# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^2+1, z^3+z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^2+1, z^3+z^2+1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t}=(t_0(z), t_1(z), t_2(z), \ldots) = (a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When* $(a,b)=(z^2+1,z^3+z^2+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{15}+z^{14}+z^{11}+z^8+z^6+z^5+z^4+z^3+1,\\
p_1(z)&=z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{11}+z^8+z^7+z^6+z^3,\\
p_2(z)&=z^{20}+z^{14}+z^6+1,\\
p_3(z)&=z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{11}+z^8+z^7+z^6+z^3,\\
p_4(z)&=z^{16}+z^{15}+z^{11}+z^8+z^5+z^4+z^3+1,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{15}+z^{14}+z^{12}+z^{11}+z^8+z^6+z^5+z^4+z^3+1,\\
p_1(z)&=z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{11}+z^8+z^7+z^6+z^3,\\
p_2(z)&=z^{20}+z^{14}+z^6+1,\\
p_3(z)&=z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{11}+z^8+z^7+z^6+z^3,\\
p_4(z)&=z^{16}+z^{15}+z^{12}+z^{11}+z^8+z^5+z^4+z^3+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(125, 125, 125, 125, 125)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 450 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 450, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 550 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 550, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:10u] &= M^e[:5u] + x^uM^e[:4u] + x^{2u}M^e[:3u] + x^{4u}M^e[:u] + x^uM^e[:5u]\\ &\quad + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^{3u}M^e[:5u]\\ &\quad + x^{4u}M^e[:4u] + x^{5u}M^e[:3u] + x^{7u}M^e[:u] + x^{8u}M^e[:2u]\\ &\quad + (x^{5u} + x^{6u} + x^{8u})R^e,\\ W^e[:10u] &= W^e[:5u] + x^uW^e[:4u] + x^{2u}W^e[:3u] + x^{4u}W^e[:u] + x^uW^e[:5u]\\ &\quad + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{3u}W^e[:5u]\end{aligned}$$

$$+ x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{7u}W^e[:u] + x^{8u}W^e[:2u] + (x^{5u} + x^{6u} + x^{8u})R^e.$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$\begin{aligned} M^o[:10u] &= M^o[:5u] + x^u M^o[:4u] + x^{2u}M^o[:3u] + x^{4u}M^o[:u] + x^u M^o[:5u] \\ &\quad + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{3u}M^o[:5u] \\ &\quad + x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{7u}M^o[:u] + x^{8u}M^o[:2u] \\ &\quad + (x^{5u} + x^{6u} + x^{8u})R^o, \\ W^o[:10u] &= W^o[:5u] + x^u W^o[:4u] + x^{2u}W^o[:3u] + x^{4u}W^o[:u] + x^u W^o[:5u] \\ &\quad + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{3u}W^o[:5u] \\ &\quad + x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{7u}W^o[:u] + x^{8u}W^o[:2u] \\ &\quad + (x^{5u} + x^{6u} + x^{8u})R^o. \end{aligned}$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:10u] &= T[:5u] + x^u T[:4u] + x^{2u}T[:3u] + x^{4u}T[:u] + x^u T[:5u] + x^{2u}T[:4u] \\ &\quad + x^{3u}T[:3u] + x^{5u}T[:u] + x^{3u}T[:5u] + x^{4u}T[:4u] + x^{5u}T[:3u] \\ &\quad + x^{7u}T[:u] + x^{8u}T[:2u] + (x^{5u} + x^{6u} + x^{8u}). \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 5 parts:

$$\begin{aligned} x^5u &= x^{4u}T[u:2u] + x^{2u}T[3u:4u] + x^u T[4u:5u] + T[5u:6u], \\ x^6u &= x^{5u}T[u:2u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] + T[6u:7u], \\ 0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] \\ &\quad + T[7u:8u], \\ x^8u &= x^{8u}T[:u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + T[9u:10u], \end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2], \tag{2.7}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[1000w]_2], \tag{2.10}$$

$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.11}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2], \tag{2.12}$$

$$1 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$1 = T[[w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.16}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 548 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.16) can be written as

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)), \tag{2.17}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.18}$$
$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.19}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1000)), \tag{2.20}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 1001)), \tag{2.21}$$

for all $s \in E_{2k}$, and

$$1 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)), \tag{2.22}$$
$$1 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.23}$$
$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.24}$$
$$1 = \tau(A(s, \epsilon)) + \tau(A(s, 1000)), \tag{2.25}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 1001)), \tag{2.26}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{16}), E_{16}) = (A(i, 0^{14}), E_{14})$, so that we only have to verify that identities (2.17) through (2.26) hold for $2 \leq k \leq 8$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:5u] + x^u M^e[:4u] + x^{2u} M^e[:3u] + x^{4u} M^e[:u] + x^{5u} R^e,$$
$$W_{2k} = W^e[:5u] + x^u W^e[:4u] + x^{2u} W^e[:3u] + x^{4u} W^e[:u] + x^{5u} R^e.$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$M_{2k+1} = M^o[:5u] + x^u M^o[:4u] + x^{2u} M^o[:3u] + x^{4u} M^o[:u] + x^{5u} R^o,$$
$$W_{2k+1} = W^o[:5u] + x^u W^o[:4u] + x^{2u} W^o[:3u] + x^{4u} W^o[:u] + x^{5u} R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:5v] + x^v W^e[:4v] + x^{2v} W^e[:3v] + x^{4v} W^e[:v] + x^{5u} R^e\big) \times \big( \\ &M^e[:5v] + x^v M^e[:4v] + x^{2v} M^e[:3v] + x^{4v} M^e[:v] + x^{5u} R^e\big); \end{aligned} \tag{2.27}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{10v}R^o$. Therefore we only need to prove that their difference is $O(x^{10v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{10v}$, to

$$\begin{aligned} W^e[:10v]M^e[:10v] + x^{2v}W^e[:8v]M^e[:8v] + x^{4v}W^e[:6v]M^e[:6v] \\ + x^{8v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \le 10$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.27) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{10}, b_1, ..., b_{10}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{10})$, which completes the proof. $\square$

