Some tools for "focusing" variational data assimilation with applications to ocean modelling

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Outline

- On the use of reduced bases in variational DA
  - a taxinomy of particular vectors
  - how can they be useful for VDA ?
- On the use of zoom techniques in variational DA
  - mathematical formulation
  - an illustration

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Rationale Data assimilation methods are looking for an optimal correction to some background value in a space of huge dimension $\longrightarrow$ try to describe (most of) this correction in a subspace of low dimension.

The subspace must represent most of the natural "variability" of the system. But several definitions of the variability can be thought of:

- Statistical approach: POD (or EOFs, PCA)
  - variability = variance
- Dynamical systems: vectors of maximum growth
  - variability = "most dangerous" perturbations
- Spectral analysis:
  - variability = energy
- ...

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EOFs : Empirical Orthogonal Functions
(principal components, Proper Orthogonal Decomposition)

Sample of a model trajectory : $(\mathrm{x}(t_1), \ldots, \mathrm{x}(t_p))$

L$_1$, ..., L$_r$: directions in which the variance is maximum

They are the first eigenvectors of the empirical correlation matrix $\mathrm{X}\mathrm{X}^T$ with $\mathrm{X} = (\mathrm{X}_1, \ldots, \mathrm{X}_p)$

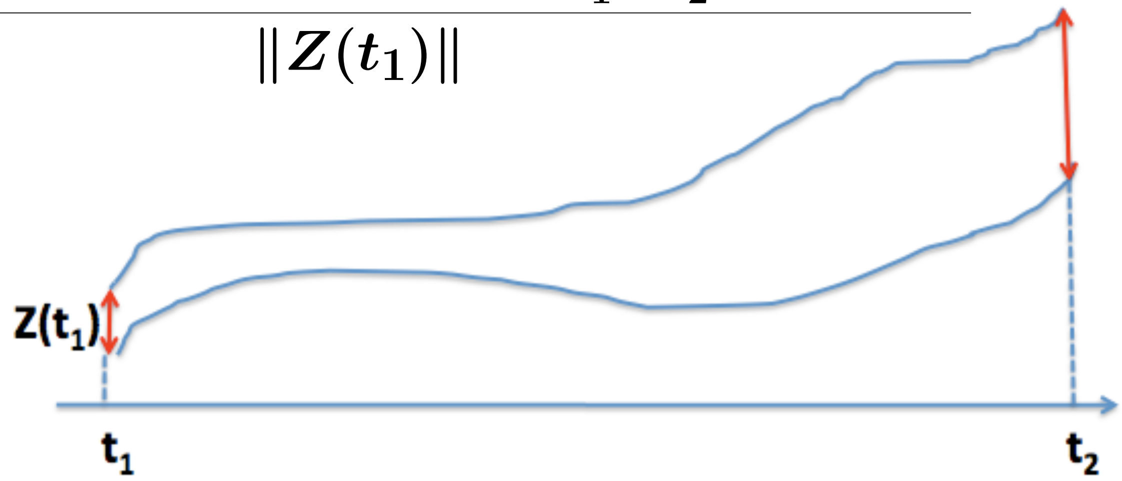
$$\mathrm{X}_j(i) = \frac{1}{\sigma_i}[\mathrm{x}(t_j) - \bar{\mathrm{x}}]$$

where $\bar{\mathrm{x}} = \frac{1}{p} \sum_{j=1}^{p} \mathrm{x}(t_j)$ $\quad \sigma_i^2 = \frac{1}{p} \sum_{j=1}^{p} (\mathrm{X}_j(i))^2$

Vectors of maximal growth

Amplification rate of some perturbation $Z(t_l)$:

$$\rho\left(Z(t_1)\right) = \frac{\| M_{t_1 \to t_2}\left(X(t_1) + Z(t_1)\right) - M_{t_1 \to t_2}\left(X(t_1)\right) \|}{\|Z(t_1)\|}$$



Find $Z_1^*(t_1)$ such that $\rho\left(Z_1^*(t_1)\right) = \max_{Z(t_1)} \rho\left(Z(t_1)\right)$

Degrees of freedom : $[t_l, t_2]$, M , $\| \cdot \|$, forward / backward

Vectors of maximal growth (2)

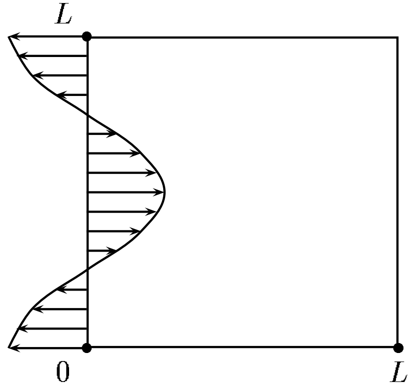
	Tangent linear approximation	Full (nonlinear) model
$[t_1,t_2]$ finite	*singular vectors*	*non-linear singular vectors (Mu et al.)*
$[t_1,t_2]$ infinite	*Lyapunov vectors*	*breeding vectors (Kalnay et al.)*

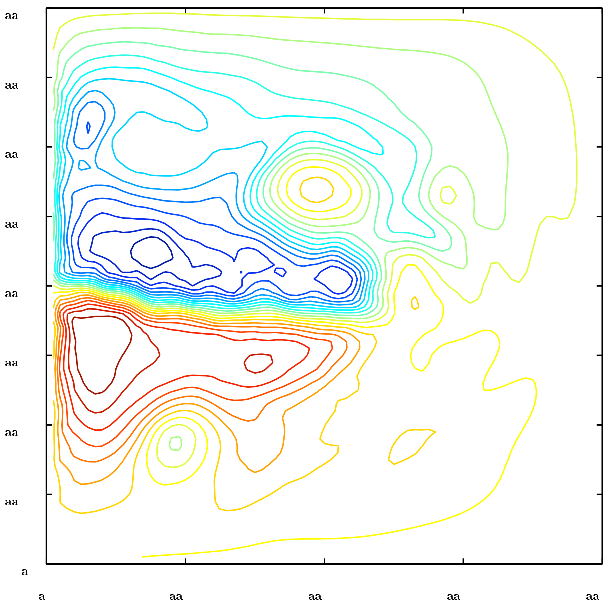
Such vectors are used in particular for stability analysis and for ensemble simulations.

Illustration in the context of an idealized shallow water model (Durbiano, 2001; Blayo et al., 2004)

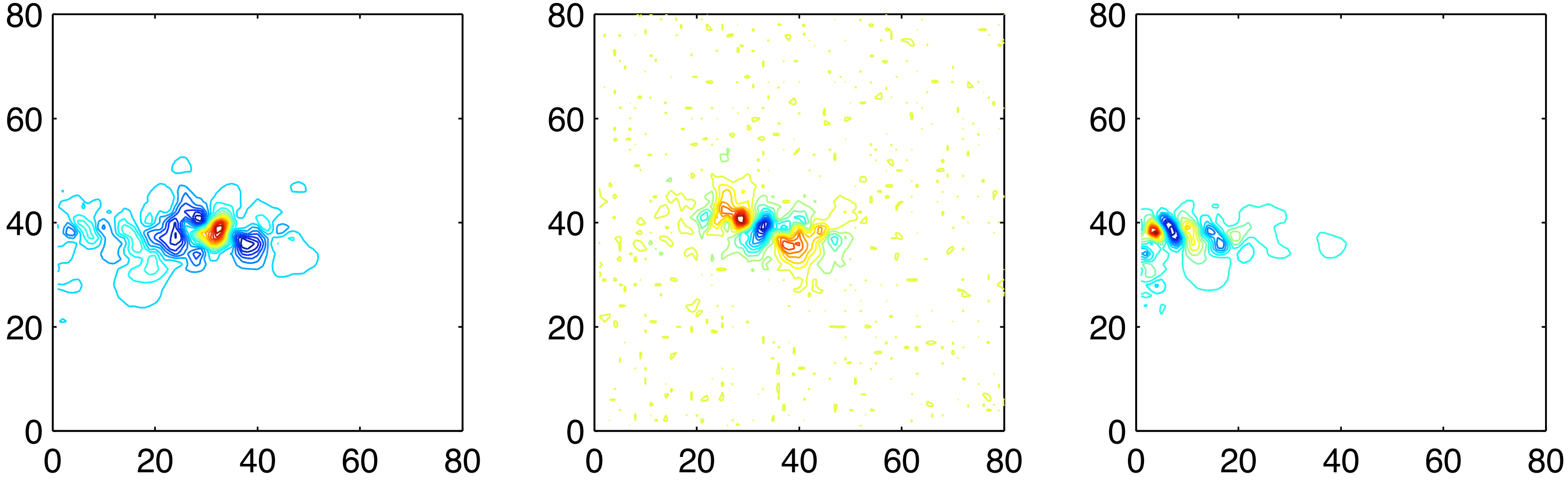
$$\left\{\begin{array}{l}
\dfrac{\partial u}{\partial t} + u\,\dfrac{\partial u}{\partial x} + v\,\dfrac{\partial u}{\partial y} - fv + g\,\dfrac{\partial h}{\partial x} + D_x = F_x \\
\dfrac{\partial v}{\partial t} + u\,\dfrac{\partial v}{\partial x} + v\,\dfrac{\partial v}{\partial y} + fu + g\,\dfrac{\partial h}{\partial y} + D_y = F_y \\
\dfrac{\partial h}{\partial t} + u\,\dfrac{\partial h}{\partial x} + v\,\dfrac{\partial h}{\partial y} + h\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0
\end{array}\right.$$

