

PROOF OF THE CONTINUED FRACTION CONJECTURE FOR

$$(a, b) = (z, z^3 + z^2 + 1)$$

GUO-NIU HAN* AND YINING HU*

ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b) -Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z, z^3 + z^2 + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$(1.1) \quad \text{CF}(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\ddots}}}}$$

as the limit of the finite continued fractions

$$(1.2) \quad \text{CF}_n(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{\ddots + \frac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)).$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$(1.3) \quad c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \dots$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b) -Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$(1.4) \quad \text{CF}(a, b) := \text{CF}(\mathbf{t}).$$

Date: 9/04/2020 00:01:02 .

2010 Mathematics Subject Classification. 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 5.5 minutes CPU time used.

Theorem 1.1. *When $(a, b) = (z, z^3 + z^2 + 1)$, the two power series $\text{CF}(a, b)$ and $\text{CF}(b, a)$ are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\text{CF}(a, b)$,

$$\begin{aligned} p_0(z) &= z^{11} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1, \\ p_1(z) &= z^{15} + z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^2 + z, \\ p_2(z) &= z^{16} + z^{14} + z^{12} + z^8 + z^2, \\ p_3(z) &= z^{15} + z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^2 + z, \\ p_4(z) &= z^{14} + z^{12} + z^{11} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z, \end{aligned}$$

and for $\text{CF}(b, a)$,

$$\begin{aligned} p_0(z) &= z^{12} + z^{11} + z^9 + z^7 + z^6 + z^5 + z^3 + z^2 + z, \\ p_1(z) &= z^{15} + z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^2 + z, \\ p_2(z) &= z^{16} + z^{14} + z^{12} + z^8 + z^2, \\ p_3(z) &= z^{15} + z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^2 + z, \\ p_4(z) &= z^{14} + z^{11} + z^9 + z^7 + z^6 + z^5 + z^3 + z + 1. \end{aligned}$$

2. PROOF

Define the sequences of polynomials $(P_n(z))_{n \geq 0}$ and $(Q_n(z))_{n \geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$(2.1) \quad \text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)).$$

As formula (2.1) is not efficient for calculating $\text{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\ W_n(x) &= x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $\bar{t}$ is the (b, a) -Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n . Hence,

$$(2.2) \quad \text{CF}_{2^{2^n}-1}(a, b) = \frac{\tilde{P}_{2^{2^n}-1}(x)}{\tilde{Q}_{2^{2^n}-1}(x)} = \frac{M_{2^n}(x)_{0,1}}{M_{2^n}(x)_{0,0}}.$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\text{CF}(a, b)$ will be established if it is shown that both $M_{2^n}(x)_{0,1}$ and $M_{2^n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i, j \leq 1$ the four sequences $(M_{2^n}(x)_{i,j})_n$, $(M_{2^{n+1}}(x)_{i,j})_n$, $(W_{2^n}(x)_{i,j})_n$, and $(W_{2^{n+1}}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i, j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions; the polynomials $\phi(T, i, j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2^n}(x))_n$, $(M_{2^{n+1}}(x))_n$, $(W_{2^n}(x))_n$, and $(W_{2^{n+1}}(x))_n$.

Let us explain how the polynomials $\phi(T, i, j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i, j \leq 1$, the coefficients of the polynomial $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$) are the Padé-Hermite approximants of type

$$(100, 100, 100, 100, 100)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x, y) \in \mathbb{F}_2[x, y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o . The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$(2.3) \quad M^e = W^o \cdot M^o,$$

$$(2.4) \quad M^o = W^e \cdot M^e,$$

$$(2.5) \quad W^e = M^o \cdot W^o,$$

$$(2.6) \quad W^o = M^e \cdot W^e.$$

Proof. We give the proof of the identity

$$M_{0,0}^e = W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o,$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W_{0,0}^o M_{0,0}^o$ and $W_{0,1}^o M_{1,0}^o$. We know that

$$P(x, y) = \text{Res}_z (\phi(W^o, 0, 0)(x, z), z^{12} \cdot \phi(M^o, 0, 0)(x, y/z))$$

is an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W_{0,0}^o M_{0,0}^o$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We verify the first point directly. For the second point, we truncate $W_{0,0}^o M_{0,0}^o$ to order 360 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 360, which proves that $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We find the minimal polynomial $\phi_1(x, y)$ of $W_{0,1}^o M_{1,0}^o$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We know that

$$S(x, y) = \text{Res}_z (\phi_0(x, z), \phi_1(x, y + z))$$

is an annihilating of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ , and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. To see the last point, we truncate $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ to order 440 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 440, and therefore $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$.

