The geometry of Flag manifolds II

SRNI 45th School

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The slides are still available at
`https://irma.math.unistra.fr/~guichard/srni`

The linear algebra exercise of the day

Let $V = \mathrm{Sym}^{d-1}\mathbf{R}^2 \simeq \mathbf{R}_{d-1}[X,Y]$ be the space of homogenous polynomials in 2 variables.

V has a natural basis $e_i = Y^{i-1}X^{d-i}$ $(i = 1, \dots, d)$.

V and V^* bear a natural action of $\mathrm{SL}_2(\mathbf{R})$.

Exercise

$t \mapsto \left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot e_1^*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot e_1 \right\rangle$ is a non zero multiple of $t \mapsto t^d$.

The cross-ratio on the projective line, the collar lemma

The cross-ratio on the projective line $\mathbb{P}^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$ is defined by the formula $[x, y, X, Y] = \dfrac{X - x}{X - y}\dfrac{Y - y}{Y - x}$

The normalization is so that $[\infty, 0, 1, t] = t$. This means that $[x, y, X, Y]$ belongs to $[0, 1]$ if Y is between y and X, $[x, y, X, Y] \geq 1$ if Y is between X and x, etc.

Let M be a complete hyperbolic surface.
Let α, β be intersecting geodesics on M and $\ell(\alpha)$, $\ell(\beta)$ their lengths.

Theorem (Collar lemma)

$$\frac{1}{\exp(\ell(\alpha))} + \frac{1}{\exp(\ell(\beta))} \leq 1.$$

Collar lemma (the proof)

[Two nice drawings should go here]

Let A and B in $\mathrm{SL}_2(\mathbf{R})$ be the holonomies of α and β respectively. Those are diagonalizable and thus admit attracting a^+, b^+ and repelling a^-, b^- fixed points in $\mathbb{P}^1(\mathbf{R})$.

The relation between the length and the cross-ratio is the following $\exp(\ell(\alpha)) = [a^+, a^-, x, A(x)]$

Magical relation $[a^-, b^+, a^+, A(b^+)] + [a^-, a^+, b^+, A(b^+)] = 1.$

One has $[a^-, a^+, b^+, A(b^+)] = \exp(-\ell(\beta))$.

$$\begin{aligned}[a^-, b^+, a^+, A(b^+)] &= [a^-, b^-, a^+, A(b^+)][b^-, b^+, a^+, A(b^+)] \\ &\geq [b^-, b^+, a^+, B(a^+)][b^-, b^+, B(a^+), A(b^+)] \\ &\geq [b^-, b^+, a^+, B(a^+)] = \exp(-\ell(\beta))\end{aligned}$$

Projective cross-ratios

V a real vector space
On $\mathcal{O}_V = \{(x, y, X, Y) \in \mathbb{P}(V)^2 \times \mathbb{P}^*(V)^2 \mid X \pitchfork y \text{ and } Y \pitchfork x\}$,
we define

$$b_V(x, y, X, Y) = \frac{\langle \dot{X}, \dot{x} \rangle}{\langle \dot{X}, \dot{y} \rangle} \frac{\langle \dot{Y}, \dot{y} \rangle}{\langle \dot{Y}, \dot{x} \rangle}$$

Cocycle relations $b_V(x, y, X, Y) b_V(y, z, X, Y) = b_V(x, z, X, Y)$

For a loxodromic element $A \in \mathrm{GL}(V)$,
$b_V(a^+, a^-, X, A \cdot X) = \lambda_{\max}(A)/\lambda_{\min}(A)$

Symplectic interpretation

Let ω^V be the natural symplectic form on $V \times V^* = T^*V$.
The map $\mu\colon V \times V^* \to \mathbf{R} \mid (v, \varphi) \mapsto \langle \varphi, v \rangle$ is a moment for the $\mathbf{R}^*$-action $\lambda \cdot (v, \varphi) = (\lambda v, \lambda^{-1}\varphi)$.

The symplectic reduction $\mu^{-1}(1)/\mathbf{R}^*$ carries a symplectic form ω and is isomorphic to $\mathcal{U}_V = \{(x, X) \in \mathbb{P}(V) \times \mathbb{P}^*(V) \mid X \pitchfork x\}$.

Proposition

Let $f\colon [0,1]^2 \to \mathcal{U}_V$ be a C^1 map such that, $\forall t \in [0,1]$, $f(0,t) = (x, *)$, $f(1,t) = (y, *)$ and $\forall s \in [0,1]$, $f(s,0) = (*, X)$, $f(s,1) = (*, Y)$, then

$$b_V(x, y, X, Y) = \exp\Big(\int_{[0,1]^2} f^*\omega\Big)$$

Constructing cross-ratio on flag manifolds

Let G, Θ, P_Θ, etc. as in lecture 1.
Suppose that $\tau\colon G \to \mathrm{GL}(V)$ is a (continuous) representation such that there exists a one-to-one τ-equivariant map

$$i_\Theta\colon \mathcal{F}_\Theta \longrightarrow \mathbb{P}(V)$$

Proposition

Then there is $i_\Theta^{\mathrm{opp}}\colon \mathcal{F}_\Theta^{\mathrm{opp}} \longrightarrow \mathbb{P}^*(V)$ (one-to-one, equivariant) with the property that $P \pitchfork Q \Rightarrow i_\Theta(P) \pitchfork i_\Theta^{\mathrm{opp}}(Q)$.