Wind





Snapshot : h

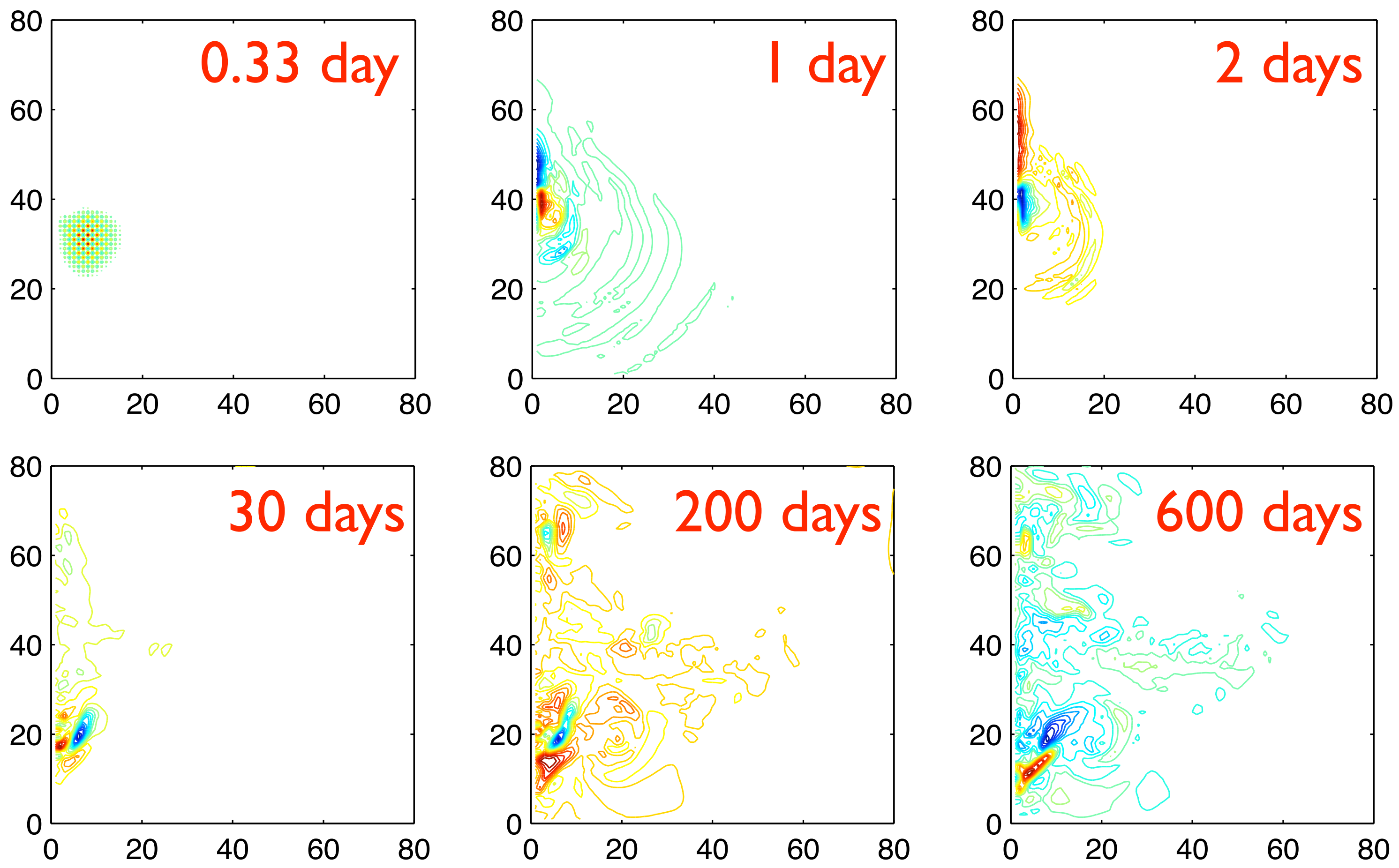


euclidian norm $u^2 + v^2 + h^2$

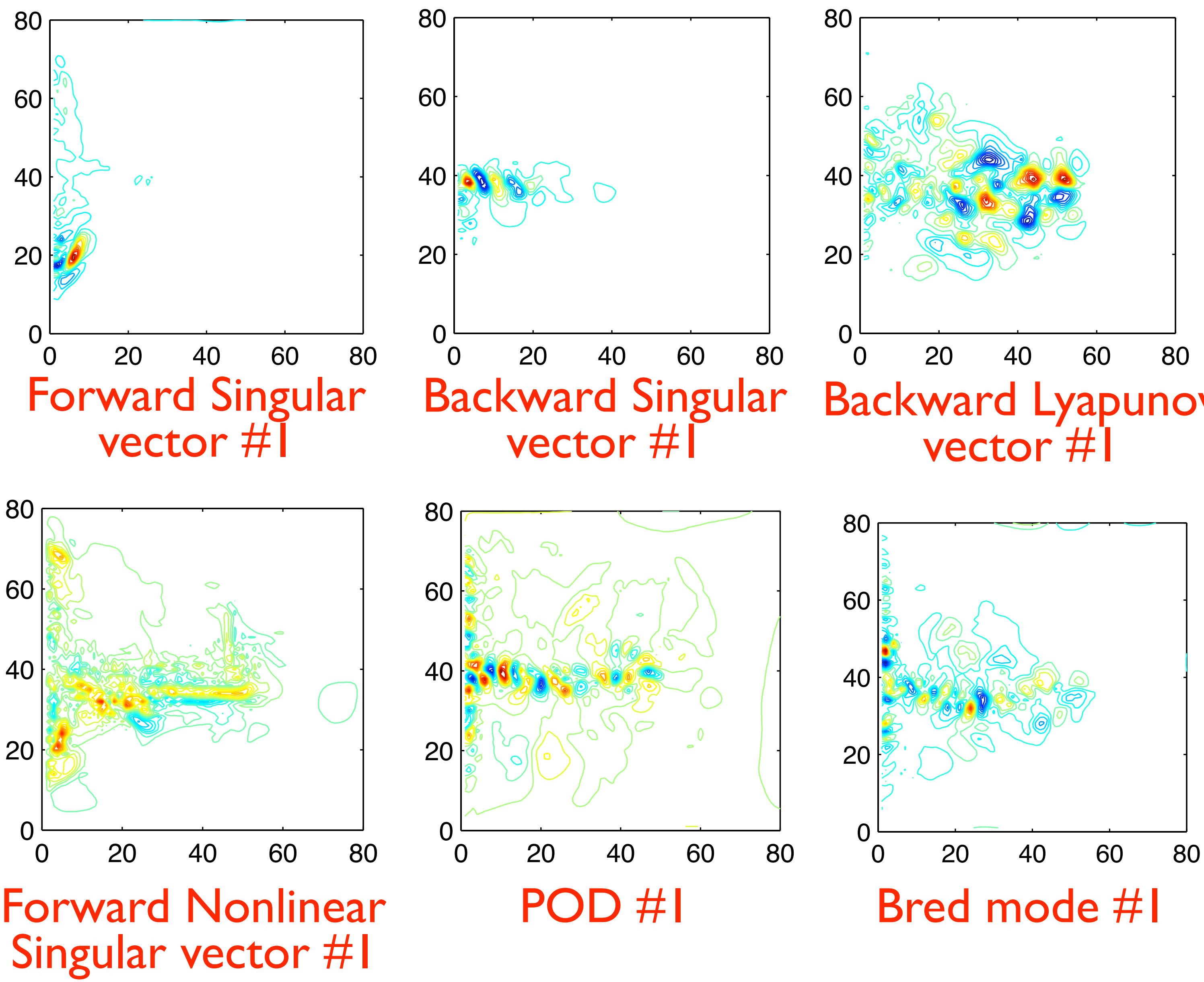
velocity norm $\frac{1}{2}(u^2 + v^2)$

energy norm $\frac{1}{2}(u^2 + v^2) + gh$

Backward singular vector #1 for different norms (*h*-component)



Forward singular vector #1 for different lengths of the time-window (*h*-component)



Forward Singular vector #1

Backward Singular vector #1

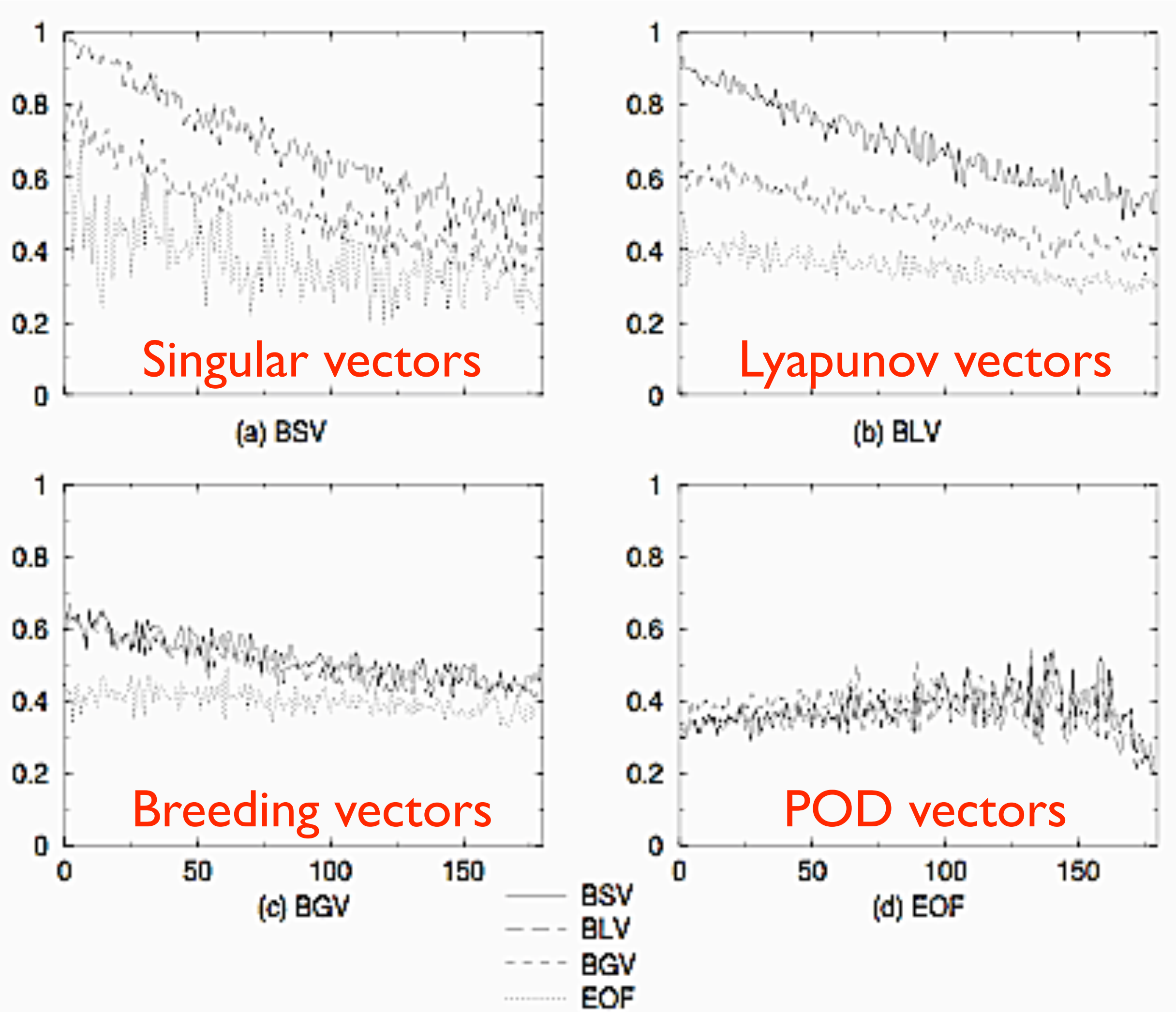
Backward Lyapunov vector #1

Forward Nonlinear Singular vector #1

POD #1

Bred mode #1

Colinearity of the different families of vectors



x-axis : # of the vector
y-axis : projection ratio on another family (180 members)

- Impact of non linearities (breeding vectors vs Lyapunov vectors)
- Information contained in the PODs is quite “different”

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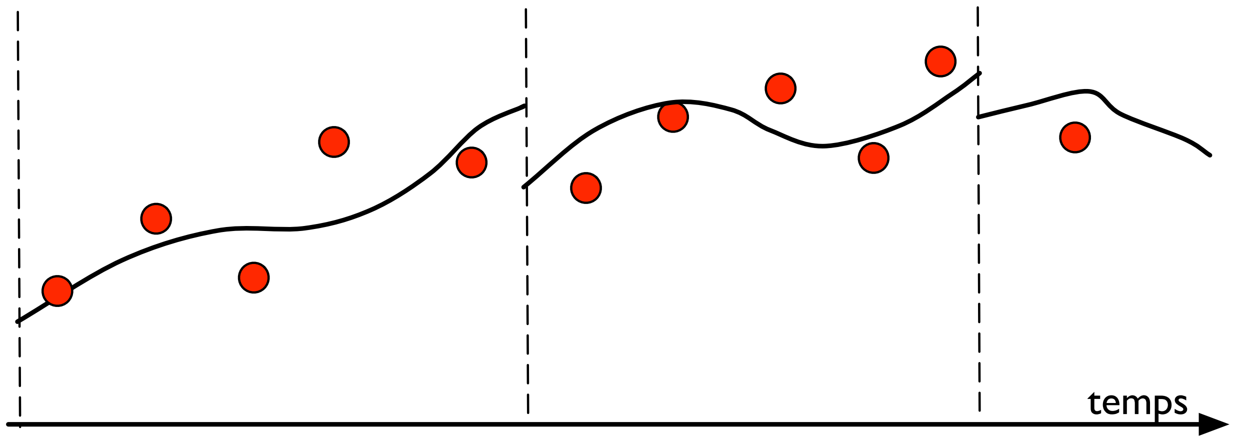
Reduced order methods in data assimilation

- Can we (significantly) reduce the cost of data assimilation in the context of ocean/atmosphere simulation without (significantly) degrading the results ?
- More generally, can the concept of "order reduction" lead to improvements in data assimilation methods ?