Finally, the first 8 terms of $M_{0,0}^e$ and $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:8u] &= M^e[:4u] + x^u M^e[:3u] + x^{2u} M^e[:2u] + x^u M^e[:4u] + x^{2u} M^e[:3u] \\ &\quad + x^{3u} M^e[:2u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ &\quad + (x^{4u} + x^{5u} + x^{7u}) R^e, \\ W^e[:8u] &= W^e[:4u] + x^u W^e[:3u] + x^{2u} W^e[:2u] + x^u W^e[:4u] + x^{2u} W^e[:3u] \\ &\quad + x^{3u} W^e[:2u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] \\ &\quad + (x^{4u} + x^{5u} + x^{7u}) R^e. \end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:8u] &= M^o[:4u] + x^u M^o[:3u] + x^{2u} M^o[:2u] + x^u M^o[:4u] + x^{2u} M^o[:3u] \\
&\quad + x^{3u} M^o[:2u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] \\
&\quad + (x^{4u} + x^{5u} + x^{7u}) R^o, \\
W^o[:8u] &= W^o[:4u] + x^u W^o[:3u] + x^{2u} W^o[:2u] + x^u W^o[:4u] + x^{2u} W^o[:3u] \\
&\quad + x^{3u} W^o[:2u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] \\
&\quad + (x^{4u} + x^{5u} + x^{7u}) R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k 's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M_{0,0}^o$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:8u] &= T[:4u] + x^u T[:3u] + x^{2u} T[:2u] + x^u T[:4u] + x^{2u} T[:3u] + x^{3u} T[:2u] \\
&\quad + x^{3u} T[:4u] + x^{4u} T[:3u] + x^{5u} T[:2u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 4 parts:

$$\begin{aligned}
0 &= x^{4u} T[:u] + x^{2u} T[2u:3u] + x^u T[3u:4u] + T[4u:5u], \\
0 &= x^{5u} T[:u] + x^{4u} T[u:2u] + x^{3u} T[2u:3u] + T[5u:6u], \\
0 &= x^{5u} T[u:2u] + x^{4u} T[2u:3u] + x^{3u} T[3u:4u] + T[6u:7u], \\
0 &= T[7u:8u],
\end{aligned}$$

which can be rewritten as

$$(2.7) \quad 0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2],$$

$$(2.8) \quad 0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2],$$

$$(2.9) \quad 0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2],$$

$$(2.10) \quad 0 = T[[111w]_2],$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$(2.11) \quad 0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2],$$

$$(2.12) \quad 0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2],$$

$$(2.13) \quad 0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2],$$

$$(2.14) \quad 0 = T[[111w]_2],$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 48 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s , and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.14) can be written as

$$(2.15) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)),$$

$$(2.16) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 101)),$$

$$(2.17) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)),$$

$$(2.18) \quad 0 = \tau(A(s, 111)),$$

for all $s \in E_{2k}$, and

$$(2.19) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)),$$

$$(2.20) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 101)),$$

$$(2.21) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)),$$

$$(2.22) \quad 0 = \tau(A(s, 111)),$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{10}), E_{10}) = (A(i, 0^8), E_8)$, so that we only have to verify that identities (2.15) through (2.22) hold for $2 \leq k \leq 5$, which turns out to be true. $\square$

In the following lemma, we express M_{2k} , M_{2k+1} , W_{2k} , and W_{2k+1} in terms of M^e , M^o , W^e , and W^o .