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define
$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$
where
$$\begin{aligned}
q_0(x) &= x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^9 + x^6 + x^5,\\
q_1(x) &= x^{17} + x^{14} + x^{13} + x^{12} + x^9 + x^7 + x^6 + x^5 + x^3 + x^2,\\
q_2(x) &= x^{20} + x^{14} + x^6 + 1,\\
q_3(x) &= x^{17} + x^{14} + x^{13} + x^{12} + x^9 + x^7 + x^6 + x^5 + x^3 + x^2,\\
q_4(x) &= x^{20} + x^{17} + x^{16} + x^{15} + x^{12} + x^9 + x^5 + x^4.
\end{aligned}$$
The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 156, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 156, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form
$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$
The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:
$$\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{102} + x^{100} + x^{94} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78}\\
&\quad + x^{76} + x^{72} + x^{62} + x^{60} + x^{58} + x^{56} + x^{44} + x^{32} + x^{26} + x^{24},\\
p_3(x) &= x^{102} + x^{98} + x^{90} + x^{86} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{62} + x^{60}\\
&\quad + x^{56} + x^{48} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14},\\
p_6(x) &= x^{102} + x^{100} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{74} + x^{66} + x^{64} + x^{60}\\
&\quad + x^{48} + x^{44} + x^{42} + x^{36} + x^{32} + x^{24} + x^{22} + x^{20} + x^{14} + x^{10} + x^8\\
&\quad + x^6 + 1,\\
p_9(x) &= x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{72} + x^{64} + x^{62} + x^{56}\\
&\quad + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{26} + x^{24}\\
&\quad + x^{20} + x^{12} + x^6 + x^4,\\
p_{12}(x) &= x^{108} + x^{104} + x^{92} + x^{88} + x^{84} + x^{80} + x^{44} + x^{40} + x^{20} + x^{16} + x^{12}\\
&\quad + x^8 + x^4 + 1,
\end{aligned}$$
and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^e,0,1)$:
$$\begin{aligned}
p_0(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{88} + x^{85} + x^{84} + x^{80}\\
&\quad + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{62} + x^{61} + x^{58} + x^{55} + x^{54} + x^{51} + x^{49} + x^{45} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{30} + x^{28} + x^{27},\\
p_3(x) &= x^{78} + x^{72} + x^{64} + x^{58} + x^{46} + x^{40} + x^{38} + x^{26} + x^{24} + x^{18},\\
p_6(x) &= x^{87} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71}\\
&+ x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50}\\
&+ x^{49} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29}\\
&+ x^{28} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{96} + x^{88} + x^{86} + x^{84} + x^{76} + x^{74} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{42} + x^{34} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{8} + x^{6},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18}\\
&+ x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{88} + x^{85} + x^{84} + x^{80}\\
&+ x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64}\\
&+ x^{62} + x^{61} + x^{58} + x^{55} + x^{54} + x^{51} + x^{49} + x^{45} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{30} + x^{28} + x^{27},\\
p_3(x) &= x^{78} + x^{72} + x^{64} + x^{58} + x^{46} + x^{40} + x^{38} + x^{26} + x^{24} + x^{18},\\
p_6(x) &= x^{87} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71}\\
&+ x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50}\\
&+ x^{49} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29}\\
&+ x^{28} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{96} + x^{88} + x^{86} + x^{84} + x^{76} + x^{74} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{42} + x^{34} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{8} + x^{6},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18}\\
&+ x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{98} + x^{94} + x^{84} + x^{82} + x^{76} + x^{74} + x^{70} + x^{62} + x^{60} + x^{54}\\
&+ x^{50} + x^{48} + x^{36} + x^{30},\\
p_3(x) &= x^{96} + x^{92} + x^{90} + x^{78} + x^{74} + x^{72} + x^{70} + x^{64} + x^{62} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{50} + x^{46} + x^{42} + x^{40} + x^{34} + x^{32} + x^{20} + x^{14} + x^{12},\\
p_6(x) &= x^{96} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{66} + x^{60}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{42} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28}\\
&+ x^{26} + x^{16} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68}\\
&+ x^{66} + x^{64} + x^{54} + x^{46} + x^{44} + x^{38} + x^{34} + x^{32} + x^{30} + x^{20} + x^{12}\\
&+ x^{10} + x^{4},\\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{86} + x^{82} + x^{80} + x^{38} + x^{34} + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{98} + x^{94} + x^{90} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68}\\
&+ x^{66} + x^{62} + x^{58} + x^{54} + x^{44} + x^{42} + x^{36} + x^{34} + x^{30} + x^{24},\\
p_3(x) &= x^{96} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{62}\\
&+ x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{28} + x^{24}\\
&+ x^{20} + x^{14},\\
p_6(x) &= x^{96} + x^{94} + x^{90} + x^{88} + x^{82} + x^{72} + x^{70} + x^{62} + x^{58} + x^{52} + x^{50}\\
&+ x^{46} + x^{44} + x^{38} + x^{30} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{6} + x^{4} + x^{2}\\
&+ 1,\\
p_9(x) &= x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68}\\
&+ x^{66} + x^{64} + x^{54} + x^{46} + x^{44} + x^{38} + x^{34} + x^{32} + x^{30} + x^{20} + x^{12}\\
&+ x^{10} + x^{4},\\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{86} + x^{82} + x^{80} + x^{38} + x^{34} + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{88} + x^{87} + x^{85} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{76} + x^{75} + x^{71} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60}\\
&+ x^{58} + x^{55} + x^{53} + x^{48} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38}\\
&+ x^{37} + x^{31} + x^{28} + x^{27},\\
p_3(x) &= x^{78} + x^{72} + x^{64} + x^{58} + x^{46} + x^{40} + x^{38} + x^{26} + x^{24} + x^{18},\\
p_6(x) &= x^{87} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71}\\
&+ x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50}\\
&+ x^{49} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29}\\
&+ x^{28} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{96} + x^{88} + x^{86} + x^{84} + x^{76} + x^{74} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{42} + x^{34} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{8} + x^{6},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18}
\end{aligned}
$$

