PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^3, z^3+z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^3, z^3+z^2+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^3, z^3+z^2+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{16}+z^{13}+z^{12}+z^{11}+z^8+z^7+z^5+z^3+1,\\
p_1(z)&=z^{20}+z^{19}+z^{17}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^6+z^3,\\
p_2(z)&=z^{24}+z^{20}+z^{16}+z^{14}+z^6,\\
p_3(z)&=z^{20}+z^{19}+z^{17}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^6+z^3,\\
p_4(z)&=z^{18}+z^{17}+z^{14}+z^{13}+z^{11}+z^7+z^6+z^5+z^3,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{16}+z^{13}+z^{12}+z^{11}+z^7+z^5+z^3,\\
p_1(z)&=z^{20}+z^{19}+z^{17}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^6+z^3,\\
p_2(z)&=z^{24}+z^{20}+z^{16}+z^{14}+z^6,\\
p_3(z)&=z^{20}+z^{19}+z^{17}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^6+z^3,\\
p_4(z)&=z^{18}+z^{17}+z^{14}+z^{13}+z^{11}+z^8+z^7+z^6+z^5+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 24 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] &= M^e[:6u] + x^uM^e[:5u] + x^{3u}M^e[:3u] + x^uM^e[:6u] + x^{2u}M^e[:5u] \\
&\quad + x^{4u}M^e[:3u] + x^{2u}M^e[:6u] + x^{3u}M^e[:5u] + x^{5u}M^e[:3u] \\
&\quad + x^{4u}M^e[:6u] + x^{5u}M^e[:5u] + x^{7u}M^e[:3u] + x^{7u}M^e[:5u] \\
&\quad + x^{8u}M^e[:4u] + x^{10u}M^e[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^e, \\
W^e[:12u] &= W^e[:6u] + x^uW^e[:5u] + x^{3u}W^e[:3u] + x^uW^e[:6u] + x^{2u}W^e[:5u] \\
&\quad + x^{4u}W^e[:3u] + x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{5u}W^e[:3u]
\end{aligned}$$

$$+ x^{4u}W^e[:6u] + x^{5u}W^e[:5u] + x^{7u}W^e[:3u] + x^{7u}W^e[:5u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^u M^o[:5u] + x^{3u}M^o[:3u] + x^u M^o[:6u] + x^{2u}M^o[:5u] \\ &\quad + x^{4u}M^o[:3u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{5u}M^o[:3u] \\ &\quad + x^{4u}M^o[:6u] + x^{5u}M^o[:5u] + x^{7u}M^o[:3u] + x^{7u}M^o[:5u] \\ &\quad + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^u W^o[:5u] + x^{3u}W^o[:3u] + x^u W^o[:6u] + x^{2u}W^o[:5u] \\ &\quad + x^{4u}W^o[:3u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{5u}W^o[:3u] \\ &\quad + x^{4u}W^o[:6u] + x^{5u}W^o[:5u] + x^{7u}W^o[:3u] + x^{7u}W^o[:5u] \\ &\quad + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] &= T[:6u] + x^u T[:5u] + x^{3u}T[:3u] + x^u T[:6u] + x^{2u}T[:5u] + x^{4u}T[:3u] \\ &\quad + x^{2u}T[:6u] + x^{3u}T[:5u] + x^{5u}T[:3u] + x^{4u}T[:6u] + x^{5u}T[:5u] \\ &\quad + x^{7u}T[:3u] + x^{7u}T[:5u] + x^{8u}T[:4u] + x^{10u}T[:2u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{3u}T[3u:4u] + x^u T[5u:6u] + T[6u:7u], \\ 0 &= x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + x^{5u}T[4u:5u] + x^{4u}T[5u:6u] + T[9u:10u], \\ 0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + T[10u:11u], \\ 0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1118 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.19}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.20}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.21}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.23}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.25}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.26}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.27}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.29}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{24}), E_{24}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 12$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{6u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^v W^e[:5v] + x^{3v} W^e[:3v] + x^{6u} R^e\big) \times \big(M^e[:6v] \\ &+ x^v M^e[:5v] + x^{3v} M^e[:3v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{2v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24}+x^{21}+x^{19}+x^{17}+x^{16}+x^{13}+x^{12}+x^{11}+x^8+x^7+x^6,\\
q_1(x) &= x^{21}+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{10}+x^9+x^8+x^7+x^5+x^4,\\
q_2(x) &= x^{18}+x^{10}+x^8+x^4+1,\\
q_3(x) &= x^{21}+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{10}+x^9+x^8+x^7+x^5+x^4,\\
q_4(x) &= x^{21}+x^{19}+x^{18}+x^{17}+x^{13}+x^{11}+x^{10}+x^7+x^6.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 24 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) = {}& x^{126}+x^{122}+x^{120}+x^{112}+x^{108}+x^{106}+x^{98}+x^{94}+x^{92}+x^{90}\\
&+x^{88}+x^{86}+x^{84}+x^{82}+x^{78}+x^{76}+x^{72}+x^{66}+x^{64}+x^{62}+x^{56}\\
&+x^{52}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36},\\
p_3(x) = {}& x^{116}+x^{114}+x^{112}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{96}+x^{90}\\
&+x^{88}+x^{84}+x^{80}+x^{78}+x^{76}+x^{74}+x^{70}+x^{68}+x^{66}+x^{64}+x^{60}\\
&+x^{56}+x^{54}+x^{52}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{28}+x^{20},\\
p_6(x) = {}& x^{116}+x^{112}+x^{110}+x^{104}+x^{100}+x^{98}+x^{96}+x^{94}+x^{80}+x^{74}+x^{72}\\
&+x^{70}+x^{66}+x^{64}+x^{60}+x^{58}+x^{54}+x^{52}+x^{50}+x^{38}+x^{36}+x^{30}\\
&+x^{22}+x^{20}+x^{18}+x^{12}+x^{10}+1,\\
p_9(x) = {}& x^{120}+x^{114}+x^{102}+x^{100}+x^{98}+x^{96}+x^{90}+x^{88}+x^{86}+x^{74}+x^{70}\\
&+x^{68}+x^{66}+x^{62}+x^{60}+x^{58}+x^{56}+x^{54}+x^{44}+x^{42}+x^{34}+x^{28}\\
&+x^{18}+x^{14}+x^{10}+x^8,
\end{aligned}$$

