# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^3,z^3+z+1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^3,z^3+z+1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

*Date*: 9/04/2020 09:28:14 .
2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.
*Key words and phrases.* automatic sequence, continued fraction, Thue-Morse sequence.
* This paper was automatically generated from a computer program developed by the authors, with 22.0 minutes CPU time used.

**Theorem 1.1.** *When $(a,b)=(z^3,z^3+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^9+z^6+z^5+z^3+1,\\
p_1(z)&=z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^8+z^6+z^3,\\
p_2(z)&=z^{24}+z^{16}+z^{14}+z^{10}+z^6,\\
p_3(z)&=z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^8+z^6+z^3,\\
p_4(z)&=z^{18}+z^{16}+z^{15}+z^{12}+z^{11}+z^9+z^5+z^4+z^3,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^9+z^6+z^5+z^4+z^3,\\
p_1(z)&=z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^8+z^6+z^3,\\
p_2(z)&=z^{24}+z^{16}+z^{14}+z^{10}+z^6,\\
p_3(z)&=z^{19}+z^{18}+z^{17}+z^{14}+z^{12}+z^8+z^6+z^3,\\
p_4(z)&=z^{18}+z^{16}+z^{15}+z^{12}+z^{11}+z^9+z^5+z^3+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$
$$M^o = W^e \cdot M^e, \tag{2.4}$$
$$W^e = M^o \cdot W^o, \tag{2.5}$$
$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 24 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ∎

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] = {} & M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{2u}M^e[:6u] + x^{4u}M^e[:4u] \\
& + x^{5u}M^e[:3u] + x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{6u}M^e[:3u] \\
& + x^{4u}M^e[:6u] + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{7u}M^e[:5u] \\
& + x^{10u}M^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^e, \\
W^e[:12u] = {} & W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] \\
& + x^{5u}W^e[:3u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u]
\end{aligned}$$

$$+ x^{4u}W^e[:6u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{7u}W^e[:5u] + x^{10u}W^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^e.$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$\begin{aligned} M^o[:12u] = {} & M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] \\ & + x^{5u}M^o[:3u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u] \\ & + x^{4u}M^o[:6u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{7u}M^o[:5u] \\ & + x^{10u}M^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^o, \\ W^o[:12u] = {} & W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] \\ & + x^{5u}W^o[:3u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u] \\ & + x^{4u}W^o[:6u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{7u}W^o[:5u] \\ & + x^{10u}W^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^o. \end{aligned}$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] = {} & T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{2u}T[:6u] + x^{4u}T[:4u] + x^{5u}T[:3u] \\ & + x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{4u}T[:6u] + x^{6u}T[:4u] \\ & + x^{7u}T[:3u] + x^{7u}T[:5u] + x^{10u}T[:2u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{3u}T[3u:4u] + x^{2u}T[4u:5u] + T[6u:7u], \\ 0 &= x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u], \\ 0 &= x^{5u}T[3u:4u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u], \\ 0 &= x^{10u}T[:u] + x^{7u}T[3u:4u] + T[10u:11u], \\ 0 &= x^{10u}T[u:2u] + x^{7u}T[4u:5u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[1w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 976 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 110)), \tag{2.19}$$
$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.20}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.21}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.23}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 110)), \tag{2.25}$$
$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.26}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.27}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.29}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{32}), E_{32}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 16$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^{2u} W^e[:4u] + x^{3u} W^e[:3u] + x^{6u} R^e.$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$M_{2k+1} = M^o[:6u] + x^{2u} M^o[:4u] + x^{3u} M^o[:3u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{2u} W^o[:4u] + x^{3u} W^o[:3u] + x^{6u} R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^{2v}W^e[:4v] + x^{3v}W^e[:3v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ &+ x^{2v}M^e[:4v] + x^{3v}M^e[:3v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{12v}$, to

$$W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{6v}W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.31) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24}+x^{21}+x^{19}+x^{18}+x^{15}+x^{14}+x^{13}+x^{10}+x^9+x^8+x^6,\\
q_1(x) &= x^{21}+x^{18}+x^{16}+x^{12}+x^{10}+x^7+x^6+x^5,\\
q_2(x) &= x^{18}+x^{14}+x^{10}+x^8+1,\\
q_3(x) &= x^{21}+x^{18}+x^{16}+x^{12}+x^{10}+x^7+x^6+x^5,\\
q_4(x) &= x^{21}+x^{20}+x^{19}+x^{15}+x^{13}+x^{12}+x^9+x^8+x^6.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 24 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{120}+x^{116}+x^{108}+x^{104}+x^{102}+x^{100}+x^{92}+x^{90}\\
&\quad+x^{84}+x^{82}+x^{76}+x^{74}+x^{72}+x^{68}+x^{66}+x^{64}+x^{62}+x^{58}+x^{56}\\
&\quad+x^{52}+x^{48}+x^{44}+x^{42}+x^{40}+x^{36},\\
p_3(x) &= x^{118}+x^{116}+x^{112}+x^{110}+x^{104}+x^{102}+x^{100}+x^{96}+x^{94}+x^{92}\\
&\quad+x^{86}+x^{84}+x^{82}+x^{78}+x^{74}+x^{72}+x^{70}+x^{68}+x^{64}+x^{62}+x^{58}\\
&\quad+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{40}+x^{36}+x^{34}+x^{32}+x^{30}\\
&\quad+x^{24}+x^{22},\\
p_6(x) &= x^{118}+x^{116}+x^{114}+x^{104}+x^{102}+x^{100}+x^{96}+x^{90}+x^{84}+x^{76}\\
&\quad+x^{74}+x^{72}+x^{70}+x^{68}+x^{62}+x^{60}+x^{58}+x^{56}+x^{50}+x^{48}+x^{40}\\
&\quad+x^{38}+x^{36}+x^{34}+x^{30}+x^{26}+x^{22}+x^{20}+x^{18}+x^{14}+x^{10}+1,\\
p_9(x) &= x^{120}+x^{114}+x^{112}+x^{100}+x^{98}+x^{96}+x^{86}+x^{74}+x^{70}+x^{66}+x^{64}\\
&\quad+x^{60}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{40}+x^{32}+x^{28}+x^{14}
\end{aligned}$$

$$+ x^{12} + x^{10},$$
$$p_{12}(x) = x^{120} + x^{112} + x^{104} + x^{80} + x^{72} + x^{64} + x^{56} + x^{32} + x^{24} + x^{16} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{112} \\
& + x^{111} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{93} \\
& + x^{92} + x^{91} + x^{87} + x^{85} + x^{84} + x^{81} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} \\
& + x^{65} + x^{61} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{45} + x^{43} + x^{42} \\
& + x^{39} + x^{38} + x^{36}, \\
p_3(x) = {} & x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109} + x^{104} + x^{103} \\
& + x^{102} + x^{101} + x^{100} + x^{99} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{80} \\
& + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{60} + x^{59} \\
& + x^{56} + x^{55} + x^{46} + x^{45}, \\
p_6(x) = {} & x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} \\
& + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{85} + x^{84} + x^{82} \\
& + x^{81} + x^{80} + x^{79} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{49} + x^{48} \\
& + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30}, \\
p_9(x) = {} & x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} \\
& + x^{100} + x^{99} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
& + x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
& + x^{62} + x^{61} + x^{56} + x^{55} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} \\
& + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} \\
& + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15}, \\
p_{12}(x) = {} & x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{95} \\
& + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{63} + x^{62} + x^{60} \\
& + x^{59} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} \\
& + x^{40} + x^{35} + x^{34} + x^{32} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2} \\
& + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{112} \\
& + x^{111} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{93} \\
& + x^{92} + x^{91} + x^{87} + x^{85} + x^{84} + x^{81} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} \\
& + x^{65} + x^{61} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{45} + x^{43} + x^{42}
\end{aligned}$$

$$\begin{aligned}
&+ x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109} + x^{104} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{80}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{60} + x^{59}\\
&+ x^{56} + x^{55} + x^{46} + x^{45},\\
p_6(x) &= x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{85} + x^{84} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30},\\
p_9(x) &= x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{99} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&+ x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{56} + x^{55} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15},\\
p_{12}(x) &= x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{95}\\
&+ x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42}\\
&+ x^{40} + x^{35} + x^{34} + x^{32} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{110} + x^{96} + x^{90} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72}\\
&+ x^{68} + x^{66} + x^{62} + x^{60} + x^{52} + x^{48} + x^{44} + x^{36},\\
p_3(x) &= x^{108} + x^{106} + x^{100} + x^{94} + x^{92} + x^{82} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66}\\
&+ x^{64} + x^{62} + x^{58} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} + x^{34} + x^{30} + x^{26}\\
&+ x^{24} + x^{22},\\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{78} + x^{70}\\
&+ x^{68} + x^{66} + x^{60} + x^{48} + x^{46} + x^{44} + x^{40} + x^{34} + x^{32} + x^{30} + x^{26}\\
&+ x^{24} + x^{22} + x^{10} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{102} + x^{100} + x^{94} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66}\\
&+ x^{64} + x^{62} + x^{58} + x^{56} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{26} + x^{20}\\
&+ x^{12} + x^{10},\\
p_{12}(x) &= x^{102} + x^{100} + x^{96} + x^{86} + x^{84} + x^{80} + x^{54} + x^{52} + x^{48} + x^{38} + x^{36}
\end{aligned}$$

$$+ x^{32} + x^{6} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{114} + x^{112} + x^{110} + x^{108} + x^{102} + x^{100} + x^{98} + x^{92} \\
&\quad + x^{90} + x^{86} + x^{74} + x^{68} + x^{66} + x^{62} + x^{54} + x^{52} + x^{50} + x^{48} + x^{36}, \\
p_3(x) &= x^{108} + x^{106} + x^{100} + x^{96} + x^{92} + x^{84} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{60} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} + x^{32} + x^{26} \\
&\quad + x^{24} + x^{22}, \\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{96} + x^{92} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{58} \\
&\quad + x^{54} + x^{50} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{10} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{102} + x^{100} + x^{94} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{58} + x^{56} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{26} + x^{20} \\
&\quad + x^{12} + x^{10}, \\
p_{12}(x) &= x^{102} + x^{100} + x^{96} + x^{86} + x^{84} + x^{80} + x^{54} + x^{52} + x^{48} + x^{38} + x^{36} \\
&\quad + x^{32} + x^{6} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{117} + x^{116} + x^{112} + x^{111} + x^{108} + x^{102} + x^{101} + x^{97} \\
&\quad + x^{95} + x^{91} + x^{89} + x^{88} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{74} + x^{71} \\
&\quad + x^{70} + x^{68} + x^{67} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} \\
&\quad + x^{50} + x^{45} + x^{43} + x^{42} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{80} \\
&\quad + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{60} + x^{59} \\
&\quad + x^{56} + x^{55} + x^{46} + x^{45}, \\
p_6(x) &= x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} \\
&\quad + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{85} + x^{84} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{49} + x^{48} \\
&\quad + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30}, \\
p_9(x) &= x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} \\
&\quad + x^{100} + x^{99} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{61} + x^{56} + x^{55} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15},\\
p_{12}(x) &= x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{95}\\
&+ x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42}\\
&+ x^{40} + x^{35} + x^{34} + x^{32} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{117} + x^{116} + x^{112} + x^{111} + x^{108} + x^{102} + x^{101} + x^{97}\\
&+ x^{95} + x^{91} + x^{89} + x^{88} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{74} + x^{71}\\
&+ x^{70} + x^{68} + x^{67} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51}\\
&+ x^{50} + x^{45} + x^{43} + x^{42} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109} + x^{104} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{80}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{60} + x^{59}\\
&+ x^{56} + x^{55} + x^{46} + x^{45},\\
p_6(x) &= x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{85} + x^{84} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30},\\
p_9(x) &= x^{114} + x^{113} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{99} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&+ x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{56} + x^{55} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15},\\
p_{12}(x) &= x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{95}\\
&+ x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42}\\
&+ x^{40} + x^{35} + x^{34} + x^{32} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1].

