ON THE ALGEBRAICITY OF THUE-MORSE CONTINUED FRACTIONS AND STIELTJES CONTINUED FRACTIONS

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Abstract. We put forward several general conjectures concerning the algebraicity or transcendence of continued fractions and Stieltjes continued fractions defined by the Thue-Morse and period-doubling sequences in characteristic 2. We present our Guess'n'Prove method, in which we exploit the structure of automata, for proving some of our conjectures in special cases.

1. Introduction

1.1. **Background.** We are interested in the continued fractions and Stieltjes continued fractions defined by automatic sequences in finite characteristic, and more precisely their algebraicity or transcendence. We give here the background and motivation for studying such problems. The definitions of related notions will given in subsection 1.2.

The link of automaticity and algebraicity goes back to the well-known Theorem of Christol, Kamae, Mendès France and Rauzy [8] which states that a formal power series in $\mathbb{F}_q[[x]]$ is algebraic over $\mathbb{F}_q(x)$ if and only if the sequence of its coefficients is q-automatic. The situation is completely different for real numbers. In 2007, Adamczewski and Bugeaud [1] proved that for an integer $b \geq 2$, if the b-ary expansion of an irrational real number u form an automatic sequence, then u must be transcendental. In 2013, Bugeaud [7] proved that the continued fraction expansion of an algebraic real number of degree at least 3 is not automatic.

As with real numbers, a formal Laurent series can also be represented by a continued fraction whose partial quotients are polynomials. Unlike for real numbers, the continued fraction expansion of an algebraic Laurent series of degree at least 3 may or may not have automatic partial quotients [4, 5, 14, 2, 15, 11, 12, 13]; see also the introduction of [10].

We could also ask the converse question: what can we say about the algebraicity of a continued fraction whose partial quotients form an automatic sequence? To our knowledge, little has been done in this direction. The authors [10] proved that the Stieltjes continued fractions defined by the Thue-Morse sequence and the period-doubling sequence in $\mathbb{Z}[[x]]$ are congruent, modulo 4, to algebraic series in $\mathbb{Z}[[x]]$. In 2020, Wu [16] obtained similar results concerning the Stieltjes continued fractions defined by the paperfolding sequence and the Golay-Shaprio-Rudin sequence.

In this article we propose to approach this problem with the most classical example of automatic sequences, the Thue-Morse sequence.

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1.2. **Preliminaries.** We introduce the necessary notions for stating the conjectures and our main results.

1.2.1. *Automatic sequnces.* A sequence is said to be *k-automatic* if it can be generated by a k-DFAO (*deterministic finite automaton with output*). For an integer $k \geq 2$, a k-DFAO is defined to be a 6-tuple

$$M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$$

where Q is the set of states with $q_0 \in Q$ being the initial state, $\Sigma = \{0, 1, \ldots, k-1\}$ the input alphabet, $\delta : Q \times \Sigma \to Q$ the transition function, Δ the output alphabet, and $\tau : Q \to \Delta$ the output function. The k-DFAO M generates a sequence $(c_n)_{n\geq 0}$ in the following way: for each non-negative integer n, the base-k expansion of n is read by M from right to left starting from the initial state q_0, and the automaton moves from state to state according to its transition funciton δ. When the end of the string is reached, the automaton halts in a state q, and the automaton outputs the symbol $c_n = \tau(q)$.

A necessary and sufficient condition [9] for a sequence to be k-automatic is that its *k-kernel*, defined as

$$\{(u_{k^d n+j})_{n\geq 0} \mid d \in \mathbb{N},\, 0 \leq j \leq k^d - 1\},$$

is finite. If we let $\Lambda_i^{(k)}$ denote the operator that sends a sequence $(u(n))_{n\geq 0}$ to its subsequence $(u(kn+i))_{n\geq 0}$, then the k-kernel can be defined alternatively as the smallest set containing $\mathbf{u}$ that is stable under $\Lambda_i^{(k)}$ for $0 \leq i < k-1$. We write Λ_i instead of $\Lambda_i^{(k)}$ when the value of k is clear from the context. We will use the fact that for an integer $m \geq 1$, a sequence is k-automatic if and only if it is k^m-automatic [9].

For a k-automatic sequence $\mathbf{u}$, we can construct a k-DFAO that generates it from its k-kernel. The set of states Q will be in bijection with the k-kernel, so we choose to identify them. For $q \in Q$ and $0 \leq j < k$, the value of the transition function $\delta(q, j)$ is defined as $\Lambda_j q$; the output function τ maps q to the 0-th term of q. This automaton has the property that leading 0's in the input does not change the output. It is minimal among k-automata with this property that generates $\mathbf{u}$.

We refer the readers to [3] for a comprehensive exposition of automatic sequences.

In this article we will consider the Thue-Morse sequence and the period-doubling sequence. For two distinct element a and b from an alphabet, the (a,b)-Thue-Morse sequence is the sequence $\mathbf{t}$ defined as the fixed point $s^\infty(a)$ of the substitution $s : a \mapsto ab,\ b \mapsto ba$; the (a,b)-period-doubling sequence $\mathbf{p}$ is defined as the fixed point $\sigma(a)$ of the substitution $\sigma : a \mapsto ab,\ b \mapsto aa$.

1.2.2. *Continued fractions.* Let K be a field. Given a sequence of polynomials $a_j(z) \in K[z]\backslash K$, we may define the infinite continued fraction

$$\text{CF}(\mathbf{a}(z)) := \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in K((1/z)). \tag{1.2}$$

The existence of the limit is guaranteed by the convergence theorem, whose proof is completely analogous to that for the classical continued fracions with positive integer partial quotients.

Define the sequences $(P_n(z))$ and $(Q_n(z))$ by

(1.3)
$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} a_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $n \geq 0$, then

$$\mathrm{CF}_n(z; a, b) = P_n(z)/Q_n(z) \in K((1/z)),$$

for $n \geq 0$. Note that here rational fractions are expanded in $1/z$. The unsimplified fraction $P_n(z)/Q_n(z)$ is called the n-th *convergent* of $\mathrm{CF}(x; \mathbf{a})$.

Conversely, let

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0 + c_{-1} z^{-1} + \cdots \tag{1.4}$$

be an arbitrary element of $K((1/z))$. Define the integer part of $f(z)$ as

$$[f(z)] = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0. \tag{1.5}$$

Set $f_0 = f$, $a_0 = [f_0]$, $f_0 = a_0 + 1/f_1$, $a_1 = [f_1]$, $f_1 = a_1 + 1/f_2$, $a_2 = [f_1]$... Then $a_0 \in K[z]$ and $a_j \in K[z] \backslash K$ for $j \geq 1$, and $f(z)$ admits the following continued fraction expansion

$$f(z) = a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{a_3(z) + \cfrac{1}{\ddots}}}}. \tag{1.6}$$

1.2.3. *Stieltjes continued fractions.* Let $(u_j)_{j \geq 0}$ be a sequence taking value in $K^\times$, the Stieltjes continued fraction

$$\mathrm{Stiel}(x; \mathbf{u}) := \cfrac{u_0}{1 + \cfrac{u_1 x}{1 + \cfrac{u_2 x}{1 + \cfrac{u_3 x}{\ddots}}}} \tag{1.7}$$

is defined to be the limit of the finite Stieltjes continued fractions

$$\mathrm{Stiel}_n(x;\mathbf{u}) := \cfrac{u_0}{1+\cfrac{u_1x}{1+\cfrac{u_2x}{\ddots\, 1+\cfrac{}{1+u_nx}}}} \in K[[x]]. \tag{1.8}$$

It can be easily shown that the sequence $\mathrm{Stiel}_n(x;\mathbf{u})$ is convergent.

Define the sequence $(P_n(x))$ and $(Q_n(x))$ by

$$\begin{pmatrix} P_n(x) & Q_n(x) \\ P_{n-1}(x) & Q_{n-1}(x) \end{pmatrix} := \begin{pmatrix} 1 & u_nx \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u_{n-1}x \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & u_0x \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/x & 0 \end{pmatrix} \tag{1.9}$$

for $n \geq 0$. Then

$$\mathrm{Stiel}_n(x;\mathbf{u}) = \frac{P_n(x)}{Q_n(x)},$$

for $n \geq 0$. The unsimplified fraction $P_n(x)/Q_n(x)$ is called the n-th *convergent* of $\mathrm{Stiel}(x;\mathbf{u})$.

Unlike continued fractions, every formal power series in $K[[x]]$ can not be expanded as a Stieltjes continued fraction.

1.3. **Conjectures.** We put forward the following conjectures concerning the Thue-Morse and period-doubling continued fractions and Stieltjes continued fractions.

Conjecture 1.1. *Let a, b be two distinct elements from $\mathbb{F}_2[z]\backslash\mathbb{F}_2$. Let $\mathbf{u}(z)$ be the (a,b)-Thue-Morse sequence. The continued fraction $\mathrm{CF}(\mathbf{u}(z))$ is algebraic of degree 4 over $\mathbb{F}_2(z)$.*

Conjecture 1.2. *Let $k \geq 2$ be an integer. Let a, b be two distinct elements from $\mathbb{F}_{2^k}^{\times}$. Let $\mathbf{u}$ be the (a,b)-Thue-Morse sequence. The Stieltjes continued fraction $\mathrm{Stiel}(x;\mathbf{u})$ is algebraic over $\mathbb{F}_2^k(x)$. Its minimal polynomial is*

$$p_0(x) + p_1(x)y + p_2(x)y^2 + p_4(x)y^4,$$

where

$$\begin{aligned} p_0(x) &= (a^2b^4 + b^6)/a^4x^2 + b^5/(a^5 + a^4b), \\ p_1(x) &= ((ab^4 + b^5)/a^5)x + b^4/a^5, \\ p_2(x) &= b^4/a^5x + b^4/(a^6 + a^5b), \\ p_4(x) &= (b^4/(a^6 + a^5b))x^2. \end{aligned}$$

Let $\mathbf{v}$ be the $(a/b, 1)$-Thue-Morse sequence, then

$$\mathrm{Stiel}(x;\mathbf{u}) = b \cdot \mathrm{Stiel}(bx;\mathbf{v}).$$

Therefore conjecture 1.2 admits the following equivalent form:

Conjecture 1.2a. Let $k \geq 2$ be an integer. Let a an elements from $\mathbb{F}_{2^k}^{\times}$ distinct from 1. Let $\mathbf{u}$ be the $(a,1)$-Thue-Morse sequence. The Stieltjes continued fraction $\mathrm{Stiel}(x;\mathbf{u})$ is algebraic over $\mathbb{F}_2^k(x)$. Its minimal polynomial is

$$p_0(x) + p_1(x)y + p_2(x)y^2 + p_4(x)y^4,$$

where

$$\begin{aligned} p_0(x) &= (a^2+1)/a^4)x^2 + 1/(a^5+a^4), \\ p_1(x) &= ((a+1)/a^5)x + 1/a^5, \\ p_2(x) &= 1/a^5x + 1/(a^6+a^5), \\ p_4(x) &= (1/(a^6+a^5))x^2. \end{aligned}$$

Or still

Conjecture 1.2b. We regard a as a formal variable. Let $\mathbf{u}$ be the $(a,1)$-Thue-Morse sequence. Then the Stieljtes continued fraction $\mathrm{Stiel}(x;\mathbf{u}) \in \mathbb{F}_2(a)[[x]]$ is algebraic over $\mathbb{F}_2(a)(x)$. Its minimal polynomial is

$$p_0(x) + p_1(x)y + p_2(x)y^2 + p_4(x)y^4,$$

where

$$\begin{aligned} p_0(x) &= (a^2+1)/a^4)x^2 + 1/(a^5+a^4), \\ p_1(x) &= ((a+1)/a^5)x + 1/a^5, \\ p_2(x) &= 1/a^5x + 1/(a^6+a^5), \\ p_4(x) &= (1/(a^6+a^5))x^2. \end{aligned}$$

It is clear that conjecture 1.2b implies conjecture 1.2a, noticing that the only roots of the denominators of the coefficients of $p_j(x)$, $j = 0, 1, 2, 4$, are 0 and 1. On the other hand, if conjecture 1.2b does not hold, then

$$0 \neq p_0(x) + p_1(x)\,\mathrm{Stiel}(x;\mathbf{u}) + p_2(x)\,\mathrm{Stiel}(x;\mathbf{u})^2 + p_4(x)\,\mathrm{Stiel}(x;\mathbf{u})^4 =: \sum_{n=0}^{\infty} c_n(a)x^n,$$

and there exists an $n \in \mathbb{N}$ for which $c_n(a) \in \mathbb{F}_2(a)$ is not the zero. Necessarily there exists a $k \geq 2$ and an element $u \in \mathbb{F}_{2^k}\backslash\{0,1\}$ that is not a root of the numerator of $c_n(a)$, and consequently conjecture 1.2a does not hold for $a = u$.

Based on our calculation, we believe that the period-doubling continued fractions are also algebraic. However, the period-doubling Stieltjes continued fractions seem to be transcendental.

Conjecture 1.3. *Let a, b be two distinct elements from $\mathbb{F}_2[z]\backslash\mathbb{F}_2$. Let $\mathbf{u}(z)$ be the (a,b)-period-doubling sequence. The continued fraction $\mathrm{CF}(\mathbf{u}(z))$ is algebraic over $\mathbb{F}_2(z)$.*

Conjecture 1.4. *Let $J \in \mathbf{F}_4\backslash\{0,1\}$. Let $\mathbf{u}$ be the $(1,J)$-period-doubling sequence. The Stieltjes continued fraction $\mathrm{Stiel}(x;\mathbf{u})$ is transcendental over $\mathbb{F}_2(z)$.*

1.4. **Main results.** We developped a method for checking conjecture 1.1 and 1.2a and implemented it. Using this method, we checked that conjecture 1.1 holds for all pairs (a,b) of elements from $\mathbb{F}_2[x]\backslash\mathbb{F}_2$ such that $\deg a + \deg b \leq 7$, and that conjecture 1.2a holds for all $a \in \mathbb{F}_{2^k}\backslash\{0,1\}$ for $k = 2, 3, 4$.