$$p_{12}(x) = x^{120} + x^{104} + x^{96} + x^{88} + x^{64} + x^{56} + x^{48} + x^{40} + x^{16} + x^{8} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{91} + x^{88} + x^{85} + x^{77} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} \\
&\quad + x^{63} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36}, \\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} \\
&\quad + x^{86} + x^{84} + x^{82} + x^{78} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} \\
&\quad + x^{46} + x^{42} + x^{40} + x^{38} + x^{36}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{111} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99} \\
&\quad + x^{96} + x^{94} + x^{93} + x^{92} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{65} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{26} + x^{25} + x^{24}, \\
p_9(x) &= x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} \\
&\quad + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18} \\
&\quad + x^{12}, \\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79} \\
&\quad + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} \\
&\quad + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21} \\
&\quad + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{91} + x^{88} + x^{85} + x^{77} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} \\
&\quad + x^{63} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36}, \\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} \\
&\quad + x^{86} + x^{84} + x^{82} + x^{78} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} \\
&\quad + x^{46} + x^{42} + x^{40} + x^{38} + x^{36},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{111} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99} \\
&\quad + x^{96} + x^{94} + x^{93} + x^{92} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{65} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{26} + x^{25} + x^{24}, \\
p_9(x) &= x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} \\
&\quad + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18} \\
&\quad + x^{12}, \\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79} \\
&\quad + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} \\
&\quad + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21} \\
&\quad + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{108} + x^{96} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} \\
&\quad + x^{74} + x^{70} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{44} + x^{40} + x^{36}, \\
p_3(x) &= x^{108} + x^{104} + x^{98} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} \\
&\quad + x^{62} + x^{58} + x^{50} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{20}, \\
p_6(x) &= x^{108} + x^{104} + x^{102} + x^{92} + x^{86} + x^{84} + x^{82} + x^{76} + x^{72} + x^{70} + x^{66} \\
&\quad + x^{62} + x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} + x^{26} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{102} + x^{98} + x^{92} + x^{90} + x^{86} + x^{84} + x^{76} + x^{66} + x^{56} + x^{52} + x^{50} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{16} \\
&\quad + x^{12} + x^{8}, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{70} + x^{66} + x^{64} + x^{54} + x^{50} + x^{48} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{114} + x^{110} + x^{102} + x^{98} + x^{96} + x^{94} + x^{88} + x^{86} + x^{82} \\
&\quad + x^{80} + x^{78} + x^{70} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} \\
&\quad + x^{36}, \\
p_3(x) &= x^{108} + x^{104} + x^{98} + x^{96} + x^{90} + x^{78} + x^{74} + x^{70} + x^{62} + x^{60} + x^{58}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{38} + x^{26} + x^{22} + x^{20},\\
p_6(x) &= x^{108} + x^{104} + x^{102} + x^{96} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{66}\\
&+ x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} + x^{34}\\
&+ x^{28} + x^{24} + x^{8} + x^{6} + x^{2} + 1,\\
p_9(x) &= x^{102} + x^{98} + x^{92} + x^{90} + x^{86} + x^{84} + x^{76} + x^{66} + x^{56} + x^{52} + x^{50}\\
&+ x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{18} + x^{16}\\
&+ x^{12} + x^{8},\\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{70} + x^{66} + x^{64} + x^{54} + x^{50} + x^{48} + x^{22} + x^{18}\\
&+ x^{16} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{117} + x^{115} + x^{111} + x^{110} + x^{109} + x^{102} + x^{98} + x^{95}\\
&+ x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{81} + x^{80} + x^{78} + x^{73}\\
&+ x^{66} + x^{60} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{48} + x^{47} + x^{46} + x^{44}\\
&+ x^{42} + x^{41} + x^{40} + x^{37} + x^{36},\\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94}\\
&+ x^{86} + x^{84} + x^{82} + x^{78} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50}\\
&+ x^{46} + x^{42} + x^{40} + x^{38} + x^{36},\\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{111} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99}\\
&+ x^{96} + x^{94} + x^{93} + x^{92} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72}\\
&+ x^{65} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{26} + x^{25} + x^{24},\\
p_9(x) &= x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84}\\
&+ x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{52}\\
&+ x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18}\\
&+ x^{12},\\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79}\\
&+ x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21}\\
&+ x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
p_0(x) = x^{126} + x^{122} + x^{117} + x^{115} + x^{111} + x^{110} + x^{109} + x^{102} + x^{98} + x^{95}
$$

$$
\begin{aligned}
&+ x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{81} + x^{80} + x^{78} + x^{73}\\
&+ x^{66} + x^{60} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{48} + x^{47} + x^{46} + x^{44}\\
&+ x^{42} + x^{41} + x^{40} + x^{37} + x^{36},\\
p_3(x) = {}& x^{120} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94}\\
&+ x^{86} + x^{84} + x^{82} + x^{78} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50}\\
&+ x^{46} + x^{42} + x^{40} + x^{38} + x^{36},\\
p_6(x) = {}& x^{117} + x^{115} + x^{114} + x^{111} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99}\\
&+ x^{96} + x^{94} + x^{93} + x^{92} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72}\\
&+ x^{65} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{26} + x^{25} + x^{24},\\
p_9(x) = {}& x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84}\\
&+ x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{58} + x^{54} + x^{52}\\
&+ x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18}\\
&+ x^{12},\\
p_{12}(x) = {}& x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79}\\
&+ x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21}\\
&+ x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{122} + x^{114} + x^{110} + x^{108} + x^{96} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80}\\
&+ x^{76} + x^{74} + x^{72} + x^{66} + x^{62} + x^{58} + x^{52} + x^{42} + x^{38} + x^{36},\\
p_3(x) = {}& x^{116} + x^{112} + x^{106} + x^{102} + x^{98} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{72} + x^{70} + x^{68} + x^{64} + x^{54} + x^{46} + x^{42} + x^{38} + x^{36} + x^{32} + x^{26}\\
&+ x^{24} + x^{20},\\
p_6(x) = {}& x^{116} + x^{114} + x^{112} + x^{110} + x^{96} + x^{94} + x^{92} + x^{88} + x^{82} + x^{80} + x^{70}\\
&+ x^{68} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{44} + x^{38} + x^{30} + x^{28} + x^{26}\\
&+ x^{22} + x^{18} + x^{16} + x^{12} + x^{10} + 1,\\
p_9(x) = {}& x^{120} + x^{114} + x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{86} + x^{74} + x^{70}\\
&+ x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{44} + x^{42} + x^{34} + x^{28}\\
&+ x^{18} + x^{14} + x^{10} + x^{8},\\
p_{12}(x) = {}& x^{120} + x^{104} + x^{96} + x^{88} + x^{64} + x^{56} + x^{48} + x^{40} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{113} + x^{112} + x^{110} \\
& + x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} \\
& + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{78} + x^{76} + x^{74} + x^{72} + x^{67} + x^{65} \\
& + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{49} + x^{45} \\
& + x^{44} + x^{41} + x^{38} + x^{37} + x^{36}, \\
p_3(x) = {} & x^{123} + x^{121} + x^{117} + x^{114} + x^{113} + x^{110} + x^{109} + x^{104} + x^{100} + x^{98} \\
& + x^{97} + x^{89} + x^{84} + x^{83} + x^{80} + x^{77} + x^{75} + x^{72} + x^{70} + x^{62} + x^{61} \\
& + x^{57} + x^{56} + x^{50} + x^{48} + x^{43} + x^{42} + x^{40} + x^{37} + x^{36} + x^{33} + x^{30} \\
& + x^{29} + x^{28}, \\
p_6(x) = {} & x^{123} + x^{121} + x^{117} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} \\
& + x^{103} + x^{102} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{84} \\
& + x^{82} + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} + x^{69} + x^{68} + x^{58} + x^{57} + x^{56} \\
& + x^{55} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} + x^{37} + x^{35} + x^{30} + x^{28} + x^{27} \\
& + x^{26} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{7} + x^{5} + x^{4} \\
& + x^{3} + x + 1, \\
p_9(x) = {} & x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{91} + x^{89} + x^{88} + x^{83} \\
& + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{43} + x^{41} + x^{40} \\
& + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16}, \\
p_{12}(x) = {} & x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{103} \\
& + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} \\
& + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} \\
& + x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{35} \\
& + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} \\
& + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{132} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} \\
& + x^{113} + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{100} + x^{98} + x^{97} + x^{91} \\
& + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{75} + x^{70} + x^{69} \\
& + x^{68} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{54} + x^{51} + x^{50} + x^{49} \\
& + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36}, \\
p_3(x) = {} & x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{87} + x^{84} \\
& + x^{82} + x^{81} + x^{80} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{60}+x^{58}+x^{57}+x^{56}+x^{53}+x^{50}+x^{49}+x^{48}+x^{45}+x^{42}+x^{41}\\
&+x^{40}+x^{39}+x^{37}+x^{36},\\
p_6(x)=&\,x^{114}+x^{108}+x^{106}+x^{104}+x^{102}+x^{96}+x^{94}+x^{92}+x^{78}+x^{74}+x^{72}\\
&+x^{62}+x^{58}+x^{56}+x^{54}+x^{48}+x^{44}+x^{40}+x^{36}+x^{34}+x^{32}+x^{30}\\
&+x^{26}+x^{24},\\
p_9(x)=&\,x^{117}+x^{114}+x^{112}+x^{110}+x^{109}+x^{108}+x^{85}+x^{82}+x^{80}+x^{78}\\
&+x^{77}+x^{76}+x^{69}+x^{66}+x^{64}+x^{62}+x^{61}+x^{60}+x^{37}+x^{34}+x^{32}\\
&+x^{30}+x^{29}+x^{28}+x^{21}+x^{18}+x^{16}+x^{14}+x^{13}+x^{12},\\
p_{12}(x)=&\,x^{120}+x^{112}+x^{108}+x^{100}+x^{96}+x^{88}+x^{80}+x^{76}+x^{72}+x^{68}+x^{60}\\
&+x^{52}+x^{48}+x^{40}+x^{32}+x^{28}+x^{24}+x^{20}+x^{12}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1].