4D-Var data assimilation

Model $\begin{cases} \dfrac{d\mathrm{x}}{dt} = F(\mathrm{x}) & t \in [t_0, t_f] \\ \mathrm{x}(t_0) \end{cases}$

Observations $y_1, \ldots, y_N$



Incremental 4D-Var : find dx that minimizes

$$J(\delta\mathrm{x}) = \frac{1}{2} \sum_{i=1}^{N} (\mathrm{H}_i \mathrm{M}_{t_i,t_0} \delta\mathrm{x} - \mathrm{d}_i)^T \mathrm{R}_i^{-1} (\mathrm{H}_i \mathrm{M}_{t_i,t_0} \delta\mathrm{x} - \mathrm{d}_i)$$
$$+ \frac{1}{2} (\delta\mathrm{x})^T \mathrm{B}^{-1} \delta\mathrm{x}$$

where $\delta\mathrm{x} = \mathrm{x}_0 - \mathrm{x}^b$ and $\mathrm{d}_i = \mathrm{y}_i - H(x^b(t_i))$

Reduced order version

Control space Span (L_1, ..., L_r)

$$\delta \mathbf{x} = \mathbf{x}_0 - \mathbf{x}^b = \sum_{i=1}^{r} w_i \mathbf{L}_i = \mathbf{L}\mathbf{w}$$

Cost Function

$$J_b(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{B}_w^{-1} \mathbf{w}$$

with $\mathbf{B}_w = \mathbf{E}\left[(\mathbf{w} - \bar{\mathbf{w}})(\mathbf{w} - \bar{\mathbf{w}})^T\right]$

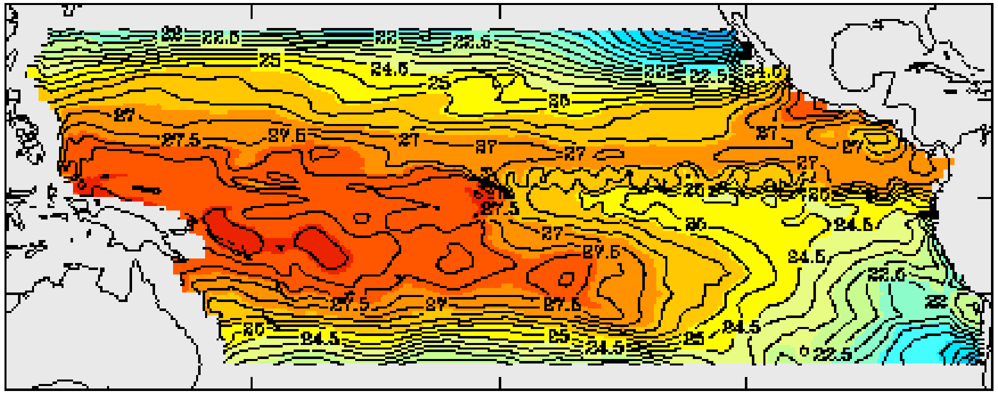
Covariance matrix in the full space

$$\begin{aligned} \mathbf{B}_r &= \mathbf{E}\left[(\delta\mathbf{x} - \delta\bar{\mathbf{x}})(\delta\mathbf{x} - \delta\bar{\mathbf{x}})^T\right] \\ &= \mathbf{L}\,\mathbf{E}\left[(\mathbf{w} - \bar{\mathbf{w}})(\mathbf{w} - \bar{\mathbf{w}})^T\right]\mathbf{L}^T \\ &= \mathbf{L}\mathbf{B}_w\mathbf{L}^T \end{aligned}$$

(singular low-rank matrix)

+ Minimization in a space of dimension *r* << [x]
+ Almost no modification to the algorithm
- Choice of (L_1, ..., L_r) and estimation of B_w

Numerical experiment: use of a POD basis for the control of the initial condition in a model of the Tropical Pacific ocean
(Durbiano, 2001; Robert et al., 2005, 2006)



OPA - TDH model
(Weaver et al.)

Primitive Equations

Momentum

$$\frac{\partial u}{\partial t} + \mathrm{U} \cdot \nabla u - \nu \Delta u - f v + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} + \mathrm{U} \cdot \nabla v - \nu \Delta v + f u + \frac{1}{\rho_0} \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\rho g$$ (hydrostatic approximation)

Conservation of mass

$$\mathrm{div}\ \mathrm{U} = 0$$ (Boussinesq approximation)

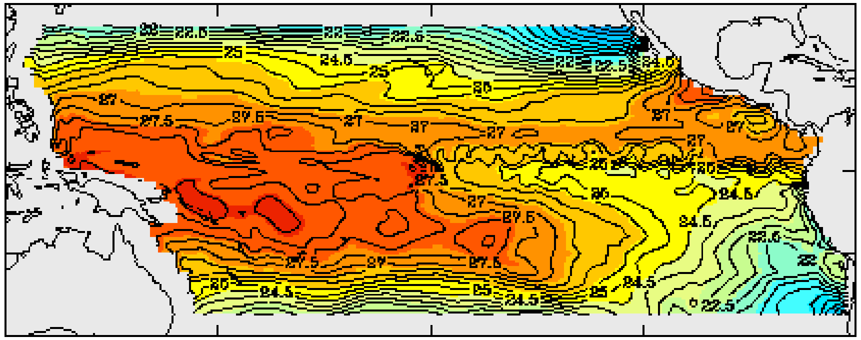
Equations for tracers

$$\frac{\partial T}{\partial t} + \mathrm{U} \cdot \nabla T = K_T\, \Delta T$$
$$\frac{\partial S}{\partial t} + \mathrm{U} \cdot \nabla S = K_S\, \Delta S$$

Equation of state

$$\rho = \rho(T, S, p)$$

+ boundary conditions



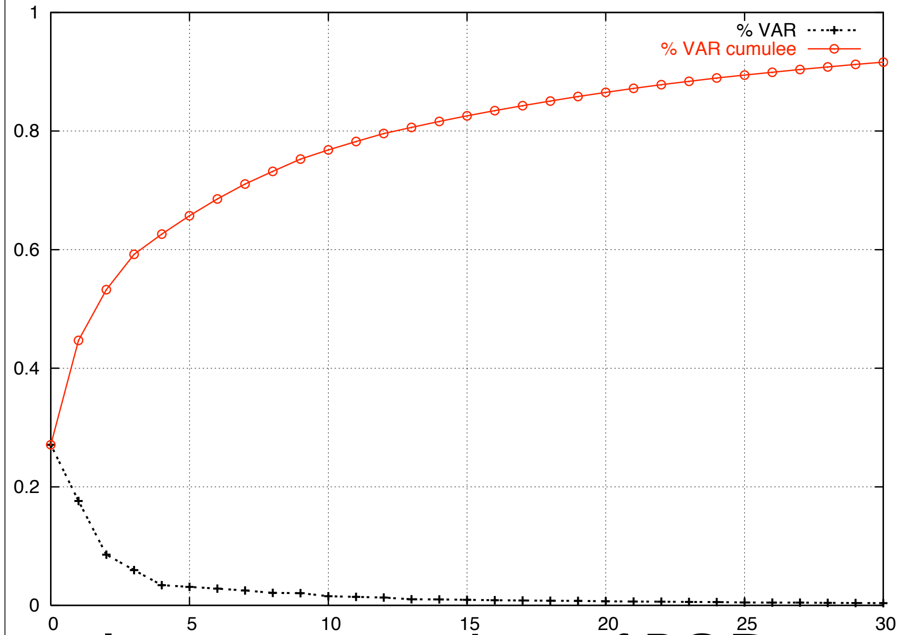
Resolution: 1°x1/2°-2°x 25 levels
State variable : [x] ~10^6
Timestep = a few minutes

Background error covariance matrix :

Full rank 4D-Var: standard “bell-shaped” spatial covariance *(Weaver et al, 2001)*

Reduced-4D-Var: due to the definition of PODs, the covariance matrix in this basis is diagonal : $\mathrm{B}_w = \mathrm{diag}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r)$

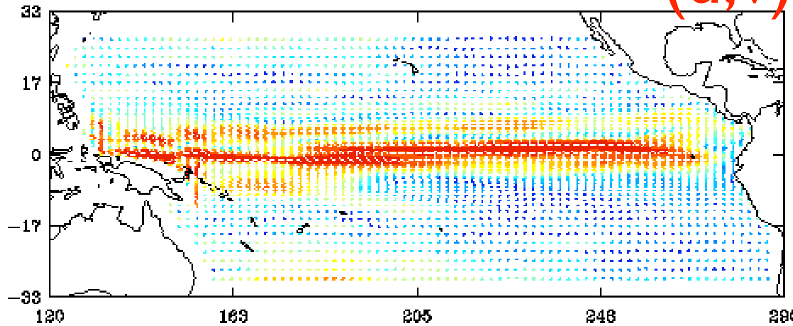
POD analysis of a one-year trajectory of the model



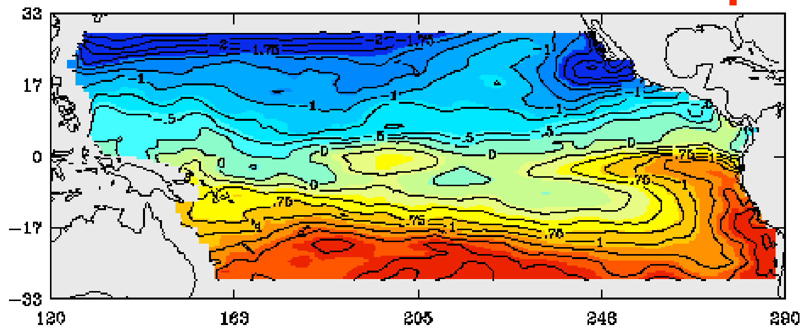
Inertia vs number of PODs

POD #1 at z=5m

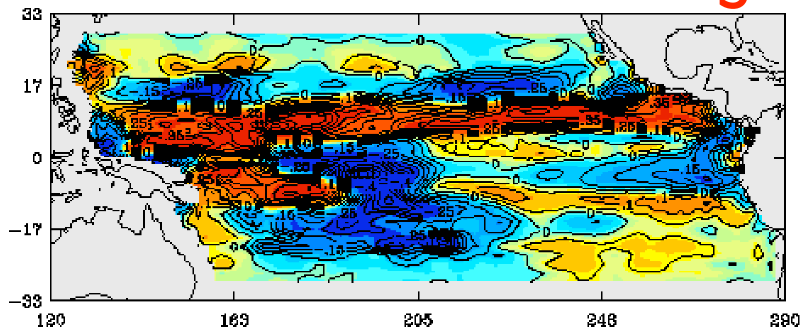
(u,v)



T



S



Structure of B : assimilation of a single observation

Innovation of 1°C, located on the equator at 160°W, in the thermocline, at the end of a one-month assimilation window

