The geometry of Flag manifolds I

SRNI 45th School

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The slides are available at
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The linear algebra exercise of the day

Let V be a real vector space of dimension $4n + 2$ (n is an integer) equipped with a quadratic form q of signature $(2n + 1, 2n + 1)$. Let E and F be two maximal isotropic subspaces of V. This means that $q(v) = 0$ for every $v \in E \cup F$ and $\dim E = \dim F = 2n + 1$.

There exists thus an element g in the orthogonal group $\mathrm{O}(V, q)$ such that $g(E) = F$ (Witt's theorem).

Exercise

If E and F are transverse (that is, if $E \cap F = \{0\}$), then $g \notin \mathrm{SO}(q)$ (that is, $\det(g) = -1$).

Lie algebra setting

G a semisimple Lie group ; $\mathfrak{g}$ its Lie algebra. For example, $G = \mathrm{O}(p, p+k)$ is the orthogonal group of a quadratic form q of signature $(p, p+k)$ [p and k are positive integers]. For definiteness we will realize $\mathrm{O}(p, p+k)$ as a subgroup of $\mathrm{GL}_{2p+k}(\mathbf{R})$ and q will be the form

$$q(x_1, \ldots, x_{2p+k}) = 2\sum_{i=1}^{p}(-1)^{i+p}x_i x_{2p+k+1-i} - \sum_{i=1}^{k} x_{p+i}^2.$$

K is a maximal compact subgroup of G ; $\mathfrak{k}$ its Lie algebra. One can take $K = G \cap \mathrm{O}(2p+k)$.

A *Cartan subspace* $\mathfrak{a}$ is a maximal (Abelian) subalgebra orthogonal to $\mathfrak{k}$ with respect to the Killing form.

One can take $\mathfrak{a}$ to be the space of matrices of the form

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\lambda_p, \ldots, -\lambda_1),\ (\lambda_1, \ldots, \lambda_p) \in \mathbf{R}^p$$

Lie algebra setting (continued)

For $\beta \in \mathfrak{a}^*$, set $\mathfrak{g}_\beta = \{X \in \mathfrak{g} \mid [A, X] = \beta(A)X, \forall A \in \mathfrak{a}\}$ and $\Sigma = \{\beta \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\beta \neq 0\}$.
The maps $\varepsilon_i \colon \mathfrak{a} \to \mathbf{R}$ [i varies from 1 to p] defined by $\varepsilon_i(\mathrm{diag}(\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\lambda_p, \ldots, -\lambda_1)) = \lambda_i$ are linear and form a basis of $\mathfrak{a}^*$. the roots are the $\pm\varepsilon_i \pm \varepsilon_j$ (for $i < j$) and the $\pm\varepsilon_i$.

Choosing $<_{\mathfrak{a}^*}$ a total linear ordering (the lexicographic order), one defines $\Sigma^+ = \{\alpha \in \Sigma \mid 0 <_{\mathfrak{a}^*} \alpha\}$ the positive roots. Here $\varepsilon_i \pm \varepsilon_j$, $i < j$ and $+\varepsilon_i$.

Let α belongs to Σ^+, when there are β, γ in Σ^+ such that $\alpha = \beta + \gamma$, one has $\mathfrak{g}_\alpha = [\mathfrak{g}_\beta, \mathfrak{g}_\gamma]$ and the root α is called *decomposable*, it is called *simple* otherwise. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha_p = \varepsilon_p$.

Denote $\Delta \subset \Sigma^+$ the set of simple roots.
Every positive root decomposes $\beta = \sum_\Delta n_\alpha \alpha$ where $n_\alpha \geq 0$.

The Weyl group

It is the automorphism group W of $\Sigma \subset \mathfrak{a}^*$. It is the group of signed permutation matrices, isomorphic to $\{\pm 1\}^p \rtimes S_p$.
For each α in Σ there is a unique hyperplane reflection contained in W such that $s_\alpha(\alpha) = -\alpha$. $s_i = s_{\alpha_i}$, s_p changes the sign of the last coordinate and s_i exchanges the coordinates in the indices i and $i+1$.

W is generated by $\{s_\alpha\}_{\alpha \in \Delta}$.
There is a unique element $w_{\max}$ of W sending Σ^+ to $\Sigma^- = -\Sigma^+ = \Sigma \smallsetminus \Sigma^+$. It is the longest length element.
$w_{\max} = -\operatorname{Id}$.

The map $\iota \colon \alpha \mapsto -w_{\max}(\alpha)$ sends Σ^+ to Σ^+ and Δ to Δ. It is called the *opposition involution*. The opposition involution is trivial.

W is isomorphic to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. For w in W, we will sometimes denote $\dot{w}$ a representative of w in $N_K(\mathfrak{a})$.

$\mathfrak{sl}_2$-triples, fundamental weights

Those are triples (x, y, h) in $\mathfrak{g}$ such that $[x, y] = h$, $[h, x] = 2x$
For example $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathfrak{sl}_2(\mathbf{R})$.

For all α in Δ we will choose an $\mathfrak{sl}_2$-triple $(x_\alpha, x_{-\alpha}, h_\alpha)$ with $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$.
If $i < p$, one can set $x_i = E_{i,i+1} + E_{2p+k-i,2p+k+1-i}$ and $x_{-i} = {}^t x_i$, and $x_p = E_{p,p+1} + E_{p+1,p+k+1}$, $x_{-p} = {}^t x_p$.

