Multiplication-addition theorems for self-conjugate partitions

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Summary

Ferrers diagram and hooks of partitions

$\lambda \in \mathcal{P}(n)$: finite nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

7	6	4	1
5	4	2	
4	3	1	
2	1		

(a) $(4, 3, 3, 2) \in \mathcal{P}$

7	5	4	1
5	3	2	
4	2	1	
1		$\mathcal{H}_3$	

(b) $(4, 3, 3, 1) \in \mathcal{SC}$

$+$	$-$	$+$	$-$
$-$	$+$	$-$	
$+$	$-$	$+$	
$-$			

(c) BG-rank $= -1$

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$-$			

(c) BG-rank $= -1$

- $\mathcal{H}(\lambda) := \{\text{hook-length}\}$
- for $t \in \mathbb{N}^*$, $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$
- BG-rank of Berkovich-Garvan (2008): sum of signs

Formal power series and partitions

- Generating series of partitions:

$$\sum_{n\in\mathbb{N}} p(n)q^n = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} = \frac{1}{(q;q)_\infty}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

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- Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z^2}{h^2}\right) = (q;q)_\infty^{z^2-1}.$$

Littlewood decomposition: an example for $t = 3$

$$\begin{array}{|c|c|c|c|}\hline 7 & 6 & 4 & 1 \\ \hline 5 & 4 & 2 \\ \cline{1-3} 4 & 3 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2}\end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|c|}\hline 7 & 4 & 3 & 1 \\ \hline 5 & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad \rightarrow \quad \begin{array}{|c|c|}\hline 7 & 2 \\ \hline 4 & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$\lambda = (4, 3, 3, 2)$ $\qquad$ $\omega_3(\lambda) = (2, 2, 1, 1)$

$$\lambda \mapsto \left(\omega_3, \left(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}\right)\right) \in \mathcal{P}_{(3)} \times \mathcal{P}^3.$$

Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega_t \in \mathcal{A} \mid \mathcal{H}_t(\omega_t) = \emptyset\}$

1. partitions: $\lambda \in \mathcal{P} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$

$$\mathcal{H}_t(\lambda) = t \bigcup_{i=0}^{t-1} \mathcal{H}(\nu^{(i)}),$$

$$|\lambda| = |\omega_t| + t \sum_{i=0}^{t-1} |\nu^{(i)}|$$

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2. Self-conjugate partitions:
   (a) for t even: $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{t/2}$
   (b) for t odd: $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC}$

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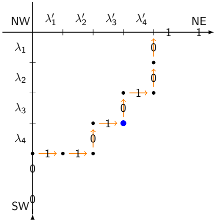
Cho–Huh–Sohn (2019) $\lambda \in \mathcal{SC}^{(\mathrm{BG})} \mapsto \kappa \in \mathcal{P}$ bijection such that $|\lambda| = 4|\kappa| + \mathrm{BG}(\lambda)(2\,\mathrm{BG}(\lambda) - 1)$

An example $\lambda = (4, 4, 3, 2) \in \mathcal{SC}$ and $t = 4$



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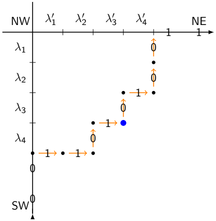

$$s(\lambda) = (c_i)_{i \in \mathbb{Z}}$$

$$= \underbrace{\cdots 00001101}_{\text{number of “1”}'s} \;\Big|\; \underbrace{01001111 \cdots}_{\text{number of “0”}'s}$$

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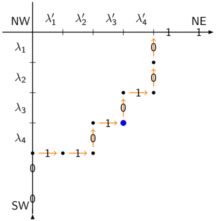

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$$\begin{aligned} s_0 &= \cdots 001|011\cdots \\ s_1 &= \cdots 001|111\cdots \\ s_2 &= \cdots 000|011\cdots \\ s_3 &= \cdots 001|011\cdots \end{aligned} \quad \xrightarrow[10\mapsto 01]{} \quad \begin{aligned} s'_0 &= \cdots 000|111\cdots \\ s'_1 &= \cdots 001|111\cdots \\ s'_2 &= \cdots 000|011\cdots \\ s'_3 &= \cdots 000|111\cdots \end{aligned}$$

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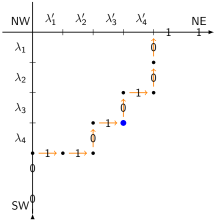

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$$\to s(\omega_4) = \cdots 00000100|11011111\cdots \to \omega_4 = (3,1,1) \in \mathcal{SC}_{(4)}$$

$$\left(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \nu^{(3)}\right) = ((1), \emptyset, \emptyset, (1))$$