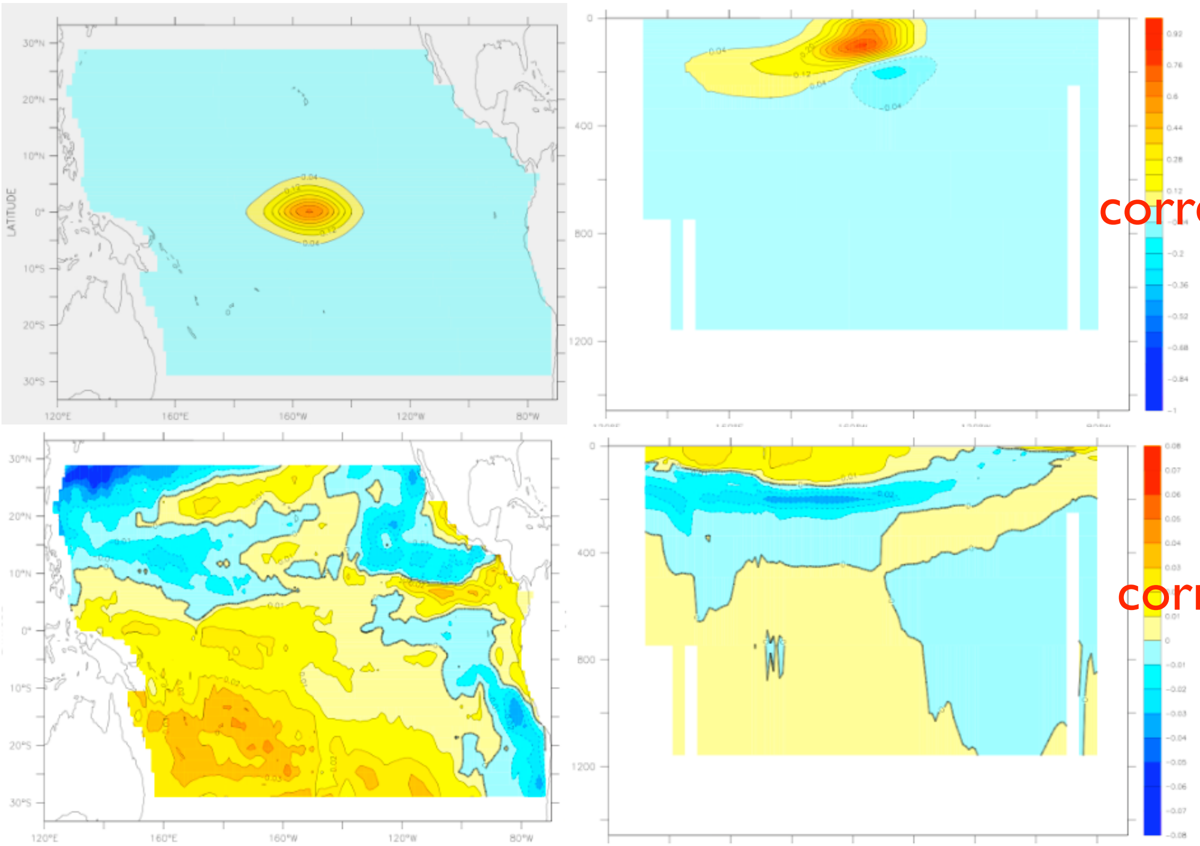
Temperature component of δx

z = 5m

vertical section

Full 4D-Var

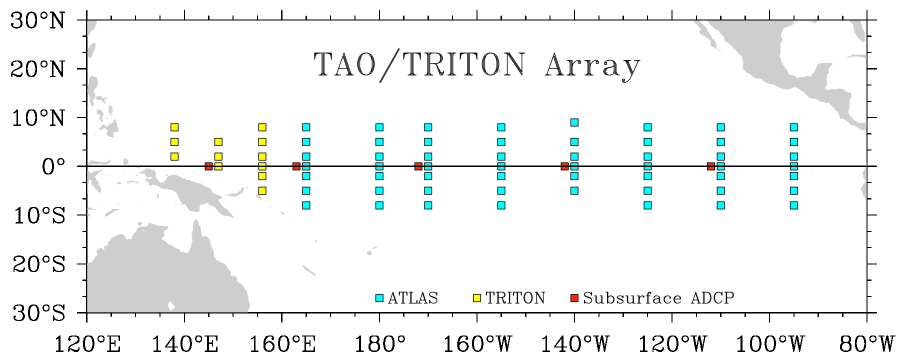
Reduced 4D-Var



maximal correction : 0,94 °C

maximal correction : 0,06 °C

Twin experiments : assimilation of simulated observations



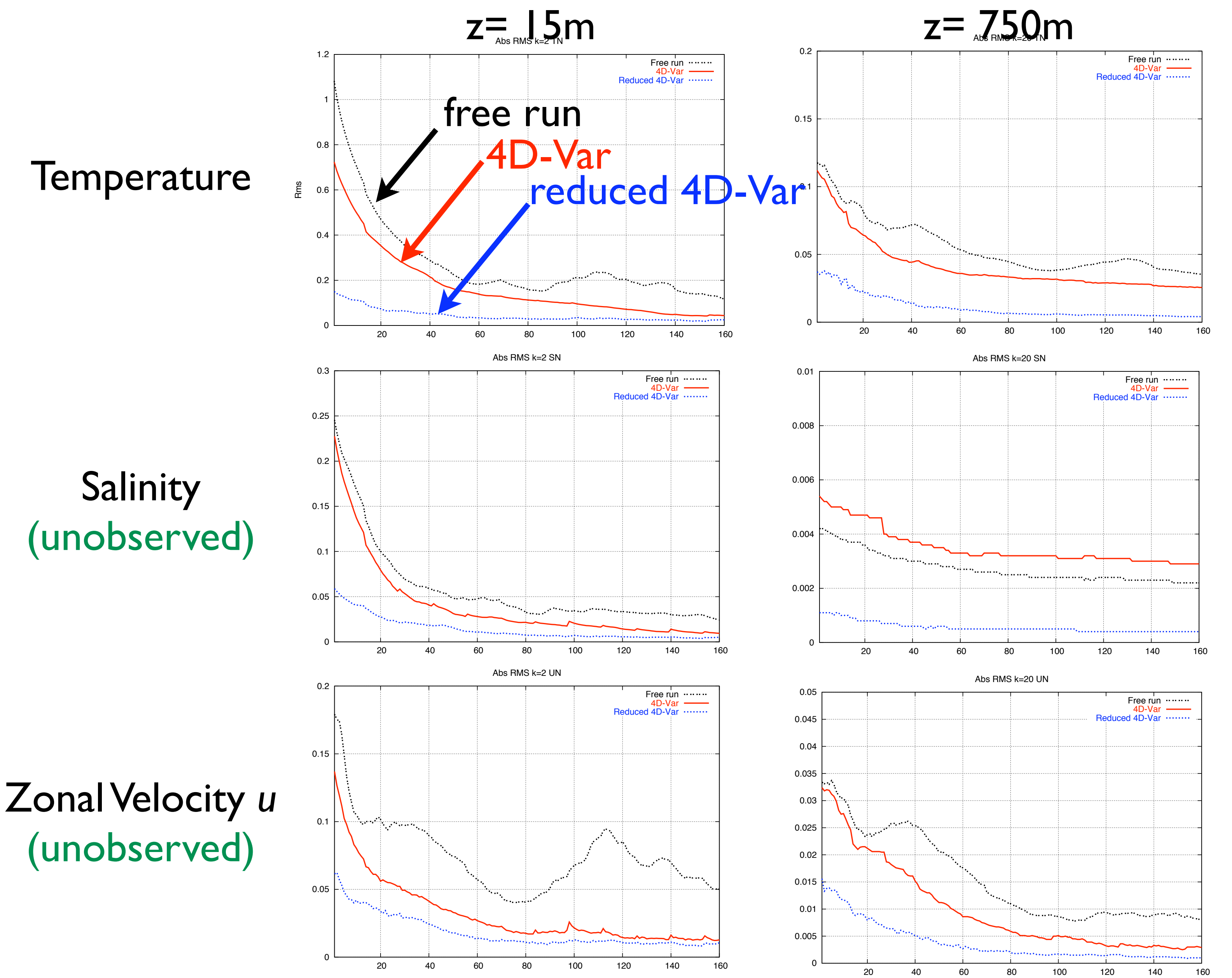
Reference simulation   one-year experiment

Simulated data   70 TAO moorings : vertical sampling of T in the 500 first meters (0,17% of [x]), every 6h + gaussian noise

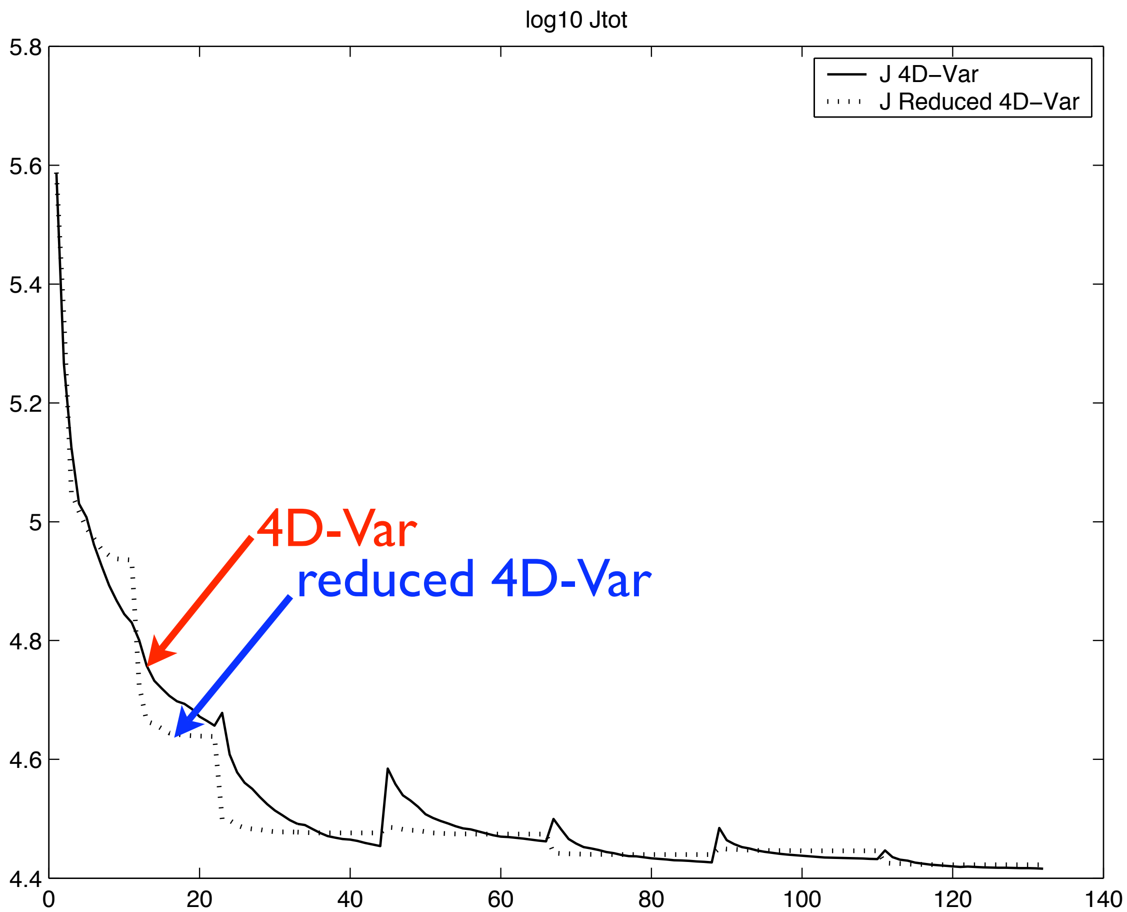
Background x^b a model state three months before

Numerical experiment   12 one-month assimilation windows

L^2 - norm of the error as a function of time



Cost function
(ln J vs iteration #)



6 one-month windows, 22 iterations each

The necessary number of iterations is divided by a factor of 4-5

Assimilation of real data : the role of model error

The model error makes unefficient the POD basis obtained by analysis of a model run.

- Compute PODs from a simulation using data assimilation
  - limited improvement

or

- Use Reduced-4D-Var as a preconditionner for full 4D-Var ("two-step 4D-Var")
  - the number of iterations is divided by a factor of 2

or

- Weak constraint optimization: explicit control of (part of) the model error.

Explicit control of the model error

$$\begin{cases} \mathrm{x}_{i+1} = M_{i\rightarrow i+1}(\mathrm{x}_i) + \mathrm{e}_{i+1} \\ \mathrm{x}_0 = \mathrm{x}^b + \delta\mathrm{x} \end{cases}$$

$$J(\delta\mathrm{x}, \mathrm{e}_1, \ldots, \mathrm{e}_N) = \frac{1}{2}\sum_{i=1}^{N}(H(\mathrm{x}_i) - \mathrm{y}_i)^T \mathrm{R}_i^{-1}(H(\mathrm{x}_i) - \mathrm{y}_i)$$

$$+\frac{1}{2}(\delta\mathrm{x})^T \mathrm{B}^{-1}\delta\mathrm{x} + \frac{1}{2}\sum_{i=1}^{N}\mathrm{e}_i^T \mathrm{Q}_i^{-1}\mathrm{e}_i$$

$$\begin{cases} \nabla_{\delta\mathrm{x}} J = -p_0 + \mathrm{B}^{-1}\delta\mathrm{x} \\ \nabla_{\mathrm{e}_i} J = -p_i + \mathrm{Q}_i^{-1}\mathrm{e}_i \end{cases}$$

Difficulties

- Dimension of the control space : N x [x]   !!
- Estimation of Q_i

- Dual approach - minimization in the observation space : *representers (Bennett 92), 4D-PSAS (Amodei 95, Courtier 97, Louvel 01, Auroux 02, 07)*

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- Reduced order modelling of e_i :
  - systematic bias *(Vidard 01, Vidard et al. 04, Griffith and Nichols 01, 06, D'Andréa and Vautard 01, Bell et al 02)* : $\mathrm{e}_i = \bar{\mathrm{e}}$

Control of the model bias

$$\begin{cases} \mathbf{x}_{i+1} = M_{i\rightarrow i+1}(\mathbf{x}_i) + \bar{\mathbf{e}} \\ \mathbf{x}_0 = \mathbf{x}^b + \delta\mathbf{x} \end{cases}$$