For $\phi(M^o,1,0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{115}+x^{113}+x^{112}+x^{111}\\
&+x^{109}+x^{108}+x^{102}+x^{100}+x^{98}+x^{97}+x^{96}+x^{94}+x^{93}+x^{90}\\
&+x^{85}+x^{84}+x^{82}+x^{81}+x^{78}+x^{74}+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}\\
&+x^{62}+x^{56}+x^{51}+x^{50}+x^{49}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}\\
&+x^{36},\\
p_3(x)=&\,x^{123}+x^{120}+x^{118}+x^{117}+x^{116}+x^{113}+x^{111}+x^{110}+x^{109}+x^{107}\\
&+x^{106}+x^{103}+x^{101}+x^{100}+x^{99}+x^{96}+x^{94}+x^{93}+x^{92}+x^{91}\\
&+x^{89}+x^{88}+x^{87}+x^{84}+x^{83}+x^{82}+x^{81}+x^{78}+x^{76}+x^{74}+x^{72}\\
&+x^{71}+x^{70}+x^{67}+x^{65}+x^{64}+x^{63}+x^{60}+x^{58}+x^{57}+x^{56}+x^{53}\\
&+x^{51}+x^{50}+x^{49}+x^{46}+x^{44}+x^{43}+x^{42}+x^{39}+x^{37}+x^{36},\\
p_6(x)=&\,x^{126}+x^{120}+x^{116}+x^{112}+x^{110}+x^{108}+x^{106}+x^{102}+x^{100}+x^{98}\\
&+x^{94}+x^{92}+x^{90}+x^{86}+x^{84}+x^{82}+x^{76}+x^{72}+x^{70}+x^{68}+x^{54}\\
&+x^{48}+x^{46}+x^{44}+x^{42}+x^{36}+x^{32}+x^{28}+x^{26}+x^{24},\\
p_9(x)=&\,x^{129}+x^{126}+x^{124}+x^{122}+x^{120}+x^{118}+x^{117}+x^{116}+x^{113}+x^{110}\\
&+x^{109}+x^{108}+x^{97}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{85}+x^{84}+x^{77}\\
&+x^{74}+x^{72}+x^{70}+x^{69}+x^{68}+x^{65}+x^{62}+x^{61}+x^{60}+x^{49}+x^{46}\\
&+x^{44}+x^{42}+x^{40}+x^{38}+x^{37}+x^{36}+x^{29}+x^{26}+x^{24}+x^{22}+x^{21}\\
&+x^{20}+x^{17}+x^{14}+x^{13}+x^{12},\\
p_{12}(x)=&\,x^{132}+x^{116}+x^{112}+x^{104}+x^{100}+x^{96}+x^{80}+x^{72}+x^{68}+x^{56}+x^{52}\\
&+x^{48}+x^{32}+x^{24}+x^{20}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{122} + x^{120} + x^{116} + x^{111} + x^{110} + x^{109} + x^{104} + x^{98} + x^{92} \\
& + x^{91} + x^{90} + x^{89} + x^{88} + x^{83} + x^{82} + x^{78} + x^{76} + x^{71} + x^{68} + x^{64} \\
& + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{47} + x^{46} \\
& + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36}, \\
p_3(x) = {} & x^{123} + x^{121} + x^{115} + x^{113} + x^{110} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} \\
& + x^{98} + x^{96} + x^{93} + x^{91} + x^{90} + x^{89} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} \\
& + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} \\
& + x^{55} + x^{53} + x^{52} + x^{50} + x^{48} + x^{47} + x^{44} + x^{40} + x^{39} + x^{35} + x^{32} \\
& + x^{30} + x^{29} + x^{28}, \\
p_6(x) = {} & x^{123} + x^{121} + x^{115} + x^{113} + x^{110} + x^{108} + x^{106} + x^{105} + x^{103} + x^{101} \\
& + x^{100} + x^{99} + x^{94} + x^{90} + x^{87} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{72} \\
& + x^{71} + x^{65} + x^{62} + x^{60} + x^{57} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{40} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{23} + x^{22} \\
& + x^{20} + x^{18} + x^{16} + x^{15} + x^{14} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) = {} & x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{91} + x^{89} + x^{88} + x^{83} \\
& + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{43} + x^{41} + x^{40} \\
& + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16}, \\
p_{12}(x) = {} & x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{103} \\
& + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} \\
& + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61} \\
& + x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{35} \\
& + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} \\
& + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{113} + x^{112} + x^{110} \\
& + x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} \\
& + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{78} + x^{76} + x^{74} + x^{72} + x^{67} + x^{65} \\
& + x^{64} + x^{63} + x^{61} + x^{60} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{49} + x^{45} \\
& + x^{44} + x^{41} + x^{38} + x^{37} + x^{36}, \\
p_3(x) = {} & x^{123} + x^{121} + x^{117} + x^{114} + x^{113} + x^{110} + x^{109} + x^{104} + x^{100} + x^{98} \\
& + x^{97} + x^{89} + x^{84} + x^{83} + x^{80} + x^{77} + x^{75} + x^{72} + x^{70} + x^{62} + x^{61} \\
& + x^{57} + x^{56} + x^{50} + x^{48} + x^{43} + x^{42} + x^{40} + x^{37} + x^{36} + x^{33} + x^{30}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{29} + x^{28},\\
p_6(x) &= x^{123} + x^{121} + x^{117} + x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104}\\
&+ x^{103} + x^{102} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{84}\\
&+ x^{82} + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} + x^{69} + x^{68} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} + x^{37} + x^{35} + x^{30} + x^{28} + x^{27}\\
&+ x^{26} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{7} + x^{5} + x^{4}\\
&+ x^{3} + x + 1,\\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{91} + x^{89} + x^{88} + x^{83}\\
&+ x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{43} + x^{41} + x^{40}\\
&+ x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16},\\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{103}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61}\\
&+ x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{35}\\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20}\\
&+ x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111}\\
&+ x^{109} + x^{108} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90}\\
&+ x^{85} + x^{84} + x^{82} + x^{81} + x^{78} + x^{74} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64}\\
&+ x^{62} + x^{56} + x^{51} + x^{50} + x^{49} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39}\\
&+ x^{36},\\
p_3(x) &= x^{123} + x^{120} + x^{118} + x^{117} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107}\\
&+ x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91}\\
&+ x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76} + x^{74} + x^{72}\\
&+ x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{53}\\
&+ x^{51} + x^{50} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{39} + x^{37} + x^{36},\\
p_6(x) &= x^{126} + x^{120} + x^{116} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98}\\
&+ x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{76} + x^{72} + x^{70} + x^{68} + x^{54}\\
&+ x^{48} + x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24},\\
p_9(x) &= x^{129} + x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{117} + x^{116} + x^{113} + x^{110}\\
&+ x^{109} + x^{108} + x^{97} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{77}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62} + x^{61} + x^{60} + x^{49} + x^{46}
\end{aligned}
$$

$$+ x^{44} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} + x^{17} + x^{14} + x^{13} + x^{12},$$

$$p_{12}(x) = x^{132} + x^{116} + x^{112} + x^{104} + x^{100} + x^{96} + x^{80} + x^{72} + x^{68} + x^{56} + x^{52} + x^{48} + x^{32} + x^{24} + x^{20} + x^{8} + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$p_0(x) = x^{132} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{109} + x^{107} + x^{106} + x^{105} + x^{100} + x^{98} + x^{97} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{75} + x^{70} + x^{69} + x^{68} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36},$$