Multiplication-addition theorem for partitions

Theorem [Han–Ji (2009)]

Set $t \in \mathbb{N}^*$ and let ρ_1, ρ_2 be two functions defined over $\mathbb{N}$

$$f_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)$$

$$g_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th) \sum_{h \in \mathcal{H}(\lambda)} \rho_2(th)$$

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Then

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h)$$

$$= t \left(f_t(xq^t)\right)^{t-1} g_t(xq^t) \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}$$

Multiplication-addition theorem for $\mathcal{SC}$ and t even

Theorem [W. (2021)]

Set $t \in 2\mathbb{N}^*$ and let ρ_1, ρ_2 be two functions defined over $\mathbb{N}$

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1(th)^2$$

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Then

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h)$$
$$= t \left(f_t(x^2 q^{2t}) \right)^{t/2-1} g_t(x^2 q^{2t}) \left(q^{2t}; q^{2t} \right)_\infty^{t/2}$$
$$\times \left(-bq; q^4 \right)_\infty \left(-q^3/b; q^4 \right)_\infty$$

Sketch of proof

First we compute

$$\sum_{\substack{\lambda\in\mathcal{SC}\\ core_t(\lambda)=\omega}} q^{|\lambda|}x^{|\mathcal{H}_t(\lambda)|}b^{\mathrm{BG}(\lambda)}\prod_{h\in\mathcal{H}_t(\lambda)}\rho_1(h)\sum_{h\in\mathcal{H}_t(\lambda)}\rho_2(h)$$

Sketch of proof

$$b^{\mathrm{BG}(\omega)} q^{|\omega|} \sum_{\substack{\lambda \in \mathcal{SC} \\ core_t(\lambda)=\omega}} q^{|\lambda|-|\omega|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h)$$

- $\mathrm{BG}(\lambda) = \mathrm{BG}(\omega_t)$
- Littlewood decomposition to $\lambda \rightarrow$ separates t-core from t-quotient

Sketch of proof

$$b^{\mathrm{BG}(\omega)} q^{|\omega|} \sum_{\underline{\nu} \in \mathcal{P}^t} q^{t|\underline{\nu}|} x^{|\underline{\nu}|} \prod_{h \in \mathcal{H}(\underline{\nu})} \rho_1(th) \sum_{h \in \mathcal{H}(\underline{\nu})} \rho_2(th)$$

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- Regroup components of the t-quotient with its conjugate

Sketch of proof

$$2b^{\mathrm{BG}(\omega)}q^{|\omega|}\left(\sum_{\nu\in\mathcal{P}} q^{2t|\nu|}x^{2|\nu|}\prod_{h\in\mathcal{H}(\nu)}\rho_1^2(th)\right)^{t/2-1}$$
$$\times\sum_{j=0}^{t/2-1}\left(\sum_{\nu^{(i)}\in\mathcal{P}} q^{2t|\nu^{(i)}|}x^{2|\nu^{(i)}|}\prod_{h\in\mathcal{H}(\nu^{(i)})}\rho_1^2(th)\sum_{h\in\mathcal{H}(\nu^{(i)})}\rho_2(th)\right)$$

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- Littlewood decomposition to $\lambda\rightarrow$ separates t-core from t-quotient
- Regroup components of the t-quotient with its conjugate
- Compute the sum depending on ω_t with Cho–Huh–Sohn (2019)

Applications for t even

1. $\rho_1(h) = \rho_2(h) = 1$: trivariate generating function of $\mathcal{SC}$

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} = \frac{\phi(q,b,t)}{(x^2 q^{2t}; x^2 q^{2t})_\infty^{t/2}}$$

where $\phi(q,b,t) := (q^{2t}; q^{2t})_\infty^{t/2} (-bq; q^4)_\infty (-q^3/b; q^4)_\infty$

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2. $\rho_1(h) = 1/\sqrt{h}$ and $\rho_2(h) = 1$: hook-length formula

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{\sqrt{h}}$$
$$= \phi(q,b,t) \exp\left(\frac{x^2q^{2t}}{2} + \frac{x^4q^{4t}}{4t}\right)$$

A modular Nekrasov–Okounkov formula for t even

- $\rho_1(h) = \sqrt{1 - z/h^2}$ and $\rho_2(h) = 1$: modular Nekrasov–Okounkov

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \sqrt{1 - \frac{z}{h^2}}$$
$$= \phi(q, b, t) \left(x^2 q^{2t}; x^2 q^{2t}\right)^{(z/t - t)/2}_\infty$$