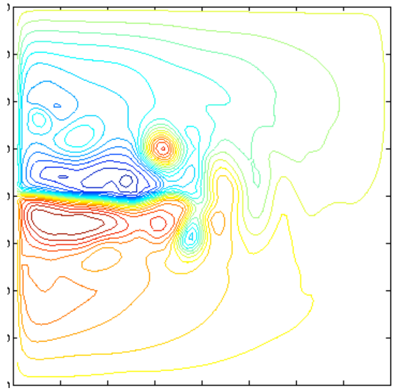
$$J(\delta\mathbf{x}, \bar{\mathbf{e}}) = \frac{1}{2}\sum_{i=1}^{N}(H(\mathbf{x}_i) - \mathbf{y}_i)^T \mathbf{R}_i^{-1}(H(\mathbf{x}_i) - \mathbf{y}_i)$$

$$+\frac{1}{2}(\delta\mathbf{x})^T \mathbf{B}^{-1}\delta\mathbf{x} + \frac{N}{2}\,\bar{\mathbf{e}}^T \mathbf{S}^{-1}\bar{\mathbf{e}}$$

$$\begin{cases} \nabla_{\delta\mathbf{x}} J = -p_0 + \mathbf{B}^{-1}\delta\mathbf{x} \\ \nabla_{\bar{\mathbf{e}}} J = -\sum_{i=1}^{N} p_i + N\,\mathbf{S}^{-1}\bar{\mathbf{e}} \end{cases}$$

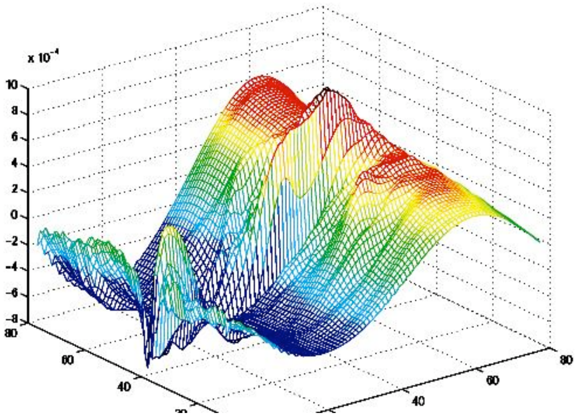
Default choice : S = B

Results with the shallow-water model (Vidard et al. 04)

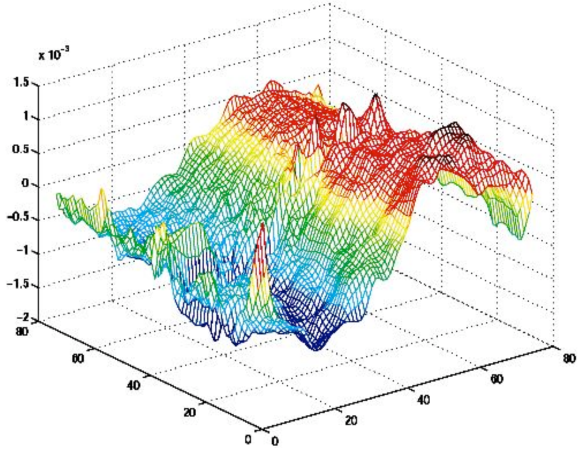


- "Cousin" experiments (a reference model and a perturbed model)
- Obs : sub-sampling of h

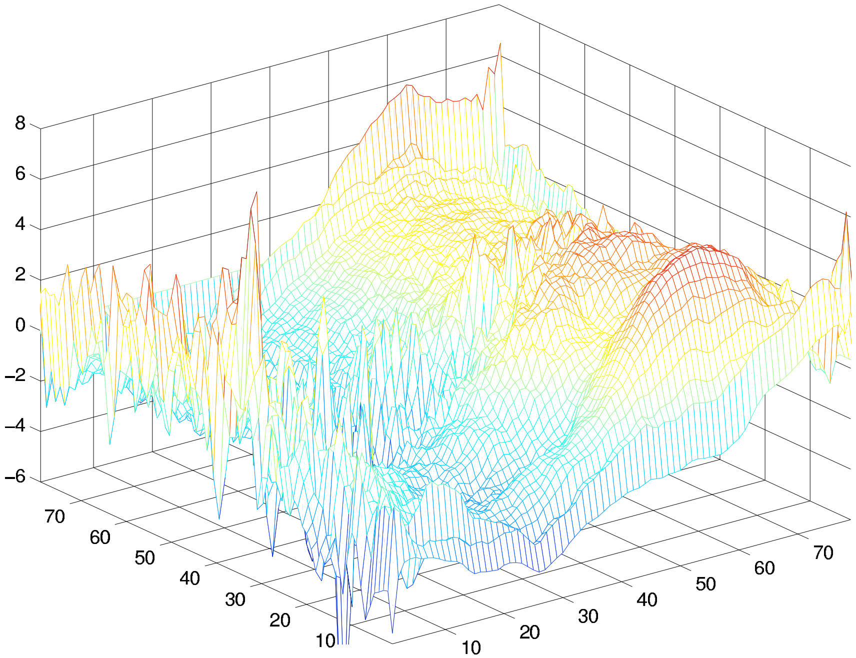
exact bias



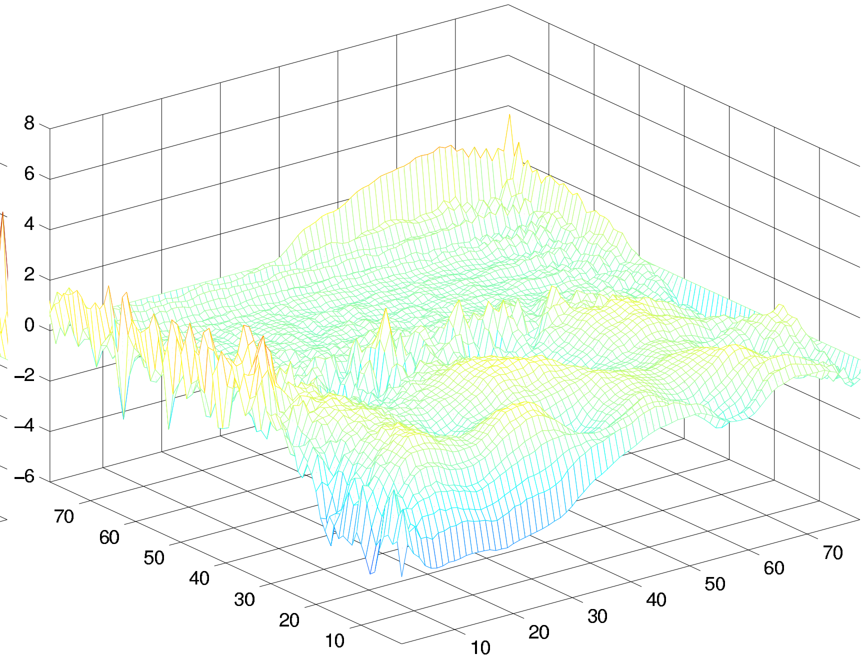
identified bias



Error on the initial correction

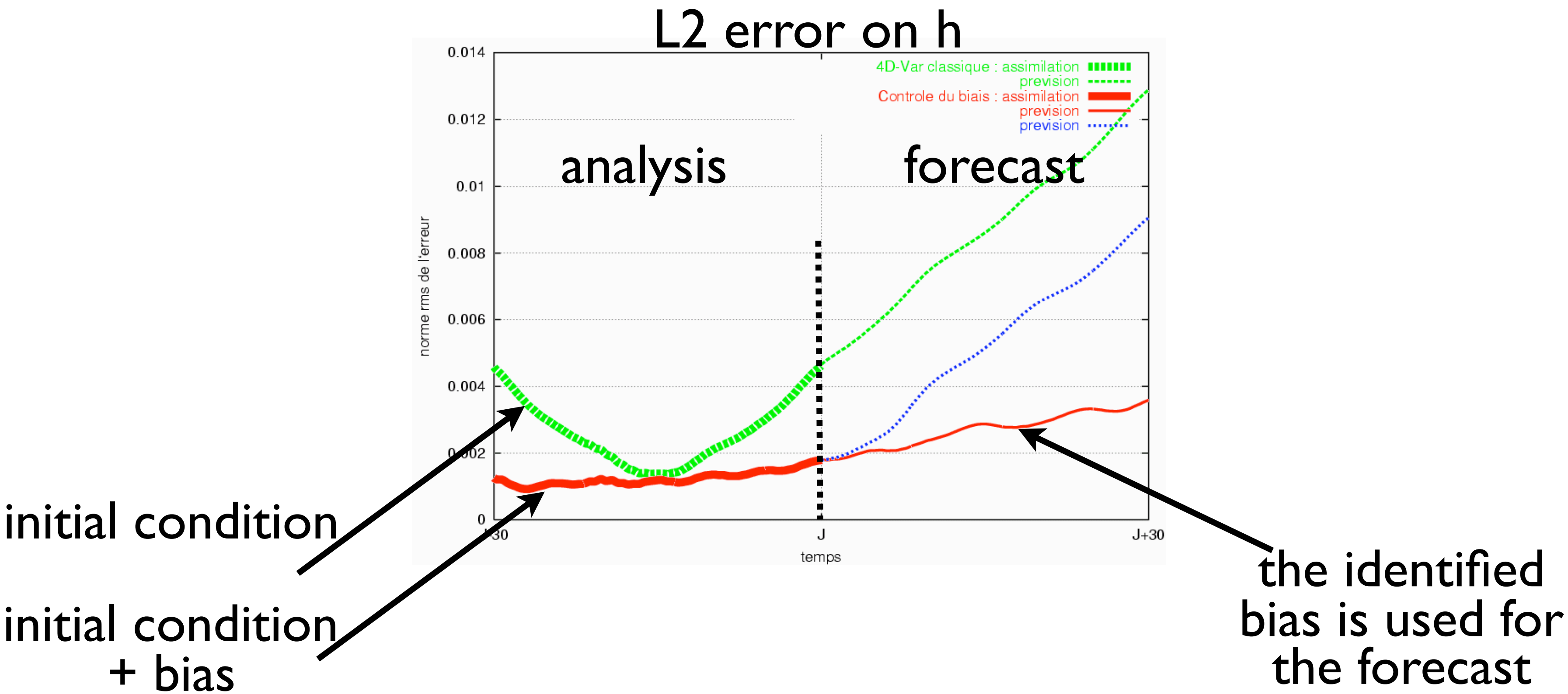


Control of the initial condition only



Control of the initial condition + bias

The use of the identified bias significantly improves the forecast.



- Dual approach - minimization in the observation space : *representers* *(Bennett 92)*, *4D-PSAS* *(Amodei 95, Courtier 97, Louvel 01, Auroux 02, 07)*
- Reduced order modelling of e_i :
  - systematic bias *(Vidard 01, Vidard et al. 04, Griffith and Nichols 01, 06, D'Andréa and Vautard 01, Bell et al 02)* : $\mathrm{e}_i = \bar{\mathrm{e}}$
  - decomposition in a low-rank basis *(Durbiano 01, Blayo et al. 04, Vidard et al. 04)* : $\mathrm{e}_i = \bar{\mathrm{e}} + \sum_{j=1}^{p} c_j^i \mathrm{L}_j$

Control of the model error in a reduced space

$$\begin{cases} \mathrm{x}_{i+1} = M_{i\rightarrow i+1}(\mathrm{x}_i) + \bar{\mathrm{e}} + \sum_{j=1}^{p} c_j^i \mathrm{L}_j \\ \mathrm{x}_0 = \mathrm{x}^b + \delta\mathrm{x} \end{cases}$$

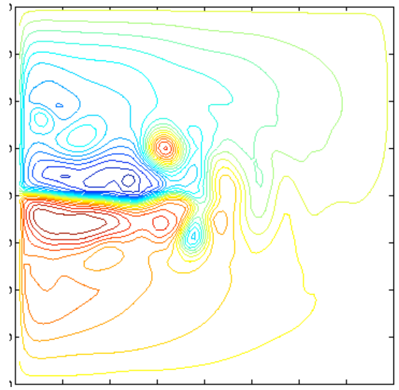
$$J(\delta\mathrm{x}, \bar{\mathrm{e}}, \mathrm{c}^1, \ldots, \mathrm{c}^N) = \frac{1}{2}\sum_{i=1}^{N}(H(\mathrm{x}_i) - \mathrm{y}_i)^T \mathrm{R}_i^{-1}(H(\mathrm{x}_i) - \mathrm{y}_i)$$

$$+\frac{1}{2}(\delta\mathrm{x})^T \mathrm{B}^{-1}\delta\mathrm{x} + \frac{N}{2}\bar{\mathrm{e}}^T \mathrm{S}^{-1}\bar{\mathrm{e}}$$

$$+\frac{1}{2}\sum_{i=1}^{N} \mathrm{c}^{iT} \mathrm{Q}_p^{-1} \mathrm{c}^i$$

$$\begin{cases} \nabla_{\delta\mathrm{x}} J = -p_0 + \mathrm{B}^{-1}\delta\mathrm{x} \\ \nabla_{\bar{\mathrm{e}}} J = -\sum_{i=1}^{N} p_i + N\,\mathrm{S}^{-1}\bar{\mathrm{e}} \\ \nabla_{\mathrm{c}_i} J = -\mathrm{L}^T p_i + \mathrm{Q}_p^{-1}\mathrm{c}_i \end{cases}$$