For $\phi(W^e,1,1)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{114}+x^{112}+x^{110}+x^{96}+x^{94}+x^{92}+x^{90}+x^{82}+x^{80}\\
&\quad +x^{74}+x^{72}+x^{70}+x^{66}+x^{64}+x^{62}+x^{60}+x^{42}+x^{40}+x^{36},\\
p_3(x) &= x^{118}+x^{116}+x^{114}+x^{112}+x^{110}+x^{108}+x^{100}+x^{98}+x^{96}+x^{94}\\
&\quad +x^{88}+x^{84}+x^{78}+x^{76}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}\\
&\quad +x^{42}+x^{40}+x^{34}+x^{24}+x^{22},\\
p_6(x) &= x^{118}+x^{116}+x^{98}+x^{92}+x^{90}+x^{88}+x^{86}+x^{82}+x^{80}+x^{60}+x^{58}\\
&\quad +x^{52}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}\\
&\quad +x^{24}+x^{22}+x^{18}+x^{14}+x^{10}+1,\\
p_9(x) &= x^{120}+x^{114}+x^{112}+x^{100}+x^{98}+x^{96}+x^{86}+x^{74}+x^{70}+x^{66}+x^{64}\\
&\quad +x^{60}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{40}+x^{32}+x^{28}+x^{14}\\
&\quad +x^{12}+x^{10},\\
p_{12}(x) &= x^{120}+x^{112}+x^{104}+x^{80}+x^{72}+x^{64}+x^{56}+x^{32}+x^{24}+x^{16}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{123}+x^{122}+x^{121}+x^{116}+x^{115}+x^{114}+x^{113}+x^{111}\\
&\quad +x^{106}+x^{104}+x^{101}+x^{99}+x^{97}+x^{90}+x^{88}+x^{82}+x^{78}+x^{74}+x^{73}\\
&\quad +x^{70}+x^{67}+x^{66}+x^{64}+x^{63}+x^{59}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}\\
&\quad +x^{50}+x^{48}+x^{45}+x^{44}+x^{43}+x^{40}+x^{39}+x^{38}+x^{36},\\
p_3(x) &= x^{123}+x^{122}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{109}+x^{108}\\
&\quad +x^{106}+x^{105}+x^{101}+x^{99}+x^{97}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}+x^{89}\\
&\quad +x^{87}+x^{84}+x^{80}+x^{78}+x^{77}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}\\
&\quad +x^{68}+x^{67}+x^{65}+x^{62}+x^{61}+x^{59}+x^{57}+x^{56}+x^{55}+x^{53}+x^{50}\\
&\quad +x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{43}+x^{41}+x^{38}+x^{35}+x^{34}+x^{32},\\
p_6(x) &= x^{123}+x^{122}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{109}+x^{106}\\
&\quad +x^{102}+x^{99}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{86}+x^{84}+x^{81}+x^{80}\\
&\quad +x^{77}+x^{74}+x^{73}+x^{71}+x^{68}+x^{67}+x^{66}+x^{64}+x^{62}+x^{55}+x^{54}\\
&\quad +x^{53}+x^{52}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{39}+x^{35}+x^{34}+x^{33}\\
&\quad +x^{27}+x^{24}+x^{23}+x^{22}+x^{21}+x^{19}+x^{16}+x^{11}+x^{10}+x^{8}+x^{3}+x^{2}\\
&\quad +1,\\
p_9(x) &= x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{107}+x^{106}+x^{104}+x^{103}\\
&\quad +x^{102}+x^{100}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{59}+x^{58}+x^{56}\\
&\quad +x^{55}+x^{54}+x^{52}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20},\\
p_{12}(x) &= x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{111}+x^{110}+x^{108}+x^{103}
\end{aligned}$$

$$+ x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{35} + x^{34} + x^{32} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{115} + x^{112} + x^{111} \\
&\quad + x^{109} + x^{107} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{91} + x^{89} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{74} + x^{72} \\
&\quad + x^{71} + x^{69} + x^{67} + x^{61} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{45} + x^{44} + x^{43} + x^{42} + x^{36}, \\
p_3(x) &= x^{111} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{95} + x^{92} + x^{90} + x^{86} \\
&\quad + x^{85} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} + x^{71} + x^{68} + x^{66} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{47} + x^{46} + x^{45}, \\
p_6(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{96} + x^{92} + x^{90} + x^{82} + x^{78} + x^{76} \\
&\quad + x^{70} + x^{68} + x^{66} + x^{46} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30}, \\
p_9(x) &= x^{117} + x^{114} + x^{112} + x^{111} + x^{101} + x^{98} + x^{96} + x^{95} + x^{69} + x^{66} + x^{64} \\
&\quad + x^{63} + x^{53} + x^{50} + x^{48} + x^{47} + x^{21} + x^{18} + x^{16} + x^{15}, \\
p_{12}(x) &= x^{120} + x^{116} + x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{60} \\
&\quad + x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{110} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} \\
&\quad + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{45} + x^{43} + x^{42} + x^{36}, \\
p_3(x) &= x^{123} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{110} + x^{109} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} \\
&\quad + x^{90} + x^{89} + x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{69} + x^{66} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{52} \\
&\quad + x^{50} + x^{47} + x^{46} + x^{45}, \\
p_6(x) &= x^{126} + x^{120} + x^{116} + x^{114} + x^{106} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{58}\\
&+ x^{54} + x^{52} + x^{50} + x^{42} + x^{36} + x^{32} + x^{30},\\
p_9(x) &= x^{129} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{117} + x^{114}\\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104}\\
&+ x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74}\\
&+ x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59}\\
&+ x^{58} + x^{56} + x^{55} + x^{53} + x^{50} + x^{48} + x^{47} + x^{33} + x^{30} + x^{29} + x^{28}\\
&+ x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{18} + x^{16} + x^{15},\\
p_{12}(x) &= x^{132} + x^{124} + x^{112} + x^{108} + x^{100} + x^{84} + x^{80} + x^{76} + x^{64} + x^{60} + x^{52}\\
&+ x^{36} + x^{32} + x^{28} + x^{20} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{112} + x^{111} + x^{108} + x^{106} + x^{104} + x^{102}\\
&+ x^{100} + x^{98} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{81} + x^{79}\\
&+ x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{66} + x^{62} + x^{61} + x^{59} + x^{58}\\
&+ x^{57} + x^{53} + x^{52} + x^{47} + x^{46} + x^{45} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{112} + x^{110} + x^{107} + x^{106} + x^{105} + x^{104}\\
&+ x^{100} + x^{95} + x^{93} + x^{92} + x^{91} + x^{87} + x^{86} + x^{85} + x^{82} + x^{80} + x^{78}\\
&+ x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{61} + x^{59} + x^{55}\\
&+ x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38}\\
&+ x^{35} + x^{34} + x^{32},\\
p_6(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{112} + x^{110} + x^{108} + x^{107} + x^{106} + x^{105}\\