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- By asymptotic for z and identification of coefficients:

$$\sum_{\substack{\lambda \in \mathcal{SC}, \lambda \vdash 2tn + j(2j-1) \\ |\mathcal{H}_t(\lambda)| = 2n \\ \mathrm{BG}(\lambda) = j}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{1}{n! 2^n t^n}$$

A modular Stanley–Panova formula for t even

$\rho_1(h) = 1/h$ and $\rho_2(h) = h^{2k}$: modular Stanley–Panova

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} h^{2k} = \phi(q,b,t)$$

$$\times t^{2k+1} \exp\left(\frac{x^2 q^{2t}}{2t}\right) \sum_{i=0}^{k} T(k+1, i+1) C(i) \left(\frac{x^2 q^{2t}}{t^2}\right)^{k+1}$$

where $T(k,i)$ is a central factorial number:

$$T(k,0) = T(0,i) = 0, \quad T(1,1) = 1,$$
$$T(k,i) = i^2 T(k-1,i) + T(k-1,i-1) \quad \text{for} \quad (k,i) \neq (1,1)$$

and

$$C(i) := \frac{1}{2(i+1)^2} \binom{2i}{i} \binom{2i+2}{i+1}$$

Signs coming from algebra

- Littlewood (1940), King (1989), Pétréolle (2016): $\varepsilon_u = \varepsilon_{(i,j)} = \mathrm{sign}(i-j)$ and $\delta_\lambda = (-1)^d$

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$+$			

$d = 3$

- Nekrasov–Okounkov formula for $\mathcal{SC}$ (Pétréolle, 2016):

$$\sum_{\lambda \in \mathcal{SC}} \delta_\lambda q^{|\lambda|} \prod_{\substack{u\in\lambda \\ h_u \in \mathcal{H}(\lambda)}} \left(1 - \frac{2z}{h_u \varepsilon_u}\right) = \left(\frac{(q^2;q^2)_\infty^{z+1}}{(q;q)_\infty}\right)^{2z-1}.$$

A signed multiplication theorem for $\mathcal{SC}$ and t even

Theorem [W. (2021)]

Set $t \in 2\mathbb{N}^*$ and let $\tilde{\rho}_1$ be a function defined over $\mathbb{Z} \times \{-1, 1\}$

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \tilde{\rho}_1(th, 1)\tilde{\rho}_1(th, -1),$$

Then

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_u, \varepsilon_u)$$

$$= \left(q^{2t}; q^{2t}\right)_\infty^{t/2} \left(-bq; q^4\right)_\infty \left(-q^3/b; q^4\right)_\infty \left(f_t(x^2 q^{2t})\right)^{t/2}$$

Nekrasov–Okounkov analogues

$\tilde{\rho}_1(a,\varepsilon) = 1 - z/(a\varepsilon)$: modular Nekrasov–Okounkov

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \left(1 - \frac{z}{h_u \varepsilon_u}\right)$$

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Extraction of coefficients:

$$\sum_{\substack{\lambda \in \mathcal{SC}, \lambda \vdash 2nt + j(2j-1) \\ \mathrm{BG}(\lambda) = j}} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} \frac{h^2}{2} = \frac{t + 3n - 3}{2^n t^{n-1} (n-1)!}$$

The t odd case

- Littlewood decomposition for t odd:
  $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC}.$

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Theorem [W. (2021)]

Set $t \in 2\mathbb{N}+1$ and let $\tilde{\rho}_1$ be a function defined on $\mathbb{Z} \times \{-1, 1\}$ Then

$$\sum_{\lambda \in \mathrm{BG}_t} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_u, \varepsilon_u)$$

$$= \frac{(q^{2t}; q^{2t})_\infty^{(t-1)/2} (-q; q^2)_\infty}{(-q^t; q^{2t})_\infty} \left(f_t(x^2 q^{2t})\right)^{(t-1)/2}$$

Some Applications

A bivariate generating function (case $x = 1$ Bessenrodt (1991)):

$$\sum_{\lambda \in \mathrm{BG}_t} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} = \frac{(q^{2t}; q^{2t})_\infty^{(t-1)/2} (-q; q^2)_\infty}{(x^2 q^{2t}; x^2 q^{2t})_\infty^{(t-1)/2} (-q^t; q^{2t})_\infty}$$

Nekrasov–Okounkov analogue:

$$\sum_{\lambda \in \mathrm{BG}_t} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \left(1 - \frac{z}{h_u \varepsilon_u}\right)$$

$$= \frac{(q^{2t}; q^{2t})_\infty^{(t-1)/2} (-q; q^2)_\infty}{(-q^t; q^{2t})_\infty} \left(x^2 q^{2t}; x^2 q^{2t}\right)_\infty^{(t-1)(z^2/t^2-1)/2}$$