The element h_α does not depends on the choices. The family $\{h_\alpha\}_{\alpha \in \Delta}$ is a basis of $\mathfrak{a}$. The dual basis $\{\omega_\alpha\}_{\alpha \in \Delta}$ of $\mathfrak{a}^*$ is called the *fundamental weights*. $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$.

Let $\exp \colon \mathfrak{g} \to G$ be the exponential. For every α, one can choose $\dot{s}_\alpha = \exp\left(\pi/2(x_\alpha - x_{-\alpha})\right)$ to represent in $N_K(\mathfrak{a})$ the element s_α.

Parabolic subgroups, flag manifolds

- The subspace $\mathfrak{u}_\Delta = \sum_{\beta \in \Sigma^+} \mathfrak{g}_\beta$ is a nilpotent subalgebra generated by $\bigcup_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Similarly $\mathfrak{u}_\Delta^{\mathrm{opp}} = \sum_{\beta \in \Sigma^+} \mathfrak{g}_{-\beta}$.
- For every $\Theta \subset \Delta$ we let $\mathfrak{u}_\Theta$ to be the ideal of $\mathfrak{u}_\Delta$ generated by $\bigcup_{\alpha \in \Theta} \mathfrak{g}_\alpha$. One has $\mathfrak{u}_\Theta = \sum_{\alpha \in \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_\alpha$. Similarly set $\mathfrak{u}_\Theta^{\mathrm{opp}} = \sum_{\alpha \in \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_{-\alpha}$.
- The *standard parabolic subgroups* are $P_\Theta = N_G(\mathfrak{u}_\Theta)$, $P_\Theta^{\mathrm{opp}} = N_G(\mathfrak{u}_\Theta^{\mathrm{opp}})$.
- The unipotent radical of P_Θ (resp. P_Θ^{opp}) is $U_\Theta = \exp(\mathfrak{u}_\Theta)$ (resp. $U_\Theta^{\mathrm{opp}} = \exp(\mathfrak{u}_\Theta^{\mathrm{opp}})$).
- $L_\Theta = P_\Theta \cap P_\Theta^{\mathrm{opp}}$ is called a *Levi factor*. One has $P_\Theta = U_\Theta \rtimes L_\Theta$.
- $\mathcal{F}_\Theta$ is the space of parabolic groups conjugated to P_Θ ; $\mathcal{F}_\Theta^{\mathrm{opp}}$ is the space of parabolic groups conjugated to P_Θ^{opp}. As P_Θ^{opp} is conjugated to P_Θ (by $\dot{w}_{\max}$), $\mathcal{F}_{\iota(\Theta)} = \mathcal{F}_\Theta^{\mathrm{opp}}$.
- As $P_\Theta = N_G(P_\Theta)$, $\mathcal{F}_\Theta \simeq G/P_\Theta$.

Parabolic subgroups (continued)

For all $i \leq p$, P_i (resp. P_i^{opp}) is the stabilizer of the (isotropic) i-dimensional space generated by the i first (resp. last) basis vectors.
$\mathcal{F}_i = \mathcal{F}_i^{\mathrm{opp}}$ is naturally isomorphic to the space of isotropic i-planes.
More generally, $\mathcal{F}_{i_1 < \cdots < i_\ell}$ is the space of partial flags $(E_1 \subset \cdots \subset E_\ell)$ with $\dim E_m = i_m$ and E_ℓ isotropic.

A pair (P, Q) of parabolic subgroups is *transverse* if it is conjugated to $(P_\Theta, P_\Theta^{\mathrm{opp}})$. This is equivalent to $P \cap Q$ being reductive.
Two isotropic i-dimensional space E and F in $\mathcal{F}_i$ are transverse if and only if they are ...transverse! that is $E^{\perp_q} \cap F = 0$.

Lemma

The map $\mathfrak{u}_\Theta^{\mathrm{opp}} \to \mathcal{F}_\Theta \mid X \mapsto \exp(X) \cdot P_\Theta$ is one-to-one onto the space of elements transverse to P_Θ^{opp}

Embeddings into projective space

Let $\eta = \sum_\Delta k_\alpha \omega_\alpha$ be a dominant weight and let $\tau \colon G \to \mathrm{GL}(V)$ be the associated irreducible representation.

If $\eta = \omega_i$ take $V = \bigwedge^i \mathbf{R}^{2p+k}$.

We denote by V_η the eigenspace of $\mathfrak{a}$ (with respect to τ) relative to the eigenvalue η. This is a line in V. Denote by V_η° the $\mathfrak{a}$-invariant supplementary hyperplane.

Lemma

Let $\Theta = \{\alpha \in \Delta \mid k_\alpha = 0\}$. Then the stabilizer of V_η in G is P_Θ, the stabilizer of V_η° is P_Θ^{opp}.

We can therefore build (one-to-one) maps

$$i_\Theta \colon \mathcal{F}_\Theta \longrightarrow \mathbb{P}(V) \mid g \cdot P_\Theta \longmapsto \tau(g) \cdot V_\eta$$
$$i_\Theta^{\mathrm{opp}} \colon \mathcal{F}_\Theta^{\mathrm{opp}} \longrightarrow \mathbb{P}^*(V) \mid g \cdot P_\Theta^{\mathrm{opp}} \longmapsto \tau(g) \cdot V_\eta^\circ$$

Lemma

$(P, Q) \in \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}}$ are transverse if and only if $(i_\Theta(P), i_\Theta^{\mathrm{opp}}(Q)) \in \mathbb{P}(V) \times \mathbb{P}^*(V)$ are transverse.