&+ x^{98} + x^{96} + x^{95} + x^{92} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{78} + x^{77} + x^{75} + x^{71} + x^{70} + x^{69} + x^{67} + x^{64} + x^{62} + x^{59} + x^{58}\\
&+ x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{41} + x^{40} + x^{36} + x^{30} + x^{26} + x^{21} + x^{20} + x^{19} + x^{16} + x^{11} + x^{10}\\
&+ x^{8} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{107} + x^{106} + x^{104} + x^{103}\\
&+ x^{102} + x^{100} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{59} + x^{58} + x^{56}\\
&+ x^{55} + x^{54} + x^{52} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20},\\
p_{12}(x) &= x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{111} + x^{110} + x^{108} + x^{103}\\
&+ x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88}\\
&+ x^{83} + x^{82} + x^{80} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62}\\
&+ x^{60} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43}
\end{aligned}
$$

$$+x^{42}+x^{40}+x^{35}+x^{34}+x^{32}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}$$
$$+x^{15}+x^{14}+x^{12}+x^{11}+x^{10}+x^{8}+x^{3}+x^{2}+1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{123}+x^{122}+x^{121}+x^{116}+x^{115}+x^{114}+x^{113}+x^{111}\\
&\quad+x^{106}+x^{104}+x^{101}+x^{99}+x^{97}+x^{90}+x^{88}+x^{82}+x^{78}+x^{74}+x^{73}\\
&\quad+x^{70}+x^{67}+x^{66}+x^{64}+x^{63}+x^{59}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}\\
&\quad+x^{50}+x^{48}+x^{45}+x^{44}+x^{43}+x^{40}+x^{39}+x^{38}+x^{36},\\
p_3(x) &= x^{123}+x^{122}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{109}+x^{108}\\
&\quad+x^{106}+x^{105}+x^{101}+x^{99}+x^{97}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}+x^{89}\\
&\quad+x^{87}+x^{84}+x^{80}+x^{78}+x^{77}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}\\
&\quad+x^{68}+x^{67}+x^{65}+x^{62}+x^{61}+x^{59}+x^{57}+x^{56}+x^{55}+x^{53}+x^{50}\\
&\quad+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{43}+x^{41}+x^{38}+x^{35}+x^{34}+x^{32},\\
p_6(x) &= x^{123}+x^{122}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{109}+x^{106}\\
&\quad+x^{102}+x^{99}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{86}+x^{84}+x^{81}+x^{80}\\
&\quad+x^{77}+x^{74}+x^{73}+x^{71}+x^{68}+x^{67}+x^{66}+x^{64}+x^{62}+x^{55}+x^{54}\\
&\quad+x^{53}+x^{52}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{39}+x^{35}+x^{34}+x^{33}\\
&\quad+x^{27}+x^{24}+x^{23}+x^{22}+x^{21}+x^{19}+x^{16}+x^{11}+x^{10}+x^{8}+x^{3}+x^{2}\\
&\quad+1,\\
p_9(x) &= x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{107}+x^{106}+x^{104}+x^{103}\\
&\quad+x^{102}+x^{100}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{59}+x^{58}+x^{56}\\
&\quad+x^{55}+x^{54}+x^{52}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20},\\
p_{12}(x) &= x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{111}+x^{110}+x^{108}+x^{103}\\
&\quad+x^{102}+x^{100}+x^{99}+x^{98}+x^{96}+x^{95}+x^{94}+x^{92}+x^{91}+x^{90}+x^{88}\\
&\quad+x^{83}+x^{82}+x^{80}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{63}+x^{62}\\
&\quad+x^{60}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}+x^{44}+x^{43}\\
&\quad+x^{42}+x^{40}+x^{35}+x^{34}+x^{32}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}\\
&\quad+x^{15}+x^{14}+x^{12}+x^{11}+x^{10}+x^{8}+x^{3}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{123}+x^{122}+x^{121}+x^{117}+x^{115}+x^{114}+x^{113}+x^{112}\\
&\quad+x^{110}+x^{109}+x^{108}+x^{103}+x^{101}+x^{100}+x^{99}+x^{98}+x^{96}+x^{95}\\
&\quad+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{85}+x^{84}+x^{82}+x^{81}\\
&\quad+x^{80}+x^{78}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{68}+x^{66}+x^{65}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{54} + x^{51} + x^{45} + x^{43} + x^{42} + x^{36},\\
p_3(x) &= x^{123} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91}\\
&+ x^{90} + x^{89} + x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72}\\
&+ x^{71} + x^{69} + x^{66} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{52}\\
&+ x^{50} + x^{47} + x^{46} + x^{45},\\
p_6(x) &= x^{126} + x^{120} + x^{116} + x^{114} + x^{106} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{58}\\
&+ x^{54} + x^{52} + x^{50} + x^{42} + x^{36} + x^{32} + x^{30},\\
p_9(x) &= x^{129} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{117} + x^{114}\\
&+ x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104}\\
&+ x^{103} + x^{101} + x^{98} + x^{96} + x^{95} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74}\\
&+ x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59}\\
&+ x^{58} + x^{56} + x^{55} + x^{53} + x^{50} + x^{48} + x^{47} + x^{33} + x^{30} + x^{29} + x^{28}\\
&+ x^{27} + x^{26} + x^{24} + x^{23} + x^{21} + x^{18} + x^{16} + x^{15},\\
p_{12}(x) &= x^{132} + x^{124} + x^{112} + x^{108} + x^{100} + x^{84} + x^{80} + x^{76} + x^{64} + x^{60} + x^{52}\\
&+ x^{36} + x^{32} + x^{28} + x^{20} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{115} + x^{112} + x^{111}\\
&+ x^{109} + x^{107} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94}\\
&+ x^{93} + x^{91} + x^{89} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{74} + x^{72}\\
&+ x^{71} + x^{69} + x^{67} + x^{61} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48}\\
&+ x^{45} + x^{44} + x^{43} + x^{42} + x^{36},\\
p_3(x) &= x^{111} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{95} + x^{92} + x^{90} + x^{86}\\
&+ x^{85} + x^{84} + x^{82} + x^{79} + x^{78} + x^{77} + x^{71} + x^{68} + x^{66} + x^{62} + x^{61}\\
&+ x^{60} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{47} + x^{46} + x^{45},\\
p_6(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{96} + x^{92} + x^{90} + x^{82} + x^{78} + x^{76}\\
&+ x^{70} + x^{68} + x^{66} + x^{46} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30},\\
p_9(x) &= x^{117} + x^{114} + x^{112} + x^{111} + x^{101} + x^{98} + x^{96} + x^{95} + x^{69} + x^{66} + x^{64}\\
&+ x^{63} + x^{53} + x^{50} + x^{48} + x^{47} + x^{21} + x^{18} + x^{16} + x^{15},\\
p_{12}(x) &= x^{120} + x^{116} + x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{60}\\
&+ x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1].