For the verification of conjectures 1.1 and 1.2a, we use the Guess'n'Prove method. For conjecture 1.1, our program takes the pair (a,b) as input, and, for the (a,b)-Thue-Morse sequence $\mathbf{u}$, uses the Derksen algorithm for Padé-Hermite approximants [6] to guess the minimal polynomial of $\mathrm{CF}(\mathbf{u})(z)$. To prove that the guess is correct, it then guesses and proves several lemmas, whose forms depend on the choice of (a,b), that would lead to the final result. In Section 2 we illustrate our method with an example of computer generated proof. The proofs for the other pairs that we have tested can be found on the personal web page of the authors [1].

For conjecture 1.2a, the situation is similar, except that we choose to regard a as a formal variable whenever we can. In this way we prepare a common part for all a, and to prove that conjecture 1.2a holds for a certain a, we only need to fill in the rest of the proof for this specific a.

In the proofs, we exploit the structure of the automata that generate the algebraic series in question. For this, we need to first obtain a k-automaton of an algebraic series from an annihilating polynomial of it. In Section 5 we explain this part of our program. The algorithm is based on the proof of theorem 1 of [8].

Our method for checking conjecture 1.1 can be adpated for the verification of conjecture 1.3. We give two examples in Section 4.

2. Thue-Morse Continued Fraction

Our program tests conjecture 1.1 for a given pair of distinct elements (a,b) from $\mathbb{F}_2[x]\backslash\mathbb{F}_2$. We have checked that the conjecture holds in the case where $\deg a + \deg b \le 7$.

The following is an example of proof that conjecture 1.1 holds for $(a,b)=(z,z^2+z+1)$. Both the statement of the theorem and its proof are generated automatically by our program. The exact statement of theorem 2.1, lemma 2.3 and 2.4 depends on the choice of (a,b).

2.1. Statement of the theorem for $(a,b)=(z,z^2+z+1)$.

Theorem 2.1. *Let* $(a,b)=(z,z^2+z+1)\in(\mathbb{F}_2[z]\backslash\mathbb{F}_2)^2$. *Let* $\mathbf{t}$ *be the* (a,b)*-Thue-Morse sequence, and* $\bar{\mathbf{t}}$*, the* (b,a)*-Thue-Morse sequence. The two power series* $\mathrm{CF}(\mathbf{t}(z))$ *and* $\mathrm{CF}(\bar{\mathbf{t}}(z))$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0.$$

For $\mathrm{CF}(\mathbf{t}(z))$

$$\begin{aligned}
p_0(z)&=z^9+z^7+z^6+z^5+z^4+z+1,\\
p_1(z)&=z^{11}+z^{10}+z^8+z^6+z^5+z^3+z^2+z,\\
p_2(z)&=z^{12}+z^{10}+z^2,\\
p_3(z)&=z^{11}+z^{10}+z^8+z^6+z^5+z^3+z^2+z,\\
p_4(z)&=z^{10}+z^9+z^7+z^6+z^5+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(\bar{\mathbf{t}}(z))$

$$\begin{aligned}
p_0(z)&=z^9+z^8+z^7+z^6+z^5+z^4+z,\\
p_1(z)&=z^{11}+z^{10}+z^8+z^6+z^5+z^3+z^2+z,\\
p_2(z)&=z^{12}+z^{10}+z^2,\\
p_3(z)&=z^{11}+z^{10}+z^8+z^6+z^5+z^3+z^2+z,\\
p_4(z)&=z^{10}+z^9+z^8+z^7+z^6+z^5+z^2+z+1.
\end{aligned}$$

[1] `http://irma.math.unistra.fr/~guoniu/frconj/`

2.2. **Proof.** Define

$$M_n(x) = x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix}\cdots x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$W_n(x) = x^{\deg(\bar{t}_{2^n-1})}\begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} x^{\deg(\bar{t}_{2^n-2})}\begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix}\cdots x^{\deg(\bar{t}_0)}\begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the property of the Thue-Morse sequence, we have for all $n \geq 0$

$$M_{n+1}(x) = W_n(x)\cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x)\cdot W_n(x).$$

Define $x := 1/z$. For an non-zero polynomial $P(z)$, we define $\tilde{P}(x)$ to be $P(1/x)$. Then

$$\mathrm{CF}_n(\mathbf{t}(z)) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with definition (1.3), we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n, and

$$\mathrm{CF}_{2^{2n}-1}(\mathbf{t}(z)) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.1}$$

Our strategy is to first prove that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$, and then use their minimal polynomials to obtain that of $\mathrm{CF}_n(\mathbf{t}(z))$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Subsection 6.1. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(75, 75, 75, 75, 75)$$

of the vector

$$(1, f^3, f^6, f^9, f^{12}),$$

where $f = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$). See Chapter 7 of [6] for a description of the Derksen algorithm that is used here to find the Padé-Hermite approximants. We take the first eight terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as the initial conditions for $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x, y) \in \mathbb{F}_2[x, y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.2. *We have*

$$M^e = W^o \cdot M^o, \tag{2.2}$$

$$M^o = W^e \cdot M^e, \tag{2.3}$$

$$W^e = M^o \cdot W^o, \tag{2.4}$$

$$W^o = M^e \cdot W^e. \tag{2.5}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z) \right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$; here Res_z means the resultant with respect to the variable z (see Chapter 6 of [6]). We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 270 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 270, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z) \right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 330 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 330, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 8 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.3. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:6u] &= M^e[:3u] + x^u M^e[:2u] + x^u M^e[:3u] + x^{2u} M^e[:2u] + x^{2u} M^e[:3u] \\ &\quad + x^{3u} M^e[:2u] + (x^{3u} + x^{4u} + x^{5u}) R^e, \\ W^e[:6u] &= W^e[:3u] + x^u W^e[:2u] + x^u W^e[:3u] + x^{2u} W^e[:2u] + x^{2u} W^e[:3u] \\ &\quad + x^{3u} W^e[:2u] + (x^{3u} + x^{4u} + x^{5u}) R^e. \end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} M^o[:6u] &= M^o[:3u] + x^u M^o[:2u] + x^u M^o[:3u] + x^{2u} M^o[:2u] + x^{2u} M^o[:3u] \\ &\quad + x^{3u} M^o[:2u] + (x^{3u} + x^{4u} + x^{5u}) R^o, \\ W^o[:6u] &= W^o[:3u] + x^u W^o[:2u] + x^u W^o[:3u] + x^{2u} W^o[:2u] + x^{2u} W^o[:3u] \\ &\quad + x^{3u} W^o[:2u] + (x^{3u} + x^{4u} + x^{5u}) R^o. \end{aligned}$$

Proof. To prove Lemma 2.3 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{1,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:6u] = T[:3u] + x^u T[:2u] + x^u T[:3u] + x^{2u} T[:2u] + x^{2u} T[:3u] + x^{3u} T[:2u].$$

The proofs of the other 15 cases are similar. We break down the above identity into 3 parts:

$$\begin{aligned} 0 &= x^{3u} T[:u] + x^u T[2u:3u] + T[3u:4u], \\ 0 &= x^{3u} T[u:2u] + x^{2u} T[2u:3u] + T[4u:5u], \\ 0 &= T[5u:6u], \end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2], \tag{2.6}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2], \tag{2.7}$$

$$0 = T[[101w]_2], \tag{2.8}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2], \tag{2.9}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2], \tag{2.10}$$

$$0 = T[[101w]_2], \tag{2.11}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 124 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.6) through (2.11) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)), \tag{2.12}$$

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)), \tag{2.13}$$

$$0 = \tau(A(s,101)), \tag{2.14}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)), \tag{2.15}$$

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)), \tag{2.16}$$

$$0 = \tau(A(s,101)), \tag{2.17}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{24}), E_{24}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.12) through (2.17) hold for $2 \leq k \leq 12$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.4. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:3u] + x^u M^e[:2u] + x^{3u} R^e,$$
$$W_{2k} = W^e[:3u] + x^u W^e[:2u] + x^{3u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:3u] + x^u M^o[:2u] + x^{3u} R^o,$$
$$W_{2k+1} = W^o[:3u] + x^u W^o[:2u] + x^{3u} R^o.$$

Proof. Call the four identities in Lemma 2.4 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the proofs of the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \left(W^e[:3v] + x^v W^e[:2v] + x^{3u} R^e\right) \times \left(M^e[:3v] + x^v M^e[:2v] + x^{3u} R^e\right). \tag{2.18}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{6v} R^o$. Therefore we only need to prove that their difference is $O(x^{6v})$. Using Lemma 2.2 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{6v}, to

$$W^e[:6v]M^e[:6v] + x^{2v} W^e[:4v]M^e[:4v].$$

For all $n \leq 6$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.3 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.18) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_6, b_1, ..., b_6, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^6)$, which completes the proof. $\square$

Proof of Theorem 2.1. We prove the theorem for $\mathrm{CF}(\mathbf{t}(z))$; for $\mathrm{CF}(\bar{\mathbf{t}}(z))$, the proof is similar. By Lemma 2.4, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (4.1),

$$\mathrm{CF}(\mathbf{t}(z)) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, that $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{12} + x^{11} + x^8 + x^7 + x^6 + x^5 + x^3, \\ q_1(x) &= x^{11} + x^{10} + x^9 + x^7 + x^6 + x^4 + x^2 + x, \\ q_2(x) &= x^{10} + x^2 + 1, \\ q_3(x) &= x^{11} + x^{10} + x^9 + x^7 + x^6 + x^4 + x^2 + x, \\ q_4(x) &= x^{11} + x^{10} + x^7 + x^6 + x^5 + x^3 + x^2. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate

$f(x)$ to order 96, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 96, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(\mathbf{t}(z)) = f(1/z)$. □

3. Thue-Morse Stieltjes Continued Fraction

Using our program, we checked that conjecture 1.2a holds for all $a \in \mathbb{F}_{2^k}\backslash\{0,1\}$ for $k = 2, 3, 4$. In this section we present our method.

3.1. **Testing of the conjecture.** For $k \geq 2$, instead of all a in $\mathbb{F}_{2^k}\backslash\{0,1\}$, we only need to test one a in each of the orbits of the Frobenius morphism $\phi : a \mapsto a^2$, because if we let $\mathbf{t}$ denote the $(a, 1)$-Thue-Morse seuqence and $\phi(\mathbf{t})$ the $(\phi(a), 1)$-Thue-Morse sequence, then

$$\mathrm{Stiel}(x; \phi(\mathbf{t})) = \phi(\mathrm{Stiel}(x; \mathbf{t})),$$

and they are either both algebraic or both transcendental.

For example, $\mathbb{F}_8 \cong \mathbb{F}_2[u]/ < u^3 + u + 1 >$ is partitioned into orbits

$$\{0\}, \{1\},\ \{\bar{u}, \bar{u}^2,\ \bar{u}^4\},\ \text{and}\ \{\bar{u}^3, \bar{u}^6, \bar{u}^5\}.$$

Therefore for $\mathbb{F}_8$, we only have to test the conjecture for $a = \bar{u}$ and $a = \bar{u}^3$. Furthermore, we only have to test those a in $\mathbb{F}_{2^k}\backslash\{0,1\}$ whose orbit contains k elements, because elements whose orbit has size $l < k$ are already tested in $\mathbb{F}_{2^l}$. For example, for $\mathbb{F}_{16} \cong \mathbb{F}_2[u]/ < u^4 + u + 1 >$, the orbit of $a = \bar{u}^5$ contains only itself and a^2. This means that $a^4 = a$, and therefore it is already treated in $\mathbb{F}_4$.

3.2. **Our method.** The same method for testing conjecture 1.1 can be used here to test conjecture 1.2a for $a \in \mathbb{F}_{2^k}\backslash\{0,1\}$ $(k \geq 2)$, with only slight modifications. As most of the following have a uniform expression for all a, we first regard a as a formal variable.

As in Section 2, we define

$$M_n = \begin{pmatrix} 1 & t_{2^n-1}x \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t_{2^n-2}x \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & t_0 x \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

and

$$W_n = \begin{pmatrix} 1 & \bar{t}_{2^n-1}x \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \bar{t}_{2^n-2}x \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & \bar{t}_0 x \\ 1 & 0 \end{pmatrix}, \tag{3.2}$$

where $\mathbf{t}$ is the $(a, 1)$-Thue-Morse sequence, and $\bar{\mathbf{t}}$, $(1, a)$-Thue-Morse sequence. We have $M_{n+1} = W_n \cdot M_n$ and $W_{n+1} = M_n \cdot W_n$ for all n.

We define four 2×2 matrices M^e, M^o, W^e and W^o as follows: For all $T \in \{M^e, M^o, W^e, W^o\}$, and all $i, j \in \{0, 1\}$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2(a)[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions. The polynomials $\phi(T, i, j)$ and initial conditions can be found in the annex. The reason for defining these matrices and how the polynomials $\phi(T, i, j)$ and initial conditions are found are similar to those given in Section 2.

As expected, the following Lemma holds:

Lemma 3.1.

$$M^e = W^o \cdot M^o, \tag{3.3}$$

$$M^o = W^e \cdot M^e, \tag{3.4}$$

$$W^e = M^o \cdot W^o, \tag{3.5}$$

$$W^o = M^e \cdot W^e. \tag{3.6}$$

Proof. Similar to the proof of Lemma 2.2. □

We have the following observation concerning the structure of the four matrices.

