_p & \\ & -\mathrm{Id}_q\end{pmatrix}$.

An element g in $\mathrm{End}_{K[X]}(V)$ can be considered as a matrix $(g_{i,j})_{1\leqslant i,j\leqslant r}$ where, for all i,j, $g_{i,j}$ belongs to $\mathrm{Hom}_{K[X]}(K_{[m_j]}^{p_j+q_j},K_{[m_i]}^{p_i+q_i})$. The element $g_{i,j}$ can itself be considered as a matrix of type $(p_i+q_i)\times(p_j+q_j)$ with entries in $\mathrm{Hom}_{K[X]}(K_{[m_j]}, K_{[m_i]})$. As $K_{[m_j]}$ is a cyclic $K[X]$-module, one has $\mathrm{Hom}_{K[X]}(K_{[m_j]}, K_{[m_i]}) \simeq \{x \in K_{[m_i]} \mid X^{m_j}x = 0\}$ which is equal to $K_{[m_i]}$ if $m_j \geqslant m_i$ (i.e. if $j \leqslant i$) and to $X^{m_i-m_j}K_{[m_j]}$ otherwise (i.e. if $i < j$). If $i > j$ we denote by $h_{i,j}$ the $(p_i+q_i)\times(p_j+q_j)$ matrix with entries in $K_{[m_j]}$ such that $g_{i,j} = X^{m_i-m_j}h_{i,j}$.

The element g belongs to the orthogonal group G with respect to the bilinear form $b_{K\langle\langle X^{-1}\rangle\rangle}$ if and only if the following equalities hold in $K\langle\langle X^{-1}\rangle\rangle$

- for all j,

$$\sum_{i>j} X^{1-m_i}\, {}^T g_{i,j} I_{p_i,q_i} g_{i,j} + X^{1-m_j}\, {}^T g_{j,j} I_{p_j,q_j} g_{j,j} + \sum_{i<j} X^{1+m_i-2m_j}\, {}^T h_{i,j} I_{p_i,q_i} h_{i,j} = X^{1-m_j} I_{p_j,q_j}$$

- for all $j \neq j'$,

$$\sum_i X^{1-m_i}\, {}^T g_{i,j} I_{p_i,q_i} g_{i,j'} = 0.$$

For every integer $s \geqslant 0$, let us denote by $G_{(s)}$ the subgroup of G consisting of the elements g equal to the identity modulo X^s.

Recall also the augmentation map $\epsilon\colon K_{[m]} \to K$ that we promote to maps between spaces of matrices. Thanks to the above description of the elements of G, one has

Lemma A.22. *The assignment $\pi\colon g \mapsto \epsilon(g)$ induces an isomorphism between $G/G_{(1)}$ and the group $H = \prod_{i=1}^r \mathrm{O}(I_{p_i,q_i})$. The inclusions $K \hookrightarrow K_{[m]}$ induce a morphism $\iota\colon H \to G$ that is a section of π.*

In particular, the pairs (p_i, q_i) are determined by G and this implies the thought for uniqueness in Theorem A.20.

Lemma A.23. *The assignment $\mathrm{Id} + X^s A \mapsto \epsilon(A)$ induces an isomorphism between the group $G_{(s)}/G_{(s+1)}$ and the abelian group underlying the vector space of block matrices $a = (a_{i,j})$ (i.e. for all i, j, $a_{i,j} \in M_{p_i+q_i,p_j+q_j}(K)$) satisfying:*

- *a is antisymmetric with respect to $\mathrm{diag}(I_{p_1,q_1}, \dots, I_{p_r,q_r})$;*
- *for all i and j, if $m_i \leqslant s$, then $a_{i,j} = 0$;*
- *for all i and j, if $m_i - m_j > s$, then $a_{i,j} = 0$.*

Thus $G_{(1)}$ is a nilpotent Lie group, contained in the unipotent radical of G and the Levi factor of G is a product of orthogonal groups:

Proposition A.24. *The group G of automorphisms of the pair (b_0, f) is the semi-direct product of a Levi factor $\iota\big(\prod_{p_i+q_i>2} \mathrm{O}(I_{p_i,q_i})\big)$ and its unipotent radical $\pi^{-1}\big(\prod_{p_i+q_i\leqslant 2} \mathrm{O}(I_{p_i,q_i})\big)$; its radical is equal to its unipotent radical.*

References

[1] Daniele Alessandrini and Brian Collier. The geometry of maximal components of the PSp(4, ℝ) character variety. *Geom. Topol.*, 23(3):1251–1337, 2019.