For $\phi(W^o,1,1)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{120}+x^{118}+x^{112}+x^{111}+x^{108}+x^{106}+x^{104}+x^{102}\\
&+x^{100}+x^{98}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}+x^{84}+x^{83}+x^{81}+x^{79}\\
&+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{66}+x^{62}+x^{61}+x^{59}+x^{58}\\
&+x^{57}+x^{53}+x^{52}+x^{47}+x^{46}+x^{45}+x^{43}+x^{40}+x^{39}+x^{38}+x^{36},\\
p_3(x) &= x^{123}+x^{122}+x^{119}+x^{118}+x^{112}+x^{110}+x^{107}+x^{106}+x^{105}+x^{104}\\
&+x^{100}+x^{95}+x^{93}+x^{92}+x^{91}+x^{87}+x^{86}+x^{85}+x^{82}+x^{80}+x^{78}\\
&+x^{76}+x^{75}+x^{74}+x^{72}+x^{71}+x^{69}+x^{68}+x^{66}+x^{61}+x^{59}+x^{55}\\
&+x^{54}+x^{52}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{42}+x^{41}+x^{40}+x^{38}\\
&+x^{35}+x^{34}+x^{32},\\
p_6(x) &= x^{123}+x^{122}+x^{119}+x^{118}+x^{112}+x^{110}+x^{108}+x^{107}+x^{106}+x^{105}\\
&+x^{98}+x^{96}+x^{95}+x^{92}+x^{90}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}+x^{80}\\
&+x^{78}+x^{77}+x^{75}+x^{71}+x^{70}+x^{69}+x^{67}+x^{64}+x^{62}+x^{59}+x^{58}\\
&+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}\\
&+x^{41}+x^{40}+x^{36}+x^{30}+x^{26}+x^{21}+x^{20}+x^{19}+x^{16}+x^{11}+x^{10}\\
&+x^{8}+x^{3}+x^{2}+1,\\
p_9(x) &= x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{107}+x^{106}+x^{104}+x^{103}\\
&+x^{102}+x^{100}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{59}+x^{58}+x^{56}\\
&+x^{55}+x^{54}+x^{52}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20},\\
p_{12}(x) &= x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{111}+x^{110}+x^{108}+x^{103}\\
&+x^{102}+x^{100}+x^{99}+x^{98}+x^{96}+x^{95}+x^{94}+x^{92}+x^{91}+x^{90}+x^{88}\\
&+x^{83}+x^{82}+x^{80}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{63}+x^{62}\\
&+x^{60}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}+x^{44}+x^{43}\\
&+x^{42}+x^{40}+x^{35}+x^{34}+x^{32}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}\\
&+x^{15}+x^{14}+x^{12}+x^{11}+x^{10}+x^{8}+x^{3}+x^{2}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 196 | [336, 337] | 392 | [584, 81] | 588 | [177, 52] | 784 | [896, 862] |
| 1 | [3, 4] | 197 | [338, 339] | 393 | [585, 199] | 589 | [731, 774] | 785 | [275, 252] |
| 2 | [5, 6] | 198 | [340, 341] | 394 | [586, 20] | 590 | [775, 741] | 786 | [897, 898] |
| 3 | [7, 8] | 199 | [342, 343] | 395 | [587, 588] | 591 | [416, 728] | 787 | [899, 622] |
| 4 | [9, 10] | 200 | [344, 345] | 396 | [514, 282] | 592 | [776, 490] | 788 | [900, 620] |
| 5 | [11, 12] | 201 | [346, 347] | 397 | [589, 590] | 593 | [486, 45] | 789 | [901, 895] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | [13, 14] | 202 | [348, 125] | 398 | [591, 529] | 594 | [136, 777] | 790 | [289, 902] |
| 7 | [15, 16] | 203 | [349, 350] | 399 | [592, 204] | 595 | [160, 778] | 791 | [21, 903] |
| 8 | [17, 18] | 204 | [351, 352] | 400 | [322, 593] | 596 | [407, 146] | 792 | [668, 904] |
| 9 | [19, 20] | 205 | [353, 354] | 401 | [237, 594] | 597 | [382, 731] | 793 | [517, 905] |
| 10 | [21, 22] | 206 | [355, 356] | 402 | [112, 595] | 598 | [492, 155] | 794 | [906, 840] |
| 11 | [17, 23] | 207 | [357, 37] | 403 | [270, 517] | 599 | [385, 726] | 795 | [907, 704] |
| 12 | [24, 25] | 208 | [196, 358] | 404 | [596, 227] | 600 | [190, 142] | 796 | [527, 559] |
| 13 | [26, 27] | 209 | [359, 360] | 405 | [566, 597] | 601 | [408, 490] | 797 | [708, 908] |
| 14 | [28, 29] | 210 | [361, 362] | 406 | [598, 599] | 602 | [147, 411] | 798 | [106, 600] |
| 15 | [30, 31] | 211 | [33, 158] | 407 | [323, 257] | 603 | [144, 39] | 799 | [832, 336] |
| 16 | [32, 33] | 212 | [31, 360] | 408 | [253, 315] | 604 | [469, 779] | 800 | [515, 909] |
| 17 | [34, 35] | 213 | [39, 146] | 409 | [258, 531] | 605 | [395, 738] | 801 | [104, 578] |
| 18 | [36, 37] | 214 | [363, 364] | 410 | [600, 254] | 606 | [762, 162] | 802 | [910, 61] |
| 19 | [38, 39] | 215 | [164, 118] | 411 | [315, 601] | 607 | [780, 399] | 803 | [110, 911] |
| 20 | [40, 41] | 216 | [365, 366] | 412 | [602, 600] | 608 | [410, 420] | 804 | [912, 331] |
| 21 | [42, 43] | 217 | [367, 138] | 413 | [603, 213] | 609 | [143, 420] | 805 | [913, 591] |
| 22 | [44, 45] | 218 | [368, 183] | 414 | [201, 604] | 610 | [412, 413] | 806 | [582, 914] |
| 23 | [46, 42] | 219 | [53, 369] | 415 | [240, 605] | 611 | [781, 159] | 807 | [915, 916] |
| 24 | [47, 48] | 220 | [370, 371] | 416 | [550, 606] | 612 | [354, 427] | 808 | [917, 677] |
| 25 | [49, 50] | 221 | [372, 164] | 417 | [248, 341] | 613 | [114, 447] | 809 | [633, 585] |
| 26 | [51, 52] | 222 | [156, 131] | 418 | [607, 226] | 614 | [494, 422] | 810 | [904, 646] |
| 27 | [53, 54] | 223 | [373, 49] | 419 | [263, 608] | 615 | [782, 783] | 811 | [66, 715] |
| 28 | [55, 43] | 224 | [374, 119] | 420 | [609, 610] | 616 | [784, 785] | 812 | [918, 593] |
| 29 | [56, 57] | 225 | [375, 371] | 421 | [611, 69] | 617 | [122, 394] | 813 | [919, 920] |
| 30 | [58, 59] | 226 | [376, 377] | 422 | [612, 613] | 618 | [154, 14] | 814 | [259, 921] |
| 31 | [60, 61] | 227 | [40, 378] | 423 | [614, 615] | 619 | [719, 469] | 815 | [922, 568] |