Observation 3.2. *For $k \geq 2$ and $u = 2^{2k-1}$ the following identities hold:*

$$M^e[u\!:\!2u] = x^u \cdot (a^u + 1) \cdot M^e[:u] + a^{u/2}x^u \cdot I_2, \tag{3.7}$$

$$W^e[u\!:\!2u] = x^u \cdot (a^u + 1) \cdot W^e[:u] + a^{u/2}x^u \cdot I_2; \tag{3.8}$$

for $k \geq 2$ and $u = 2^{2k}$,

$$M^o[u\!:\!2u] = x^u \cdot (a^u + 1) \cdot M^o[:u] + a^{u/2}x^u \cdot I_2, \tag{3.9}$$

$$W^o[u\!:\!2u] = x^u \cdot (a^u + 1) \cdot W^o[:u] + a^{u/2}x^u \cdot I_2. \tag{3.10}$$

Observation 3.3. *For $k \geq 2$ and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:u] + a^{u/2}x^u \cdot I_2,$$
$$W_{2k} = W^e[:u] + a^{u/2}x^u \cdot I_2;$$

for $k \geq 2$ and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:u] + a^{u/2}x^u \cdot I_2,$$
$$W_{2k+1} = W^o[:u] + a^{u/2}x^u \cdot I_2.$$

Lemma 3.4. *Observation 3.2 implies observation 3.3.*

Proof. Let us call the four identities in observation 3.3 also by the name M_{2k}, W_{2k}, M_{2k+1} and W_{2k+1}. Suppose observation 3.3 is true. We want to prove observation 3.2 by induction. For $n = 2$, the identities are verified directly. The inductive step is

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

Let us show for example how to prove

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1}. \tag{3.11}$$

By definition, the left side of the identity M_{2k+1} is equal to

$$W_{2k} \cdot M_{2k},$$

which, by induction hypothesis, is equal to

$$(W^e[:u] + a^{u/2}x^u I_2) \cdot (M^e[:u] + a^{u/2}x^u I_2),$$

where $u = 2^{2k-1}$. Therefore only need to prove that

$$W^e[:u] \cdot M^e[:u] + a^{u/2}x^u(W^e[:u] + M^e[:u]) - M^o[:2u] \tag{3.12}$$

is equal to zero. As the degree of the above polynomial is at most $2u - 1$, we only need to prove that it is $O(x^{2^{2n}})$. By (3.4),

$$\begin{aligned}
&M^o[:2u] \\
&\equiv W^e[:2u] \cdot M^e[:2u] \mod x^{2u} \\
&\equiv W^e[:u] \cdot M^e[:u] + W^e[u\!:\!2u] \cdot M^e[:u] + W^e[:u] \cdot M^e[u\!:\!2u] \mod x^{2u}
\end{aligned}$$

Therefore (3.12) is congruent modulo x^{2u} to

$$a^{u/2}x^u(W^e[:u] + M^e[:u]) + W^e[u:2u] \cdot M^e[:u] + W^e[:u] \cdot M^e[u:2u].$$

Substitute $W^e[u:2u]$ and $M^e[u:2u]$ by the expressions in observation 3.2 and we obtain that the quantity above is $O(x^{2u})$. That is, expression (3.12) is congruent to 0 modulo x^{2u}; since it has no term of order higher than $2u-1$, it is equal to 0. □

Proposition 3.5. *Observation 3.3 implies conjecture 1.2b.*

Proof. First, taking the limit of the identities in observation 3.3, we have for all $j, k \in \{0, 1\}$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Therefore

$$\text{Stiel}(x; \mathbf{t}) = \lim_{n\to\infty} \frac{P_n}{Q_n} = \lim_{n\to\infty} \frac{P_{2^{2n}-1}}{Q_{2^{2n}-1}} = \lim_{n\to\infty} \frac{M_{2n,0,1}/x}{M_{2n,0,0}} = \frac{M^e_{0,1}/x}{M^e_{0,0}}.$$

We obtain the minimal polynomial of $\text{Stiel}(x; \mathbf{t})$ from those of $M^e_{0,1}$ and $M^e_{0,0}$, using the same method described in the proof of Theorem 2.1. □

Remark 3.1. The above proposition says that observation 3.3 implies conjecture 1.2 when a is regarded as a formal variable. The implication also holds when a specializes as an element in $\mathbb{F}_{2^k}\backslash\{0,1\}$ ($k \geq 2$).

Therefore, to prove that conjecture 1.2a holds for a certain $a \in \mathbb{F}_{2^k}\backslash\{0,1\}$ ($k \geq 2$), we only need to prove that observation 3.2 holds for a. Because of the following argument, we only have to check (3.7) through (3.10) for finitely many k's instead of for all $k \geq 2$:

For $k \geq 2$ and $u = 2^{2k-1}$, identities (3.7) and (3.8) can be written as

$$T[[1w]_2] = (a^u + 1) \cdot T[[w]_2] \tag{3.13}$$

for every component T of M^e and W^e and all binary words w of length $2k-1$ and $w \neq 0^{2k-1}$; and

$$T[[1w]_2] = (a^u + 1) \cdot T[[w]_2] + a^{u/2} \tag{3.14}$$

for $w = 0^{2k-1}$.

We calculate the an automaton of T from the algebraic equation that defines it, following the method in [8] (see Section 5). Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k-1} = \{A(i, w) \mid |w| = 2k-1, w \neq 0^{2k-1}\}.$$

Identity (3.13) and (3.14) can be written as

$$\tau(A(s, 1)) = (a^u + 1) \cdot \tau(A(s, \epsilon)) \tag{3.15}$$

for all $s \in E_{2k-1}$, and

$$\tau(A(s, 1)) = (a^u + 1) \cdot \tau(A(s, \epsilon)) + a^{u/2} \tag{3.16}$$

for $s = A(i, 0^{2k-1})$.

As E_{2k+1} is completely determined by E_{2k-1}, the sequence $(E_{2k+1})_k$ is ultimately periodic. The sequences $(A(i, 0^{2k-1}))_k$ and $a^{2^{2k-1}}$ are also periodic. Therefore we only need to check (3.15) and (3.16) for finitely many k's.

3.3. **An example.** For $a \in \mathbb{F}_4 \backslash \{0, 1\}$ and $T = Me_{0,0}$, we find that the minimal 2-DFAO of T has as transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	5	[2, 8]	10	[7, 8]	15	[13, 17]
1	[3, 4]	6	[9, 4]	11	[8, 13]	16	[18, 4]
2	[5, 6]	7	[10, 4]	12	[14, 4]	17	[19, 12]
3	[1, 7]	8	[11, 6]	13	[15, 16]	18	[16, 6]
4	[4, 4]	9	[6, 12]	14	[12, 16]	19	[17, 8]

and output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	1	3	1	6	$a+1$	9	$a+1$	12	1	15	1	18	a
1	1	4	0	7	0	10	0	13	1	16	a	19	$a+1$
2	a	5	a	8	a	11	a	14	1	17	$a+1$		

The tuple $(A(i, 0^{2k-1}), E_{2k-1})$ has the following values:

$$\begin{aligned}
k &= 3 : (1, \{2, 4, 7, 8, 9\}), \\
k &= 5 : (1, \{2, 4, 7, 8, 9, 13, 14\}), \\
k &= 7 : (1, \{2, 4, 7, 8, 9, 13, 14, 17, 18\}), \\
k &= 9 : (1, \{2, 4, 7, 8, 9, 13, 14, 17, 18\}).
\end{aligned}$$

For all $k \geq 1$, $a^{2^{2k-1}} = a+1$. Therefore we only have to check that identity (3.7) holds for $k = 3, 5, 7$, which turns out to be true.

4. Period-doubling Continued Fractions

The method for checking conjecture 1.1 can be adapted for the verification of conjecture 1.3. In this section, we give two examples. First we introduce the notation.

For $(a, b) \in (\mathbb{F}_2[z] \backslash \mathbb{F}_2)^2$. Let $\mathbf{p}$ be the (a, b)-period-doubling sequence. Define two sequence $A_n(x)$ and $B_n(x)$ by

$$\begin{aligned}
A_0(x) &= x^{\deg(a)} \begin{pmatrix} a(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\
B_0(x) &= x^{\deg(b)} \begin{pmatrix} b(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\
A_{n+1}(x) &= B_n(x) A_n(x) \quad \forall n \geq 0, \\
B_{n+1}(x) &= A_n(x) A_n(x) \quad \forall n \geq 0.
\end{aligned}$$

Define $x := 1/z$. For an non-zero polynomial $P(z)$, we define $\tilde{P}(x)$ to be $P(1/x)$. Then

$$\mathrm{CF}_n(\mathbf{p}(z)) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $A_n(x)$ with definition (1.3), we see that

$$A_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$A_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n, and

$$\mathrm{CF}_{2^{2n}-1}(\mathbf{p}(z)) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{A_{2n}(x)_{0,1}}{A_{2n}(x)_{0,0}}. \tag{4.1}$$

4.1. **The (z^2, z)-period-doubling sequence.** In this subsection, we prove the following theorem.

Theorem 4.1. *Let $(a, b) = (z^2, z) \in (\mathbb{F}_2[z]\backslash\mathbb{F}_2)^2$. Let $\mathbf{p}$ be the (a, b)-period-doubling sequence. The power series $\mathrm{CF}(\mathbf{p}(z))$ is algebraic over $\mathbb{F}_2(z)$; its minimal polynomial is*

$$z^4 + x^3z^2 + (x^5 + x^4)z + x^3 + x^2 + 1 = 0.$$

We define four 2×2 matrices A^e, A^o, B^e and B^o as follows: For all $T \in \{A^e, A^o, B^e, B^o\}$, and all $i, j \in \{0, 1\}$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2(a)[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions. The polynomials $\phi(T, i, j)$ and initial conditions can be found in the annex. The reason for defining these matrices and how the polynomials $\phi(T, i, j)$ and initial conditions are found are similar to those given in Section 2.

Lemma 4.2. *The following identities hold:*

$$A^e = B^o \cdot A^o,$$
$$A^o = B^e \cdot A^e,$$
$$B^e = A^o \cdot A^o,$$
$$B^o = A^e \cdot A^e.$$

Proof. Similar to the proof of Lemma 2.2. □

Lemma 4.3. *For $n \geq 2$ even and $u = (5 \cdot 2^n + 1)/3$,*

$$A^e[: 2u] = A^e[: u] + x^u \cdot I_2$$
$$B^e[: 2u] = B^e[: u] + x^u \begin{pmatrix} x^{u-1} & x^{u-2} + x^{u-3} \\ 0 & x^{u-1} \end{pmatrix}$$

For $n \geq 1$ odd, for $u = (5 \cdot 2^n + 2)/3$,

$$A^o[: 2u - 1] = A^o[: u] + x^{2u-2} \cdot I_2$$
$$B^o[: 2u - 1] = B^o[: u] + x^u \begin{pmatrix} 1 & x^{u-2} + x^{u-3} \\ 0 & 1 \end{pmatrix}$$

Proof. We give proof of

$$A^e_{0,0}[: 2u] = A^e_{0,0}[: u] + x^u \tag{4.2}$$

for $n \geq 2$ even and $u = (5 \cdot 2^n + 1)/3$; the proofs of the other 15 cases are similar. Let $T = A^e_{0,0}$. Identity (4.2) can be written as

$$T[u : 2u] = x^u. \tag{4.3}$$

From the minimal polynomial and the first terms of T, we find its minimal automaton. Its transition function δ ($\Lambda(n) := [\delta(n,0), \delta(n,1)]$) and output function τ are as follows:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	2	[4, 5]	4	[4, 4]	6	[6, 4]
1	[3, 2]	3	[6, 6]	5	[2, 6]		

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	1	2	0	4	0	6	1
1	1	3	1	5	0		

Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s.

For $n \geq 2$ even, $k = n/2 - 1$, and $u = (5 \cdot 2^n + 1)/3$ the binary expansions of integers j in $[u, 2u[$ have the following forms:

j	$[j]_2$
$j = u$	$1(10)^k 11$
$u < j < 2^{n+1}$	$1(10)^{k-l} 11\{0,1\}^{2l}$, $1 \leq l \leq k$
$2^{n+1} \leq j < 2u - 2$	$1(10)^l 0\{0,1\}^{n-2l}$, $0 \leq l \leq k$
$j = u-1, u-2$	$1(10)^k 10\{0,1\}$

Consider the sets

$$B_0 = \{1(10)^k 11 \mid k \geq 0\},$$
$$B_1 = \{1(10)^m 11\{0,1\}^{2l} \mid l \geq 1,\ m \geq 0\},$$
$$B_2 = \{1(10)^m 0\{0,1\}^{2l} \mid l \geq 1,\ m \geq 0\},$$
$$B_3 = \{1(10)^m 10\{0,1\} \mid m \geq 0\}.$$

For $i = 0, 1, 2, 3$, define

$$E_i = \{A(0, w) \mid w \in B_i\}.$$

We find that

$$E_0 = \{6\}, \quad E_1 = \{4\}, \quad E_2 = \{4, 5\}, \quad E_3 = \{4\}.$$

We verify that for all $s \in E_0$, $\tau(s) = 1$, and for all $s \in E_i$, $i = 1, 2, 3$, $\tau(s) = 0$. This proves identity (4.3) for all $n \geq 2$ even, and $u = (5 \cdot 2^n + 1)/3$. □

Lemma 4.4. *For $n \geq 2$ even and $u = (5 \cdot 2^n + 1)/3$*

$$A_n = A^e[: u] + x^u \cdot I_2,$$
$$B_n = B^e[: u].$$

For $n \geq 1$ odd, for $u = (5 \cdot 2^n + 2)/3$

$$A_n = A^o[: u],$$
$$B_n = B^o[: u] + x^u \cdot I_2.$$

Proof. Let us call the identities involving A_n and B_n also by the name A_n and B_n. We will prove the lemma by induction. It can be verified directly that A_n and B_n holds for $n = 1,\ 2$. For the inductive step, we want to prove that for $n \geq 2$,

$$A_n \wedge B_n \Rightarrow A_{n+1} \wedge B_{n+1}.$$

We give the proof of

$$A_n \wedge B_n \Rightarrow A_{n+1}$$

when n is even. The proofs of the other cases are similar. Now suppose that for some $n \geq 2$ even and $u = (5 \cdot 2^n + 1)/3$ it holds that

$$\begin{aligned} A_n &= A^e[:u] + x^u \cdot I_2, \\ B_n &= B^e[:u]. \end{aligned}$$

We want to prove that

$$A_{n+1} = A^o[:2u]. \tag{4.4}$$

By definition and induction hypothesis,

$$A_{n+1} = B_n A_n = B^e[:u] \cdot (A^e[:u] + x^u \cdot I_2).$$

As the degrees of both A_{n+1} and $A^o[:2u]$ are at most $2u-1$, to prove that they are equal, we only need to prove that they are congruent modulo x^{2u}. By lemma 4.2 and lemma 4.3,

$$\begin{aligned} A^o[:2u] &\equiv B^e[:2u] \cdot A^e[:2u] \\ &\equiv \left(B^e[:u] + x^u \begin{pmatrix} x^{u-1} & x^{u-2} + x^{u-3} \\ 0 & x^{u-1} \end{pmatrix} \right) \cdot (A^e[:u] + x^u \cdot I_2) \mod x^{2u}. \end{aligned}$$

Therefore

$$\begin{aligned} A^o[:2u] - A_{n+1} &\equiv x^u \begin{pmatrix} x^{u-1} & x^{u-2} + x^{u-3} \\ 0 & x^{u-1} \end{pmatrix} \cdot A^e[:u] \\ &\equiv x^u \begin{pmatrix} x^{u-1} & x^{u-2} + x^{u-3} \\ 0 & x^{u-1} \end{pmatrix} \cdot (A^e[:u] \mod x^3) \\ &\equiv x^u \begin{pmatrix} x^{u-1} & x^{u-2} + x^{u-3} \\ 0 & x^{u-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & x^2 \\ x^2 & 0 \end{pmatrix} \\ &\equiv 0 \mod x^{2u}. \end{aligned}$$

□

Theorem 4.1 can be derived from lemma 4.4 and the definition of A^e, using the same method as in the proof of theorem 2.1.