$$+ x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{93} + x^{88} + x^{87} + x^{85} + x^{83}\\
&\quad + x^{82} + x^{81} + x^{80} + x^{76} + x^{75} + x^{71} + x^{68} + x^{67} + x^{65} + x^{62} + x^{60}\\
&\quad + x^{58} + x^{55} + x^{53} + x^{48} + x^{47} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38}\\
&\quad + x^{37} + x^{31} + x^{28} + x^{27},\\
p_3(x) &= x^{78} + x^{72} + x^{64} + x^{58} + x^{46} + x^{40} + x^{38} + x^{26} + x^{24} + x^{18},\\
p_6(x) &= x^{87} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71}\\
&\quad + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50}\\
&\quad + x^{49} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29}\\
&\quad + x^{28} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{96} + x^{88} + x^{86} + x^{84} + x^{76} + x^{74} + x^{68} + x^{64} + x^{60} + x^{58} + x^{56}\\
&\quad + x^{54} + x^{52} + x^{42} + x^{34} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{8} + x^{6},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{81}\\
&\quad + x^{80} + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18}\\
&\quad + x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{80} + x^{76} + x^{70} + x^{64} + x^{62} + x^{60} + x^{54} + x^{50} + x^{48} + x^{44}\\
&\quad + x^{42} + x^{40} + x^{36} + x^{32} + x^{30},\\
p_3(x) &= x^{102} + x^{98} + x^{84} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} + x^{62} + x^{60} + x^{56}\\
&\quad + x^{48} + x^{34} + x^{32} + x^{24} + x^{18} + x^{16} + x^{12},\\
p_6(x) &= x^{102} + x^{100} + x^{96} + x^{86} + x^{84} + x^{82} + x^{70} + x^{68} + x^{66} + x^{60} + x^{48}\\
&\quad + x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{12} + x^{10}\\
&\quad + x^{6} + 1,\\
p_9(x) &= x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{72} + x^{64} + x^{62} + x^{56}\\
&\quad + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{26} + x^{24}\\
&\quad + x^{20} + x^{12} + x^{6} + x^{4},\\
p_{12}(x) &= x^{108} + x^{104} + x^{92} + x^{88} + x^{84} + x^{80} + x^{44} + x^{40} + x^{20} + x^{16} + x^{12}\\
&\quad + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{90} + x^{89} + x^{87}\\
&\quad + x^{86} + x^{85} + x^{84} + x^{77} + x^{76} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{49} + x^{47} + x^{46} + x^{44} \\
&+ x^{43} + x^{40} + x^{38} + x^{35} + x^{29} + x^{27} + x^{25} + x^{24}, \\
p_3(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{78} + x^{77} + x^{75} \\
&+ x^{74} + x^{73} + x^{72} + x^{65} + x^{64} + x^{59} + x^{58} + x^{55} + x^{52} + x^{50} + x^{49} \\
&+ x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} \\
&+ x^{34} + x^{29} + x^{27} + x^{21} + x^{19} + x^{17} + x^{16}, \\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&+ x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} \\
&+ x^{62} + x^{58} + x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{41} + x^{38} \\
&+ x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{19} + x^{18} \\
&+ x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1, \\
p_9(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{29} + x^{26} + x^{25} + x^{24} + x^{13} + x^{10} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{89} + x^{86} + x^{84} + x^{82} + x^{81} + x^{80} + x^{41} \\
&+ x^{38} + x^{37} + x^{36} + x^{21} + x^{18} + x^{17} + x^{16} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{99} + x^{97} + x^{93} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{80} \\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} \\
&+ x^{62} + x^{59} + x^{58} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&+ x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27}, \\
p_3(x) &= x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} \\
&+ x^{61} + x^{59} + x^{58} + x^{49} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} \\
&+ x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{21} + x^{19} \\
&+ x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} + x^{54} + x^{50} + x^{48} + x^{38} \\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12}, \\
p_9(x) &= x^{99} + x^{96} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} + x^{86} + x^{35} + x^{32} + x^{31} \\
&+ x^{30} + x^{27} + x^{24} + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{8} \\
&+ x^{7} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} + x^{44} + x^{36} + x^{24} + x^{16} + x^{12} \\
&+ x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{112} + x^{103} + x^{101} + x^{100} + x^{97} + x^{93} + x^{92} + x^{88} + x^{86} + x^{85} + x^{84} \\
&+ x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{62}+x^{61}+x^{59}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{51}+x^{50}+x^{49}\\
&+x^{48}+x^{46}+x^{42}+x^{41}+x^{40}+x^{35}+x^{33}+x^{31}+x^{30}+x^{29}+x^{27},\\
p_3(x)&=x^{85}+x^{82}+x^{81}+x^{80}+x^{79}+x^{76}+x^{74}+x^{73}+x^{72}+x^{67}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{59}+x^{58}+x^{53}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}\\
&+x^{44}+x^{39}+x^{36}+x^{35}+x^{34}+x^{33}+x^{31}+x^{30}+x^{29}+x^{26}+x^{24}\\
&+x^{23}+x^{22}+x^{21}+x^{19}+x^{18},\\
p_6(x)&=x^{94}+x^{88}+x^{84}+x^{82}+x^{80}+x^{66}+x^{62}+x^{60}+x^{50}+x^{44}+x^{42}\\
&+x^{40}+x^{34}+x^{28}+x^{26}+x^{24}+x^{22}+x^{16}+x^{14}+x^{12},\\
p_9(x)&=x^{103}+x^{100}+x^{99}+x^{98}+x^{95}+x^{92}+x^{90}+x^{88}+x^{87}+x^{86}+x^{39}\\
&+x^{36}+x^{35}+x^{34}+x^{31}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{19}+x^{18}\\
&+x^{15}+x^{12}+x^{10}+x^{8}+x^{7}+x^{6},\\
p_{12}(x)&=x^{112}+x^{104}+x^{100}+x^{96}+x^{92}+x^{88}+x^{80}+x^{48}+x^{40}+x^{36}+x^{28}\\
&+x^{20}+x^{12}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1].

For $\phi(M^o,1,1)$:

$$
\begin{aligned}
p_0(x)&=x^{105}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{93}+x^{91}+x^{88}+x^{86}\\
&+x^{85}+x^{83}+x^{82}+x^{81}+x^{79}+x^{78}+x^{76}+x^{74}+x^{72}+x^{71}+x^{68}\\
&+x^{66}+x^{65}+x^{64}+x^{60}+x^{58}+x^{55}+x^{52}+x^{51}+x^{50}+x^{49}+x^{46}\\
&+x^{44}+x^{43}+x^{41}+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{31}+x^{30},\\
p_3(x)&=x^{93}+x^{90}+x^{89}+x^{88}+x^{87}+x^{85}+x^{84}+x^{81}+x^{79}+x^{76}+x^{74}\\
&+x^{73}+x^{71}+x^{70}+x^{67}+x^{65}+x^{63}+x^{62}+x^{61}+x^{59}+x^{57}+x^{56}\\
&+x^{55}+x^{52}+x^{50}+x^{48}+x^{45}+x^{42}+x^{38}+x^{36}+x^{35}+x^{34}+x^{26}\\
&+x^{25}+x^{23}+x^{22}+x^{19}+x^{18},\\
p_6(x)&=x^{99}+x^{97}+x^{96}+x^{91}+x^{90}+x^{89}+x^{86}+x^{85}+x^{82}+x^{81}+x^{80}\\
&+x^{77}+x^{76}+x^{74}+x^{72}+x^{70}+x^{65}+x^{64}+x^{61}+x^{60}+x^{58}+x^{57}\\
&+x^{52}+x^{50}+x^{49}+x^{48}+x^{42}+x^{40}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}\\
&+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}+x^{25}+x^{24}+x^{22}+x^{19}+x^{18}+x^{16}\\
&+x^{13}+x^{11}+x^{8}+x^{6}+x^{4}+x^{2}+x+1,\\
p_9(x)&=x^{93}+x^{90}+x^{89}+x^{88}+x^{29}+x^{26}+x^{25}+x^{24}+x^{13}+x^{10}+x^{9}+x^{8},\\
p_{12}(x)&=x^{105}+x^{102}+x^{101}+x^{100}+x^{89}+x^{86}+x^{84}+x^{82}+x^{81}+x^{80}+x^{41}\\
&+x^{38}+x^{37}+x^{36}+x^{21}+x^{18}+x^{17}+x^{16}+x^{9}+x^{6}+x^{4}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1].

For $\phi(W^o,0,0)$:

$$
\begin{aligned}
p_0(x)&=x^{105}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{91}+x^{90}+x^{89}+x^{87}\\
&+x^{86}+x^{85}+x^{84}+x^{77}+x^{76}+x^{71}+x^{70}+x^{69}+x^{67}+x^{65}+x^{64}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{49} + x^{47} + x^{46} + x^{44}\\
&+ x^{43} + x^{40} + x^{38} + x^{35} + x^{29} + x^{27} + x^{25} + x^{24},\\
p_3(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{78} + x^{77} + x^{75}\\
&+ x^{74} + x^{73} + x^{72} + x^{65} + x^{64} + x^{59} + x^{58} + x^{55} + x^{52} + x^{50} + x^{49}\\
&+ x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35}\\
&+ x^{34} + x^{29} + x^{27} + x^{21} + x^{19} + x^{17} + x^{16},\\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65}\\
&+ x^{62} + x^{58} + x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{41} + x^{38}\\
&+ x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{19} + x^{18}\\
&+ x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,\\
p_9(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{29} + x^{26} + x^{25} + x^{24} + x^{13} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{89} + x^{86} + x^{84} + x^{82} + x^{81} + x^{80} + x^{41}\\
&+ x^{38} + x^{37} + x^{36} + x^{21} + x^{18} + x^{17} + x^{16} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{112} + x^{103} + x^{101} + x^{100} + x^{97} + x^{93} + x^{92} + x^{88} + x^{86} + x^{85} + x^{84}\\
&+ x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63}\\
&+ x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49}\\
&+ x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27},\\
p_3(x) &= x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{67} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45}\\
&+ x^{44} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24}\\
&+ x^{23} + x^{22} + x^{21} + x^{19} + x^{18},\\
p_6(x) &= x^{94} + x^{88} + x^{84} + x^{82} + x^{80} + x^{66} + x^{62} + x^{60} + x^{50} + x^{44} + x^{42}\\
&+ x^{40} + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12},\\
p_9(x) &= x^{103} + x^{100} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} + x^{88} + x^{87} + x^{86} + x^{39}\\
&+ x^{36} + x^{35} + x^{34} + x^{31} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18}\\
&+ x^{15} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{112} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{48} + x^{40} + x^{36} + x^{28}\\
&+ x^{20} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{99} + x^{97} + x^{93} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} + x^{80}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{62} + x^{59} + x^{58} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&+ x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27}, \\
p_3(x) &= x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} \\
&+ x^{61} + x^{59} + x^{58} + x^{49} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} \\
&+ x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{21} + x^{19} \\
&+ x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} + x^{54} + x^{50} + x^{48} + x^{38} \\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12}, \\
p_9(x) &= x^{99} + x^{96} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} + x^{86} + x^{35} + x^{32} + x^{31} \\
&+ x^{30} + x^{27} + x^{24} + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{11} + x^{8} \\
&+ x^{7} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} + x^{44} + x^{36} + x^{24} + x^{16} + x^{12} \\
&+ x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{88} + x^{86} \\
&+ x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{74} + x^{72} + x^{71} + x^{68} \\
&+ x^{66} + x^{65} + x^{64} + x^{60} + x^{58} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{46} \\
&+ x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{31} + x^{30}, \\
p_3(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{81} + x^{79} + x^{76} + x^{74} \\
&+ x^{73} + x^{71} + x^{70} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} \\
&+ x^{55} + x^{52} + x^{50} + x^{48} + x^{45} + x^{42} + x^{38} + x^{36} + x^{35} + x^{34} + x^{26} \\
&+ x^{25} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{91} + x^{90} + x^{89} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} \\
&+ x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{65} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} \\
&+ x^{52} + x^{50} + x^{49} + x^{48} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} \\
&+ x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{19} + x^{18} + x^{16} \\
&+ x^{13} + x^{11} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1, \\
p_9(x) &= x^{93} + x^{90} + x^{89} + x^{88} + x^{29} + x^{26} + x^{25} + x^{24} + x^{13} + x^{10} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{105} + x^{102} + x^{101} + x^{100} + x^{89} + x^{86} + x^{84} + x^{82} + x^{81} + x^{80} + x^{41} \\
&+ x^{38} + x^{37} + x^{36} + x^{21} + x^{18} + x^{17} + x^{16} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 110 | [194, 54] | 220 | [132, 57] | 330 | [106, 318] | 440 | [516, 514] |
| 1 | [3, 4] | 111 | [123, 195] | 221 | [129, 338] | 331 | [431, 432] | 441 | [176, 411] |
| 2 | [5, 6] | 112 | [196, 197] | 222 | [339, 340] | 332 | [427, 433] | 442 | [390, 143] |
| 3 | [7, 8] | 113 | [198, 149] | 223 | [341, 342] | 333 | [243, 110] | 443 | [511, 422] |