| 32 | [62, 63] | 228 | [379, 380] | 424 | [283, 616] | 620 | [195, 159] | 816 | [99, 923] |
| 33 | [64, 65] | 229 | [381, 382] | 425 | [617, 618] | 621 | [786, 760] | 817 | [669, 89] |
| 34 | [66, 67] | 230 | [365, 383] | 426 | [619, 620] | 622 | [33, 440] | 818 | [924, 253] |
| 35 | [68, 69] | 231 | [139, 2] | 427 | [621, 622] | 623 | [770, 150] | 819 | [69, 925] |
| 36 | [70, 71] | 232 | [384, 148] | 428 | [287, 623] | 624 | [186, 787] | 820 | [926, 558] |
| 37 | [72, 73] | 233 | [385, 386] | 429 | [624, 625] | 625 | [372, 736] | 821 | [927, 928] |
| 38 | [74, 75] | 234 | [387, 388] | 430 | [202, 626] | 626 | [427, 457] | 822 | [929, 302] |
| 39 | [76, 77] | 235 | [389, 180] | 431 | [277, 627] | 627 | [452, 788] | 823 | [614, 708] |
| 40 | [78, 79] | 236 | [390, 347] | 432 | [568, 83] | 628 | [126, 343] | 824 | [578, 930] |
| 41 | [80, 81] | 237 | [391, 392] | 433 | [628, 333] | 629 | [34, 33] | 825 | [931, 852] |
| 42 | [82, 83] | 238 | [393, 394] | 434 | [629, 621] | 630 | [771, 789] | 826 | [932, 544] |
| 43 | [84, 85] | 239 | [113, 395] | 435 | [300, 630] | 631 | [790, 772] | 827 | [933, 695] |
| 44 | [86, 87] | 240 | [343, 396] | 436 | [631, 632] | 632 | [390, 468] | 828 | [341, 250] |
| 45 | [88, 89] | 241 | [397, 398] | 437 | [633, 634] | 633 | [791, 750] | 829 | [57, 446] |
| 46 | [90, 84] | 242 | [399, 400] | 438 | [541, 635] | 634 | [185, 344] | 830 | [381, 501] |
| 47 | [91, 92] | 243 | [401, 402] | 439 | [636, 637] | 635 | [460, 179] | 831 | [400, 934] |
| 48 | [93, 94] | 244 | [155, 403] | 440 | [67, 638] | 636 | [792, 181] | 832 | [809, 380] |
| 49 | [95, 96] | 245 | [404, 151] | 441 | [639, 640] | 637 | [348, 423] | 833 | [935, 356] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 50 | [97, 98] | 246 | [405, 360] | 442 | [641, 227] | 638 | [127, 153] | 834 | [417, 827] |
| 51 | [99, 100] | 247 | [406, 376] | 443 | [642, 643] | 639 | [793, 464] | 835 | [729, 731] |
| 52 | [101, 102] | 248 | [407, 408] | 444 | [644, 190] | 640 | [794, 501] | 836 | [776, 147] |
| 53 | [103, 104] | 249 | [409, 410] | 445 | [645, 646] | 641 | [512, 795] | 837 | [936, 794] |
| 54 | [105, 106] | 250 | [408, 147] | 446 | [110, 647] | 642 | [427, 796] | 838 | [419, 144] |
| 55 | [107, 108] | 251 | [409, 143] | 447 | [648, 647] | 643 | [732, 398] | 839 | [509, 762] |
| 56 | [109, 110] | 252 | [145, 146] | 448 | [649, 650] | 644 | [797, 724] | 840 | [467, 347] |
| 57 | [111, 112] | 253 | [39, 408] | 449 | [651, 652] | 645 | [786, 176] | 841 | [361, 773] |
| 58 | [113, 114] | 254 | [411, 409] | 450 | [653, 637] | 646 | [737, 778] | 842 | [483, 810] |
| 59 | [47, 115] | 255 | [142, 190] | 451 | [654, 655] | 647 | [195, 375] | 843 | [782, 342] |
| 60 | [116, 117] | 256 | [142, 142] | 452 | [656, 657] | 648 | [749, 798] | 844 | [166, 355] |
| 61 | [118, 119] | 257 | [147, 412] | 453 | [658, 651] | 649 | [508, 498] | 845 | [937, 444] |
| 62 | [8, 120] | 258 | [411, 413] | 454 | [659, 88] | 650 | [799, 442] | 846 | [938, 49] |
| 63 | [10, 121] | 259 | [414, 415] | 455 | [660, 661] | 651 | [135, 446] | 847 | [357, 198] |
| 64 | [122, 123] | 260 | [56, 135] | 456 | [617, 662] | 652 | [355, 121] | 848 | [939, 403] |
| 65 | [124, 125] | 261 | [416, 417] | 457 | [23, 276] | 653 | [191, 115] | 849 | [940, 478] |
| 66 | [126, 127] | 262 | [125, 418] | 458 | [663, 664] | 654 | [800, 426] | 850 | [823, 941] |
| 67 | [128, 129] | 263 | [419, 420] | 459 | [665, 666] | 655 | [801, 759] | 851 | [938, 450] |
| 68 | [130, 131] | 264 | [367, 421] | 460 | [667, 540] | 656 | [175, 760] | 852 | [718, 394] |
| 69 | [132, 52] | 265 | [360, 422] | 461 | [668, 645] | 657 | [406, 167] | 853 | [766, 784] |
| 70 | [133, 134] | 266 | [423, 418] | 462 | [669, 670] | 658 | [435, 722] | 854 | [370, 496] |
| 71 | [135, 136] | 267 | [424, 425] | 463 | [522, 671] | 659 | [481, 167] | 855 | [188, 57] |
| 72 | [137, 138] | 268 | [426, 427] | 464 | [672, 238] | 660 | [455, 409] | 856 | [165, 9] |
| 73 | [139, 140] | 269 | [428, 429] | 465 | [86, 244] | 661 | [490, 411] | 857 | [727, 377] |
| 74 | [141, 142] | 270 | [430, 386] | 466 | [673, 254] | 662 | [358, 802] | 858 | [940, 789] |
| 75 | [143, 144] | 271 | [431, 432] | 467 | [674, 675] | 663 | [803, 136] | 859 | [814, 934] |
| 76 | [144, 145] | 272 | [433, 434] | 468 | [676, 553] | 664 | [804, 192] | 860 | [942, 461] |
| 77 | [146, 147] | 273 | [435, 436] | 469 | [107, 677] | 665 | [805, 429] | 861 | [816, 766] |
| 78 | [148, 149] | 274 | [437, 378] | 470 | [678, 679] | 666 | [181, 484] | 862 | [376, 168] |
| 79 | [150, 151] | 275 | [438, 439] | 471 | [680, 681] | 667 | [752, 143] | 863 | [763, 148] |
| 80 | [152, 153] | 276 | [35, 440] | 472 | [660, 231] | 668 | [197, 737] | 864 | [119, 484] |
| 81 | [154, 155] | 277 | [441, 442] | 473 | [680, 302] | 669 | [402, 754] | 865 | [420, 145] |
| 82 | [156, 157] | 278 | [124, 423] | 474 | [521, 266] | 670 | [806, 381] | 866 | [179, 510] |
| 83 | [35, 158] | 279 | [443, 444] | 475 | [74, 682] | 671 | [807, 504] | 867 | [46, 762] |
| 84 | [120, 159] | 280 | [359, 445] | 476 | [666, 292] | 672 | [790, 129] | 868 | [943, 171] |