4.2. **The (z^3, z^2+z+1)-period-doubling sequence.** In this subsection, we prove the following theorem.

Theorem 4.5. *Let $(a, b) = (z^3, z^2 + z + 1) \in (\mathbb{F}_2[z] \backslash \mathbb{F}_2)^2$. Let $\mathbf{p}$ be the (a, b)-period-doubling sequence. The power series $\mathrm{CF}(\mathbf{p}(z))$ is algebraic over $\mathbb{F}_2(z)$; its minimal polynomial is*

$$z^4 + (x^5 + x^4 + x^3)z^2 + (x^8 + x^6 + x^5 + x^3)z + x^5 + x^3 + x^2 = 0.$$

We define four 2×2 matrices A^e, A^o, B^e and B^o as follows: For all $T \in \{A^e, A^o, B^e, B^o\}$, and all $i, j \in \{0, 1\}$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2(a)[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions. The polynomials $\phi(T, i, j)$ and initial conditions can be found in the annex. The reason for defining these matrices and how the polynomials $\phi(T, i, j)$ and initial conditions are found are similar to those given in Section 2.

Lemma 4.6. *The following identities hold:*

$$\begin{aligned} A^e &= B^o \cdot A^o, \\ A^o &= B^e \cdot A^e, \\ B^e &= A^o \cdot A^o, \\ B^o &= A^e \cdot A^e. \end{aligned}$$

Proof. Similar to the proof of Lemma 2.2. □

Lemma 4.7. *For $n \geq 2$ even, $u = (2^{n+3} + 1)/3$, $v = 2^n$,*

$$\begin{aligned} A^e[:2u] &= (1 + x^v + x^{2v}) \cdot (A^e[:u] + x^v A^e[:u - v]) + x^{4v} A^e[:2u - 4v] + \\ &\quad (x^u + x^{u+v} + x^{u+2v}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ B^e[:2u] &= (1 + x^v + x^{2v}) \cdot (B^e[:u] + x^v B^e[:u - v]) + x^{4v} B^e[:2u - 4v] + \\ &\quad \begin{pmatrix} x^{2u-1} & x^{2u-1} + x^{2u-4} \\ 0 & x^{2u-1} \end{pmatrix}. \end{aligned}$$

For $n \geq 1$ odd, $u = (2^{n+3} + 2)/3$, $v = 2^n$,

$$\begin{aligned} A^o[:2u - 1] &= (1 + x^v + x^{2v}) \cdot (A^o[:u] + x^v A^o[:u - v]) + x^{4v} A^o[:2u - 4v - 1] + \\ &\quad (x^{2u-2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ B^o[:2u - 1] &= (1 + x^v + x^{2v}) \cdot (B^o[:u] + x^v B^o[:u - v]) + x^{4v} B^o[:2u - 4v - 1] + \\ &\quad \begin{pmatrix} x^u + x^{u+v} + x^{u+2v} & x^{2u-2} + x^{2u-4} \\ 0 & x^u + x^{u+v} + x^{u+2v} \end{pmatrix}. \end{aligned}$$

Proof. We give the proof of the $(0, 0)$-th component of the first identity. The proofs of the other 15 identities are similar. Let $T = A^e_{0,0}$. We want to prove that

$$T[:2u] = (1+x^v+x^{2v})\cdot(T[:u]+x^vT[:u-v])+x^{4v}T[:2u-4v]+x^u(1+x^v+x^{2v}). \tag{4.5}$$

for all $n \geq 2$ even, $u = (2^{n+3} + 1)/3$, and $v = 2^n$. Equation (4.5) can be rewritten as

$$\begin{aligned} T[u:2u] &= x^v T[u - v:u] + x^{2v} T[u - v:u] + x^{3v} T[:u - v] + \\ &\quad x^{4v} T[:2u - 4v] + x^u + x^{u+v} + x^{u+2v} \\ &= (x^u + x^v T[u - v:2v]) + (x^v T[2v:u] + x^{3v} T[:u - 2v]) + \\ &\quad (x^{u+v} + x^{3v} T[u - 2v:v] + x^{2v} T[u - v:2v]) + \\ &\quad (x^{2v} T[2v:u] + x^{3v} T[v:u - v] + x^{4v} T[:u - 2v]) + \\ &\quad (x^{u+2v} + x^{4v} T[u - 2v:2u - 4v]) \end{aligned}$$

noting that $u < 3v < u + v < 4v < u + 2v < 2u$. The above identity can be decomposed into five parts:

$$\begin{aligned}
T[u : 3v] &= x^u + x^v T[u - v : 2v] \\
T[3v : u + v] &= x^v T[2v : u] + x^{3v} T[: u - 2v] \\
T[u + v : 4v] &= x^{u+v} + x^{3v} T[u - 2v : v] + x^{2v} T[u - v : 2v] \\
T[4v : u + 2v] &= x^{2v} T[2v : u] + x^{3v} T[v : u - v] + x^{4v} T[: u - 2v] \\
T[u + 2v : 2u] &= x^{u+2v} + x^{4v} T[u - 2v : 2u - 4v]
\end{aligned}$$

That the above five identities hold for all $n \geq 2$, $u = (2^{n+3} + 1)/3$, and $v = 2^n$ is equivalent to the following identities:

$$T[[10w]_2] = 1 + T[[1w]_2] \qquad \forall w \in L_0 \tag{4.6}$$

$$T[[10w]_2] = T[[1w]_2] \qquad \forall w \in L_1 \tag{4.7}$$

$$T[[11w]_2] = T[[10w]_2] + T[[w]_2] \qquad \forall w \in L_2 \tag{4.8}$$

$$T[[11w]_2] = 1 + T[[w]_2] + T[[1w]_2] \qquad \forall w \in L_0 \tag{4.9}$$

$$T[[11w]_2] = T[[w]_2] + T[[1w]_2] \qquad \forall w \in L_1 \tag{4.10}$$

$$T[[100w]_2] = T[[10w]_2] + T[[1w]_2] + T[[w]_2] \qquad \forall w \in L_2 \tag{4.11}$$

$$T[[100w]_2] = 1 + T[[w]_2] \qquad \forall w \in L_0 \tag{4.12}$$

$$T[[100w]_2] = T[[w]_2] \qquad \forall w \in L_1 \tag{4.13}$$

$$T[[10w]_2] = T[[w]_2] \qquad \forall w \in L_3 \tag{4.14}$$

$$T[[10w]_2] = T[[w]_2] \qquad \forall w \in L_4 \tag{4.15}$$

where

$$\begin{aligned}
L_0 &= L((10)^*11), \\
L_1 &= L((10)^*11\{00, 01, 10, 11\}^+), \\
L_2 &= L((10)^*0\{0, 1\}\{00, 01, 10, 11\}^* + (10)^+), \\
L_3 &= L((10)^+0\{00, 01, 10, 11\}^+), \\
L_4 &= L((10)^+\{0, 1\}).
\end{aligned}$$

From the minimal polynomial and the first terms of T, we find its minimal automaton. Its transition function δ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$) and output function τ are as follows:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	6	[11, 12]	12	[15, 20]	18	[23, 12]	24	[27, 15]
1	[3, 4]	7	[13, 14]	13	[9, 21]	19	[24, 23]	25	[15, 18]
2	[5, 6]	8	[5, 15]	14	[14, 14]	20	[14, 26]	26	[17, 20]
3	[7, 8]	9	[16, 14]	15	[22, 10]	21	[14, 16]	27	[28, 14]
4	[9, 10]	10	[17, 18]	16	[23, 9]	22	[5, 17]	28	[27, 17]
5	[8, 9]	11	[19, 6]	17	[24, 25]	23	[16, 16]		

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	1	5	1	10	0	15	1	20	0	25	1
1	1	6	0	11	0	16	1	21	0	26	0
2	1	7	1	12	1	17	0	22	1	27	0
3	1	8	1	13	1	18	1	23	1	28	0
4	1	9	1	14	0	19	0	24	0		

Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s. For $j = 0, 1, \dots, 4$, define

$$E_j = \{A(0,w) \mid w \in L_j\}.$$

We can compute E_j explicitly and find

$$\begin{aligned} E_0 &= \{6\} \\ E_1 &= \{14, 15, 16, 17, 18, 20\} \\ E_2 &= \{3, 4, 5, 13, 14, 15, 16, 17, 19, 27\} \\ E_3 &= \{9, 14, 21, 23\} \\ E_4 &= \{8, 9\}. \end{aligned}$$

Equations (4.6) through (4.15) can be written as

$$\tau(A(s,10)) = 1 + \tau(A(s,1)) \qquad \forall s \in E_0 \tag{4.16}$$

$$\tau(A(s,10)) = \tau(A(s,1)) \qquad \forall s \in E_1 \tag{4.17}$$

$$\tau(A(s,11)) = \tau(A(s,10)) + \tau(s) \qquad \forall s \in E_2 \tag{4.18}$$

$$\tau(A(s,11)) = 1 + \tau(s) + \tau(A(s,1)) \qquad \forall s \in E_0 \tag{4.19}$$

$$\tau(A(s,11)) = \tau(s) + \tau(A(s,1)) \qquad \forall s \in E_1 \tag{4.20}$$

$$\tau(A(s,100)) = \tau(A(s,10)) + \tau(A(s,1)) + \tau(s) \qquad \forall s \in E_2 \tag{4.21}$$

$$\tau(A(s,100)) = 1 + \tau(s) \qquad \forall s \in E_0 \tag{4.22}$$

$$\tau(A(s,100)) = \tau(s) \qquad \forall s \in E_1 \tag{4.23}$$

$$\tau(A(s,10)) = \tau(s) \qquad \forall s \in E_3 \tag{4.24}$$

$$\tau(A(s,10)) = \tau(s) \qquad \forall s \in E_4 \tag{4.25}$$

We verify equations (4.16) through (4.25) directly. □

Lemma 4.8. *For $n \geq 2$ even, $u = (2^{n+3}+1)/3$, $v = 2^n$,*

$$\begin{aligned} A_n &= A^e[:u] + x^v \cdot A^e[:u-v] + x^u \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ B_n &= B^e[:u] + x^v \cdot B^e[:u-v] \end{aligned}$$

For $n \geq 1$ odd, $u = (2^{n+3}+2)/3$, $v = 2^n$,

$$\begin{aligned} A_n &= A^o[:u] + x^v \cdot A^o[:u-v] \\ B_n &= B^o[:u] + x^v \cdot B^o[:u-v] + x^u \cdot I_2 \end{aligned}$$

Proof. For $n \geq 2$ even, we prove that

$$A_n \wedge B_n \Rightarrow A_{n+1}.$$

Set $u = (2^{n+3}+1)/3$, $v = 2^n$. Identity A_{n+1} can be written as

$$A_{n+1} = A^o[: 2u] + x^{2v} \cdot A^o[2u - 2v] \tag{4.26}$$

The left side of Eq. (4.26) is

$$\begin{aligned}
&B_n A_n \\
&= (B^e[: u] + x^v \cdot B^e[: u - v]) \left(A^e[: u] + x^v \cdot A^e[: u - v] + x^u \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \\
&= (B^e[: u] + x^v \cdot B^e[: u - v])(A^e[: u] + x^v \cdot A^e[: u - v]) \\
&\quad x^u B^e[: u] \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + x^{u+v} B^e[: u - v] \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

By lemma 4.6, the right side of Eq. (4.26) is congruent, modulo x^{2u} to

$$(1 + x^{2v})A^o[: 2u] = (1 + x^{2v})B^e[: 2u]A^e[: 2u].$$

We recall that

$$\begin{aligned}
A^e[: 2u] &= (1 + x^v + x^{2v}) \cdot (A^e[: u] + x^v A^e[: u - v]) + x^{4v} A^e[: 2u - 4v] + \\
&\quad (x^u + x^{u+v} + x^{u+2v}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\
B^e[: 2u] &= (1 + x^v + x^{2v}) \cdot (B^e[: u] + x^v B^e[: u - v]) + x^{4v} B^e[: 2u - 4v] + \\
&\quad \begin{pmatrix} x^{2u-1} & x^{2u-1} + x^{2u-4} \\ 0 & x^{2u-1} \end{pmatrix}.
\end{aligned}$$

Noticing that

$$\begin{pmatrix} x^{2u-1} & x^{2u-1} + x^{2u-4} \\ 0 & x^{2u-1} \end{pmatrix} A^e[: 2u] \equiv 0 \mod x^{2u},$$

and

$$(1 + x^{2v})(1 + x^v + x^{2v})^2 = 1 + x^{6v} \equiv 0 \mod x^{2u},$$

we have

$$\begin{aligned}
&(1 + x^{2v}) \cdot B^e[: 2u]A^e[: 2u] \\
&\equiv (1 + x^{2v}) \cdot \big((1 + x^v + x^{2v}) \cdot (B^e[: u] + x^v B^e[: u - v]) + x^{4v} B^e[: 2u - 4v]\big) \\
&\quad ((1 + x^v + x^{2v}) \cdot (A^e[: u] + x^v A^e[: u - v]) + x^{4v} A^e[: 2u - 4v]) \\
&\quad (1 + x^{2v}) \cdot \big((1 + x^v + x^{2v}) \cdot (B^e[: u] + x^v B^e[: u - v]) + x^{4v} B^e[: 2u - 4v]\big) \\
&\quad (x^u + x^{u+v} + x^{u+2v}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&\equiv (B^e[: u] + x^v B^e[: u - v]) \cdot (A^e[: u] + x^v A^e[: u - v]) \\
&\quad x^u \cdot (B^e[: u] + x^v B^e[: u - v]) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mod x^{2u}.
\end{aligned}$$

Thus we have proved that both sides of Eq. (4.26) are congruent modulo x^{2u}, so that they must be equal as both have degree at most $2u$. □

Theorem 4.1 can be derived from lemma 4.8 and the definition of A^e, using the same method as in the proof of theorem 2.1.