$$p_3(x) = x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{53} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36},$$

$$p_6(x) = x^{114} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{78} + x^{74} + x^{72} + x^{62} + x^{58} + x^{56} + x^{54} + x^{48} + x^{44} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24},$$

$$p_9(x) = x^{117} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108} + x^{85} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{69} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{37} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{21} + x^{18} + x^{16} + x^{14} + x^{13} + x^{12},$$

$$p_{12}(x) = x^{120} + x^{112} + x^{108} + x^{100} + x^{96} + x^{88} + x^{80} + x^{76} + x^{72} + x^{68} + x^{60} + x^{52} + x^{48} + x^{40} + x^{32} + x^{28} + x^{24} + x^{20} + x^{12} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$p_0(x) = x^{126} + x^{122} + x^{120} + x^{116} + x^{111} + x^{110} + x^{109} + x^{104} + x^{98} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{83} + x^{82} + x^{78} + x^{76} + x^{71} + x^{68} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36},$$

$$p_3(x) = x^{123} + x^{121} + x^{115} + x^{113} + x^{110} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{98} + x^{96} + x^{93} + x^{91} + x^{90} + x^{89} + x^{84} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{55} + x^{53} + x^{52} + x^{50} + x^{48} + x^{47} + x^{44} + x^{40} + x^{39} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28},$$

$$\begin{aligned}
p_6(x) &= x^{123} + x^{121} + x^{115} + x^{113} + x^{110} + x^{108} + x^{106} + x^{105} + x^{103} + x^{101}\\
&\quad + x^{100} + x^{99} + x^{94} + x^{90} + x^{87} + x^{82} + x^{81} + x^{75} + x^{74} + x^{73} + x^{72}\\
&\quad + x^{71} + x^{65} + x^{62} + x^{60} + x^{57} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43}\\
&\quad + x^{42} + x^{41} + x^{40} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{23} + x^{22}\\
&\quad + x^{20} + x^{18} + x^{16} + x^{15} + x^{14} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{91} + x^{89} + x^{88} + x^{83}\\
&\quad + x^{81} + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{43} + x^{41} + x^{40}\\
&\quad + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16},\\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{103}\\
&\quad + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80}\\
&\quad + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{63} + x^{61}\\
&\quad + x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{35}\\
&\quad + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20}\\
&\quad + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	280	[458, 459]	560	[778, 779]	840	[576, 971]
1	[3, 4]	281	[165, 222]	561	[370, 759]	841	[958, 931]
2	[5, 6]	282	[460, 32]	562	[780, 704]	842	[972, 688]
3	[7, 8]	283	[186, 461]	563	[101, 116]	843	[968, 781]
4	[9, 10]	284	[462, 463]	564	[781, 691]	844	[651, 636]
5	[11, 12]	285	[464, 139]	565	[782, 303]	845	[973, 237]
6	[13, 14]	286	[465, 466]	566	[783, 481]	846	[974, 975]
7	[15, 16]	287	[181, 9]	567	[784, 197]	847	[75, 976]
8	[17, 18]	288	[467, 468]	568	[436, 552]	848	[977, 978]
9	[19, 20]	289	[469, 470]	569	[498, 785]	849	[979, 287]
10	[21, 22]	290	[471, 160]	570	[786, 152]	850	[233, 654]
11	[23, 24]	291	[472, 473]	571	[787, 145]	851	[980, 749]
12	[25, 26]	292	[474, 475]	572	[788, 789]	852	[677, 583]
13	[27, 28]	293	[476, 477]	573	[790, 164]	853	[981, 982]
14	[29, 30]	294	[478, 213]	574	[791, 561]	854	[983, 984]
15	[31, 32]	295	[479, 424]	575	[792, 125]	855	[985, 71]
16	[33, 34]	296	[54, 212]	576	[496, 414]	856	[986, 987]
17	[35, 36]	297	[182, 220]	577	[793, 794]	857	[988, 948]
18	[37, 38]	298	[480, 423]	578	[194, 420]	858	[989, 748]
19	[39, 40]	299	[481, 482]	579	[503, 561]	859	[990, 991]

20	[41, 42]	300	[483, 38]	580	[795, 796]	860	[992, 959]
21	[43, 44]	301	[484, 408]	581	[797, 798]	861	[993, 994]
22	[40, 45]	302	[485, 486]	582	[534, 449]	862	[995, 947]
23	[46, 47]	303	[172, 487]	583	[140, 178]	863	[584, 591]
24	[48, 49]	304	[488, 51]	584	[463, 411]	864	[996, 23]
25	[50, 51]	305	[406, 489]	585	[159, 799]	865	[997, 366]
26	[52, 53]	306	[490, 459]	586	[800, 542]	866	[961, 30]
27	[54, 55]	307	[491, 492]	587	[801, 536]	867	[998, 264]
28	[56, 57]	308	[493, 494]	588	[161, 799]	868	[999, 1000]
29	[58, 59]	309	[139, 174]	589	[502, 14]	869	[1001, 388]
30	[60, 61]	310	[495, 219]	590	[60, 460]	870	[1002, 639]
31	[62, 63]	311	[155, 36]	591	[9, 220]	871	[367, 579]
32	[64, 65]	312	[187, 405]	592	[802, 543]	872	[1003, 1004]
33	[66, 67]	313	[496, 497]	593	[417, 418]	873	[1005, 976]
34	[68, 69]	314	[127, 498]	594	[197, 197]	874	[1006, 81]
35	[70, 71]	315	[499, 417]	595	[419, 543]	875	[1007, 630]
36	[72, 73]	316	[500, 501]	596	[198, 197]	876	[1008, 720]
37	[74, 75]	317	[502, 448]	597	[536, 131]	877	[1009, 1010]
38	[76, 77]	318	[205, 503]	598	[803, 804]	878	[1011, 1012]
39	[78, 79]	319	[126, 44]	599	[177, 141]	879	[1013, 716]
40	[80, 81]	320	[504, 461]	600	[805, 806]	880	[87, 1014]
41	[82, 83]	321	[166, 505]	601	[39, 184]	881	[1014, 1015]
42	[84, 85]	322	[218, 506]	602	[460, 191]	882	[1016, 391]
43	[86, 87]	323	[176, 212]	603	[807, 129]	883	[1017, 662]
44	[88, 89]	324	[507, 508]	604	[808, 2]	884	[1018, 937]
45	[90, 91]	325	[131, 208]	605	[196, 535]	885	[1019, 957]
46	[92, 93]	326	[463, 509]	606	[424, 540]	886	[1020, 1010]
47	[94, 95]	327	[510, 158]	607	[439, 804]	887	[247, 95]
48	[96, 97]	328	[481, 511]	608	[210, 49]	888	[1021, 694]
49	[98, 99]	329	[153, 512]	609	[809, 426]	889	[976, 702]
50	[100, 101]	330	[513, 222]	610	[810, 428]	890	[978, 984]
51	[102, 103]	331	[211, 434]	611	[531, 503]	891	[258, 941]
52	[26, 104]	332	[514, 222]	612	[201, 811]	892	[1022, 683]
53	[105, 106]	333	[515, 516]	613	[812, 133]	893	[568, 594]
54	[107, 108]	334	[517, 174]	614	[813, 221]	894	[1023, 337]
55	[109, 110]	335	[518, 519]	615	[814, 477]	895	[398, 646]
56	[111, 112]	336	[224, 520]	616	[153, 149]	896	[227, 1024]
57	[113, 114]	337	[449, 445]	617	[32, 789]	897	[1025, 18]
58	[115, 116]	338	[521, 522]	618	[815, 169]	898	[1000, 978]
59	[117, 118]	339	[35, 156]	619	[521, 816]	899	[1026, 340]
60	[119, 120]	340	[523, 125]	620	[817, 511]	900	[279, 1027]
61	[121, 122]	341	[524, 133]	621	[211, 546]	901	[638, 16]
62	[123, 124]	342	[525, 526]	622	[818, 40]	902	[1028, 1029]
63	[41, 125]	343	[527, 475]	623	[819, 820]	903	[349, 1030]