| 85 | [160, 161] | 281 | [446, 402] | 477 | [683, 684] | 673 | [797, 808] | 869 | [937, 828] |
| 86 | [55, 162] | 282 | [395, 447] | 478 | [685, 514] | 674 | [809, 746] | 870 | [14, 812] |
| 87 | [163, 164] | 283 | [448, 46] | 479 | [686, 588] | 675 | [342, 127] | 871 | [944, 350] |
| 88 | [165, 166] | 284 | [449, 450] | 480 | [292, 687] | 676 | [742, 352] | 872 | [173, 140] |
| 89 | [167, 168] | 285 | [451, 452] | 481 | [665, 291] | 677 | [13, 155] | 873 | [803, 446] |
| 90 | [169, 160] | 286 | [453, 176] | 482 | [570, 688] | 678 | [739, 364] | 874 | [945, 941] |
| 91 | [170, 171] | 287 | [454, 44] | 483 | [207, 689] | 679 | [783, 127] | 875 | [450, 491] |
| 92 | [172, 46] | 288 | [455, 413] | 484 | [558, 536] | 680 | [181, 810] | 876 | [368, 743] |
| 93 | [173, 2] | 289 | [456, 457] | 485 | [654, 690] | 681 | [811, 172] | 877 | [946, 9] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 94 | [174, 140] | 290 | [458, 342] | 486 | [691, 692] | 682 | [767, 138] | 878 | [456, 796] |
| 95 | [175, 176] | 291 | [459, 460] | 487 | [693, 694] | 683 | [155, 812] | 879 | [433, 353] |
| 96 | [177, 178] | 292 | [429, 398] | 488 | [695, 4] | 684 | [813, 722] | 880 | [38, 145] |
| 97 | [179, 180] | 293 | [461, 359] | 489 | [696, 618] | 685 | [814, 345] | 881 | [404, 128] |
| 98 | [118, 181] | 294 | [462, 418] | 490 | [77, 697] | 686 | [405, 445] | 882 | [947, 35] |
| 99 | [182, 183] | 295 | [463, 464] | 491 | [698, 212] | 687 | [460, 476] | 883 | [391, 117] |
| 100 | [184, 185] | 296 | [465, 466] | 492 | [677, 699] | 688 | [12, 796] | 884 | [948, 949] |
| 101 | [186, 187] | 297 | [467, 468] | 493 | [700, 642] | 689 | [117, 815] | 885 | [733, 383] |
| 102 | [188, 135] | 298 | [451, 469] | 494 | [68, 701] | 690 | [804, 358] | 886 | [723, 808] |
| 103 | [189, 167] | 299 | [470, 198] | 495 | [332, 702] | 691 | [431, 774] | 887 | [446, 777] |
| 104 | [42, 162] | 300 | [471, 45] | 496 | [212, 211] | 692 | [458, 783] | 888 | [800, 354] |
| 105 | [141, 190] | 301 | [472, 473] | 497 | [19, 603] | 693 | [816, 472] | 889 | [947, 33] |
| 106 | [174, 2] | 302 | [423, 474] | 498 | [703, 704] | 694 | [125, 474] | 890 | [413, 143] |
| 107 | [191, 48] | 303 | [475, 476] | 499 | [705, 209] | 695 | [44, 487] | 891 | [506, 442] |
| 108 | [192, 193] | 304 | [477, 478] | 500 | [60, 706] | 696 | [401, 777] | 892 | [473, 785] |
| 109 | [194, 195] | 305 | [434, 426] | 501 | [649, 707] | 697 | [145, 408] | 893 | [119, 810] |
| 110 | [196, 192] | 306 | [349, 479] | 502 | [202, 693] | 698 | [817, 818] | 894 | [761, 192] |
| 111 | [197, 160] | 307 | [480, 378] | 503 | [708, 678] | 699 | [819, 475] | 895 | [470, 37] |
| 112 | [36, 198] | 308 | [481, 376] | 504 | [709, 621] | 700 | [805, 732] | 896 | [396, 787] |
| 113 | [199, 200] | 309 | [482, 54] | 505 | [710, 711] | 701 | [425, 820] | 897 | [463, 949] |
| 114 | [201, 202] | 310 | [483, 484] | 506 | [630, 712] | 702 | [737, 161] | 898 | [783, 343] |
| 115 | [203, 204] | 311 | [485, 353] | 507 | [713, 264] | 703 | [130, 157] | 899 | [501, 731] |
| 116 | [205, 206] | 312 | [396, 187] | 508 | [71, 714] | 704 | [189, 376] | 900 | [443, 828] |
| 117 | [207, 103] | 313 | [486, 487] | 509 | [715, 638] | 705 | [749, 759] | 901 | [758, 798] |
| 118 | [208, 209] | 314 | [488, 489] | 510 | [682, 256] | 706 | [821, 48] | 902 | [52, 750] |
| 119 | [210, 211] | 315 | [146, 490] | 511 | [16, 210] | 707 | [127, 396] | 903 | [7, 826] |
| 120 | [16, 212] | 316 | [49, 491] | 512 | [294, 716] | 708 | [178, 822] | 904 | [501, 730] |
| 121 | [20, 213] | 317 | [492, 14] | 513 | [717, 395] | 709 | [823, 818] | 905 | [950, 740] |
| 122 | [214, 215] | 318 | [493, 417] | 514 | [718, 123] | 710 | [824, 825] | 906 | [717, 114] |
| 123 | [216, 217] | 319 | [494, 495] | 515 | [719, 452] | 711 | [769, 31] | 907 | [781, 375] |
| 124 | [218, 219] | 320 | [485, 434] | 516 | [360, 495] | 712 | [826, 195] | 908 | [726, 817] |
| 125 | [220, 221] | 321 | [454, 486] | 517 | [429, 720] | 713 | [137, 421] | 909 | [951, 773] |
| 126 | [90, 222] | 322 | [375, 496] | 518 | [721, 722] | 714 | [827, 828] | 910 | [128, 772] |
| 127 | [223, 224] | 323 | [497, 411] | 519 | [723, 724] | 715 | [116, 392] | 911 | [952, 157] |
| 128 | [225, 226] | 324 | [498, 499] | 520 | [725, 423] | 716 | [369, 801] | 912 | [461, 31] |
| 129 | [227, 77] | 325 | [500, 501] | 521 | [430, 726] | 717 | [109, 648] | 913 | [428, 732] |
| 130 | [228, 229] | 326 | [449, 49] | 522 | [727, 168] | 718 | [563, 829] | 914 | [953, 117] |
| 131 | [230, 231] | 327 | [502, 486] | 523 | [728, 443] | 719 | [830, 831] | 915 | [397, 720] |
| 132 | [232, 233] | 328 | [482, 369] | 524 | [11, 427] | 720 | [623, 22] | 916 | [948, 464] |
| 133 | [234, 70] | 329 | [503, 504] | 525 | [150, 128] | 721 | [302, 832] | 917 | [792, 119] |
| 134 | [235, 236] | 330 | [505, 362] | 526 | [729, 730] | 722 | [833, 645] | 918 | [780, 185] |
| 135 | [237, 238] | 331 | [10, 356] | 527 | [343, 153] | 723 | [234, 834] | 919 | [747, 744] |
| 136 | [239, 240] | 332 | [31, 445] | 528 | [731, 432] | 724 | [835, 657] | 920 | [943, 735] |