5. From equation to automaton

In this section we give an description of the algorithm that we use to calculate a p-automaton of an algebraic series $T(x)$ in $\mathbb{F}_q[[x]]$ from an annihilating polynomial of it, where $\mathbb{F}_q$ is a finite field of characteristic p. The algorithm is based on the proof of theorem 1 in [8].

Step one: Normalization
Input: an annihilating polynomial $P(x,y) \in \mathbb{F}_q(x)[y]$ of $T(x)$.
Output: an annihilating polynomial $Q(x,y) \in \mathbb{F}_q(x)[y]$ of $T(x)$ the form

$$y + \frac{a_1(x)}{b_1(x)} y^{p^1} + \frac{a_2(x)}{b_2(x)} y^{p^2} + \cdots + \frac{a_n(x)}{b_n(x)} y^{p^n}.$$

Method: use the relation $P(x,T(x)) = 0$ to express $T(x)^{p^j}$ as $\mathbb{F}_p(x)$-linear combination of $T(x)^k$, $k = 0,1,\ldots,d-1$, where d is the degree of $P(x,y)$ as a polynomial in y. In practice, to find the expression of $T(x)^{p^j}$, we first calculate that of $T(x)^{p^{j-1}}$, then raise it to the p-th power, and finally reduce again using the relation $P(x,T(x)) = 0$.

We know that the family $T(x)^{p^j}$, $j = 0,1,\ldots,d$ is necessarily linearly dependent. However, as it can be costly to compute $T(x)^{p^j}$ when j is large, in reality we stop once the rank of the family $T(x)^{p^j}$, $k = 0,1,\ldots,j_0$ is less than $j_0 + 1$.

Step two: From normalized equation to kernel
Input: the relation

$$T(x) = \frac{a_1(x)}{b_1(x)} T(x)^{p^1} + \frac{a_2(x)}{b_2(x)} T(x)^{p^2} + \cdots + \frac{a_n(x)}{b_n(x)} T(x)^{p^n}. \tag{5.1}$$

Output: the p-kernel of $T(x)$.
Method: We let ϕ denote the Frobenius morphism and Λ_j the Cartier operator that maps $\sum a_l x^l$ to $\sum a_{pl+j} x^l$ for $j = 0,1,\ldots,p-1$. We recall that for a series $f(x) = \sum_{l \geq l_0} c_l x^l \in \mathbb{F}_q((x))$ and polynomials $a(x)$ and $b(x)$,

$$\Lambda_j(a(x)f(x)^p) = \Lambda_j(a(x))\Lambda_0(f(x)^p)$$

for $j = 0,1,\ldots,p$ and

$$\Lambda_0(f(x)^p) = \Lambda_0 \sum_{l \geq l_0} c_l^p x^{p \cdot l} = \sum_{l \geq l_0} c_l^p x^l = \phi(f)(x).$$

Combining the above two identities and we get

$$\Lambda_j \left(\frac{a(x)}{b(x)} f(x)^p \right) = \Lambda_j \left(a(x)b(x)^{p-1} \frac{f(x)^p}{b(x)^p} \right) = \Lambda_j(a(x)b(x)^{p-1}) \frac{\phi(f)(x)}{\phi(b)(x)}. \tag{5.2}$$

When we apply repeatedly Λ_j, $j = 0,1,\ldots,p-1$ to both sides of (5.1) using the above computation rule and rewrite $\phi^k(T)(x)$ using relation (5.1), we always get an expression of the form (this will be illustrated by example 5.1 below)

$$\frac{c_1(x)}{d_1(x)} \phi^k(T)(x)^{p^1} + \frac{c_2(x)}{d_2(x)} \phi^k(T)(x)^{p^2} + \cdots + \frac{c_n(x)}{d_n(x)} \phi^k(T)(x)^{p^n}, \tag{5.3}$$

where $k = 0,1,\ldots,\log q/\log p - 1$, and $c_j(x)$ and $d_j(x)$ are polynomial of bounded degree for $j = 1,2,\ldots,n$. To see the last point, note that for $d_j(x)$ is always a factor of

$$\prod_{l=1}^{n} \phi^k(b_l(x))$$

for some $0 \le k < \log q / \log p$, and

$$\deg c_j \le \deg d_j + \max\{\deg a_l - \deg b_l \mid l = 1, 2, \ldots, n\}.$$

The set of expression of the form (5.3) is therefore finite and the process must terminate. In the end we get a finite set that is the p-kernel of $T(x)$.

In our program, the expression (5.3) is encoded by the tuple

$$(\{1 : c_1(x)/d_1(x), 2 : c_2(x)/d_2(x), \ldots, n : c_n(x)/d_n(x)\}, k).$$

Remark 5.1. Note that the reason we use powers of p instead of powers of q in (5.1) is that the latter usually needs much larger coefficients.

Example 5.1. Set $a = \bar{u} \in \mathbb{F}_4 = \mathbb{F}_2[u]/ < u^2 + u + 1 >$. Let $T(x)$ be the unique solution in $\mathbb{F}_4[[x]]$ of

$$(x^2 + ax)y^3 + y + a + x = 0.$$

We write the equation in the normalized form, which is really easy for this example:

$$T(x) = \frac{1}{x+a} T(x)^2 + xT(x)^4. \tag{5.4}$$

We use computation rule (5.2) to calculate $\Lambda_0 T(x)$ and $\Lambda_0\Lambda_0 T(x)$ to illustrate of this process:

$$\begin{aligned}\Lambda_0 T(x) &= \Lambda_0(\frac{1}{x+a} T(x)^2) + \Lambda_0(xT(x)^4) \\ &= \Lambda_0 \left((x+a) \frac{T(x)^2}{(x+a)^2} \right) + \Lambda_0(x) \cdot \phi(T)(x)^2 \\ &= a \frac{\phi(T)(x)}{x+a+1}.\end{aligned}$$

To calculate $\Lambda_0\Lambda_0 T(x)$, we need to first put the above expression into form (5.3). Applying ϕ to both sides of (5.4) we get

$$\phi(T)(x) = \frac{1}{x+a+1} \phi(T)(x)^2 + x\phi(T)(x)^4.$$

Therefore

$$\Lambda_0 T(x) = \frac{a}{(x+a+1)^2} \phi(T)(x)^2 + \frac{ax}{(x+a+1)} \phi(T)(x)^4,$$

and

$$\begin{aligned}\Lambda_0\Lambda_0 T(x) = \Lambda_0 &\left(\frac{a}{(x+a+1)^2} \phi(T)(x)^2 \right) + \Lambda_0 \left(\frac{ax(x+a+1)}{(x+a+1)^2} \phi(T)(x)^4, \right) \\ &= \frac{a}{x+a} \phi^2(T)(x) + \frac{a}{x+a} \phi^2(T)(x)^2 \\ &= \frac{a}{x+a} T(x) + \frac{ax}{x+a} T(x)^2.\end{aligned}$$

In our program, the series $T(x)$, $\Lambda_0 T(x)$ and $\Lambda_0\Lambda_0 T(x)$ are encoded by

$$(\{0 : 1\},\ 0),$$

$$(\{0 : a/(x + a + 1)\},\ 1),$$

and

$$(\{0 : a/(x + a),\ 1 : ax/(x + a)\},\ 0).$$

Step three: Output function In step two, each element from the p-kernel of $T(x)$ is expressed as an $\mathbb{F}_q(x)$-linear combination of powers of $\phi^k(T)(x)$, for some $0 \leq k < \log q/\log p$. The output function maps the corresponding state to the constant term of the series. To calculate it, we simply plug in $T(x) \mod x^{D+1}$, where D is the maximum of 0 and the minus of the orders of the coefficients of the linear combination.

6. Annex

6.1. **Data for Section 2.** All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 8 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{36} + x^{32} + x^{30} \\
&\quad + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{36} \\
&\quad + x^{32} + x^{28} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12}, \\
p_6(x) &= x^{56} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{16} \\
&\quad + x^{14} + x^2 + 1, \\
p_9(x) &= x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{42} + x^{32} + x^{30} + x^{26} \\
&\quad + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^8 + x^2, \\
p_{12}(x) &= x^{64} + x^{56} + x^{40} + x^{32} + x^{16} + x^8 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{63} + x^{62} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} \\
&\quad + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{15}, \\
p_3(x) &= x^{61} + x^{59} + x^{51} + x^{49} + x^{43} + x^{41} + x^{37} + x^{33} + x^{21} + x^{19} + x^{13} + x^9, \\
p_6(x) &= x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{43} + x^{42} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^6, \\
p_9(x) &= x^{59} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{37} + x^{29} + x^{27} \\
&\quad + x^{25} + x^{21} + x^{15} + x^{13} + x^7 + x^5 + x^3, \\
p_{12}(x) &= x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{10} + x^9 + x^8 + x^6 + x^5 \\
&\quad + x^4 + x^2 + x + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{63} + x^{62} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} \\
&\quad + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{15}, \\
p_3(x) &= x^{61} + x^{59} + x^{51} + x^{49} + x^{43} + x^{41} + x^{37} + x^{33} + x^{21} + x^{19} + x^{13} + x^9, \\
p_6(x) &= x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{43} + x^{42} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} \\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{59} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{37} + x^{29} + x^{27} \\
&+ x^{25} + x^{21} + x^{15} + x^{13} + x^{7} + x^{5} + x^{3}, \\
p_{12}(x) &= x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{42} + x^{41} \\
&+ x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} \\
&+ x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{38} + x^{34} + x^{30} + x^{28} + x^{26} \\
&+ x^{20} + x^{18}, \\
p_3(x) &= x^{54} + x^{50} + x^{44} + x^{38} + x^{28} + x^{20} + x^{18} + x^{12} + x^{10} + x^{6}, \\
p_6(x) &= x^{54} + x^{52} + x^{50} + x^{38} + x^{36} + x^{24} + x^{18} + x^{16} + x^{10} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{52} + x^{44} + x^{42} + x^{40} + x^{30} + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^{8} + x^{6} \\
&+ x^{4} + x^{2}, \\
p_{12}(x) &= x^{52} + x^{50} + x^{48} + x^{36} + x^{34} + x^{32} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{60} + x^{50} + x^{44} + x^{40} + x^{30} + x^{28} + x^{24} + x^{18} + x^{12}, \\
p_3(x) &= x^{54} + x^{44} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{14} + x^{12} + x^{10}, \\
p_6(x) &= x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{38} + x^{34} + x^{32} + x^{30} + x^{22} + x^{20} \\
&+ x^{18} + x^{14} + x^{12} + x^{6} + 1, \\
p_9(x) &= x^{52} + x^{44} + x^{42} + x^{40} + x^{30} + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^{8} + x^{6} \\
&+ x^{4} + x^{2}, \\
p_{12}(x) &= x^{52} + x^{50} + x^{48} + x^{36} + x^{34} + x^{32} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{63} + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{50} + x^{47} + x^{43} + x^{39} + x^{36} \\
&+ x^{35} + x^{28} + x^{25} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15}, \\
p_3(x) &= x^{61} + x^{59} + x^{51} + x^{49} + x^{43} + x^{41} + x^{37} + x^{33} + x^{21} + x^{19} + x^{13} + x^{9}, \\
p_6(x) &= x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} \\
&+ x^{43} + x^{42} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} \\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{59} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{37} + x^{29} + x^{27} \\
&+ x^{25} + x^{21} + x^{15} + x^{13} + x^{7} + x^{5} + x^{3}, \\
p_{12}(x) &= x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{42} + x^{41}
\end{aligned}
$$