[2] Arkady Berenstein and Vladimir Retakh. Noncommutative marked surfaces. *Adv. Math.*, 328:1010–1087, 2018.

[3] Dieter Biallas. Verallgemeinerte Doppelverhältnisse und Endomorphismen von Vektorräumen. *Abh. Math. Sem. Univ. Hamburg*, 29:263–291, 1966.

[4] Ada Boralevi and Jarosł aw Buczyński. Secants of Lagrangian Grassmannians. *Ann. Mat. Pura Appl. (4)*, 190(4):725–739, 2011.

[5] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen. Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. *Geom. Dedicata*, 122:185–213, 2006.

[6] Marc Burger, Alessandra Iozzi, François Labourie, and Anna Wienhard. Maximal representations of surface groups: symplectic Anosov structures. *Pure Appl. Math. Q.*, 1(3, Special Issue: In memory of Armand Borel. Part 2):543–590, 2005.

[7] Marc Burger, Alessandra Iozzi, and Anna Wienhard. Surface group representations with maximal Toledo invariant. *Ann. of Math. (2)*, 172(1):517–566, 2010.

[8] Jean-Louis Clerc and Khalid Koufany. Primitive du cocycle de Maslov généralisé. *Math. Ann.*, 337(1):91–138, 2007.
[9] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.
[10] Oscar García-Prada, Peter B. Gothen, and Ignasi Mundet i Riera. Higgs bundles and surface group representations in the real symplectic group. *J. Topol.*, 6(1):64–118, 2013.
[11] Peter B. Gothen. Components of spaces of representations and stable triples. *Topology*, 40(4):823–850, 2001.
[12] Olivier Guichard, François Labourie, and Anna Wienhard. Postitive representations. in preparation.
[13] Olivier Guichard and Anna Wienhard. θ-positivity. in preparation.
[14] Olivier Guichard and Anna Wienhard. Positivity and higher Teichmüller theory. In *European Congress of Mathematics*, pages 289–310. Eur. Math. Soc., Zürich, 2018.
[15] Tobias Hartnick and Tobias Strubel. Cross ratios, translation lengths and maximal representations. *Geom. Dedicata*, 161:285–322, 2012.
[16] Loo-Keng Hua. Geometries of matrices. I. Generalizations of von Staudt's theorem. *Trans. Amer. Math. Soc.*, 57:441–481, 1945.
[17] Rinat M. Kashaev. Coordinates for the moduli space of flat $\mathrm{PSL}(2,\mathbb{R})$-connections. *Math. Res. Lett.*, 12(1):23–36, 2005.
[18] Gérard Lion and Michèle Vergne. *The Weil representation, Maslov index and theta series*, volume 6 of *Progress in Mathematics*. Birkhäuser, Boston, Mass., 1980.
[19] Yurii A. Neretin. *Lectures on Gaussian integral operators and classical groups*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011.
[20] Robert C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.
[21] Vladimir Retakh. Noncommutative cross-ratios. *J. Geom. Phys.*, 82:13–17, 2014.
[22] Vladimir Retakh, Vladimir Rubtsov, and Georgy Sharygin. Noncommutative cross-ratio and schwarz derivative. *arXiv:1905.01366*, pages 1–24, 2019.
[23] Carl Ludwig Siegel. Symplectic geometry. *Amer. J. Math.*, 65:1–86, 1943.
[24] Jean-Marie Souriau. Construction explicite de l'indice de Maslov. Applications. In *Group theoretical methods in physics (Fourth Internat. Colloq., Nijmegen, 1975)*, pages 117–148. Lecture Notes in Phys., Vol. 50. 1976.
[25] Tobias Strubel. Fenchel-Nielsen coordinates for maximal representations. *Geom. Dedicata*, 176:45–86, 2015.
[26] Anna Wienhard. An invitation to higher Teichmüller theory. *Proceedings of the ICM 2018*, 2018.
[27] Mikhail I. Zelikin. Geometry of the cross ratio of operators. *Mat. Sb.*, 197(1):39–54, 2006.

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