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:4u] + x^u M^e[:3u] + x^{2u} M^e[:2u] + x^{4u} R^e,$$

$$W_{2k} = W^e[:4u] + x^u W^e[:3u] + x^{2u} W^e[:2u] + x^{4u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:4u] + x^u M^o[:3u] + x^{2u} M^o[:2u] + x^{4u} R^o,$$

$$W_{2k+1} = W^o[:4u] + x^u W^o[:3u] + x^{2u} W^o[:2u] + x^{4u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k} , W_{2k} , M_{2k+1} , and W_{2k+1} . For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$

$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$(2.23) \quad \begin{aligned} W_{2k} M_{2k} = & (W^e[:4v] + x^v W^e[:3v] + x^{2v} W^e[:2v] + x^{4v} R^e) \times (M^e[:4v] \\ & + x^v M^e[:3v] + x^{2v} M^e[:2v] + x^{4v} R^e); \end{aligned}$$

Call this expression *lhs*. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{8v} R^o$. Therefore we only need to prove that their difference is $O(x^{8v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{8v} , to

$$W^e[:8v] M^e[:8v] + x^{2v} W^e[:6v] M^e[:6v] + x^{4v} W^e[:4v] M^e[:4v].$$

For all $n \leq 8$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$X := x^v,$$

$$\begin{aligned}
a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\
b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\
c &:= R^e.
\end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.23) and *rhs* as polynomials in $\mathbb{F}_2[a_1, \dots, a_8, b_1, \dots, b_8, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e , because in the expressions concerned, the products of W^e -terms and M^e -terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^8)$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\text{CF}(a, b)$; for $\text{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned}
\lim_{n \rightarrow \infty} M_{2n,j,k} &= M_{j,k}^e, \\
\lim_{n \rightarrow \infty} M_{2n+1,j,k} &= M_{j,k}^o, \\
\lim_{n \rightarrow \infty} W_{2n,j,k} &= W_{j,k}^e, \\
\lim_{n \rightarrow \infty} W_{2n+1,j,k} &= W_{j,k}^o.
\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\text{CF}(a, b) = \frac{M_{0,1}^e(x)}{M_{0,0}^e(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M_{0,1}^e$ and $M_{0,0}^e$. Therefore

$$P(x, y) = \text{Res}_t (\phi(M^e, 0, 1)(x, t), y^{12} \phi(M^e, 0, 0)(x, t/y))$$

is an annihilating polynomial of $f(x) = M_{0,1}^e/M_{0,0}^e$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^5, \\
q_1(x) &= x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x, \\
q_2(x) &= x^{14} + x^8 + x^4 + x^2 + 1, \\
q_3(x) &= x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x, \\
q_4(x) &= x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^5 + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 126, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 126, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\text{CF}(a, b) = f(1/z)$. $\square$

3. ANNEXE

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 8 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned} p_0(x) &= x^{90} + x^{86} + x^{84} + x^{80} + x^{78} + x^{70} + x^{66} + x^{56} + x^{54} + x^{52} + x^{50} \\ &\quad + x^{46} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12}, \\ p_3(x) &= x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} \\ &\quad + x^{50} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} + x^{26} + x^{18} + x^{16} + x^{14} + x^{12} \\ &\quad + x^{10} + x^8, \\ p_6(x) &= x^{86} + x^{80} + x^{74} + x^{72} + x^{68} + x^{66} + x^{62} + x^{52} + x^{50} + x^{42} + x^{40} \\ &\quad + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1, \\ p_9(x) &= x^{88} + x^{86} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} \\ &\quad + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{44} + x^{42} + x^{32} + x^{30} \\ &\quad + x^{20} + x^{18} + x^{16} + x^{14} + x^4 + x^2, \\ p_{12}(x) &= x^{88} + x^{72} + x^{64} + x^{56} + x^{40} + x^{32} + x^{24} + x^8 + 1, \end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 0]$.

For $\phi(M^e, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{84} + x^{83} + x^{79} + x^{78} + x^{77} + x^{75} + x^{67} + x^{66} + x^{62} + x^{60} + x^{57} \\ &\quad + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} \\ &\quad + x^{33} + x^{32} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21}, \\ p_3(x) &= x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{68} + x^{67} + x^{64} + x^{63} + x^{58} \\ &\quad + x^{57} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{20} + x^{19} + x^{16} + x^{15} \\ &\quad + x^{10} + x^9, \\ p_6(x) &= x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} \\ &\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} \\ &\quad + x^{20} + x^{19} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6, \\ p_9(x) &= x^{80} + x^{79} + x^{76} + x^{75} + x^{64} + x^{63} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} \\ &\quad + x^{47} + x^{40} + x^{39} + x^{28} + x^{27} + x^{24} + x^{23} + x^4 + x^3, \\ p_{12}(x) &= x^{79} + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} \\ &\quad + x^{44} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^7 \\ &\quad + x^5 + x^4 + x^3 + x + 1, \end{aligned}$$

and the initial terms are $[0, 1, 0, 0, 0, 0, 1, 0]$.