$$+ x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{63} + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{50} + x^{47} + x^{43} + x^{39} + x^{36} \\
&\quad + x^{35} + x^{28} + x^{25} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15}, \\
p_3(x) &= x^{61} + x^{59} + x^{51} + x^{49} + x^{43} + x^{41} + x^{37} + x^{33} + x^{21} + x^{19} + x^{13} + x^{9}, \\
p_6(x) &= x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{43} + x^{42} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{59} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} + x^{41} + x^{37} + x^{29} + x^{27} \\
&\quad + x^{25} + x^{21} + x^{15} + x^{13} + x^{7} + x^{5} + x^{3}, \\
p_{12}(x) &= x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} \\
&\quad + x^{4} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{66} + x^{60} + x^{58} + x^{56} + x^{52} + x^{48} + x^{42} + x^{40} + x^{36} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{20} + x^{18}, \\
p_3(x) &= x^{58} + x^{50} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{32} + x^{30} + x^{24} + x^{20} \\
&\quad + x^{18} + x^{8} + x^{6}, \\
p_6(x) &= x^{62} + x^{56} + x^{52} + x^{46} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{42} + x^{32} + x^{30} + x^{26} \\
&\quad + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{2}, \\
p_{12}(x) &= x^{64} + x^{56} + x^{40} + x^{32} + x^{16} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{31} + x^{30} + x^{28} \\
&\quad + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{62} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{46} + x^{45} + x^{41} \\
&\quad + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{25} \\
&\quad + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8}, \\
p_6(x) &= x^{62} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{48} + x^{44} + x^{42} + x^{41} \\
&\quad + x^{37} + x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{18}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{17} + x^{14} + x^{13} + x^{12} + x^{10} + x^{8} + x^{7} + x^{2} + x + 1,\\
p_9(x) &= x^{62} + x^{61} + x^{60} + x^{54} + x^{53} + x^{52} + x^{46} + x^{45} + x^{44} + x^{38} + x^{37}\\
&\quad + x^{36} + x^{14} + x^{13} + x^{12} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45}\\
&\quad + x^{44} + x^{42} + x^{41} + x^{40} + x^{34} + x^{33} + x^{32} + x^{14} + x^{13} + x^{12} + x^{10}\\
&\quad + x^{9} + x^{8} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 1, 0, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{64} + x^{61} + x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46}\\
&\quad + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30}\\
&\quad + x^{29} + x^{28} + x^{26} + x^{23} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15},\\
p_3(x) &= x^{57} + x^{56} + x^{55} + x^{51} + x^{50} + x^{48} + x^{47} + x^{43} + x^{42} + x^{40} + x^{39}\\
&\quad + x^{35} + x^{34} + x^{33} + x^{17} + x^{16} + x^{15} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{30} + x^{28} + x^{24} + x^{20}\\
&\quad + x^{18} + x^{14} + x^{12} + x^{8} + x^{6},\\
p_9(x) &= x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{43} + x^{42} + x^{40} + x^{38} + x^{36}\\
&\quad + x^{35} + x^{11} + x^{10} + x^{8} + x^{6} + x^{4} + x^{3},\\
p_{12}(x) &= x^{60} + x^{56} + x^{48} + x^{44} + x^{40} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{54} + x^{51} + x^{46} + x^{45}\\
&\quad + x^{43} + x^{42} + x^{41} + x^{40} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{23}\\
&\quad + x^{22} + x^{21} + x^{20} + x^{18} + x^{15},\\
p_3(x) &= x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51}\\
&\quad + x^{47} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33}\\
&\quad + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10}\\
&\quad + x^{9},\\
p_6(x) &= x^{66} + x^{64} + x^{60} + x^{58} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{30}\\
&\quad + x^{28} + x^{26} + x^{10} + x^{8} + x^{6},\\
p_9(x) &= x^{67} + x^{66} + x^{64} + x^{63} + x^{55} + x^{54} + x^{52} + x^{50} + x^{48} + x^{47} + x^{39}\\
&\quad + x^{38} + x^{36} + x^{35} + x^{19} + x^{18} + x^{16} + x^{15} + x^{7} + x^{6} + x^{4} + x^{3},\\
p_{12}(x) &= x^{68} + x^{48} + x^{36} + x^{32} + x^{20} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 0, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{49} + x^{48} + x^{45}\\
&\quad + x^{44} + x^{43} + x^{42} + x^{35} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{20}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{18},\\
p_3(x) &= x^{62}+x^{60}+x^{56}+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{46}\\
&+x^{45}+x^{42}+x^{40}+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}+x^{31}+x^{30}\\
&+x^{28}+x^{27}+x^{25}+x^{23}+x^{19}+x^{17}+x^{16}+x^{13}+x^{12}+x^{10},\\
p_6(x) &= x^{62}+x^{60}+x^{57}+x^{56}+x^{55}+x^{49}+x^{46}+x^{45}+x^{44}+x^{43}+x^{41}\\
&+x^{38}+x^{35}+x^{34}+x^{30}+x^{28}+x^{25}+x^{23}+x^{20}+x^{19}+x^{18}+x^{15}\\
&+x^{13}+x^{12}+x^{10}+x^{5}+x^{4}+x^{2}+x+1,\\
p_9(x) &= x^{62}+x^{61}+x^{60}+x^{54}+x^{53}+x^{52}+x^{46}+x^{45}+x^{44}+x^{38}+x^{37}\\
&+x^{36}+x^{14}+x^{13}+x^{12}+x^{6}+x^{5}+x^{4},\\
p_{12}(x) &= x^{62}+x^{61}+x^{60}+x^{58}+x^{57}+x^{56}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}\\
&+x^{44}+x^{42}+x^{41}+x^{40}+x^{34}+x^{33}+x^{32}+x^{14}+x^{13}+x^{12}+x^{10}\\
&+x^{9}+x^{8}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{57}+x^{55}+x^{53}+x^{51}+x^{50}+x^{49}\\
&+x^{47}+x^{45}+x^{44}+x^{42}+x^{40}+x^{39}+x^{37}+x^{36}+x^{31}+x^{30}+x^{28}\\
&+x^{27}+x^{26}+x^{24}+x^{23}+x^{21}+x^{18}+x^{17}+x^{15}+x^{14}+x^{13}+x^{12},\\
p_3(x) &= x^{62}+x^{59}+x^{58}+x^{56}+x^{55}+x^{53}+x^{52}+x^{51}+x^{46}+x^{45}+x^{41}\\
&+x^{39}+x^{38}+x^{37}+x^{36}+x^{34}+x^{32}+x^{31}+x^{30}+x^{28}+x^{27}+x^{25}\\
&+x^{23}+x^{20}+x^{18}+x^{17}+x^{16}+x^{13}+x^{11}+x^{10}+x^{9}+x^{8},\\
p_6(x) &= x^{62}+x^{59}+x^{58}+x^{57}+x^{54}+x^{52}+x^{51}+x^{48}+x^{44}+x^{42}+x^{41}\\
&+x^{37}+x^{36}+x^{35}+x^{33}+x^{31}+x^{30}+x^{29}+x^{26}+x^{24}+x^{23}+x^{18}\\
&+x^{17}+x^{14}+x^{13}+x^{12}+x^{10}+x^{8}+x^{7}+x^{2}+x+1,\\
p_9(x) &= x^{62}+x^{61}+x^{60}+x^{54}+x^{53}+x^{52}+x^{46}+x^{45}+x^{44}+x^{38}+x^{37}\\
&+x^{36}+x^{14}+x^{13}+x^{12}+x^{6}+x^{5}+x^{4},\\
p_{12}(x) &= x^{62}+x^{61}+x^{60}+x^{58}+x^{57}+x^{56}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}\\
&+x^{44}+x^{42}+x^{41}+x^{40}+x^{34}+x^{33}+x^{32}+x^{14}+x^{13}+x^{12}+x^{10}\\
&+x^{9}+x^{8}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 1, 0, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{66}+x^{65}+x^{64}+x^{62}+x^{60}+x^{59}+x^{58}+x^{54}+x^{51}+x^{46}+x^{45}\\
&+x^{43}+x^{42}+x^{41}+x^{40}+x^{35}+x^{33}+x^{31}+x^{29}+x^{28}+x^{27}+x^{23}\\
&+x^{22}+x^{21}+x^{20}+x^{18}+x^{15},\\
p_3(x) &= x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}+x^{51}\\
&+x^{47}+x^{46}+x^{44}+x^{43}+x^{41}+x^{40}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} \\
&+ x^{9}, \\
p_6(x) &= x^{66} + x^{64} + x^{60} + x^{58} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{30} \\
&+ x^{28} + x^{26} + x^{10} + x^{8} + x^{6}, \\
p_9(x) &= x^{67} + x^{66} + x^{64} + x^{63} + x^{55} + x^{54} + x^{52} + x^{50} + x^{48} + x^{47} + x^{39} \\
&+ x^{38} + x^{36} + x^{35} + x^{19} + x^{18} + x^{16} + x^{15} + x^{7} + x^{6} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{68} + x^{48} + x^{36} + x^{32} + x^{20} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 0, 0, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{64} + x^{61} + x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} \\
&+ x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} \\
&+ x^{29} + x^{28} + x^{26} + x^{23} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}, \\
p_3(x) &= x^{57} + x^{56} + x^{55} + x^{51} + x^{50} + x^{48} + x^{47} + x^{43} + x^{42} + x^{40} + x^{39} \\
&+ x^{35} + x^{34} + x^{33} + x^{17} + x^{16} + x^{15} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{30} + x^{28} + x^{24} + x^{20} \\
&+ x^{18} + x^{14} + x^{12} + x^{8} + x^{6}, \\
p_9(x) &= x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{43} + x^{42} + x^{40} + x^{38} + x^{36} \\
&+ x^{35} + x^{11} + x^{10} + x^{8} + x^{6} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{60} + x^{56} + x^{48} + x^{44} + x^{40} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{49} + x^{48} + x^{45} \\
&+ x^{44} + x^{43} + x^{42} + x^{35} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{20} \\
&+ x^{18}, \\
p_3(x) &= x^{62} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} \\
&+ x^{45} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} \\
&+ x^{28} + x^{27} + x^{25} + x^{23} + x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{10}, \\
p_6(x) &= x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{49} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} \\
&+ x^{38} + x^{35} + x^{34} + x^{30} + x^{28} + x^{25} + x^{23} + x^{20} + x^{19} + x^{18} + x^{15} \\
&+ x^{13} + x^{12} + x^{10} + x^{5} + x^{4} + x^{2} + x + 1, \\
p_9(x) &= x^{62} + x^{61} + x^{60} + x^{54} + x^{53} + x^{52} + x^{46} + x^{45} + x^{44} + x^{38} + x^{37} \\
&+ x^{36} + x^{14} + x^{13} + x^{12} + x^{6} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} \\
&+ x^{44} + x^{42} + x^{41} + x^{40} + x^{34} + x^{33} + x^{32} + x^{14} + x^{13} + x^{12} + x^{10} \\
&+ x^{9} + x^{8} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	25	[14, 42]	50	[71, 44]	75	[99, 51]	100	[10, 39]
1	[3, 4]	26	[26, 26]	51	[78, 29]	76	[100, 97]	101	[28, 9]
2	[5, 6]	27	[43, 44]	52	[32, 14]	77	[16, 101]	102	[84, 36]
3	[7, 8]	28	[45, 46]	53	[44, 79]	78	[102, 101]	103	[79, 41]
4	[9, 10]	29	[47, 48]	54	[80, 81]	79	[103, 104]	104	[115, 40]
5	[11, 12]	30	[49, 50]	55	[82, 36]	80	[54, 95]	105	[81, 93]
6	[13, 14]	31	[51, 52]	56	[83, 84]	81	[64, 105]	106	[87, 81]
7	[15, 16]	32	[53, 54]	57	[30, 71]	82	[48, 60]	107	[116, 42]
8	[17, 18]	33	[55, 20]	58	[85, 79]	83	[61, 49]	108	[117, 35]
9	[19, 20]	34	[56, 51]	59	[86, 87]	84	[97, 23]	109	[118, 41]
10	[20, 21]	35	[57, 49]	60	[88, 37]	85	[23, 19]	110	[52, 56]
11	[22, 23]	36	[58, 59]	61	[73, 31]	86	[106, 64]	111	[95, 59]
12	[24, 19]	37	[60, 61]	62	[89, 75]	87	[59, 68]	112	[101, 104]
13	[25, 26]	38	[62, 56]	63	[40, 2]	88	[107, 105]	113	[69, 18]
14	[27, 4]	39	[63, 64]	64	[90, 26]	89	[108, 98]	114	[18, 27]
15	[12, 28]	40	[21, 16]	65	[1, 75]	90	[105, 54]	115	[119, 106]
16	[29, 30]	41	[65, 47]	66	[42, 12]	91	[6, 47]	116	[104, 98]
17	[31, 32]	42	[66, 60]	67	[91, 92]	92	[109, 27]	117	[120, 61]
18	[33, 9]	43	[67, 68]	68	[93, 87]	93	[68, 106]	118	[121, 21]
19	[34, 30]	44	[50, 69]	69	[77, 84]	94	[110, 2]	119	[37, 92]
20	[35, 29]	45	[70, 71]	70	[46, 6]	95	[111, 93]	120	[122, 28]
21	[36, 37]	46	[72, 73]	71	[94, 95]	96	[112, 39]	121	[92, 89]
22	[2, 38]	47	[74, 10]	72	[96, 48]	97	[8, 77]	122	[123, 69]
23	[39, 40]	48	[75, 73]	73	[4, 97]	98	[113, 31]	123	[38, 8]
24	[41, 35]	49	[76, 77]	74	[98, 46]	99	[114, 32]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	18	0	36	1	54	1	72	1	90	1	108	1
1	0	19	1	37	0	55	0	73	1	91	1	109	1
2	0	20	0	38	1	56	1	74	0	92	1	110	0
3	0	21	1	39	1	57	0	75	0	93	1	111	0
4	1	22	0	40	1	58	1	76	0	94	0	112	1
5	0	23	1	41	0	59	0	77	0	95	0	113	0
6	1	24	0	42	1	60	0	78	1	96	1	114	0
7	0	25	1	43	1	61	1	79	0	97	1	115	0
8	1	26	0	44	0	62	1	80	1	98	0	116	0
9	1	27	1	45	1	63	1	81	1	99	0	117	1

10	0	28	1	46	1	64	1	82	0	100	0	118	1
11	0	29	0	47	0	65	0	83	1	101	1	119	0
12	0	30	0	48	0	66	1	84	1	102	1	120	1
13	1	31	1	49	0	67	1	85	1	103	0	121	1
14	1	32	0	50	0	68	1	86	0	104	0	122	1
15	0	33	0	51	1	69	0	87	0	105	1	123	1
16	0	34	1	52	0	70	1	88	0	106	0		
17	1	35	0	53	0	71	0	89	1	107	0		

6.2. **Data for Section 3.** All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

For $(i,j) = (1,0)$ and all $T \in \{M^e, M^o, W^e, W^o\}$,

$$p_6(x) = p_9(x) = p_{12}(x) = 0.$$

The coefficients $p_j(x)$ and the 2 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= \left(a^{12}+a^{11}+a^4+a^3\right)x^{12} + \left(a^{10}+a^9+a^7+a^5+a^3+a^2\right)x^{10} \\
&\quad + \left(a^8+a^7+a^2+a\right)x^8 + \left(a^6+a^5+a^3+a+1\right)x^6, \\
p_3(x) &= \left(a^9+a\right)x^9 + \left(a^8+1\right)x^8 + \left(a^6+a^5+a^2+a\right)x^7 + \left(a^5+a\right)x^5, \\
&\quad + \left(a^4+1\right)x^4 + \left(a^2+a\right)x^3, \\
p_6(x) &= \left(a^9+a\right)x^9 + \left(a^7+a^6+a^5+a^4+a^3+a^2+a+1\right)x^8 \\
&\quad + \left(a^6+a^5+a^2+a\right)x^7 + \left(a^6+a^5+a^4+a^2+a+1\right)x^6 + \left(a^4+1\right)x^5 \\
&\quad + \left(a^4+a^3+a^2+a\right)x^4 + \left(a^2+a\right)x^3 + \left(a^2+a+1\right)x^2 + (a+1)\,x + 1, \\
p_9(x) &= \left(a^8+1\right)x^8 + \left(a^7+a^6+a^5+a^4+a^3+a^2+a+1\right)x^7 \\
&\quad + \left(a^6+a^4+a^2+1\right)x^6 + \left(a^5+a^4+a+1\right)x^5 + \left(a^4+1\right)x^4 \\
&\quad + \left(a^3+a^2+a+1\right)x^3 + \left(a^2+1\right)x^2 + (a+1)\,x, \\
p_{12}(x) &= \left(a^8+1\right)x^8 + 1,
\end{aligned}$$

and the initial terms are $[1, a]$.