64	[126, 127]	344	[528, 408]	624	[201, 470]	904	[762, 1031]
65	[128, 129]	345	[529, 530]	625	[821, 405]	905	[738, 588]
66	[130, 131]	346	[531, 206]	626	[561, 475]	906	[31, 191]
67	[132, 133]	347	[441, 468]	627	[536, 418]	907	[802, 209]
68	[134, 14]	348	[14, 156]	628	[14, 36]	908	[127, 880]
69	[135, 9]	349	[222, 432]	629	[465, 822]	909	[839, 186]
70	[136, 137]	350	[532, 533]	630	[823, 152]	910	[1032, 438]
71	[138, 139]	351	[534, 422]	631	[212, 444]	911	[1033, 563]
72	[140, 141]	352	[535, 536]	632	[786, 824]	912	[124, 896]
73	[142, 143]	353	[537, 463]	633	[200, 469]	913	[1034, 167]
74	[144, 145]	354	[538, 57]	634	[810, 825]	914	[1035, 522]
75	[146, 147]	355	[417, 131]	635	[548, 826]	915	[879, 495]
76	[148, 149]	356	[131, 419]	636	[827, 43]	916	[876, 1036]
77	[150, 151]	357	[535, 417]	637	[514, 8]	917	[458, 794]
78	[152, 153]	358	[199, 536]	638	[412, 494]	918	[801, 417]
79	[33, 154]	359	[209, 52]	639	[828, 805]	919	[1037, 212]
80	[155, 156]	360	[53, 199]	640	[829, 469]	920	[817, 482]
81	[157, 158]	361	[537, 539]	641	[123, 216]	921	[546, 544]
82	[159, 160]	362	[193, 540]	642	[429, 38]	922	[446, 810]
83	[161, 160]	363	[541, 59]	643	[787, 552]	923	[1038, 52]
84	[162, 149]	364	[135, 182]	644	[808, 51]	924	[1038, 195]
85	[163, 164]	365	[132, 542]	645	[423, 136]	925	[1039, 494]
86	[165, 8]	366	[543, 195]	646	[830, 831]	926	[44, 880]
87	[166, 167]	367	[448, 36]	647	[832, 196]	927	[529, 180]
88	[168, 169]	368	[434, 544]	648	[134, 448]	928	[819, 551]
89	[170, 171]	369	[545, 492]	649	[833, 158]	929	[486, 843]
90	[172, 173]	370	[182, 10]	650	[834, 556]	930	[560, 190]
91	[174, 175]	371	[435, 504]	651	[447, 142]	931	[539, 411]
92	[176, 55]	372	[523, 42]	652	[185, 504]	932	[152, 415]
93	[177, 178]	373	[47, 519]	653	[157, 835]	933	[427, 420]
94	[179, 180]	374	[49, 546]	654	[456, 836]	934	[893, 1040]
95	[181, 182]	375	[208, 543]	655	[10, 489]	935	[871, 799]
96	[183, 184]	376	[52, 196]	656	[837, 426]	936	[418, 208]
97	[61, 32]	377	[513, 8]	657	[467, 442]	937	[507, 1041]
98	[185, 186]	378	[223, 492]	658	[838, 549]	938	[533, 806]
99	[187, 188]	379	[547, 167]	659	[839, 504]	939	[1042, 806]
100	[189, 190]	380	[548, 549]	660	[504, 840]	940	[1043, 153]
101	[61, 191]	381	[1, 422]	661	[191, 554]	941	[186, 840]
102	[192, 193]	382	[550, 551]	662	[419, 209]	942	[905, 225]
103	[194, 189]	383	[500, 526]	663	[805, 841]	943	[1044, 216]
104	[195, 196]	384	[207, 552]	664	[838, 826]	944	[1045, 212]
105	[197, 198]	385	[553, 554]	665	[464, 168]	945	[557, 509]
106	[196, 199]	386	[555, 556]	666	[550, 820]	946	[1046, 44]
107	[200, 201]	387	[557, 411]	667	[842, 843]	947	[53, 535]

108	[202, 203]	388	[558, 154]	668	[844, 507]	948	[437, 460]
109	[58, 204]	389	[434, 559]	669	[485, 426]	949	[469, 811]
110	[205, 206]	390	[560, 421]	670	[845, 546]	950	[445, 797]
111	[207, 145]	391	[402, 487]	671	[128, 846]	951	[433, 492]
112	[208, 209]	392	[561, 465]	672	[453, 520]	952	[824, 153]
113	[210, 211]	393	[168, 174]	673	[495, 506]	953	[409, 40]
114	[212, 213]	394	[562, 143]	674	[847, 125]	954	[833, 835]
115	[214, 215]	395	[493, 215]	675	[848, 849]	955	[32, 554]
116	[216, 217]	396	[206, 563]	676	[850, 851]	956	[545, 170]
117	[218, 219]	397	[524, 542]	677	[852, 463]	957	[852, 539]
118	[220, 221]	398	[564, 473]	678	[136, 798]	958	[1047, 508]
119	[7, 222]	399	[148, 512]	679	[853, 807]	959	[451, 448]
120	[139, 169]	400	[565, 556]	680	[403, 421]	960	[844, 1047]
121	[223, 170]	401	[566, 336]	681	[413, 497]	961	[1048, 189]
122	[224, 225]	402	[567, 568]	682	[191, 789]	962	[1049, 199]
123	[226, 227]	403	[569, 570]	683	[854, 855]	963	[488, 2]
124	[228, 229]	404	[571, 111]	684	[829, 201]	964	[553, 789]
125	[230, 231]	405	[572, 573]	685	[179, 530]	965	[1050, 151]
126	[232, 233]	406	[574, 322]	686	[823, 824]	966	[462, 539]
127	[234, 235]	407	[575, 576]	687	[856, 539]	967	[867, 47]
128	[236, 237]	408	[577, 578]	688	[1, 449]	968	[1047, 1041]
129	[238, 239]	409	[579, 580]	689	[857, 470]	969	[1051, 835]
130	[240, 241]	410	[581, 582]	690	[483, 430]	970	[401, 10]
131	[242, 243]	411	[583, 584]	691	[824, 415]	971	[546, 559]
132	[244, 245]	412	[585, 586]	692	[854, 849]	972	[554, 903]
133	[246, 247]	413	[587, 588]	693	[518, 143]	973	[1052, 218]
134	[248, 249]	414	[589, 590]	694	[547, 505]	974	[873, 836]
135	[250, 21]	415	[591, 117]	695	[858, 51]	975	[169, 175]
136	[251, 252]	416	[592, 593]	696	[830, 147]	976	[564, 1053]
137	[253, 254]	417	[594, 595]	697	[474, 465]	977	[499, 536]
138	[255, 256]	418	[596, 376]	698	[859, 860]	978	[472, 1053]
139	[257, 258]	419	[597, 355]	699	[410, 509]	979	[1054, 469]
140	[259, 241]	420	[598, 351]	700	[515, 861]	980	[901, 556]
141	[260, 261]	421	[599, 600]	701	[862, 198]	981	[1046, 127]
142	[262, 263]	422	[601, 251]	702	[863, 864]	982	[848, 855]
143	[264, 265]	423	[602, 603]	703	[856, 463]	983	[784, 198]
144	[266, 267]	424	[604, 605]	704	[827, 126]	984	[50, 2]
145	[268, 269]	425	[296, 606]	705	[807, 846]	985	[1037, 55]
146	[270, 271]	426	[607, 608]	706	[865, 535]	986	[1055, 805]
147	[272, 273]	427	[609, 610]	707	[209, 195]	987	[56, 1056]
148	[274, 275]	428	[611, 612]	708	[866, 507]	988	[475, 466]
149	[276, 277]	429	[613, 365]	709	[867, 142]	989	[1057, 841]
150	[278, 279]	430	[614, 615]	710	[491, 170]	990	[1058, 430]
151	[280, 27]	431	[79, 616]	711	[868, 516]	991	[1059, 877]