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{84} + x^{83} + x^{79} + x^{78} + x^{77} + x^{75} + x^{67} + x^{66} + x^{62} + x^{60} + x^{57}$$

$$\begin{aligned}
& + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} \\
& + x^{33} + x^{32} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21}, \\
p_3(x) &= x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{68} + x^{67} + x^{64} + x^{63} + x^{58} \\
& + x^{57} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{20} + x^{19} + x^{16} + x^{15} \\
& + x^{10} + x^9, \\
p_6(x) &= x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} \\
& + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} \\
& + x^{20} + x^{19} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6, \\
p_9(x) &= x^{80} + x^{79} + x^{76} + x^{75} + x^{64} + x^{63} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} \\
& + x^{47} + x^{40} + x^{39} + x^{28} + x^{27} + x^{24} + x^{23} + x^4 + x^3, \\
p_{12}(x) &= x^{79} + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} \\
& + x^{44} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^7 \\
& + x^5 + x^4 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 1, 0, 0, 0, 0, 1, 0]$.

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{62} + x^{58} + x^{52} + x^{48} + x^{46} + x^{40} \\
& + x^{36} + x^{34} + x^{32} + x^{30}, \\
p_3(x) &= x^{72} + x^{70} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{40} \\
& + x^{34} + x^{30} + x^{28} + x^{22} + x^{18} + x^{16} + x^6, \\
p_6(x) &= x^{72} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{40} \\
& + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{18} + x^{14} + x^{10} + x^6 + x^4 + 1, \\
p_9(x) &= x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{48} + x^{44} + x^{38} + x^{34} + x^{32} + x^{28} \\
& + x^{20} + x^{16} + x^8 + x^6 + x^2, \\
p_{12}(x) &= x^{70} + x^{66} + x^{64} + x^{38} + x^{34} + x^{32} + x^6 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{78} + x^{76} + x^{68} + x^{64} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{42} + x^{40} \\
& + x^{38} + x^{32} + x^{28} + x^{24} + x^{14} + x^{12}, \\
p_3(x) &= x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{50} + x^{46} + x^{44} + x^{40} \\
& + x^{34} + x^{30} + x^{28} + x^{22} + x^{20} + x^{16} + x^{12} + x^{10} + x^8, \\
p_6(x) &= x^{72} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{34} + x^{30} \\
& + x^{24} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^8 + x^6 + 1, \\
p_9(x) &= x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{48} + x^{44} + x^{38} + x^{34} + x^{32} + x^{28} \\
& + x^{20} + x^{16} + x^8 + x^6 + x^2, \\
p_{12}(x) &= x^{70} + x^{66} + x^{64} + x^{38} + x^{34} + x^{32} + x^6 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{82} + x^{81} + x^{77} + x^{76} + x^{71} + x^{68} + x^{63} + x^{62} + x^{61} + x^{58} \\
&\quad + x^{57} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{36} \\
&\quad + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{21} + x^{18}, \\
p_3(x) &= x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{68} + x^{67} + x^{64} + x^{63} + x^{58} \\
&\quad + x^{57} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{20} + x^{19} + x^{16} + x^{15} \\
&\quad + x^{10} + x^9, \\
p_6(x) &= x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} \\
&\quad + x^{20} + x^{19} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6, \\
p_9(x) &= x^{80} + x^{79} + x^{76} + x^{75} + x^{64} + x^{63} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} \\
&\quad + x^{47} + x^{40} + x^{39} + x^{28} + x^{27} + x^{24} + x^{23} + x^4 + x^3, \\
p_{12}(x) &= x^{79} + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^7 \\
&\quad + x^5 + x^4 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1, 1, 0, 1, 0]$.