| 4 | [9, 10] | 114 | [199, 200] | 224 | [166, 343] | 334 | [434, 287] | 444 | [413, 159] |
| 5 | [11, 12] | 115 | [179, 201] | 225 | [344, 161] | 335 | [435, 247] | 445 | [412, 145] |
| 6 | [13, 14] | 116 | [176, 202] | 226 | [132, 345] | 336 | [436, 437] | 446 | [513, 517] |
| 7 | [15, 16] | 117 | [162, 193] | 227 | [144, 346] | 337 | [438, 439] | 447 | [332, 37] |
| 8 | [17, 18] | 118 | [203, 204] | 228 | [347, 139] | 338 | [440, 112] | 448 | [33, 518] |
| 9 | [19, 20] | 119 | [205, 206] | 229 | [348, 162] | 339 | [441, 442] | 449 | [519, 156] |
| 10 | [21, 22] | 120 | [207, 208] | 230 | [349, 350] | 340 | [443, 444] | 450 | [506, 323] |
| 11 | [23, 24] | 121 | [209, 205] | 231 | [351, 352] | 341 | [98, 238] | 451 | [61, 182] |
| 12 | [25, 26] | 122 | [210, 211] | 232 | [150, 353] | 342 | [445, 446] | 452 | [366, 386] |
| 13 | [27, 28] | 123 | [63, 212] | 233 | [354, 201] | 343 | [267, 221] | 453 | [55, 141] |
| 14 | [29, 30] | 124 | [213, 214] | 234 | [355, 356] | 344 | [447, 441] | 454 | [191, 136] |
| 15 | [31, 32] | 125 | [68, 215] | 235 | [8, 38] | 345 | [448, 209] | 455 | [514, 340] |
| 16 | [33, 34] | 126 | [216, 217] | 236 | [357, 160] | 346 | [103, 444] | 456 | [372, 416] |
| 17 | [35, 36] | 127 | [218, 219] | 237 | [119, 195] | 347 | [63, 449] | 457 | [520, 510] |
| 18 | [37, 38] | 128 | [220, 221] | 238 | [169, 358] | 348 | [450, 309] | 458 | [377, 383] |
| 19 | [39, 40] | 129 | [79, 222] | 239 | [359, 150] | 349 | [451, 452] | 459 | [365, 175] |
| 20 | [41, 42] | 130 | [85, 223] | 240 | [360, 336] | 350 | [238, 300] | 460 | [419, 334] |
| 21 | [43, 44] | 131 | [204, 224] | 241 | [361, 352] | 351 | [91, 453] | 461 | [339, 515] |
| 22 | [45, 46] | 132 | [225, 226] | 242 | [199, 362] | 352 | [454, 76] | 462 | [354, 180] |
| 23 | [47, 48] | 133 | [27, 29] | 243 | [53, 363] | 353 | [243, 313] | 463 | [521, 359] |
| 24 | [49, 50] | 134 | [227, 228] | 244 | [364, 55] | 354 | [248, 301] | 464 | [37, 522] |
| 25 | [51, 52] | 135 | [229, 116] | 245 | [365, 125] | 355 | [228, 455] | 465 | [519, 523] |
| 26 | [53, 54] | 136 | [225, 100] | 246 | [366, 367] | 356 | [456, 262] | 466 | [12, 358] |
| 27 | [55, 40] | 137 | [230, 231] | 247 | [12, 170] | 357 | [457, 458] | 467 | [341, 524] |
| 28 | [56, 57] | 138 | [232, 233] | 248 | [368, 369] | 358 | [24, 249] | 468 | [348, 192] |
| 29 | [58, 59] | 139 | [234, 235] | 249 | [48, 37] | 359 | [64, 243] | 469 | [520, 396] |
| 30 | [60, 61] | 140 | [236, 237] | 250 | [50, 370] | 360 | [459, 460] | 470 | [418, 187] |
| 31 | [62, 63] | 141 | [238, 239] | 251 | [371, 356] | 361 | [431, 461] | 471 | [13, 417] |
| 32 | [64, 65] | 142 | [240, 241] | 252 | [137, 190] | 362 | [282, 208] | 472 | [421, 512] |
| 33 | [66, 67] | 143 | [242, 69] | 253 | [372, 329] | 363 | [107, 462] | 473 | [171, 188] |
| 34 | [68, 69] | 144 | [16, 243] | 254 | [373, 34] | 364 | [463, 238] | 474 | [364, 39] |
| 35 | [70, 71] | 145 | [244, 91] | 255 | [323, 120] | 365 | [441, 464] | 475 | [414, 147] |
| 36 | [72, 73] | 146 | [245, 246] | 256 | [374, 375] | 366 | [236, 465] | 476 | [356, 525] |
| 37 | [72, 74] | 147 | [247, 248] | 257 | [376, 14] | 367 | [278, 307] | 477 | [526, 413] |
| 38 | [16, 65] | 148 | [249, 250] | 258 | [377, 378] | 368 | [6, 466] | 478 | [507, 162] |
| 39 | [75, 76] | 149 | [251, 252] | 259 | [379, 380] | 369 | [21, 467] | 479 | [402, 55] |
| 40 | [77, 78] | 150 | [253, 220] | 260 | [61, 381] | 370 | [95, 86] | 480 | [349, 375] |
| 41 | [79, 80] | 151 | [254, 255] | 261 | [382, 383] | 371 | [468, 19] | 481 | [198, 508] |
| 42 | [81, 82] | 152 | [256, 27] | 262 | [189, 138] | 372 | [321, 75] | 482 | [527, 362] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 43 | [83, 84] | 153 | [257, 258] | 263 | [152, 338] | 373 | [469, 470] | 483 | [379, 399] |
| 44 | [85, 86] | 154 | [29, 259] | 264 | [384, 347] | 374 | [471, 472] | 484 | [398, 380] |
| 45 | [87, 88] | 155 | [260, 261] | 265 | [385, 386] | 375 | [77, 79] | 485 | [181, 381] |
| 46 | [89, 90] | 156 | [262, 263] | 266 | [158, 202] | 376 | [96, 90] | 486 | [528, 397] |
| 47 | [91, 72] | 157 | [264, 265] | 267 | [177, 59] | 377 | [23, 472] | 487 | [36, 365] |
| 48 | [92, 93] | 158 | [266, 267] | 268 | [135, 60] | 378 | [473, 271] | 488 | [404, 140] |
| 49 | [94, 95] | 159 | [268, 28] | 269 | [387, 388] | 379 | [112, 455] | 489 | [529, 171] |
| 50 | [96, 97] | 160 | [269, 270] | 270 | [389, 186] | 380 | [474, 475] | 490 | [384, 164] |
| 51 | [98, 99] | 161 | [17, 271] | 271 | [390, 391] | 381 | [269, 476] | 491 | [333, 420] |
| 52 | [100, 101] | 162 | [272, 273] | 272 | [371, 392] | 382 | [477, 452] | 492 | [355, 392] |
| 53 | [102, 103] | 163 | [274, 210] | 273 | [56, 345] | 383 | [453, 73] | 493 | [192, 163] |
| 54 | [104, 105] | 164 | [205, 236] | 274 | [393, 394] | 384 | [478, 241] | 494 | [516, 339] |
| 55 | [106, 107] | 165 | [275, 276] | 275 | [324, 323] | 385 | [429, 309] | 495 | [43, 425] |
| 56 | [108, 109] | 166 | [277, 278] | 276 | [123, 326] | 386 | [479, 22] | 496 | [409, 517] |
| 57 | [110, 111] | 167 | [279, 280] | 277 | [395, 396] | 387 | [480, 481] | 497 | [530, 150] |
| 58 | [112, 113] | 168 | [281, 268] | 278 | [49, 133] | 388 | [482, 483] | 498 | [401, 167] |
| 59 | [114, 115] | 169 | [282, 283] | 279 | [387, 397] | 389 | [304, 441] | 499 | [423, 524] |