Numerical results with a shallow-water model



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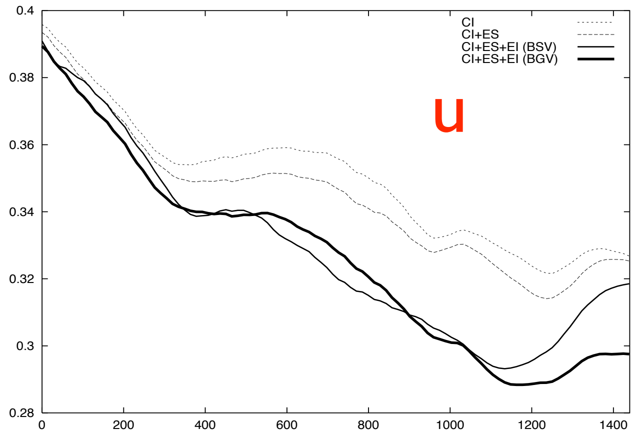
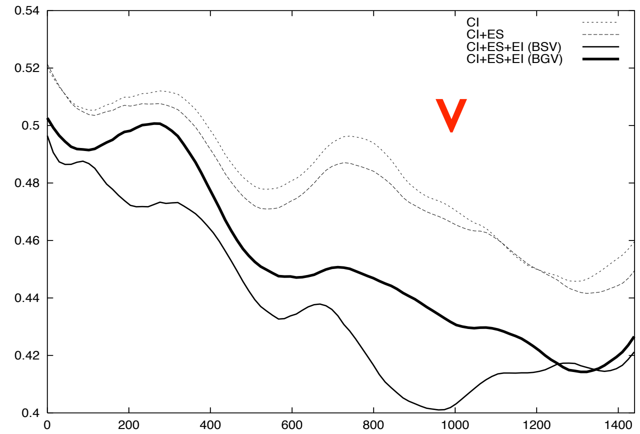
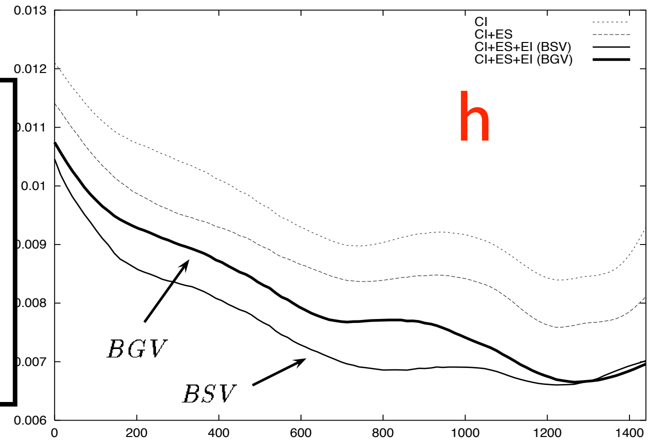
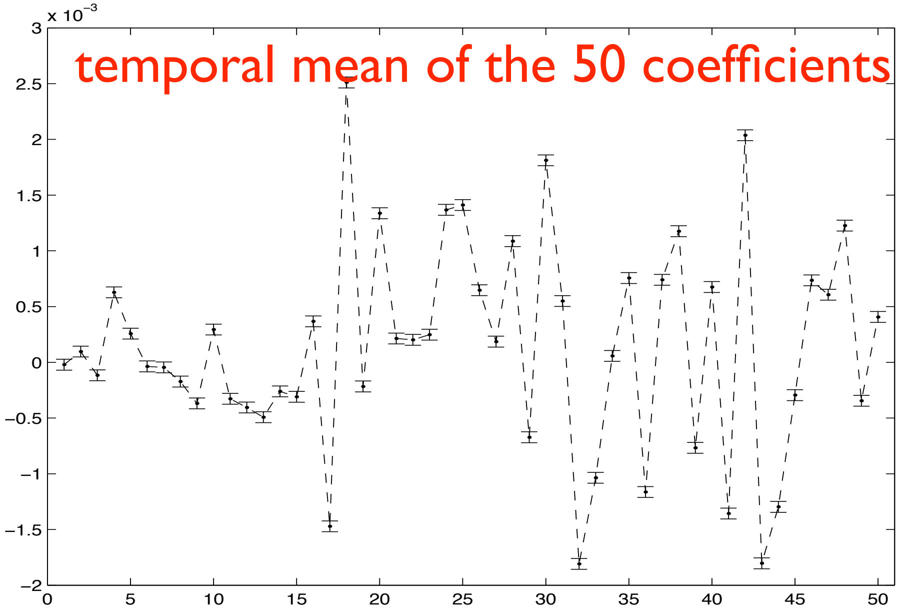
L^2 norm of the error



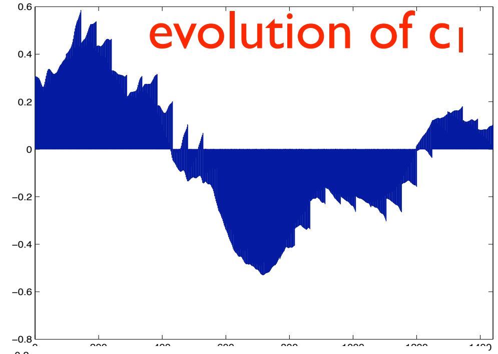


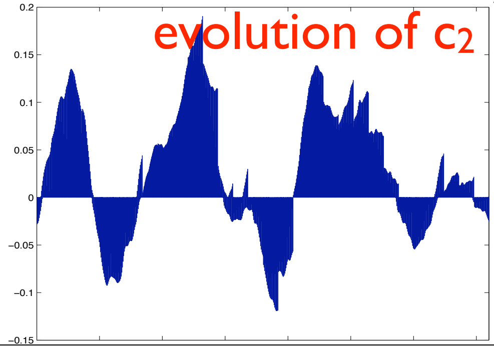


Control of :
I.C.
I.C. + bias
I.C. + bias + time-varying part



The identified part of the error is indeed unbiased.





This identification seems useless to improve the forecast.
But it can be used to improve the model itself.

Summary on reduced order approaches for variational DA

- Reduced-order methods can be implemented for variational data assimilation. An important question is therefore : which basis for which problem ?
- In our experiments, we have seen that POD vectors are relevant for the control of the initial condition, and that vectors of "maximal growth" are relevant for the control of the model error.

- There is (to my knowledge) almost no theoretical results concerning nonlinear vectors (NL Singular vectors, Bred modes - *Mu, Kalnay, Toth...*).
- Such vectors can perhaps be of interest in the context of extreme events ("most dangerous" vectors).

- A remark : reduced models are presently being developed for real time prediction. It seems clear that, at least in their present form, such models cannot predict extreme events.

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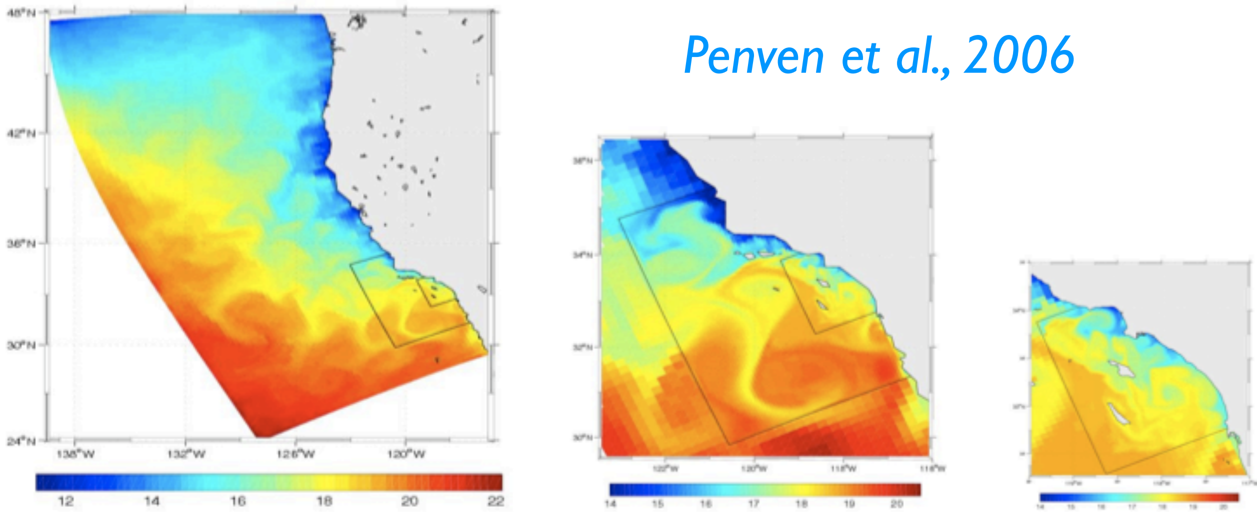
  - a taxinomy of particular vectors
  - how can they be useful for VDA ?
- On the use of zoom techniques in variational DA
  - mathematical formulation
  - an illustration

Context : nested models

Nested models give particular insight into local dynamics in regions of particular interest :

- to locally improve the numerical solution
- to improve the global solution (through some feedback)

(can be interesting in the context of extreme events)



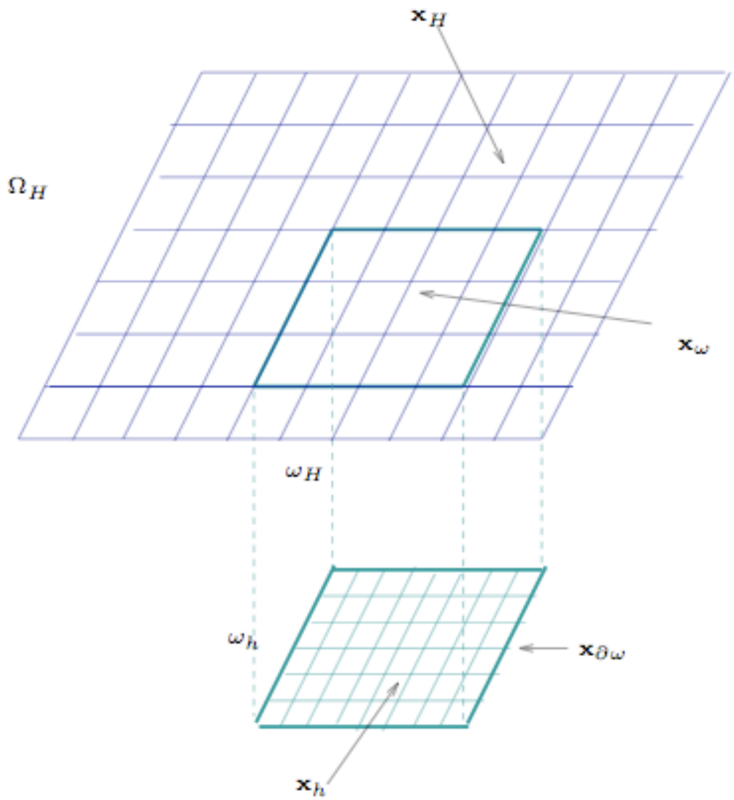
Penven et al., 2006

- How can we adapt variational data assimilation to this context ?
- Are the results improved w.r. to single-grid data assimilation with a control of boundary data ?

