The geometry of Flag manifolds III

SRNI 45th School

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The slides are again available at
`https://irma.math.unistra.fr/~guichard/srni`

The linear algebra exercise of the day

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & tF & \\ & & 1 & & & \\ & & & 1 & & \\ & A & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

$t \in \mathbf{R}$, where $A \in \mathrm{M}_{\ell,k}(\mathbf{R})$ and where $F \in \mathrm{M}_{k,\ell}(\mathbf{R})$ has rank one

Exercise

There is a unique $t \in \mathbf{R}$ such that this matrix is singular.

Geometric interpretation:

The first k columns represent an k-plane x, the last ℓ columns represent a ℓ-plane y_t;

The conclusion says that there is a unique t such that y_t is not tranverse to x.

The main result (I)

Theorem

This happens in every flag manifold.

G, $\mathcal{F}_\Theta$, $\mathcal{F}_\Theta^{\mathrm{opp}}$, P_Θ, P_Θ^{opp}, U_Θ, U_Θ^{opp}, L_Θ, $\mathfrak{p}_\Theta$, $\mathfrak{u}_\Theta$, $\mathfrak{a}$, ...

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

$\mathfrak{a}_L :=$ the centralizer of $\mathfrak{l}$ in $\mathfrak{a}$; $\mathfrak{a}_L = \bigcap_{\alpha \in \Delta \setminus \Theta} \ker \alpha.$

Proposition (Kostant, 2010)

The weight decomposition of $\mathfrak{u}_\Theta$ w.r.t. the action of $\mathfrak{a}_L$ coincides with the decomposition into irreducible L-summands:

$$\mathfrak{u}_\Theta = \bigoplus_{\aleph \in P} \mathfrak{u}_\aleph, \quad P \subset \mathfrak{a}_L^*, \quad [\mathfrak{u}_\aleph, \mathfrak{u}_\beth] = \mathfrak{u}_{\aleph + \beth}.$$

Photons

$$\mathfrak{u}_\Theta = \bigoplus_{\aleph \in P} \mathfrak{u}_\aleph, \quad P \subset \mathfrak{a}_L^*$$

In fact $P = \{\alpha|_{\mathfrak{a}_L}\}_{\alpha \in \Sigma^+ \smallsetminus \mathrm{Span}(\Delta \smallsetminus \Theta)}$; indecomposable weights $P \smallsetminus (P+P)$ naturally identifies with Θ [via $\Theta \to P \mid \alpha \mapsto \alpha|_{\mathfrak{a}_L}$]

For every $\alpha \in \Theta$, $\mathfrak{u}_\alpha \supset \mathfrak{u}_\alpha^{\text{high}} = \mathfrak{g}_\alpha$ (w.r.t. the action of $\mathfrak{a}$)

Consider $x_\alpha \in \mathfrak{u}_\alpha^{\text{high}}$

Definition

$\Phi_\alpha := \overline{\{\exp(tx_\alpha) \cdot P_\Theta^{\text{opp}}\}} \subset \mathcal{F}_\Theta^{\text{opp}}$ is the α-*photon* ; An α-*photon* is $\Phi = g \cdot \Phi_\alpha$ for some $g \in G$.

Lemma

Φ_α is homogenous under the action of $\mathrm{SL}_2(\mathbf{R})_\alpha$ *[the subgroup tangent to $\langle x_\alpha, x_{-\alpha}, h_\alpha\rangle$] and is $\simeq$ to* $\mathbb{P}^1(\mathbf{R})$.

Properties of Photons

Lemma

For all $x \in \mathcal{F}_\Theta^{\mathrm{opp}}$, so that $\mathrm{T}_x\mathcal{F}_\Theta^{\mathrm{opp}} \simeq \mathfrak{u}_\Theta$ and for all non zero v in this tangent space

- *There is Φ such that $x \in \Phi$ and $v \in \mathrm{T}_x\Phi \iff v \in L_\Theta \cdot \mathfrak{u}_\alpha^{\mathrm{high}} \subset \mathfrak{u}_\alpha \subset \mathfrak{u}_\Theta \simeq \mathrm{T}_x\mathcal{F}_\Theta^{\mathrm{opp}}$.*
- *In this case, there is a unique such Φ.*

Remark

$Z_\alpha = \mathbb{P}\big(L_\Theta \cdot x_\alpha\big) \subset \mathbb{P}(\mathfrak{u}_\alpha)$ is closed

$\Rightarrow$ the space of α-photons is closed.