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= (a^{24}+a^{22}+a^8+a^6)x^{24} + (a^{22}+a^{21}+a^6+a^5)x^{22} + (a^{20}+a^{19}+a^{18}+ \\
&\quad a^{17}+a^{15}+a^{14}+a^{13}+a^{11}+a^{10}+a^9+a^7+a^6+a^5+a^4)x^{20} + (a^{18} \\
&\quad + a^{17}+a^{16}+a^{15}+a^{14}+a^{13}+a^{12}+a^{11}+a^{10}+a^9+a^8+a^7+a^6+a^5 \\
&\quad + a^4+a^3)x^{18} + (a^{15}+a^{12}+a^{11}+a^{10}+a^8+a^7+a^6+a^3)x^{16} + (a^{12} \\
&\quad + a^{11}+a^{10}+a^9+a^6+a^5+a^4+a^3)x^{14} + (a^9+a^8+a^6+a^4+a^3)x^{12}, \\
p_3(x) &= (a^{14}+a^{12}+a^{10}+a^8+a^6+a^4+a^2+1)x^{14} + (a^{13}+a^{12}+a^9+a^8 \\
&\quad + a^5+a^4+a+1)x^{13} + (a^{10}+a^8+a^2+1)x^{10} + (a^9+a^8+a+1)x^9, \\
p_6(x) &= (a^{12}+a^8+a^4+1)x^{12} + (a^{10}+a^8+a^2+1)x^{10} + (a^8+1)x^8 \\
&\quad + (a^6+a^4+a^2+1)x^6,
\end{aligned}$$

$$\begin{aligned}p_9(x) &= (a^{10}+a^8+a^2+1)x^{10}+(a^9+a^8+a+1)x^9+(a^8+1)x^8+(a^7+a^6\\&\quad+a^5+a^4+a^3+a^2+a+1)x^7+(a^6+a^4+a^2+1)x^6+(a^5+a^4+a\\&\quad+1)x^5+(a^4+1)x^4+(a^3+a^2+a+1)x^3,\\p_{12}(x) &= (a^8+1)x^8+1,\end{aligned}$$

and the initial terms are $[0,a]$.

For $\phi(M^e,1,0)$:

$$\begin{aligned}p_0(x) &= 1,\\p_3(x) &= (a^2+1)x^2+1,\end{aligned}$$

and the initial terms are $[1,0]$.

For $\phi(M^e,1,1)$:

$$\begin{aligned}p_0(x) &= (a^{12}+a^{11}+a^4+a^3)x^{12}+(a^9+a^7+a^5+a^3)x^{10}+(a^6+a^5+a^4+a^3)\\&\quad x^8+a^3x^6,\\p_3(x) &= (a^9+a)x^9+(a^8+1)x^8+(a^7+a^4+a^3+1)x^7+(a^5+a)x^5+(a^4+1)x^4\\&\quad+(a^3+1)x^3,\\p_6(x) &= (a^9+a)x^9+(a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^8+(a^7+a^4+a^3\\&\quad+1)x^7+(a^5+a)x^6+(a^4+1)x^5+(a^4+a^3+a^2+a)x^4+(a^3+1)x^3\\&\quad+ax^2+(a+1)x+1,\\p_9(x) &= (a^8+1)x^8+(a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^7+(a^6+a^4+a^2\\&\quad+1)x^6+(a^5+a^4+a+1)x^5+(a^4+1)x^4+(a^3+a^2+a+1)x^3\\&\quad+(a^2+1)x^2+(a+1)x,\\p_{12}(x) &= (a^8+1)x^8+1,\end{aligned}$$

and the initial terms are $[0,a]$.

For $\phi(M^o,0,0)$:

$$\begin{aligned}p_0(x) &= (a^9+a^8+a+1)x^{12}+(a^8+a^7+a^5+a^3+a+1)x^{10}+(a^7+a^6+a+1)\\&\quad x^8+(a^6+a^5+a^3+a+1)x^6,\\p_3(x) &= (a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^8+(a^7+a^5+a^3+a)x^7+(a^6\\&\quad+a^2)x^6+(a^5+a^4+a^3+a^2)x^5+(a^4+a^3+a+1)x^4+(a^2+a)x^3,\\p_6(x) &= (a^6+a^4+a^2+1)x^7+(a^6+a^5+a^2+a)x^6+(a^3+a^2+a+1)x^5\\&\quad+(a^2+1)x^4+(a^3+1)x^3+ax^2+(a+1)x+1,\\p_9(x) &= (a^5+a^4+a+1)x^5+(a+1)x,\\p_{12}(x) &= (a^4+1)x^4+1,\end{aligned}$$

and the initial terms are $[1,a]$.

For $\phi(M^o,0,1)$:

$$\begin{aligned}p_0(x) &= (a^{21}+a^{19}+a^5+a^3)x^{24}+(a^{20}+a^4)x^{22}+(a^{19}+a^{16}+a^{12}+a^8+a^4\\&\quad+a^3)x^{20}+(a^{18}+a^{14}+a^{10}+a^6)x^{18}+(a^{15}+a^{14}+a^9+a^6+a^5+a^3)\\&\quad x^{16}+(a^{12}+a^{10}+a^6+a^4)x^{14}+(a^9+a^8+a^6+a^4+a^3)x^{12},\end{aligned}$$

$$
\begin{aligned}
p_3(x) = {} & (a^{16}+1)x^{16} + (a^{15}+a^{14}+a^{13}+a^{12}+a^{11}+a^{10}+a^9+a^8+a^7+a^6 \\
& + a^5+a^4+a^3+a^2+a+1)x^{15} + (a^{14}+a^{12}+a^{10}+a^8+a^6+a^4 \\
& + a^2+1)x^{14} + (a^{13}+a^{12}+a^9+a^8+a^5+a^4+a+1)x^{13} + (a^{12}+a^8 \\
& + a^4+1)x^{12} + (a^{11}+a^{10}+a^9+a^8+a^3+a^2+a+1)x^{11} + (a^{10}+a^8 \\
& + a^2+1)x^{10} + (a^9+a^8+a+1)x^9, \\
p_6(x) = {} & (a^{12}+a^8+a^4+1)x^{12} + (a^{10}+a^8+a^2+1)x^{10} + (a^8+1)x^8 + (a^6+a^4+ \\
& a^2+1)x^6, \\
p_9(x) = {} & (a^8+1)x^8 + (a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^7 + (a^4+1)x^4 + (a^3 \\
& + a^2+a+1)x^3, \\
p_{12}(x) = {} & (a^4+1)x^4+1,
\end{aligned}
$$

and the initial terms are $[0, a]$.

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= 1 \\
p_3(x) &= (a+1)x+1,
\end{aligned}
$$

and the initial terms are $[1, a+1]$.

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {} & (a^{12}+a^{11}+a^4+a^3)x^{12} + (a^9+a^7+a^5+a^3)x^{10} + (a^6+a^5+a^4+a^3) \\
& x^8 + a^3x^6, \\
p_3(x) = {} & (a^8+a^7+a^6+a^5+a^4+a^3+a^2+a)x^8 + (a^6+a^4+a^2+1)x^7 + (a^6+ \\
& a^2)x^6 + (a^5+a^4+a^3+a^2)x^5 + (a^3+a)x^4 + (a^3+1)x^3, \\
p_6(x) = {} & (a^7+a^5+a^3+a)x^7 + (a^5+a^4+a+1)x^6 + (a^3+a^2+a+1)x^5 \\
& + (a^2+1)x^4 + (a^2+a)x^3 + (a^2+a+1)x^2 + (a+1)x+1, \\
p_9(x) = {} & (a^5+a^4+a+1)x^5 + (a+1)x, \\
p_{12}(x) = {} & (a^4+1)x^4+1,
\end{aligned}
$$

and the initial terms are $[0, a]$.

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {} & (a^9+a^8+a+1)x^{12} + (a^8+a^7+a^5+a^3+a+1)x^{10} + (a^7+a^6+a+1) \\
& x^8 + (a^6+a^5+a^3+a+1)x^6, \\
p_3(x) = {} & (a^8+1)x^9 + (a^8+1)x^8 + (a^6+a^5+a^2+a)x^7 + (a^4+1)x^5 + (a^4+1)x^4 \\
& + (a^2+a)x^3, \\
p_6(x) = {} & (a^8+1)x^9 + (a^8+a^7+a^6+a^5+a^4+a^3+a^2+a)x^8 + (a^6+a^5+a^2 \\
& + a)x^7 + (a^6+a^5+a^4+a^2+a+1)x^6 + (a^5+a)x^5 + (a^3+a^2+a+1) \\
& x^4 + (a^2+a)x^3 + (a^2+a+1)x^2 + (a+1)x+1, \\
p_9(x) = {} & (a^8+1)x^8 + (a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^7 + (a^6+a^4+a^2 \\
& + 1)x^6 + (a^5+a^4+a+1)x^5 + (a^4+1)x^4 + (a^3+a^2+a+1)x^3 \\
& + (a^2+1)x^2 + (a+1)x,
\end{aligned}
$$

$p_{12}(x) = (a^8+1)x^8+1,$

and the initial terms are $[1,1]$.

For $\phi(W^e,0,1)$:

$$\begin{aligned}
p_0(x) &= (a^{18}+a^{16}+a^2+1)x^{24}+(a^{17}+a^{16}+a+1)x^{22}+(a^{16}+a^{15}+a^{14}+a^{13}\\
&\quad +a^{11}+a^{10}+a^9+a^7+a^6+a^5+a^3+a^2+a+1)x^{20}+(a^{15}+a^{14}+a^{13}\\
&\quad +a^{12}+a^{11}+a^{10}+a^9+a^8+a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^{18}\\
&\quad +(a^{13}+a^{10}+a^9+a^8+a^6+a^5+a^4+a)x^{16}+(a^{11}+a^{10}+a^9+a^8\\
&\quad +a^5+a^4+a^3+a^2)x^{14}+(a^9+a^8+a^6+a^4+a^3)x^{12},\\
p_3(x) &= (a^{14}+a^{12}+a^{10}+a^8+a^6+a^4+a^2+1)x^{14}+(a^{13}+a^{12}+a^9+a^8+a^5\\
&\quad +a^4+a+1)x^{13}+(a^{10}+a^8+a^2+1)x^{10}+(a^9+a^8+a+1)x^9,\\
p_6(x) &= (a^{12}+a^8+a^4+1)x^{12}+(a^{10}+a^8+a^2+1)x^{10}+(a^8+1)x^8+(a^6+a^4\\
&\quad +a^2+1)x^6,\\
p_9(x) &= (a^{10}+a^8+a^2+1)x^{10}+(a^9+a^8+a+1)x^9+(a^8+1)x^8+(a^7+a^6\\
&\quad +a^5+a^4+a^3+a^2+a+1)x^7+(a^6+a^4+a^2+1)x^6+(a^5+a^4+a\\
&\quad +1)x^5+(a^4+1)x^4+(a^3+a^2+a+1)x^3,\\
p_{12}(x) &= (a^8+1)x^8+1,
\end{aligned}$$

and the initial terms are $[0,1]$.

For $\phi(W^e,1,0)$:

$$\begin{aligned}
p_0(x) &= 1\\
p_3(x) &= (a^2+1)x^2+1,
\end{aligned}$$

and the initial terms are $[1,0]$.

For $\phi(W^e,1,1)$:

$$\begin{aligned}
p_0(x) &= (a^9+a^8+a+1)x^{12}+(a^7+a^5+a^3+a)x^{10}+(a^5+a^4+a^3+a^2)x^8\\
&\quad +a^3x^6,\\
p_3(x) &= (a^8+1)x^9+(a^8+1)x^8+(a^7+a^4+a^3+1)x^7+(a^4+1)x^5+(a^4+1)x^4\\
&\quad +(a^3+1)x^3,\\
p_6(x) &= (a^8+1)x^9+(a^8+a^7+a^6+a^5+a^4+a^3+a^2+a)x^8+(a^7+a^4+a^3\\
&\quad +1)x^7+(a^5+a)x^6+(a^5+a)x^5+(a^3+a^2+a+1)x^4+(a^3+1)x^3\\
&\quad +ax^2+(a+1)x+1,\\
p_9(x) &= (a^8+1)x^8+(a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^7+(a^6+a^4+a^2\\
&\quad +1)x^6+(a^5+a^4+a+1)x^5+(a^4+1)x^4+(a^3+a^2+a+1)x^3\\
&\quad +(a^2+1)x^2+(a+1)x,\\
p_{12}(x) &= (a^8+1)x^8+1,
\end{aligned}$$

and the initial terms are $[0,1]$.