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{82} + x^{81} + x^{77} + x^{76} + x^{71} + x^{68} + x^{63} + x^{62} + x^{61} + x^{58} \\
&\quad + x^{57} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{41} + x^{36} \\
&\quad + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{21} + x^{18}, \\
p_3(x) &= x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{68} + x^{67} + x^{64} + x^{63} + x^{58} \\
&\quad + x^{57} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28} + x^{27} + x^{20} + x^{19} + x^{16} + x^{15} \\
&\quad + x^{10} + x^9, \\
p_6(x) &= x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{23} \\
&\quad + x^{20} + x^{19} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6, \\
p_9(x) &= x^{80} + x^{79} + x^{76} + x^{75} + x^{64} + x^{63} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} \\
&\quad + x^{47} + x^{40} + x^{39} + x^{28} + x^{27} + x^{24} + x^{23} + x^4 + x^3, \\
p_{12}(x) &= x^{79} + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^7 \\
&\quad + x^5 + x^4 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1, 1, 0, 1, 0]$.

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} + x^{66} + x^{64} + x^{52} + x^{48} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{28} + x^{26} + x^{24},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{86} + x^{84} + x^{82} + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{52} + x^{50} + x^{40} \\
&\quad + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} + x^8 + x^6, \\
p_6(x) &= x^{82} + x^{78} + x^{74} + x^{70} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} \\
&\quad + x^{48} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{20} + x^{14} + x^8 \\
&\quad + x^2 + 1, \\
p_9(x) &= x^{88} + x^{86} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{44} + x^{42} + x^{32} + x^{30} \\
&\quad + x^{20} + x^{18} + x^{16} + x^{14} + x^4 + x^2, \\
p_{12}(x) &= x^{88} + x^{72} + x^{64} + x^{56} + x^{40} + x^{32} + x^{24} + x^8 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 0]$.

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} + x^{67} + x^{63} + x^{60} \\
&\quad + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{42} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18} + x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{83} + x^{82} + x^{80} + x^{78} + x^{71} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{10} + x^9 + x^8, \\
p_6(x) &= x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{56} + x^{54} \\
&\quad + x^{53} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{39} + x^{37} + x^{34} + x^{32} + x^{30} \\
&\quad + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{14} + x^{10} + x^9 + x^8 + x^7 \\
&\quad + x^6 + x^3 + x + 1, \\
p_9(x) &= x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} \\
&\quad + x^{68} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{36} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^7 \\
&\quad + x^5 + x^4, \\
p_{12}(x) &= x^{83} + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{51} + x^{49} \\
&\quad + x^{48} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{19} + x^{17} + x^{16} + x^{11} \\
&\quad + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 1, 0, 1, 1, 1, 1]$.

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{81} + x^{80} + x^{74} + x^{73} + x^{71} + x^{68} + x^{65} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{47} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} \\
&\quad + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{21}, \\
p_3(x) &= x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{67} + x^{66} + x^{64} + x^{60} + x^{59}
\end{aligned}$$

$$\begin{aligned}
& + x^{57} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{12} \\
& + x^{11} + x^9, \\
p_6(x) &= x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{30} + x^{28} + x^{26} \\
& + x^{20} + x^{18} + x^{12} + x^{10} + x^6, \\
p_9(x) &= x^{79} + x^{78} + x^{77} + x^{75} + x^{71} + x^{70} + x^{69} + x^{67} + x^{47} + x^{46} + x^{45} \\
& + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{15} + x^{14} + x^{13} + x^{11} + x^7 + x^6 + x^5 \\
& + x^3, \\
p_{12}(x) &= x^{80} + x^{72} + x^{64} + x^{48} + x^{40} + x^{32} + x^{16} + x^8 + 1,
\end{aligned}$$

and the initial terms are $[0, 1, 0, 1, 0, 0, 0, 1]$.

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} \\
& + x^{69} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{53} + x^{52} + x^{49} \\
& + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{27} \\
& + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_3(x) &= x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79} + x^{76} + x^{74} + x^{70} + x^{68} \\
& + x^{67} + x^{66} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{41} + x^{40} + x^{38} + x^{37} \\
& + x^{36} + x^{32} + x^{31} + x^{28} + x^{26} + x^{22} + x^{20} + x^{19} + x^{18} + x^{15} + x^{13} \\
& + x^{12} + x^{11} + x^9, \\
p_6(x) &= x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{62} + x^{60} \\
& + x^{54} + x^{42} + x^{40} + x^{38} + x^{34} + x^{30} + x^{28} + x^{26} + x^{20} + x^{16} + x^{14} \\
& + x^{12} + x^6, \\
p_9(x) &= x^{91} + x^{90} + x^{89} + x^{87} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} \\
& + x^{67} + x^{59} + x^{58} + x^{57} + x^{55} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{38} \\
& + x^{37} + x^{35} + x^{27} + x^{26} + x^{25} + x^{23} + x^{15} + x^{14} + x^{13} + x^{10} + x^9 \\
& + x^6 + x^5 + x^3, \\
p_{12}(x) &= x^{92} + x^{80} + x^{72} + x^{68} + x^{64} + x^{60} + x^{48} + x^{40} + x^{36} + x^{32} + x^{28} \\
& + x^{16} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1, 1, 0, 1, 0]$.