152	[281, 276]	432	[16, 617]	712	[562, 519]	992	[1060, 151]
153	[282, 283]	433	[618, 619]	713	[868, 861]	993	[527, 465]
154	[284, 285]	434	[297, 620]	714	[162, 512]	994	[1061, 459]
155	[286, 287]	435	[621, 576]	715	[869, 870]	995	[1062, 419]
156	[288, 289]	436	[348, 64]	716	[558, 34]	996	[887, 441]
157	[290, 291]	437	[622, 267]	717	[44, 498]	997	[1063, 209]
158	[292, 293]	438	[623, 624]	718	[871, 160]	998	[1064, 189]
159	[294, 97]	439	[625, 117]	719	[426, 843]	999	[144, 552]
160	[295, 296]	440	[322, 626]	720	[525, 501]	1000	[872, 885]
161	[73, 297]	441	[588, 627]	721	[872, 864]	1001	[792, 42]
162	[298, 299]	442	[590, 628]	722	[873, 457]	1002	[1065, 1066]
163	[300, 238]	443	[629, 630]	723	[445, 136]	1003	[420, 190]
164	[301, 302]	444	[324, 631]	724	[874, 220]	1004	[206, 561]
165	[245, 303]	445	[4, 253]	725	[795, 875]	1005	[1067, 418]
166	[304, 259]	446	[578, 632]	726	[876, 877]	1006	[853, 128]
167	[305, 306]	447	[633, 634]	727	[812, 542]	1007	[1054, 201]
168	[307, 308]	448	[635, 636]	728	[788, 554]	1008	[1067, 131]
169	[309, 310]	449	[637, 638]	729	[878, 870]	1009	[1045, 55]
170	[311, 312]	450	[639, 263]	730	[879, 218]	1010	[894, 126]
171	[313, 314]	451	[640, 288]	731	[880, 881]	1011	[889, 520]
172	[315, 316]	452	[641, 642]	732	[55, 444]	1012	[1068, 796]
173	[317, 318]	453	[643, 644]	733	[813, 489]	1013	[897, 44]
174	[319, 320]	454	[303, 645]	734	[882, 860]	1014	[8, 454]
175	[321, 322]	455	[646, 67]	735	[883, 196]	1015	[563, 475]
176	[323, 324]	456	[647, 604]	736	[884, 201]	1016	[455, 505]
177	[67, 325]	457	[648, 570]	737	[858, 2]	1017	[1069, 195]
178	[285, 326]	458	[649, 90]	738	[863, 885]	1018	[1070, 533]
179	[327, 328]	459	[650, 651]	739	[886, 811]	1019	[866, 1047]
180	[252, 329]	460	[652, 653]	740	[539, 509]	1020	[797, 137]
181	[330, 331]	461	[654, 655]	741	[887, 467]	1021	[783, 817]
182	[332, 333]	462	[656, 657]	742	[828, 533]	1022	[1071, 191]
183	[334, 90]	463	[658, 24]	743	[888, 445]	1023	[892, 504]
184	[335, 336]	464	[659, 319]	744	[889, 225]	1024	[220, 489]
185	[337, 338]	465	[372, 660]	745	[890, 516]	1025	[1072, 807]
186	[339, 340]	466	[116, 661]	746	[174, 891]	1026	[1072, 128]
187	[341, 342]	467	[644, 662]	747	[892, 186]	1027	[563, 465]
188	[343, 344]	468	[263, 663]	748	[503, 563]	1028	[793, 459]
189	[345, 346]	469	[664, 665]	749	[13, 448]	1029	[425, 486]
190	[347, 348]	470	[666, 270]	750	[893, 203]	1030	[169, 891]
191	[120, 349]	471	[667, 78]	751	[803, 440]	1031	[415, 512]
192	[350, 351]	472	[628, 602]	752	[894, 43]	1032	[1073, 318]
193	[83, 352]	473	[668, 669]	753	[818, 184]	1033	[1074, 649]
194	[353, 354]	474	[670, 671]	754	[12, 825]	1034	[342, 776]
195	[355, 356]	475	[672, 673]	755	[800, 133]	1035	[1075, 935]

196	[352, 357]	476	[674, 675]	756	[541, 204]	1036	[1076, 948]
197	[358, 359]	477	[676, 677]	757	[486, 438]	1037	[1077, 382]
198	[198, 198]	478	[678, 113]	758	[895, 188]	1038	[1078, 352]
199	[104, 360]	479	[302, 346]	759	[492, 171]	1039	[776, 720]
200	[361, 362]	480	[679, 312]	760	[11, 810]	1040	[608, 1079]
201	[363, 364]	481	[680, 649]	761	[543, 52]	1041	[71, 1080]
202	[365, 366]	482	[681, 682]	762	[216, 896]	1042	[1081, 285]
203	[318, 367]	483	[291, 568]	763	[897, 127]	1043	[1082, 713]
204	[275, 368]	484	[395, 683]	764	[184, 45]	1044	[1083, 1084]
205	[369, 370]	485	[684, 623]	765	[471, 799]	1045	[1085, 1086]
206	[371, 372]	486	[685, 686]	766	[898, 861]	1046	[1087, 395]
207	[373, 374]	487	[687, 688]	767	[423, 797]	1047	[984, 596]
208	[241, 375]	488	[689, 370]	768	[142, 519]	1048	[273, 347]
209	[376, 105]	489	[336, 299]	769	[479, 193]	1049	[1088, 594]
210	[377, 78]	490	[690, 79]	770	[899, 415]	1050	[1089, 692]
211	[378, 379]	491	[691, 692]	771	[170, 900]	1051	[696, 935]
212	[380, 381]	492	[693, 694]	772	[847, 42]	1052	[1090, 1091]
213	[382, 244]	493	[695, 696]	773	[901, 902]	1053	[1092, 640]
214	[383, 384]	494	[697, 698]	774	[475, 822]	1054	[1093, 1094]
215	[385, 386]	495	[699, 700]	775	[883, 53]	1055	[1095, 1096]
216	[387, 321]	496	[701, 702]	776	[146, 831]	1056	[110, 1003]
217	[388, 389]	497	[703, 704]	777	[492, 900]	1057	[1097, 69]
218	[390, 63]	498	[18, 705]	778	[789, 903]	1058	[721, 316]
219	[391, 392]	499	[706, 707]	779	[48, 211]	1059	[1098, 384]
220	[393, 328]	500	[265, 234]	780	[554, 904]	1060	[700, 313]
221	[20, 275]	501	[708, 323]	781	[47, 143]	1061	[1099, 580]
222	[394, 395]	502	[709, 72]	782	[905, 520]	1062	[1100, 112]
223	[396, 279]	503	[710, 711]	783	[906, 681]	1063	[1101, 936]
224	[397, 398]	504	[712, 713]	784	[907, 358]	1064	[1102, 599]
225	[399, 400]	505	[714, 715]	785	[235, 908]	1065	[653, 619]
226	[401, 220]	506	[716, 717]	786	[909, 282]	1066	[1103, 1104]
227	[402, 173]	507	[718, 707]	787	[910, 120]	1067	[1105, 325]
228	[403, 190]	508	[30, 100]	788	[911, 777]	1068	[1106, 646]
229	[404, 405]	509	[657, 719]	789	[63, 912]	1069	[1107, 605]
230	[406, 221]	510	[720, 721]	790	[913, 914]	1070	[1108, 928]
231	[407, 408]	511	[722, 723]	791	[915, 672]	1071	[1109, 700]
232	[409, 184]	512	[340, 671]	792	[316, 604]	1072	[1110, 974]
233	[404, 188]	513	[724, 725]	793	[711, 916]	1073	[880, 785]
234	[43, 127]	514	[256, 726]	794	[917, 353]	1074	[1044, 124]
235	[124, 217]	515	[727, 718]	795	[918, 755]	1075	[130, 418]
236	[410, 411]	516	[728, 729]	796	[919, 113]	1076	[884, 469]
237	[412, 215]	517	[730, 321]	797	[694, 920]	1077	[1064, 420]
238	[413, 414]	518	[731, 590]	798	[912, 921]	1078	[1049, 535]
239	[415, 149]	519	[93, 732]	799	[922, 350]	1079	[533, 841]