Define

$$\mathcal{O}_\Theta = \left\{(x, y, X, Y) \in (\mathcal{F}_\Theta)^2 \times (\mathcal{F}_\Theta^{\mathrm{opp}})^2 \mid X \pitchfork y \text{ and } Y \pitchfork x\right\}$$
$$b_\tau(x, y, X, Y) = b_V\big(i_\Theta(x), i_\Theta(y), i_\Theta^{\mathrm{opp}}(X), i_\Theta^{\mathrm{opp}}(Y)\big).$$

Remembering some decompositions

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

$$\mathfrak{p}_\Delta = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \left(\bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \right)$$

$$\mathfrak{p}_\Theta = \left(\bigoplus_{\alpha \in \Sigma^+ \cap \mathrm{Span}(\Delta \smallsetminus \Theta)} \mathfrak{g}_{-\alpha} \right) \oplus \left(\mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \right)$$

Let $\tau \colon G \to \mathrm{GL}(V)$, $i_\Theta \colon \mathcal{F}_\Theta \to \mathbb{P}(V)$ as above.

The line $L = i_\Theta(P_\Theta)$ is P_Θ-invariant $\Rightarrow$ L is an eigenline for the action of $\mathfrak{a}$, is cancelled by $\mathfrak{u}_\Delta$ and also by $\bigcup_{\alpha \in \Theta} \mathfrak{g}_{-\alpha}$.

The subspace $W = \langle \tau(G) \cdot L \rangle$ is an irreducible representation of G with highest weight space L. Can (and will) assume $V = W$.

Continuing the analysis of V

$V = \bigoplus_{\kappa \in P(V)} V_\kappa$, $P(V) \subset \sum_\Delta \mathbf{Z}\omega_\alpha$, denote η the weight such that $L = V_\eta$. Then $\eta \in \sum_\Delta \mathbf{N}\omega_\alpha$ and uniquely determines V.

V_η invariant by $\mathfrak{g}_{-\alpha} \Leftrightarrow \eta(h_\alpha) = 0$.

One then get $\eta \in \sum \alpha \in \Theta \mathbf{N}^* \omega_\alpha$

The hyperplane $V_\eta^\circ = \sum_{\kappa \in P(V) \smallsetminus \{\eta\}} V_\kappa$ is tranverse to V_η and is P_Θ^{opp}-stable.

The maps

$$i_\Theta \colon \mathcal{F}_\Theta \longrightarrow \mathbb{P}(V) \mid g \cdot P_\Theta \longmapsto \tau(g) \cdot V_\eta$$
$$i_\Theta^{\mathrm{opp}} \colon \mathcal{F}_\Theta^{\mathrm{opp}} \longrightarrow \mathbb{P}^*(V) \mid g \cdot P_\Theta^{\mathrm{opp}} \longmapsto \tau(g) \cdot V_\eta^\circ$$

satisfy all the wanted properties.

Denote $b^\eta(x, y, X, Y) = b_V\big(i_\Theta(x), i_\Theta(y), i_\Theta^{\mathrm{opp}}(X), i_\Theta^{\mathrm{opp}}(Y)\big)$

Naturality properties

Lemma

One has $b^{\eta_1+\eta_2} = b^{\eta_1} b^{\eta_2}$, $b^{k\eta} = (b^{\eta})^k$.

Proof:

The map $\mathbb{P}(V) \times \mathbb{P}(W) \longrightarrow \mathbb{P}(V \otimes W)$ sends $b_V\, b_W$ to $b_{V\otimes W}$

Symplectic interpretation (bis)

There is a symplectic form ω^η on $\mathcal{U}_\Theta \subset \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}}$ such that

Proposition

Let $f \colon [0,1]^2 \to \mathcal{U}_\Theta$ *be a* C^1 *map such that,* $\forall t \in [0,1]$, $f(0,t) = (x,*)$, $f(1,t) = (y,*)$ *and* $\forall s \in [0,1]$, $f(s,0) = (*,X)$, $f(s,1) = (*,Y)$, *then*

$$b^\eta(x,y,X,Y) = \exp\Big(\int_{[0,1]^2} f^*\omega^\eta\Big)$$

The tangent space at a point (x,X) to $\mathcal{U}_\Theta$ identifies with $\mathfrak{u}_\Theta^{\mathrm{opp}} \times \mathfrak{u}_\Theta$.

The formula is

$$\omega^\eta\big((v_1,v_2),(w_1,w_2)\big) = \eta\big([v_1,w_2] - [v_2,w_1]\big).$$