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{79} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{61} \\
& + x^{60} + x^{59} + x^{58} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} \\
& + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33} + x^{30} + x^{29} + x^{27}, \\
p_3(x) &= x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} \\
& + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} \\
& + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36}
\end{aligned}$$

$$\begin{aligned}
& + x^{34} + x^{29} + x^{28} + x^{25} + x^{22} + x^{21} + x^{18} + x^{15} + x^{12}, \\
p_6(x) &= x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{67} \\
& + x^{63} + x^{60} + x^{58} + x^{55} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} \\
& + x^{42} + x^{40} + x^{36} + x^{35} + x^{34} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{20} \\
& + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^8 + x^7 + x^5 + x^4 + x^3 + x + 1, \\
p_9(x) &= x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} \\
& + x^{68} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} \\
& + x^{37} + x^{36} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^7 \\
& + x^5 + x^4, \\
p_{12}(x) &= x^{83} + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{51} + x^{49} \\
& + x^{48} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{19} + x^{17} + x^{16} + x^{11} \\
& + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 0, 1]$.

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} + x^{67} + x^{63} + x^{60} \\
& + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{42} + x^{38} + x^{37} \\
& + x^{36} + x^{34} + x^{32} + x^{26} + x^{25} + x^{22} + x^{19} + x^{18} + x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{83} + x^{82} + x^{80} + x^{78} + x^{71} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} \\
& + x^{58} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
& + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} \\
& + x^{27} + x^{25} + x^{24} + x^{21} + x^{20} + x^{19} + x^{10} + x^9 + x^8, \\
p_6(x) &= x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{56} + x^{54} \\
& + x^{53} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{39} + x^{37} + x^{34} + x^{32} + x^{30} \\
& + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{19} + x^{18} + x^{14} + x^{10} + x^9 + x^8 + x^7 \\
& + x^6 + x^3 + x + 1, \\
p_9(x) &= x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} \\
& + x^{68} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} \\
& + x^{37} + x^{36} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^7 \\
& + x^5 + x^4, \\
p_{12}(x) &= x^{83} + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{51} + x^{49} \\
& + x^{48} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{19} + x^{17} + x^{16} + x^{11} \\
& + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 1, 0, 1, 1, 1, 1]$.

For $\phi(W^o, 0, 1)$:

$$p_0(x) = x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71}$$

$$\begin{aligned}
& + x^{69} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{53} + x^{52} + x^{49} \\
& + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{27} \\
& + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_3(x) & = x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79} + x^{76} + x^{74} + x^{70} + x^{68} \\
& + x^{67} + x^{66} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{41} + x^{40} + x^{38} + x^{37} \\
& + x^{36} + x^{32} + x^{31} + x^{28} + x^{26} + x^{22} + x^{20} + x^{19} + x^{18} + x^{15} + x^{13} \\
& + x^{12} + x^{11} + x^9, \\
p_6(x) & = x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{62} + x^{60} \\
& + x^{54} + x^{42} + x^{40} + x^{38} + x^{34} + x^{30} + x^{28} + x^{26} + x^{20} + x^{16} + x^{14} \\
& + x^{12} + x^6, \\
p_9(x) & = x^{91} + x^{90} + x^{89} + x^{87} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} \\
& + x^{67} + x^{59} + x^{58} + x^{57} + x^{55} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{38} \\
& + x^{37} + x^{35} + x^{27} + x^{26} + x^{25} + x^{23} + x^{15} + x^{14} + x^{13} + x^{10} + x^9 \\
& + x^6 + x^5 + x^3, \\
p_{12}(x) & = x^{92} + x^{80} + x^{72} + x^{68} + x^{64} + x^{60} + x^{48} + x^{40} + x^{36} + x^{32} + x^{28} \\
& + x^{16} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 1, 1, 0, 1, 0]$.