240	[416, 208]	520	[733, 734]	800	[326, 309]	1080	[55, 213]
241	[195, 53]	521	[735, 721]	801	[923, 596]	1081	[789, 904]
242	[199, 417]	522	[736, 611]	802	[924, 26]	1082	[1111, 481]
243	[418, 419]	523	[737, 738]	803	[925, 283]	1083	[1071, 32]
244	[420, 421]	524	[739, 256]	804	[22, 926]	1084	[1059, 1036]
245	[422, 423]	525	[740, 621]	805	[927, 260]	1085	[1112, 47]
246	[192, 424]	526	[741, 742]	806	[928, 929]	1086	[193, 1113]
247	[425, 426]	527	[743, 277]	807	[930, 642]	1087	[1111, 817]
248	[427, 189]	528	[744, 338]	808	[931, 932]	1088	[416, 419]
249	[12, 428]	529	[745, 22]	809	[933, 934]	1089	[1058, 38]
250	[183, 40]	530	[310, 746]	810	[935, 936]	1090	[815, 174]
251	[429, 430]	531	[747, 748]	811	[937, 938]	1091	[1068, 875]
252	[184, 431]	532	[749, 750]	812	[939, 319]	1092	[450, 467]
253	[8, 432]	533	[751, 752]	813	[940, 941]	1093	[809, 486]
254	[10, 221]	534	[753, 396]	814	[942, 722]	1094	[202, 1040]
255	[433, 170]	535	[627, 242]	815	[943, 229]	1095	[1055, 533]
256	[49, 434]	536	[357, 597]	816	[944, 582]	1096	[424, 1113]
257	[435, 186]	537	[6, 754]	817	[945, 672]	1097	[498, 881]
258	[37, 430]	538	[755, 593]	818	[946, 916]	1098	[832, 53]
259	[436, 145]	539	[756, 632]	819	[384, 947]	1099	[1034, 505]
260	[437, 61]	540	[346, 757]	820	[948, 949]	1100	[1062, 208]
261	[426, 438]	541	[758, 759]	821	[702, 755]	1101	[865, 199]
262	[439, 440]	542	[760, 633]	822	[277, 950]	1102	[1070, 805]
263	[138, 168]	543	[593, 761]	823	[951, 952]	1103	[834, 902]
264	[441, 442]	544	[642, 762]	824	[953, 954]	1104	[532, 805]
265	[189, 421]	545	[763, 313]	825	[24, 373]	1105	[1063, 543]
266	[443, 444]	546	[97, 764]	826	[764, 955]	1106	[862, 197]
267	[422, 445]	547	[765, 383]	827	[956, 88]	1107	[1069, 52]
268	[446, 12]	548	[766, 65]	828	[957, 958]	1108	[837, 486]
269	[447, 47]	549	[673, 767]	829	[959, 666]	1109	[1052, 495]
270	[448, 156]	550	[586, 358]	830	[768, 393]	1110	[517, 169]
271	[449, 423]	551	[665, 768]	831	[960, 961]	1111	[1114, 1115]
272	[450, 441]	552	[769, 361]	832	[962, 627]	1112	[1116, 1117]
273	[451, 14]	553	[770, 762]	833	[963, 25]	1113	[779, 740]
274	[452, 434]	554	[122, 771]	834	[312, 725]	1114	[874, 10]
275	[40, 431]	555	[772, 726]	835	[964, 965]	1115	[1035, 816]
276	[453, 225]	556	[773, 107]	836	[966, 779]	1116	[1112, 142]
277	[222, 454]	557	[774, 611]	837	[967, 968]	1117	[538, 1056]
278	[455, 167]	558	[775, 776]	838	[969, 349]		
279	[456, 457]	559	[379, 777]	839	[970, 654]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	187	0	374	1	561	1	748	1	935	0

1	0	188	0	375	1	562	0	749	1	936	0
2	0	189	1	376	0	563	0	750	1	937	0
3	0	190	1	377	1	564	0	751	0	938	0
4	0	191	0	378	0	565	1	752	1	939	1
5	0	192	1	379	0	566	0	753	1	940	1
6	1	193	1	380	0	567	0	754	1	941	1
7	0	194	1	381	0	568	1	755	1	942	1
8	1	195	1	382	1	569	0	756	0	943	1
9	0	196	0	383	1	570	0	757	0	944	1
10	0	197	1	384	0	571	1	758	0	945	1
11	0	198	0	385	0	572	1	759	1	946	1
12	1	199	1	386	0	573	0	760	0	947	1
13	1	200	1	387	1	574	0	761	1	948	1
14	1	201	0	388	1	575	1	762	1	949	1
15	0	202	0	389	1	576	1	763	0	950	0
16	0	203	0	390	1	577	0	764	1	951	1
17	1	204	0	391	0	578	1	765	0	952	1
18	0	205	0	392	1	579	1	766	0	953	1
19	0	206	0	393	1	580	0	767	1	954	1
20	1	207	0	394	0	581	0	768	1	955	1
21	0	208	1	395	0	582	1	769	0	956	0
22	0	209	0	396	0	583	1	770	0	957	0
23	0	210	1	397	0	584	1	771	0	958	1
24	0	211	0	398	0	585	1	772	0	959	0
25	1	212	0	399	0	586	1	773	1	960	1
26	0	213	1	400	1	587	0	774	1	961	0
27	1	214	1	401	0	588	1	775	1	962	0
28	0	215	0	402	0	589	0	776	0	963	1
29	1	216	1	403	0	590	0	777	1	964	0
30	0	217	1	404	1	591	0	778	1	965	0
31	0	218	1	405	1	592	0	779	0	966	0
32	1	219	0	406	0	593	1	780	0	967	0
33	0	220	1	407	1	594	1	781	0	968	1
34	1	221	1	408	0	595	0	782	1	969	1
35	1	222	0	409	1	596	0	783	0	970	0
36	1	223	0	410	0	597	0	784	0	971	0
37	0	224	0	411	1	598	0	785	0	972	0
38	0	225	0	412	1	599	0	786	0	973	1
39	0	226	0	413	0	600	1	787	1	974	1
40	0	227	0	414	0	601	0	788	1	975	0
41	1	228	0	415	0	602	1	789	1	976	0
42	1	229	1	416	0	603	1	790	0	977	1
43	0	230	0	417	1	604	0	791	0	978	1
44	1	231	1	418	0	605	0	792	1	979	1