Outline

- On the use of reduced bases in variational DA
  - a taxinomy of particular vectors
  - how can they be useful for VDA ?
- On the use of zoom techniques in variational DA
  - mathematical formulation
  - an illustration

Mathematical formulation (Debreu et al., 2008)



Formulation

- **One-way**:

Domaine Ω

$$\begin{cases} \dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H) \\ \mathbf{x}_H(x,0) = \mathbf{x}_H^0 \end{cases}$$

Domaine ω

$$\begin{cases} \dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \\ \mathbf{x}_h(x,0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H) \end{cases}$$

- **Two-way**:

Domaine Ω

$$\begin{cases} \dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H, \mathbf{x}_\omega) \\ \mathbf{x}_H(x,0) = \mathbf{x}_H^0 \\ \mathbf{x}_\omega = G_h^H(\mathbf{x}_h) \end{cases}$$

Domaine ω

$$\begin{cases} \dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \\ \mathbf{x}_h(x,0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H) \end{cases}$$

Assimilation system

- Observations on both grids
- Control of the initial condition on both grids $\mathbf{x}^0 = \begin{bmatrix} \mathbf{x}_H^0 \\ \mathbf{x}_h^0 \end{bmatrix}$

$$\begin{aligned} J &= J^b + J^{obs} \\ J^b &= (\mathbf{x}^0 - \mathbf{x}^b)^T \mathbf{B}^{-1} (\mathbf{x}^0 - \mathbf{x}^b) \\ J^{obs} &= \int_0^T \|\mathbf{x}_H - \mathbf{y}_H^{obs}\|_\Omega^2 + \int_0^T \|\mathbf{x}_h - \mathbf{y}_h^{obs}\|_\omega^2 \end{aligned}$$

Background error covariance matrix :

It can be demonstrated that

$$\mathbf{B}_{multi}^{1/2} \approx \mathbf{S}\mathbf{S}^T = \begin{bmatrix} \mathbf{J}_{\mathring{\omega}_H} \mathbf{B}_H^{1/2} & \bar{\mathbf{J}}_{\mathring{\omega}_H} \mathbf{G}_h^H \mathbf{B}_h^{1/2} \\ \mathbf{J}_{\mathring{\omega}_h} \mathbf{I}_H^h \mathbf{K}_{\omega_H} \mathbf{B}_H^{1/2} & \bar{\mathbf{J}}_{\mathring{\omega}_h} \mathbf{B}_h^{1/2} \end{bmatrix}$$

where B_H and B_h correspond to single-grid covariance matrices.

Optimality system: no interaction

$$\Omega_H \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} \dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H) \quad \text{sur} \quad \Omega_H \times [0, T] \\ \mathbf{x}_H(x,0) = \mathbf{x}_H^0 \end{array} \right. \\ \\ \left\{ \begin{array}{l} \dfrac{\partial \mathbf{P}}{\partial t} + \left[\dfrac{\partial F_H}{\partial \mathbf{x}_H} \right]^* . \mathbf{P} = \mathbf{H}_H^* (\mathbf{H}_H \mathbf{x}_H(t) - \mathbf{y}_H(t)) \\ \mathbf{P}(T) = 0 \end{array} \right. \\ \\ \nabla_{\mathbf{x}_H^0} J^{obs} = -\mathbf{P}(0) \end{array} \right.$$

$$\omega_h \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} \dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h) \quad \text{sur} \quad \omega_h \times [0, T] \\ \mathbf{x}_h(x,0) = \mathbf{x}_h^0 \end{array} \right. \\ \\ \left\{ \begin{array}{l} \dfrac{\partial \mathbf{Q}}{\partial t} + \left[\dfrac{\partial F_h}{\partial \mathbf{x}_h} \right]^* . \mathbf{Q} = \mathbf{H}_h^* (\mathbf{H}_h \mathbf{x}_h(t) - \mathbf{y}_h(t)) \\ \mathbf{Q}(T) = 0 \end{array} \right. \\ \\ \nabla_{\mathbf{x}_h^0} J^{obs} = -\mathbf{Q}(0) \end{array} \right.$$

Optimality system: one-way interaction

$$\Omega_H \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} \dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H) \quad \text{sur} \quad \Omega_H \times [0, T] \\ \mathbf{x}_H(x, 0) = \mathbf{x}_H^0 \end{array} \right. \\ \\ \left\{ \begin{array}{l} \dfrac{\partial \mathbf{P}}{\partial t} + \left[\dfrac{\partial F_H}{\partial \mathbf{x}_H} \right]^* .\mathbf{P} + \mathbf{I}_h^H \left[\dfrac{\partial F_h}{\partial \mathbf{x}_{\partial \omega}} \right]^* .\mathbf{Q} = \mathbf{H}_H^*(\mathbf{H}_H \mathbf{x}_H(t) - \mathbf{y}_H(t)) \\ \mathbf{P}(T) = 0 \end{array} \right. \\ \\ \nabla_{\mathbf{x}_H^0} J^{obs} = -\mathbf{P}(0) \end{array} \right.$$

$$\omega_h \quad \left\{ \begin{array}{l} \left\{ \begin{array}{l} \dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h, \mathbf{x}_{\partial \omega}) \quad \text{sur} \quad \omega_h \times [0, T] \\ \mathbf{x}_h(x, 0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial \omega} = I_H^h(\mathbf{x}_H) \end{array} \right. \\ \\ \left\{ \begin{array}{l} \dfrac{\partial \mathbf{Q}}{\partial t} + \left[\dfrac{\partial F_h}{\partial \mathbf{x}_h} \right]^* .\mathbf{Q} = \mathbf{H}_h^*(\mathbf{H}_h \mathbf{x}_h(t) - \mathbf{y}_h(t)) \\ \mathbf{Q}(T) = 0 \end{array} \right. \\ \\ \nabla_{\mathbf{x}_h^0} J^{obs} = -\mathbf{Q}(0) \end{array} \right.$$

Intergrid interactions in the adjoint models are in the opposite sense than in the direct models.

Optimality system: two-way interaction

$$
\Omega_H \quad \left\{
\begin{array}{l}
\left\{
\begin{array}{l}
\dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H, \mathbf{x}_\omega) \quad \text{sur} \quad \Omega_H \times [0, T] \\
\mathbf{x}_H(x, 0) = \mathbf{x}_H^0 \\
\mathbf{x}_\omega = G_h^H(\mathbf{x}_h) \quad \text{sur} \quad \mathring{\omega}_H \times [0, T]
\end{array}
\right. \\
\\
\left\{
\begin{array}{l}
\dfrac{\partial \mathbf{P}}{\partial t} + \left[\dfrac{\partial F_H}{\partial \mathbf{x}_H}\right]^* .\mathbf{P} + \mathbf{I}_h^H \left[\dfrac{\partial F_h}{\partial \mathbf{x}_{\partial\omega}}\right]^* .\mathbf{Q} = \mathbf{H}_H^*(\mathbf{H}_H \mathbf{x}_H(t) - \mathbf{y}_H(t \\
\mathbf{P}(T) = 0
\end{array}
\right. \\
\\
\nabla_{\mathbf{x}_H^0} J^{obs} = -\mathbf{P}(0)
\end{array}
\right.
$$

$$
\omega_h \quad \left\{
\begin{array}{l}
\left\{
\begin{array}{l}
\dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \quad \text{sur} \quad \omega_h \times [0, T] \\
\mathbf{x}_h(x, 0) = \mathbf{x}_h^0 \\
\mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H)
\end{array}
\right. \\
\\
\left\{
\begin{array}{l}
\dfrac{\partial \mathbf{Q}}{\partial t} + \left[\dfrac{\partial F_h}{\partial \mathbf{x}_h}\right]^* .\mathbf{Q} + \mathbf{G}_H^h \left[\dfrac{\partial F_H}{\partial \mathbf{x}_\omega}\right]^* .\mathbf{P} = \mathbf{H}_h^*(\mathbf{H}_h \mathbf{x}_h(t) - \mathbf{y}_h(t)) \\
\mathbf{Q}(T) = 0
\end{array}
\right. \\
\\
\nabla_{\mathbf{x}_h^0} J^{obs} = -\mathbf{Q}(0)
\end{array}
\right.
$$

Eventually : addition of a new term controlling the intergrid transfers

Domaine Ω

$$\begin{cases} \dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H, \mathbf{x}_\omega) \\ \mathbf{x}_H(x,0) = \mathbf{x}_H^0 \\ \mathbf{x}_\omega = G_h^H(\mathbf{x}_h) + \epsilon_\omega \end{cases}$$

Domaine ω

$$\begin{cases} \dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \\ \mathbf{x}_h(x,0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H) + \epsilon_{\partial\omega} \end{cases}$$

$$J(\mathbf{x}^0, \epsilon) \quad = \quad J^b(\mathbf{x}^0) + J^{obs}(\mathbf{x}^0, \epsilon) + J^\epsilon(\epsilon)$$

with $J^\epsilon(\epsilon) \quad = \quad ||\breve{\mathbf{C}}\epsilon||^2$

Outline

- On the use of reduced bases in variational DA
  - a taxinomy of particular vectors
  - how can they be useful for VDA ?
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  - mathematical formulation
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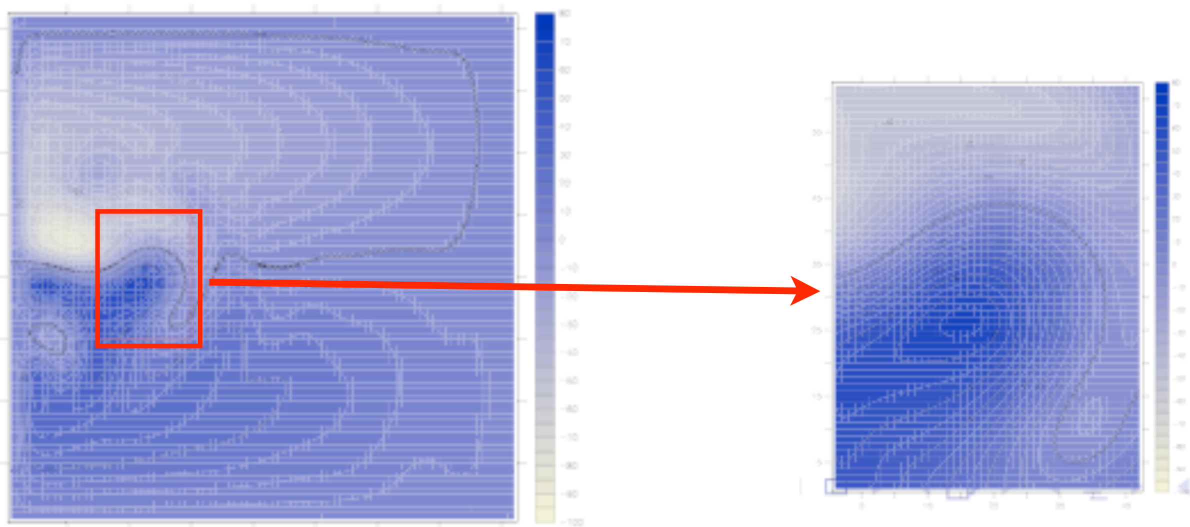
Numerical experiments (Simon 07, Simon et al. 08)

Shallow water model



dx=dy=25 km

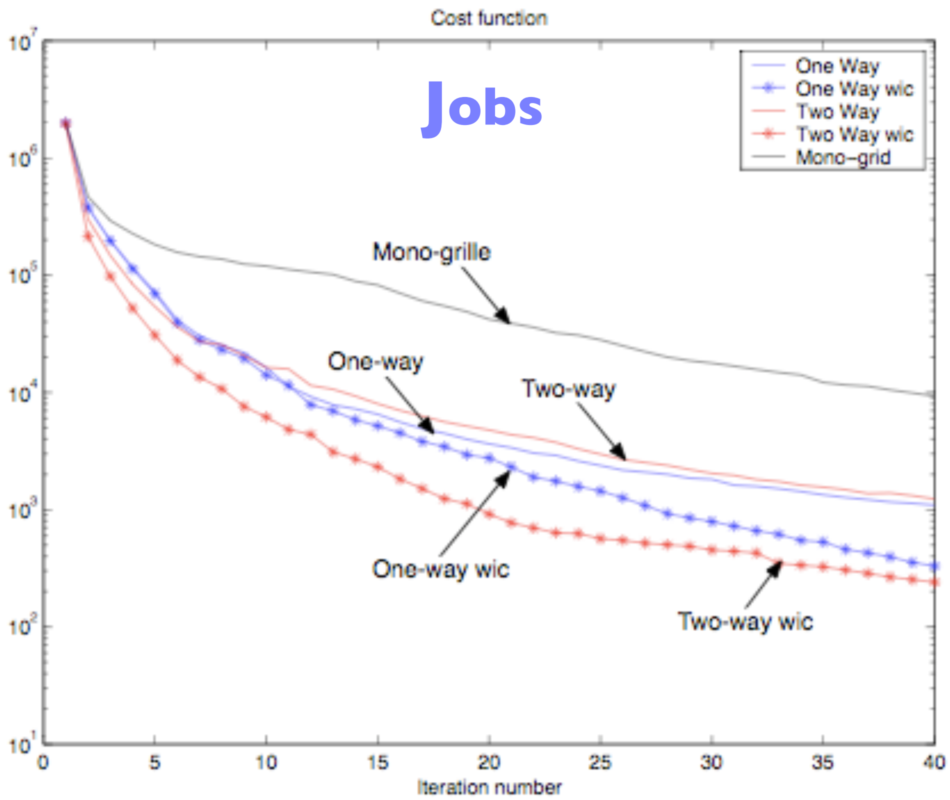
dx=dy=8.33 km

Twin experiments (true state = simulation with uniformly high resolution everywhere)