Example(s)

$G = \mathrm{O}(p, p+k)$, $\Delta = \{\alpha_1, \dots, \alpha_p\}$, choose $\Theta = \{\alpha_1, \dots, \alpha_{p-1}\}$
Then
$\mathcal{F}_\Theta = \mathcal{F}_\Theta^{\mathrm{opp}} = \big\{(E_1 \subset \cdots \subset E_{p-1}) \mid \dim E_i = i, E_{p-1} \text{ isotropic}\big\}.$

Fix $x = (E_1, \dots, E_{p-1})$

For every $i < p-1$, there is a unique α_i-photon through x :
$\Phi_i = \big\{(F_1, \dots, F_{p-1}) \in \mathcal{F}_\Theta \mid \forall j \neq i, F_j = E_j\big\}$
The isomorphism with the projective line is concrete:
$\Phi_i \to \mathbb{P}(E_{i+1}/E_{i-1}) \mid (F_1, \dots, F_{p-1}) \mapsto F_i/E_{i-1}$

For every isotropic p-plane E_p containing E_{p-1},
$\Phi_{p-1} = \big\{(F_1, \dots, F_{p-1}) \in \mathcal{F}_\Theta \mid \forall j \neq p, F_j = E_j, F_{p-1} \subset E_p\big\}$ is a α_p-photon through x (and all α_p-photon has this form)
$\Phi_{p-1} \to \mathbb{P}(E_p/E_{p-2}) \mid (F_1, \dots, F_{p-1}) \mapsto F_{p-1}/E_{p-2}$

Photon projection

Define $\mathcal{V}_\Phi = \left\{x \in \mathcal{F}_\Theta \mid \exists y \in \Phi, x \pitchfork y\right\}$

Theorem

For every x in $\mathcal{V}_\Phi$, there is a unique y in Φ such that y is not *transverse to x. Set $p_\Phi(x) = y$.*
The map $p_\Phi \colon \mathcal{V}_\Phi \to \Phi$ has connected fibers.

Proof.

Up to G-action can assume $x = P_\Theta$, $P_\Theta^{\mathrm{opp}} \in \Phi$ and $\Phi = \Phi_\alpha$.
Then one needs to have $y = \dot{s}_\alpha \cdot P_\Theta^{\mathrm{opp}}$.

Let $U = \{\exp(tx_{-\alpha})\} \subset \mathrm{SL}_2(\mathbf{R})_\alpha$, one "sees" that
$\mathcal{V}_y = \{z \in \mathcal{F}_\Theta \mid z \pitchfork y\} \simeq U \times p_{\Phi_\alpha}^{-1}(\dot{s}_\alpha \cdot y)$. □

Example(s) (continued)

$(E_1, \dots, E_{p-1}) \in \mathcal{F}_{1,\dots,p-1}$ [and choose also an isotropic p-plane E_p containing E_{p-1} in order to treat the case $i = p-1$ on an equal footing]

$$\Phi_i = \big\{ (F_1, \dots, F_{p-1}) \in \mathcal{F}_{1,\dots,p-1} \mid \forall j \neq i, F_j = E_j,\ F_i \subset E_p \big\}$$

$$\mathcal{V}_{\Phi_i} = \big\{ (F_1, \dots, F_{p-1}) \in \mathcal{F}_{1,\dots,p-1} \mid \forall j \neq i, F_j \pitchfork E_j \big\}$$

$$p_{\Phi_i} \colon \mathcal{V}_{\Phi_i} \to \Phi_i$$

$$(F_1, \dots, F_{p-1}) \mapsto \big(\dots, E_{i-1}, E_{i-1} \oplus F_i^{\perp} \cap E_{i+1}, E_{i+1}, \dots \big)$$

The main result (II)

Choose $\eta = \sum_{\alpha\in\Theta} n_\alpha \omega_\alpha$ $(n_\alpha \in \mathbf{N})$ so that b^η is defined on $\mathcal{O}_\Theta \subset \mathcal{F}_\Theta \times \mathcal{F}_\Theta \times \mathcal{F}_\Theta^{\mathrm{opp}} \times \mathcal{F}_\Theta^{\mathrm{opp}}$

Fix $\alpha \in \Theta$ and an α-photon Φ.

Theorem

Let $x, y \in \Phi$. For all z, w in $\mathcal{V}_\Phi$ such that $p_\Phi(z) = p_\Phi(w) \notin \{x, y\}$, then $b^\eta(x, y, z, w) = 1$.

Let $x, y \in \Phi$. For all z, w in $\mathcal{V}_\Phi$, with $p_\Phi(z) \neq y$ and $p_\Phi(w) \neq x$, then

$$b^\eta(x, y, z, w) = [x, y, p_\Phi(z), p_\Phi(w)]^{n_\alpha}$$