45	1	232	1	419	0	606	0	793	0	980	1
46	0	233	1	420	0	607	1	794	1	981	1
47	0	234	0	421	0	608	1	795	0	982	1
48	0	235	0	422	0	609	1	796	0	983	0
49	1	236	0	423	1	610	0	797	0	984	1
50	1	237	1	424	0	611	1	798	0	985	0
51	1	238	0	425	1	612	0	799	1	986	1
52	0	239	0	426	1	613	1	800	1	987	0
53	1	240	0	427	1	614	1	801	0	988	1
54	1	241	1	428	1	615	1	802	0	989	0
55	1	242	1	429	1	616	1	803	0	990	0
56	0	243	0	430	1	617	1	804	0	991	0
57	1	244	0	431	0	618	1	805	1	992	1
58	1	245	0	432	0	619	1	806	0	993	0
59	1	246	1	433	1	620	1	807	1	994	0
60	0	247	1	434	1	621	0	808	0	995	1
61	0	248	1	435	0	622	1	809	1	996	1
62	0	249	1	436	1	623	0	810	0	997	1
63	1	250	0	437	1	624	0	811	0	998	0
64	1	251	1	438	0	625	1	812	1	999	0
65	0	252	1	439	1	626	1	813	1	1000	0
66	0	253	1	440	1	627	0	814	1	1001	1
67	0	254	0	441	1	628	1	815	1	1002	0
68	1	255	1	442	0	629	0	816	1	1003	0
69	0	256	1	443	0	630	1	817	1	1004	0
70	1	257	0	444	0	631	0	818	1	1005	1
71	1	258	0	445	0	632	0	819	0	1006	1
72	1	259	1	446	1	633	1	820	1	1007	1
73	1	260	1	447	1	634	0	821	1	1008	1
74	0	261	1	448	0	635	0	822	0	1009	1
75	0	262	1	449	1	636	0	823	1	1010	1
76	0	263	1	450	0	637	1	824	1	1011	0
77	1	264	1	451	0	638	1	825	0	1012	1
78	0	265	1	452	0	639	0	826	1	1013	0
79	0	266	0	453	1	640	0	827	0	1014	1
80	0	267	0	454	1	641	0	828	0	1015	0
81	0	268	1	455	1	642	1	829	0	1016	1
82	1	269	1	456	0	643	1	830	1	1017	1
83	1	270	0	457	1	644	0	831	1	1018	0
84	1	271	1	458	1	645	1	832	0	1019	0
85	1	272	0	459	0	646	1	833	1	1020	0
86	0	273	0	460	1	647	0	834	0	1021	0
87	1	274	0	461	0	648	1	835	0	1022	1
88	1	275	0	462	0	649	1	836	0	1023	1

89	0	276	1	463	1	650	0	837	0	1024	1
90	1	277	0	464	0	651	1	838	1	1025	0
91	1	278	1	465	0	652	1	839	0	1026	0
92	0	279	0	466	1	653	0	840	1	1027	0
93	0	280	1	467	0	654	0	841	1	1028	0
94	0	281	0	468	1	655	0	842	0	1029	1
95	1	282	1	469	1	656	0	843	1	1030	0
96	0	283	1	470	1	657	0	844	1	1031	0
97	0	284	0	471	0	658	1	845	1	1032	1
98	1	285	0	472	1	659	0	846	1	1033	1
99	0	286	0	473	1	660	0	847	0	1034	0
100	1	287	1	474	1	661	0	848	1	1035	0
101	0	288	0	475	1	662	0	849	1	1036	0
102	1	289	1	476	0	663	1	850	1	1037	0
103	1	290	0	477	1	664	1	851	1	1038	0
104	1	291	1	478	1	665	0	852	0	1039	0
105	1	292	1	479	0	666	1	853	1	1040	1
106	0	293	0	480	1	667	0	854	0	1041	1
107	1	294	1	481	0	668	1	855	0	1042	1
108	0	295	0	482	0	669	0	856	1	1043	1
109	1	296	1	483	1	670	1	857	1	1044	1
110	0	297	1	484	0	671	0	858	0	1045	1
111	0	298	1	485	0	672	1	859	0	1046	1
112	1	299	0	486	0	673	0	860	1	1047	1
113	1	300	1	487	1	674	0	861	0	1048	0
114	0	301	0	488	1	675	1	862	1	1049	0
115	1	302	0	489	0	676	1	863	1	1050	0
116	1	303	1	490	1	677	0	864	1	1051	1
117	1	304	1	491	1	678	1	865	1	1052	1
118	1	305	0	492	1	679	1	866	0	1053	0
119	0	306	1	493	0	680	0	867	0	1054	1
120	0	307	1	494	1	681	0	868	0	1055	1
121	0	308	0	495	0	682	0	869	1	1056	0
122	0	309	0	496	1	683	0	870	0	1057	0
123	0	310	0	497	1	684	0	871	0	1058	0
124	0	311	0	498	0	685	0	872	0	1059	0
125	0	312	0	499	1	686	1	873	1	1060	1
126	1	313	1	500	1	687	1	874	1	1061	0
127	0	314	0	501	0	688	0	875	1	1062	1
128	0	315	1	502	0	689	1	876	1	1063	1
129	0	316	1	503	1	690	1	877	1	1064	0
130	0	317	0	504	0	691	1	878	0	1065	0
131	1	318	0	505	1	692	0	879	0	1066	0
132	0	319	1	506	1	693	1	880	1	1067	1

133	1	320	0	507	0	694	0	881	1	1068	1
134	1	321	1	508	0	695	0	882	1	1069	1
135	0	322	1	509	0	696	1	883	1	1070	0
136	1	323	0	510	0	697	1	884	0	1071	1
137	1	324	0	511	1	698	0	885	0	1072	0
138	1	325	1	512	0	699	0	886	0	1073	1
139	0	326	1	513	1	700	1	887	1	1074	1
140	1	327	0	514	1	701	1	888	0	1075	0
141	1	328	0	515	1	702	1	889	0	1076	0
142	1	329	1	516	1	703	1	890	1	1077	0
143	1	330	1	517	0	704	0	891	0	1078	0
144	0	331	0	518	1	705	1	892	1	1079	0
145	1	332	1	519	0	706	1	893	1	1080	1
146	0	333	1	520	1	707	0	894	1	1081	1
147	0	334	0	521	1	708	0	895	0	1082	1
148	0	335	1	522	0	709	0	896	0	1083	1
149	1	336	0	523	0	710	1	897	0	1084	0
150	1	337	1	524	0	711	0	898	0	1085	1
151	1	338	1	525	0	712	0	899	0	1086	1
152	0	339	1	526	1	713	0	900	0	1087	1
153	1	340	0	527	0	714	1	901	1	1088	0
154	0	341	0	528	0	715	1	902	0	1089	0
155	0	342	0	529	1	716	1	903	0	1090	1
156	0	343	0	530	0	717	1	904	1	1091	1
157	0	344	0	531	1	718	0	905	1	1092	0
158	1	345	1	532	1	719	1	906	0	1093	1
159	1	346	1	533	0	720	0	907	0	1094	0
160	0	347	1	534	1	721	0	908	0	1095	1
161	1	348	1	535	0	722	1	909	0	1096	0
162	1	349	0	536	0	723	0	910	1	1097	0
163	1	350	1	537	1	724	1	911	1	1098	0
164	0	351	1	538	1	725	0	912	0	1099	0
165	0	352	0	539	0	726	1	913	0	1100	1
166	1	353	1	540	1	727	1	914	0	1101	1
167	0	354	1	541	0	728	1	915	0	1102	0
168	1	355	1	542	0	729	0	916	1	1103	0
169	0	356	1	543	1	730	0	917	1	1104	1
170	0	357	0	544	1	731	1	918	0	1105	1
171	1	358	1	545	0	732	1	919	0	1106	1
172	1	359	0	546	0	733	1	920	1	1107	1
173	0	360	1	547	0	734	1	921	0	1108	0
174	1	361	1	548	0	735	1	922	1	1109	1
175	1	362	1	549	0	736	0	923	0	1110	0
176	0	363	0	550	1	737	0	924	0	1111	1

177	0	364	0	551	0	738	1	925	0	1112	1
178	0	365	0	552	0	739	0	926	1	1113	0
179	0	366	1	553	0	740	0	927	1	1114	1
180	1	367	0	554	0	741	1	928	0	1115	0
181	1	368	1	555	0	742	0	929	0	1116	1
182	1	369	0	556	1	743	0	930	1	1117	1
183	0	370	1	557	1	744	0	931	0		
184	1	371	0	558	1	745	1	932	0		
185	1	372	0	559	0	746	1	933	1		
186	1	373	0	560	1	747	1	934	1		

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