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= (a^{12}+a^{11}+a^4+a^3)x^{12}+(a^{10}+a^9+a^7+a^5+a^3+a^2)x^{10}+(a^8\\
&\quad +a^7+a^2+a)x^8+(a^6+a^5+a^3+a+1)x^6,\\
p_3(x) &= (a^8+a^7+a^6+a^5+a^4+a^3+a^2+a)x^8+(a^6+a^4+a^2+1)x^7\\
&\quad +(a^4+1)x^6+(a^3+a^2+a+1)x^5+(a^4+a^3+a+1)x^4+(a^2+a)x^3,\\
p_6(x) &= (a^7+a^5+a^3+a)x^7+(a^5+a^4+a+1)x^6+(a^5+a^4+a^3+a^2)x^5\\
&\quad +(a^4+a^2)x^4+(a^3+1)x^3+ax^2+(a+1)x+1,\\
p_9(x) &= (a^5+a^4+a+1)x^5+(a+1)x,\\
p_{12}(x) &= (a^4+1)x^4+1,
\end{aligned}$$

and the initial terms are $[1,1]$.

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= (a^{21}+a^{19}+a^5+a^3)x^{24}+(a^{18}+a^2)x^{22}+(a^{17}+a^{16}+a^{12}+a^8+a^4\\
&\quad +a)x^{20}+(a^{12}+a^8+a^4+1)x^{18}+(a^{13}+a^{11}+a^{10}+a^7+a^2+a)x^{16}\\
&\quad +(a^{10}+a^8+a^4+a^2)x^{14}+(a^9+a^8+a^6+a^4+a^3)x^{12},\\
p_3(x) &= (a^{16}+1)x^{16}+(a^{15}+a^{14}+a^{13}+a^{12}+a^{11}+a^{10}+a^9+a^8+a^7+a^6\\
&\quad +a^5+a^4+a^3+a^2+a+1)x^{15}+(a^{14}+a^{12}+a^{10}+a^8+a^6+a^4+a^2\\
&\quad +1)x^{14}+(a^{13}+a^{12}+a^9+a^8+a^5+a^4+a+1)x^{13}+(a^{12}+a^8+a^4\\
&\quad +1)x^{12}+(a^{11}+a^{10}+a^9+a^8+a^3+a^2+a+1)x^{11}+(a^{10}+a^8+a^2\\
&\quad +1)x^{10}+(a^9+a^8+a+1)x^9,\\
p_6(x) &= (a^{12}+a^8+a^4+1)x^{12}+(a^{10}+a^8+a^2+1)x^{10}+(a^8+1)x^8+(a^6+a^4\\
&\quad +a^2+1)x^6,\\
p_9(x) &= (a^8+1)x^8+(a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^7+(a^4+1)x^4+(a^3\\
&\quad +a^2+a+1)x^3,\\
p_{12}(x) &= (a^4+1)x^4+1,
\end{aligned}$$

and the initial terms are $[0,1]$.

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= 1,\\
p_3(x) &= (a+1)x+1,
\end{aligned}$$

and the initial terms are $[1, a+1]$.

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= (a^9+a^8+a+1)x^{12}+(a^7+a^5+a^3+a)x^{10}+(a^5+a^4+a^3+a^2)x^8\\
&\quad +a^3x^6,\\
p_3(x) &= (a^7+a^6+a^5+a^4+a^3+a^2+a+1)x^8+(a^7+a^5+a^3+a)x^7+(a^4\\
&\quad +1)x^6+(a^3+a^2+a+1)x^5+(a^3+a)x^4+(a^3+1)x^3,\\
p_6(x) &= (a^6+a^4+a^2+1)x^7+(a^6+a^5+a^2+a)x^6+(a^5+a^4+a^3+a^2)x^5\\
&\quad +(a^4+a^2)x^4+(a^2+a)x^3+(a^2+a+1)x^2+(a+1)x+1,
\end{aligned}$$

$$p_9(x) = (a^5 + a^4 + a + 1)x^5 + (a + 1)x,$$
$$p_{12}(x) = (a^4 + 1)x^4 + 1,$$

and the initial terms are $[0, 1]$.

6.3. **Data for subsection 4.1.** All 16 polynomials are of the form

$$p_0(x) + p_1(x)y + p_2(x)y^2 + p_4(x)y^4.$$

The coefficients $p_j(x)$ and the two initial terms to determine the solutions uniquely are given below.

For $\phi(A^e, 0, 0)$:

$$p_0(x) = x^8 + x^6 + x^5 + x^2 + 1, \quad p_1(x) = x^2 + x, \quad p_2(x) = x, \quad p_4(x) = 1.$$

and the initial terms are [1, 0].

For $\phi(A^e, 0, 1)$:

$$p_0(x) = x^{13} + x^9 + x^4 + x^3 + x^2, \quad p_1(x) = x^3 + x^2 + x + 1, \quad p_2(x) = 0, \quad p_4(x) = x^5.$$

and the initial terms are [0, 0].

For $\phi(A^e, 1, 0)$:

$$p_0(x) = x^2, \quad p_1(x) = 1.$$

and the initial terms are [0, 0].

For $\phi(A^e, 1, 1)$:

$$p_0(x) = x^8 + x^6 + x^5, \quad p_1(x) = x^2 + x, \quad p_2(x) = x, \quad p_4(x) = 1.$$

and the initial terms are [0, 0].

For $\phi(A^o, 0, 0)$:

$$p_0(x) = x^6 + x^4 + x^3 + x^2 + 1, \quad p_1(x) = x + 1, \quad p_2(x) = x, \quad p_4(x) = x^2.$$

and the initial terms are [1, 0].

For $\phi(A^o, 0, 1)$:

$$p_0(x) = x^{11} + x^5 + x^4 + x^3 + x^2, \; p_1(x) = x^3 + x^2 + x + 1, \; p_2(x) = 0, \; p_4(x) = x^7.$$

and the initial terms are [0, 0].

For $\phi(A^o, 1, 0)$:

$$p_0(x) = x, \quad p_1(x) = 1.$$

and the initial terms are [0, 1]. For $\phi(A^o, 1, 1)$:

$$p_0(x) = x^6 + x^4 + x^3, \quad p_1(x) = x + 1, \quad p_2(x) = x, \quad p_4(x) = x^2.$$

and the initial terms are [0, 0].

For all $0 \le i, j \le 1$, $\phi(B^e, i, j) = \phi(A^o, i, j)$, $\phi(B^o, i, j) = \phi(Ae, i, j)$. The corresponding initial conditions are also the same. We happen to have $Ae = Bo$ and $Ao = Be$ here.

6.4. **Data for subsection 4.2.** All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

For $(i, j) = (1, 0)$ and all $T \in \{A^e, A^o, B^e, B^o\}$,

$$p_6(x) = p_9(x) = p_{12}(x) = 0.$$

The coefficients $p_j(x)$ and the initial terms to determine the solutions uniquely are given below.

For $\phi(A^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{24} + x^{22} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^5 + x^3 + x + 1, \\
p_3(x) &= x^{15} + x^{14} + x^{12} + x^9 + x^6 + x^2, \\
p_6(x) &= x^{11} + x^9 + x^8 + x^7 + x^5 + x^3 + x^2 + x, \\
p_9(x) &= x^6 + x^4 + x^3 + x, \\
p_{12}(x) &= x^2 + x + 1.
\end{aligned}$$

and the initial terms are $[1, 1]$.

For $\phi(A^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{42} + x^{41} + x^{39} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} \\
&\quad + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9, \\
p_3(x) &= x^{29} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{13} + x^{11} + x^{10} \\
&\quad + x^8 + x^5 + x^3 + x^2 + 1, \\
p_6(x) &= x^{28} + x^{27} + x^{25} + x^{24} + x^{12} + x^{11} + x^9 + x^8, \\
p_9(x) &= x^{27} + x^{24} + x^{19} + x^{16} \\
p_{12}(x) &= x^{26} + x^{25} + x^{24}.
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1]$.

For $\phi(A^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^9, \\
p_3(x) &= x^2 + x + 1.
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1]$.

For $\phi(A^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{30} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{21}, \\
p_3(x) &= x^{21} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^9 + x^8 + x^7 + x^4 + x^3, \\
p_6(x) &= x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^7 + x^5 + x^3 + x^2, \\
p_9(x) &= x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x, \\
p_{12}(x) &= x^8 + x^4 + 1.
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 1]$.

For $\phi(A^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{14} + x^{13} + x^7 \\
&\quad + x^6 + x^5 + x^4 + 1, \\
p_3(x) &= x^{22} + x^{21} + x^{17} + x^{16} + x^{15} + x^{11} + x^8 + x^7 + x^6 + x^2 + x + 1,
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{23} + x^{22} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12} + x^9 + x^7 + x^6 + x^5 + x^2,\\
p_9(x) &= x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^4,\\
p_{12}(x) &= x^{22} + x^{14} + x^6.
\end{aligned}$$

and the initial terms are $[1, 1]$.
For $\phi(A^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{48} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33}\\
&\quad + x^{32} + x^{27} + x^{25} + x^{24} + x^{14} + x^{12} + x^{11} + x^9,\\
p_3(x) &= x^{43} + x^{42} + x^{41} + x^{40} + x^{27} + x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{17}\\
&\quad + x^{16} + x^3 + x^2 + x + 1,\\
p_6(x) &= x^{44} + x^{42} + x^{36} + x^{34} + x^{20} + x^{18} + x^{12} + x^{10},\\
p_9(x) &= x^{45} + x^{44} + x^{21} + x^{20},\\
p_{12}(x) &= x^{46} + x^{38} + x^{30}.
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1]$.
For $\phi(A^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^6,\\
p_3(x) &= x^4 + x^2 + 1.
\end{aligned}$$

and the initial terms are $[0, 0, 1]$.
For $\phi(A^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{15},\\
p_3(x) &= x^{20} + x^{19} + x^{17} + x^{14} + x^{13} + x^{11} + x^{10} + x^9 + x^6 + x^4 + x^3 + 1,\\
p_6(x) &= x^{21} + x^{20} + x^{17} + x^{13} + x^{12} + x^{11} + x^{10} + x^7 + x^6 + x^4 + x^3 + x^2,\\
p_9(x) &= x^{21} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^4,\\
p_{12}(x) &= x^{22} + x^{14} + x^6.
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 1]$.
For $\phi(B^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{14} + x^{13} + x^7\\
&\quad + x^6 + x^5 + x^4 + 1,\\
p_3(x) &= x^{20} + x^{18} + x^{17} + x^{14} + x^{12} + x^{11} + x^6 + 1,\\
p_6(x) &= x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{10} + x^6 + x^5 + x^4 + x^2,\\
p_9(x) &= x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4,\\
p_{12}(x) &= x^{14} + x^{10} + x^6.
\end{aligned}$$

and the initial terms are $[1, 0]$.
For $\phi(B^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{48} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33}\\
&\quad + x^{32} + x^{27} + x^{25} + x^{24} + x^{14} + x^{12} + x^{11} + x^9,\\
p_3(x) &= x^{41} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{23}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} + 1,\\
p_6(x) &= x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{10},\\
p_9(x) &= x^{39} + x^{36} + x^{35} + x^{32} + x^{27} + x^{24} + x^{23} + x^{20},\\
p_{12}(x) &= x^{38} + x^{34} + x^{30}.
\end{aligned}
$$

and the initial terms are $[0, 0, 0, 1]$.

For $\phi(B^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^6,\\
p_3(x) &= x^2 + x + 1.
\end{aligned}
$$

and the initial terms are $[0, 0, 1, 1]$.

For $\phi(B^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{15},\\
p_3(x) &= x^{18} + x^{16} + x^{14} + x^{13} + x^{12} + x^{11} + x^{8} + x^{6} + x^{5} + x^{4} + x + 1,\\
p_6(x) &= x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^{7} + x^{5} + x^{3} + x^{2},\\
p_9(x) &= x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{14} + x^{10} + x^{6}.
\end{aligned}
$$

and the initial terms are $[0, 0, 0, 0, 0, 1]$.

For $\phi(B^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{24} + x^{22} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{5} + x^{3} + x + 1,\\
p_3(x) &= x^{17} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x^{2},\\
p_6(x) &= x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{2} + x,\\
p_9(x) &= x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x,\\
p_{12}(x) &= x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{2} + x + 1.
\end{aligned}
$$

and the initial terms are $[1, 0]$.

For $\phi(B^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{42} + x^{41} + x^{39} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24}\\
&\quad + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9},\\
p_3(x) &= x^{31} + x^{30} + x^{25} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{15} + x^{14} + x^{9}\\
&\quad + x^{8} + x^{7} + x^{6} + x + 1,\\
p_6(x) &= x^{32} + x^{31} + x^{30} + x^{26} + x^{25} + x^{24} + x^{16} + x^{15} + x^{14} + x^{10} + x^{9} + x^{8},\\
p_9(x) &= x^{33} + x^{32} + x^{29} + x^{28} + x^{21} + x^{20} + x^{17} + x^{16},\\
p_{12}(x) &= x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24}.
\end{aligned}
$$

and the initial terms are $[0, 0, 0, 1]$.

For $\phi(B^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^9,\\
p_3(x) &= x^4 + x^2 + 1.
\end{aligned}
$$

and the initial terms are $[0, 0, 0, 1]$.

For $\phi(B^o, 1, 1)$:

$$p_0(x) = x^{30} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{21},$$
$$p_3(x) = x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^9 + x^7 + x^6 + x^3,$$
$$p_6(x) = x^{21} + x^{20} + x^{17} + x^{13} + x^{12} + x^{11} + x^{10} + x^7 + x^6 + x^4 + x^3 + x^2,$$
$$p_9(x) = x^{18} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^5 + x^4 + x,$$
$$p_{12}(x) = x^{16} + x^8 + 1.$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1]$.

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