| 60 | [108, 20] | 170 | [284, 285] | 280 | [398, 399] | 390 | [248, 456] | 500 | [58, 178] |
| 61 | [116, 117] | 171 | [4, 286] | 281 | [400, 56] | 391 | [74, 484] | 501 | [416, 330] |
| 62 | [118, 119] | 172 | [247, 287] | 282 | [153, 394] | 392 | [234, 286] | 502 | [531, 164] |
| 63 | [120, 32] | 173 | [256, 268] | 283 | [401, 343] | 393 | [485, 85] | 503 | [532, 514] |
| 64 | [121, 122] | 174 | [28, 259] | 284 | [402, 39] | 394 | [486, 82] | 504 | [532, 339] |
| 65 | [32, 123] | 175 | [73, 288] | 285 | [359, 403] | 395 | [453, 74] | 505 | [530, 403] |
| 66 | [124, 125] | 176 | [289, 290] | 286 | [142, 391] | 396 | [107, 487] | 506 | [533, 209] |
| 67 | [126, 50] | 177 | [291, 108] | 287 | [164, 404] | 397 | [488, 263] | 507 | [206, 465] |
| 68 | [127, 128] | 178 | [214, 292] | 288 | [136, 61] | 398 | [291, 19] | 508 | [305, 442] |
| 69 | [129, 130] | 179 | [240, 293] | 289 | [9, 405] | 399 | [67, 437] | 509 | [534, 436] |
| 70 | [131, 132] | 180 | [244, 294] | 290 | [41, 174] | 400 | [65, 110] | 510 | [104, 535] |
| 71 | [133, 134] | 181 | [295, 273] | 291 | [335, 406] | 401 | [245, 489] | 511 | [110, 536] |
| 72 | [135, 136] | 182 | [100, 296] | 292 | [389, 407] | 402 | [490, 491] | 512 | [537, 447] |
| 73 | [137, 138] | 183 | [297, 298] | 293 | [50, 134] | 403 | [492, 493] | 513 | [311, 538] |
| 74 | [139, 140] | 184 | [299, 280] | 294 | [368, 197] | 404 | [456, 488] | 514 | [213, 252] |
| 75 | [39, 141] | 185 | [230, 71] | 295 | [327, 405] | 405 | [315, 307] | 515 | [539, 91] |
| 76 | [142, 143] | 186 | [99, 300] | 296 | [161, 365] | 406 | [320, 475] | 516 | [540, 443] |
| 77 | [144, 145] | 187 | [301, 262] | 297 | [196, 369] | 407 | [494, 79] | 517 | [472, 541] |
| 78 | [48, 8] | 188 | [302, 303] | 298 | [351, 408] | 408 | [317, 107] | 518 | [73, 484] |
| 79 | [146, 147] | 189 | [304, 305] | 299 | [403, 151] | 409 | [102, 443] | 519 | [6, 537] |
| 80 | [148, 149] | 190 | [273, 306] | 300 | [409, 410] | 410 | [495, 496] | 520 | [473, 18] |
| 81 | [150, 151] | 191 | [307, 308] | 301 | [158, 411] | 411 | [436, 497] | 521 | [316, 253] |
| 82 | [152, 130] | 192 | [219, 117] | 302 | [412, 346] | 412 | [433, 498] | 522 | [433, 211] |
| 83 | [153, 154] | 193 | [309, 310] | 303 | [413, 46] | 413 | [275, 84] | 523 | [542, 483] |
| 84 | [155, 156] | 194 | [311, 312] | 304 | [414, 415] | 414 | [277, 315] | 524 | [466, 304] |
| 85 | [157, 158] | 195 | [65, 313] | 305 | [337, 416] | 415 | [302, 499] | 525 | [428, 226] |
| 86 | [45, 159] | 196 | [260, 314] | 306 | [185, 407] | 416 | [272, 500] | 526 | [78, 80] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 87 | [87, 87] | 197 | [315, 297] | 307 | [347, 404] | 417 | [486, 501] | 527 | [454, 543] |
| 88 | [51, 160] | 198 | [316, 266] | 308 | [133, 370] | 418 | [502, 247] | 528 | [475, 454] |
| 89 | [35, 161] | 199 | [317, 318] | 309 | [357, 52] | 419 | [254, 224] | 529 | [544, 460] |
| 90 | [162, 163] | 200 | [25, 205] | 310 | [323, 121] | 420 | [503, 303] | 530 | [210, 498] |
| 91 | [164, 139] | 201 | [103, 319] | 311 | [376, 417] | 421 | [448, 25] | 531 | [287, 456] |
| 92 | [165, 165] | 202 | [320, 321] | 312 | [418, 171] | 422 | [504, 505] | 532 | [545, 538] |
| 93 | [166, 167] | 203 | [322, 323] | 313 | [122, 119] | 423 | [457, 491] | 533 | [121, 32] |
| 94 | [168, 45] | 204 | [120, 122] | 314 | [419, 420] | 424 | [88, 88] | 534 | [521, 530] |
| 95 | [169, 170] | 205 | [32, 119] | 315 | [421, 422] | 425 | [440, 228] | 535 | [392, 525] |
| 96 | [171, 172] | 206 | [165, 87] | 316 | [400, 132] | 426 | [322, 325] | 536 | [155, 523] |
| 97 | [173, 174] | 207 | [324, 325] | 317 | [344, 36] | 427 | [506, 325] | 537 | [509, 176] |
| 98 | [124, 175] | 208 | [325, 120] | 318 | [404, 328] | 428 | [187, 172] | 538 | [385, 367] |
| 99 | [157, 176] | 209 | [87, 165] | 319 | [423, 342] | 429 | [393, 154] | 539 | [424, 44] |
| 100 | [177, 178] | 210 | [122, 123] | 320 | [424, 425] | 430 | [173, 42] | 540 | [168, 413] |
| 101 | [179, 180] | 211 | [325, 121] | 321 | [332, 8] | 431 | [507, 192] | 541 | [146, 415] |
| 102 | [181, 182] | 212 | [119, 326] | 322 | [426, 207] | 432 | [148, 508] | 542 | [183, 128] |
| 103 | [126, 133] | 213 | [327, 10] | 323 | [427, 210] | 433 | [194, 363] | 543 | [8, 522] |
| 104 | [183, 184] | 214 | [139, 328] | 324 | [313, 111] | 434 | [509, 158] | 544 | [546, 359] |
| 105 | [185, 186] | 215 | [329, 330] | 325 | [92, 310] | 435 | [395, 510] | 545 | [331, 48] |
| 106 | [187, 188] | 216 | [331, 332] | 326 | [204, 255] | 436 | [511, 512] | 546 | [249, 547] |
| 107 | [189, 190] | 217 | [333, 334] | 327 | [428, 100] | 437 | [513, 410] | 547 | [361, 408] |
| 108 | [191, 60] | 218 | [335, 336] | 328 | [429, 92] | 438 | [131, 56] | | |
| 109 | [192, 193] | 219 | [337, 329] | 329 | [219, 430] | 439 | [514, 515] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 79 | 0 | 158 | 1 | 237 | 0 | 316 | 0 | 395 | 1 | 474 | 0 |
| 1 | 0 | 80 | 1 | 159 | 0 | 238 | 0 | 317 | 0 | 396 | 1 | 475 | 1 |
| 2 | 0 | 81 | 0 | 160 | 0 | 239 | 0 | 318 | 0 | 397 | 0 | 476 | 0 |
| 3 | 0 | 82 | 1 | 161 | 1 | 240 | 0 | 319 | 0 | 398 | 1 | 477 | 1 |
| 4 | 0 | 83 | 0 | 162 | 0 | 241 | 1 | 320 | 1 | 399 | 1 | 478 | 1 |
| 5 | 0 | 84 | 0 | 163 | 1 | 242 | 0 | 321 | 0 | 400 | 0 | 479 | 1 |