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) & = x^{84} + x^{81} + x^{80} + x^{74} + x^{73} + x^{71} + x^{68} + x^{65} + x^{64} + x^{62} + x^{60} \\
& + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{47} + x^{43} + x^{42} + x^{38} + x^{36} + x^{35} \\
& + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{21}, \\
p_3(x) & = x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{67} + x^{66} + x^{64} + x^{60} + x^{59} \\
& + x^{57} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{21} + x^{19} + x^{18} + x^{16} + x^{12} \\
& + x^{11} + x^9, \\
p_6(x) & = x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{30} + x^{28} + x^{26} \\
& + x^{20} + x^{18} + x^{12} + x^{10} + x^6, \\
p_9(x) & = x^{79} + x^{78} + x^{77} + x^{75} + x^{71} + x^{70} + x^{69} + x^{67} + x^{47} + x^{46} + x^{45} \\
& + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{15} + x^{14} + x^{13} + x^{11} + x^7 + x^6 + x^5 \\
& + x^3, \\
p_{12}(x) & = x^{80} + x^{72} + x^{64} + x^{48} + x^{40} + x^{32} + x^{16} + x^8 + 1,
\end{aligned}$$

and the initial terms are $[0, 1, 0, 1, 0, 0, 0, 1]$.

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) & = x^{84} + x^{79} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{61} \\
& + x^{60} + x^{59} + x^{58} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} \\
& + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33} + x^{30} + x^{29} + x^{27},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} \\
&\quad + x^{34} + x^{29} + x^{28} + x^{25} + x^{22} + x^{21} + x^{18} + x^{15} + x^{12}, \\
p_6(x) &= x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{67} \\
&\quad + x^{63} + x^{60} + x^{58} + x^{55} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{42} + x^{40} + x^{36} + x^{35} + x^{34} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^8 + x^7 + x^5 + x^4 + x^3 + x + 1, \\
p_9(x) &= x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} \\
&\quad + x^{68} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{36} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^7 \\
&\quad + x^5 + x^4, \\
p_{12}(x) &= x^{83} + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{51} + x^{49} \\
&\quad + x^{48} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{19} + x^{17} + x^{16} + x^{11} \\
&\quad + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 0, 1]$.

Below is the transition function and output function of an 2-automaton that generates $T = M_{0,0}^o$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	10	[13, 18]	20	[32, 27]	30	[37, 38]	40	[5, 35]
1	[3, 4]	11	[18, 19]	21	[20, 29]	31	[17, 17]	41	[18, 46]
2	[5, 6]	12	[20, 17]	22	[33, 18]	32	[29, 20]	42	[37, 18]
3	[7, 8]	13	[21, 22]	23	[34, 35]	33	[26, 39]	43	[20, 20]
4	[7, 9]	14	[23, 21]	24	[15, 35]	34	[40, 39]	44	[36, 6]
5	[2, 10]	15	[24, 22]	25	[16, 20]	35	[41, 31]	45	[47, 10]
6	[11, 12]	16	[25, 26]	26	[17, 29]	36	[8, 42]	46	[42, 12]
7	[13, 14]	17	[27, 21]	27	[16, 29]	37	[43, 42]	47	[17, 20]
8	[15, 6]	18	[28, 29]	28	[36, 35]	38	[44, 27]		
9	[16, 17]	19	[30, 31]	29	[29, 29]	39	[45, 18]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	7	0	14	0	21	0	28	1	35	1	42	0
1	0	8	1	15	1	22	1	29	0	36	1	43	0
2	0	9	1	16	1	23	0	30	0	37	0	44	1
3	0	10	0	17	1	24	1	31	1	38	1	45	1
4	0	11	1	18	1	25	1	32	0	39	1	46	0
5	0	12	0	19	0	26	1	33	1	40	0	47	1
6	1	13	0	20	0	27	1	34	0	41	1		

UNIVERSITÉ DE STRASBOURG, CNRS, IRMA UMR 7501, F-67000 STRASBOURG, FRANCE

Email address: `guoniu.han@unistra.fr`

SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, WUHAN, PR CHINA

Email address: `huyining@hust.edu.cn`