Observations : sampling of h on the fine grid only

Numerical experiments (2)

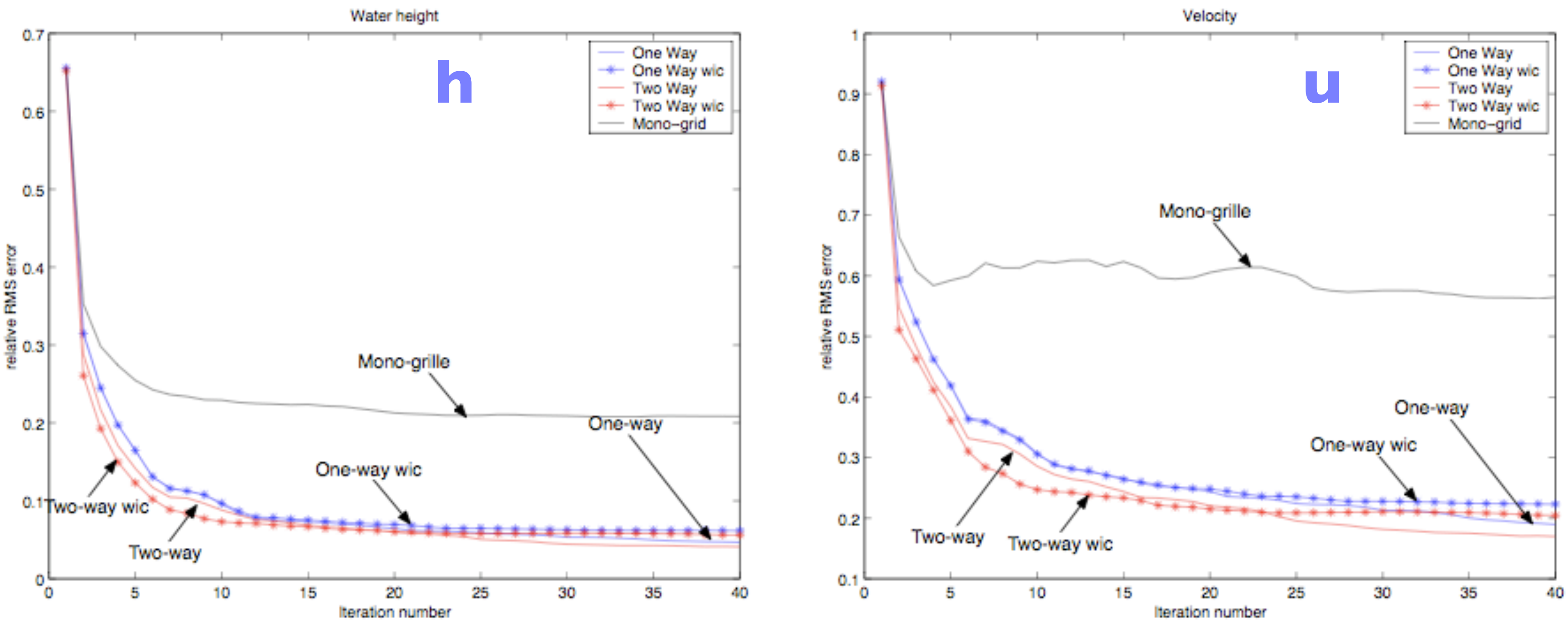
Cost function



- Better decrease of Jobs with the two-grid algorithms
- Additional control of the intergrid errors improves the decrease

Numerical experiments (3)

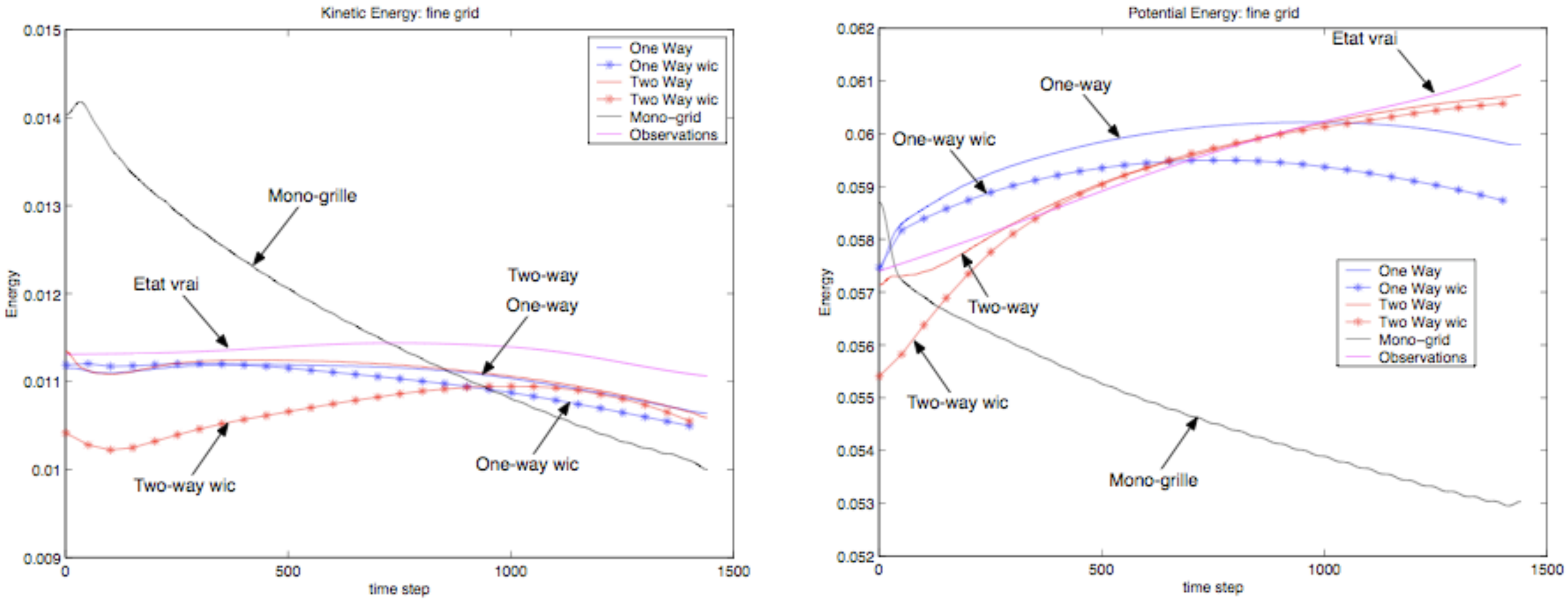
RMS error on the fine grid



- Two-grid solutions are clearly better than the single grid solution
- Two-way interaction leads to (slightly) better results than one-way interaction

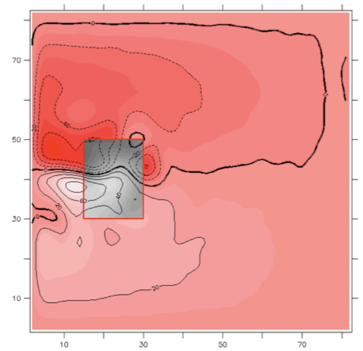
Numerical experiments (4)

Kinetic and potential energy on the fine grid

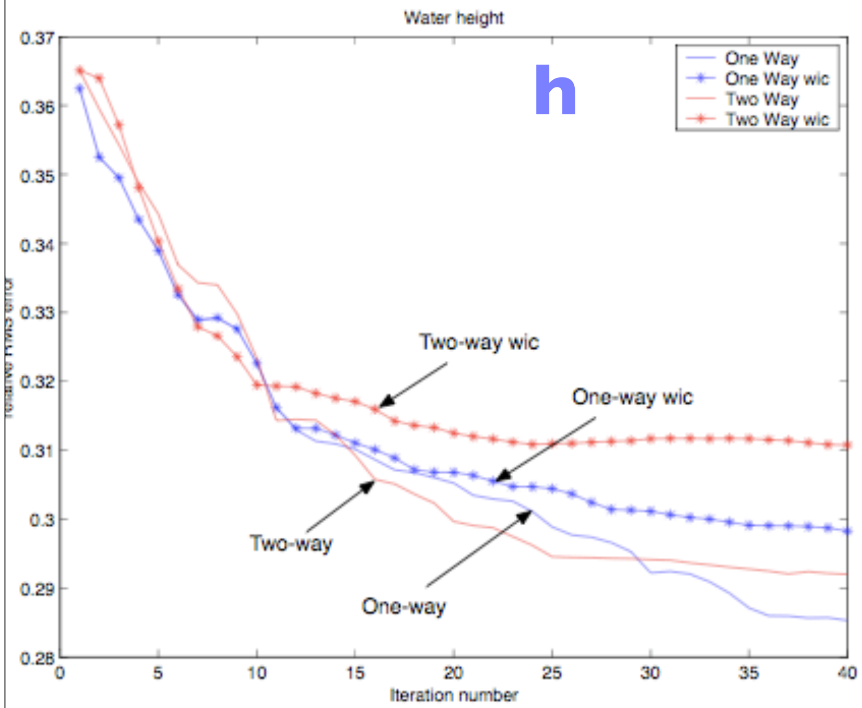


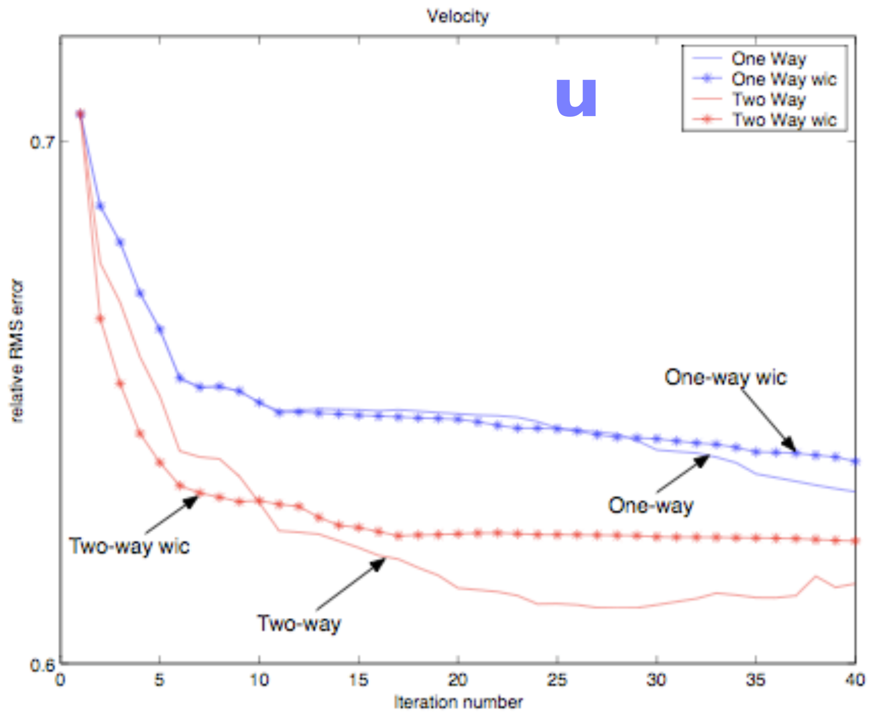
- Bad physical behaviour of the single grid optimal solution
- Energies of the two-grid solutions are close to the “truth”

Numerical experiments (5)



RMS error on the outer coarse grid





- small improvement of the coarse solution outside from the refined domain

Summary on variational DA for nested systems

Nested systems allow some focus of the simulation on regions of particular interest.

- The formulation of variational data assimilation in such systems has been derived.
- First numerical experiments indicate that such an approach leads to improved results with regard to data assimilation in a local fine resolution model with control of boundary data

- + several other technical aspects on VDA and multigrid methods (see Simon, 2007)

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