| 6 | 1 | 85 | 1 | 164 | 0 | 243 | 1 | 322 | 0 | 401 | 0 | 480 | 0 |
| 7 | 0 | 86 | 0 | 165 | 1 | 244 | 0 | 323 | 0 | 402 | 1 | 481 | 0 |
| 8 | 1 | 87 | 0 | 166 | 1 | 245 | 0 | 324 | 1 | 403 | 1 | 482 | 1 |
| 9 | 0 | 88 | 1 | 167 | 0 | 246 | 0 | 325 | 1 | 404 | 0 | 483 | 0 |
| 10 | 0 | 89 | 1 | 168 | 0 | 247 | 1 | 326 | 1 | 405 | 1 | 484 | 1 |
| 11 | 0 | 90 | 0 | 169 | 0 | 248 | 1 | 327 | 1 | 406 | 1 | 485 | 1 |
| 12 | 1 | 91 | 0 | 170 | 1 | 249 | 1 | 328 | 1 | 407 | 0 | 486 | 1 |
| 13 | 1 | 92 | 1 | 171 | 0 | 250 | 0 | 329 | 1 | 408 | 0 | 487 | 0 |
| 14 | 0 | 93 | 1 | 172 | 1 | 251 | 0 | 330 | 1 | 409 | 1 | 488 | 0 |
| 15 | 0 | 94 | 0 | 173 | 1 | 252 | 0 | 331 | 1 | 410 | 0 | 489 | 1 |
| 16 | 1 | 95 | 0 | 174 | 1 | 253 | 0 | 332 | 0 | 411 | 0 | 490 | 1 |
| 17 | 1 | 96 | 0 | 175 | 0 | 254 | 0 | 333 | 1 | 412 | 0 | 491 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 18 | 0 | 97 | 1 | 176 | 0 | 255 | 0 | 334 | 0 | 413 | 1 | 492 | 1 |
| 19 | 0 | 98 | 1 | 177 | 1 | 256 | 1 | 335 | 1 | 414 | 1 | 493 | 1 |
| 20 | 0 | 99 | 1 | 178 | 1 | 257 | 0 | 336 | 0 | 415 | 0 | 494 | 0 |
| 21 | 0 | 100 | 1 | 179 | 0 | 258 | 0 | 337 | 1 | 416 | 0 | 495 | 0 |
| 22 | 0 | 101 | 0 | 180 | 0 | 259 | 0 | 338 | 0 | 417 | 1 | 496 | 1 |
| 23 | 0 | 102 | 1 | 181 | 1 | 260 | 0 | 339 | 0 | 418 | 0 | 497 | 1 |
| 24 | 0 | 103 | 1 | 182 | 1 | 261 | 1 | 340 | 0 | 419 | 0 | 498 | 0 |
| 25 | 1 | 104 | 0 | 183 | 0 | 262 | 1 | 341 | 1 | 420 | 1 | 499 | 0 |
| 26 | 1 | 105 | 0 | 184 | 1 | 263 | 1 | 342 | 0 | 421 | 1 | 500 | 0 |
| 27 | 1 | 106 | 1 | 185 | 0 | 264 | 1 | 343 | 1 | 422 | 1 | 501 | 0 |
| 28 | 1 | 107 | 1 | 186 | 1 | 265 | 1 | 344 | 0 | 423 | 0 | 502 | 0 |
| 29 | 0 | 108 | 1 | 187 | 1 | 266 | 1 | 345 | 1 | 424 | 1 | 503 | 1 |
| 30 | 1 | 109 | 1 | 188 | 0 | 267 | 1 | 346 | 1 | 425 | 0 | 504 | 1 |
| 31 | 0 | 110 | 0 | 189 | 1 | 268 | 0 | 347 | 1 | 426 | 0 | 505 | 1 |
| 32 | 0 | 111 | 1 | 190 | 1 | 269 | 0 | 348 | 0 | 427 | 0 | 506 | 0 |
| 33 | 1 | 112 | 0 | 191 | 1 | 270 | 1 | 349 | 0 | 428 | 1 | 507 | 1 |
| 34 | 1 | 113 | 0 | 192 | 1 | 271 | 1 | 350 | 0 | 429 | 1 | 508 | 1 |
| 35 | 1 | 114 | 0 | 193 | 0 | 272 | 0 | 351 | 0 | 430 | 1 | 509 | 0 |
| 36 | 0 | 115 | 0 | 194 | 0 | 273 | 1 | 352 | 1 | 431 | 1 | 510 | 0 |
| 37 | 0 | 116 | 0 | 195 | 0 | 274 | 1 | 353 | 1 | 432 | 1 | 511 | 0 |
| 38 | 1 | 117 | 0 | 196 | 0 | 275 | 1 | 354 | 1 | 433 | 0 | 512 | 0 |
| 39 | 0 | 118 | 0 | 197 | 1 | 276 | 1 | 355 | 1 | 434 | 0 | 513 | 0 |
| 40 | 1 | 119 | 0 | 198 | 0 | 277 | 1 | 356 | 0 | 435 | 1 | 514 | 1 |
| 41 | 0 | 120 | 1 | 199 | 0 | 278 | 0 | 357 | 0 | 436 | 0 | 515 | 1 |
| 42 | 0 | 121 | 0 | 200 | 1 | 279 | 0 | 358 | 0 | 437 | 0 | 516 | 0 |
| 43 | 0 | 122 | 1 | 201 | 1 | 280 | 1 | 359 | 0 | 438 | 1 | 517 | 1 |
| 44 | 1 | 123 | 1 | 202 | 1 | 281 | 0 | 360 | 0 | 439 | 1 | 518 | 0 |
| 45 | 0 | 124 | 1 | 203 | 0 | 282 | 0 | 361 | 1 | 440 | 0 | 519 | 1 |
| 46 | 1 | 125 | 1 | 204 | 1 | 283 | 0 | 362 | 0 | 441 | 0 | 520 | 0 |
| 47 | 0 | 126 | 1 | 205 | 0 | 284 | 1 | 363 | 1 | 442 | 1 | 521 | 0 |
| 48 | 1 | 127 | 1 | 206 | 1 | 285 | 0 | 364 | 0 | 443 | 0 | 522 | 0 |
| 49 | 0 | 128 | 0 | 207 | 1 | 286 | 0 | 365 | 0 | 444 | 1 | 523 | 0 |
| 50 | 0 | 129 | 0 | 208 | 1 | 287 | 0 | 366 | 0 | 445 | 0 | 524 | 1 |
| 51 | 1 | 130 | 1 | 209 | 0 | 288 | 0 | 367 | 0 | 446 | 0 | 525 | 1 |
| 52 | 1 | 131 | 1 | 210 | 1 | 289 | 0 | 368 | 1 | 447 | 0 | 526 | 1 |
| 53 | 1 | 132 | 0 | 211 | 1 | 290 | 0 | 369 | 0 | 448 | 1 | 527 | 1 |
| 54 | 0 | 133 | 1 | 212 | 0 | 291 | 1 | 370 | 0 | 449 | 1 | 528 | 1 |
| 55 | 1 | 134 | 1 | 213 | 1 | 292 | 1 | 371 | 0 | 450 | 0 | 529 | 1 |
| 56 | 1 | 135 | 0 | 214 | 1 | 293 | 0 | 372 | 0 | 451 | 0 | 530 | 1 |
| 57 | 0 | 136 | 0 | 215 | 1 | 294 | 1 | 373 | 0 | 452 | 0 | 531 | 0 |
| 58 | 0 | 137 | 0 | 216 | 1 | 295 | 1 | 374 | 1 | 453 | 1 | 532 | 1 |
| 59 | 0 | 138 | 0 | 217 | 1 | 296 | 1 | 375 | 1 | 454 | 1 | 533 | 0 |
| 60 | 1 | 139 | 1 | 218 | 1 | 297 | 0 | 376 | 0 | 455 | 1 | 534 | 0 |
| 61 | 0 | 140 | 0 | 219 | 1 | 298 | 0 | 377 | 0 | 456 | 0 | 535 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 62 | 0 | 141 | 0 | 220 | 0 | 299 | 1 | 378 | 0 | 457 | 0 | 536 | 0 |
| 63 | 1 | 142 | 0 | 221 | 0 | 300 | 1 | 379 | 0 | 458 | 0 | 537 | 0 |
| 64 | 0 | 143 | 0 | 222 | 0 | 301 | 1 | 380 | 0 | 459 | 0 | 538 | 1 |
| 65 | 0 | 144 | 1 | 223 | 1 | 302 | 0 | 381 | 0 | 460 | 0 | 539 | 1 |
| 66 | 1 | 145 | 0 | 224 | 1 | 303 | 1 | 382 | 1 | 461 | 0 | 540 | 0 |
| 67 | 1 | 146 | 0 | 225 | 0 | 304 | 1 | 383 | 1 | 462 | 1 | 541 | 0 |
| 68 | 1 | 147 | 1 | 226 | 0 | 305 | 1 | 384 | 1 | 463 | 0 | 542 | 0 |
| 69 | 0 | 148 | 1 | 227 | 1 | 306 | 0 | 385 | 1 | 464 | 0 | 543 | 1 |
| 70 | 1 | 149 | 0 | 228 | 1 | 307 | 1 | 386 | 1 | 465 | 1 | 544 | 1 |
| 71 | 1 | 150 | 0 | 229 | 0 | 308 | 1 | 387 | 0 | 466 | 1 | 545 | 1 |
| 72 | 0 | 151 | 0 | 230 | 0 | 309 | 0 | 388 | 1 | 467 | 1 | 546 | 1 |
| 73 | 0 | 152 | 1 | 231 | 0 | 310 | 0 | 389 | 1 | 468 | 0 | 547 | 1 |
| 74 | 1 | 153 | 0 | 232 | 0 | 311 | 0 | 390 | 1 | 469 | 0 | | |
| 75 | 0 | 154 | 0 | 233 | 1 | 312 | 0 | 391 | 1 | 470 | 0 | | |
| 76 | 0 | 155 | 0 | 234 | 1 | 313 | 1 | 392 | 1 | 471 | 1 | | |
| 77 | 1 | 156 | 1 | 235 | 1 | 314 | 0 | 393 | 1 | 472 | 1 | | |
| 78 | 1 | 157 | 1 | 236 | 0 | 315 | 1 | 394 | 1 | 473 | 0 | | |

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