| 137 | [241, 242] | 333 | [373, 450] | 529 | [732, 720] | 725 | [704, 607] | 921 | [148, 388] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 138 | [243, 65] | 334 | [506, 507] | 530 | [733, 366] | 726 | [836, 307] | 922 | [817, 941] |
| 139 | [244, 245] | 335 | [508, 425] | 531 | [412, 409] | 727 | [225, 837] | 923 | [159, 496] |
| 140 | [246, 247] | 336 | [172, 509] | 532 | [734, 735] | 728 | [838, 73] | 924 | [488, 795] |
| 141 | [248, 249] | 337 | [476, 510] | 533 | [736, 374] | 729 | [298, 839] | 925 | [159, 371] |
| 142 | [249, 250] | 338 | [511, 374] | 534 | [346, 468] | 730 | [840, 613] | 926 | [344, 934] |
| 143 | [251, 252] | 339 | [473, 466] | 535 | [169, 737] | 731 | [841, 702] | 927 | [756, 825] |
| 144 | [213, 253] | 340 | [512, 489] | 536 | [114, 738] | 732 | [842, 843] | 928 | [826, 120] |
| 145 | [254, 255] | 341 | [438, 352] | 537 | [739, 740] | 733 | [683, 844] | 929 | [459, 475] |
| 146 | [75, 256] | 342 | [513, 514] | 538 | [741, 509] | 734 | [695, 273] | 930 | [954, 825] |
| 147 | [257, 258] | 343 | [515, 280] | 539 | [742, 439] | 735 | [845, 286] | 931 | [465, 785] |
| 148 | [259, 260] | 344 | [516, 517] | 540 | [480, 41] | 736 | [846, 297] | 932 | [424, 498] |
| 149 | [261, 262] | 345 | [94, 258] | 541 | [743, 744] | 737 | [847, 550] | 933 | [402, 415] |
| 150 | [263, 264] | 346 | [518, 205] | 542 | [745, 746] | 738 | [200, 240] | 934 | [955, 697] |
| 151 | [265, 266] | 347 | [519, 106] | 543 | [389, 510] | 739 | [848, 849] | 935 | [310, 541] |
| 152 | [267, 268] | 348 | [520, 521] | 544 | [393, 123] | 740 | [850, 18] | 936 | [956, 567] |
| 153 | [222, 85] | 349 | [522, 523] | 545 | [747, 748] | 741 | [851, 852] | 937 | [854, 957] |
| 154 | [269, 270] | 350 | [524, 202] | 546 | [54, 749] | 742 | [848, 853] | 938 | [15, 698] |
| 155 | [271, 272] | 351 | [22, 525] | 547 | [152, 396] | 743 | [854, 855] | 939 | [312, 958] |
| 156 | [273, 62] | 352 | [526, 96] | 548 | [178, 750] | 744 | [856, 695] | 940 | [959, 960] |
| 157 | [274, 275] | 353 | [267, 236] | 549 | [475, 179] | 745 | [857, 858] | 941 | [575, 610] |
| 158 | [63, 84] | 354 | [333, 527] | 550 | [132, 178] | 746 | [859, 840] | 942 | [961, 60] |
| 159 | [18, 276] | 355 | [528, 529] | 551 | [751, 404] | 747 | [642, 860] | 943 | [962, 963] |
| 160 | [277, 278] | 356 | [530, 531] | 552 | [752, 410] | 748 | [861, 643] | 944 | [517, 964] |
| 161 | [279, 280] | 357 | [532, 533] | 553 | [351, 439] | 749 | [589, 262] | 945 | [626, 694] |
| 162 | [281, 282] | 358 | [534, 6] | 554 | [392, 753] | 750 | [309, 862] | 946 | [6, 965] |
| 163 | [283, 284] | 359 | [535, 332] | 555 | [354, 12] | 751 | [863, 864] | 947 | [229, 966] |
| 164 | [285, 286] | 360 | [59, 536] | 556 | [30, 359] | 752 | [836, 865] | 948 | [967, 897] |
| 165 | [287, 21] | 361 | [537, 538] | 557 | [736, 118] | 753 | [866, 707] | 949 | [968, 112] |
| 166 | [288, 217] | 362 | [539, 540] | 558 | [374, 181] | 754 | [867, 595] | 950 | [969, 91] |
| 167 | [289, 290] | 363 | [541, 542] | 559 | [450, 50] | 755 | [513, 281] | 951 | [960, 970] |
| 168 | [291, 292] | 364 | [543, 544] | 560 | [414, 754] | 756 | [241, 868] | 952 | [295, 971] |
| 169 | [293, 279] | 365 | [545, 546] | 561 | [163, 736] | 757 | [869, 299] | 953 | [675, 866] |
| 170 | [294, 295] | 366 | [547, 278] | 562 | [755, 42] | 758 | [624, 870] | 954 | [972, 580] |
| 171 | [296, 297] | 367 | [548, 549] | 563 | [756, 757] | 759 | [871, 601] | 955 | [944, 479] |
| 172 | [298, 299] | 368 | [550, 551] | 564 | [758, 759] | 760 | [872, 575] | 956 | [194, 120] |
| 173 | [300, 301] | 369 | [552, 553] | 565 | [453, 760] | 761 | [873, 874] | 957 | [764, 149] |
| 174 | [302, 303] | 370 | [330, 554] | 566 | [761, 358] | 762 | [852, 875] | 958 | [973, 8] |
| 175 | [304, 305] | 371 | [555, 316] | 567 | [32, 35] | 763 | [876, 877] | 959 | [471, 487] |
| 176 | [306, 307] | 372 | [556, 208] | 568 | [120, 375] | 764 | [878, 879] | 960 | [133, 504] |
| 177 | [308, 309] | 373 | [557, 97] | 569 | [755, 762] | 765 | [637, 680] | 961 | [511, 118] |
| 178 | [310, 311] | 374 | [58, 558] | 570 | [763, 764] | 766 | [880, 275] | 962 | [939, 812] |
| 179 | [312, 313] | 375 | [334, 559] | 571 | [765, 766] | 767 | [578, 881] | 963 | [807, 134] |
| 180 | [314, 315] | 376 | [560, 561] | 572 | [182, 743] | 768 | [873, 882] | 964 | [386, 945] |
| 181 | [297, 316] | 377 | [328, 313] | 573 | [762, 43] | 769 | [883, 632] | 965 | [12, 457] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 182 | [317, 318] | 378 | [562, 103] | 574 | [497, 412] | 770 | [704, 225] | 966 | [476, 180] |
| 183 | [319, 320] | 379 | [78, 563] | 575 | [767, 421] | 771 | [545, 884] | 967 | [791, 822] |
| 184 | [321, 322] | 380 | [564, 565] | 576 | [768, 769] | 772 | [885, 255] | 968 | [935, 121] |
| 185 | [323, 94] | 381 | [566, 565] | 577 | [770, 404] | 773 | [886, 682] | 969 | [462, 474] |
| 186 | [95, 324] | 382 | [567, 568] | 578 | [167, 377] | 774 | [887, 875] | 970 | [8, 195] |
| 187 | [224, 25] | 383 | [569, 550] | 579 | [183, 748] | 775 | [888, 889] | 971 | [741, 46] |
| 188 | [325, 326] | 384 | [570, 571] | 580 | [771, 478] | 776 | [552, 890] | 972 | [387, 149] |
| 189 | [327, 328] | 385 | [572, 573] | 581 | [57, 136] | 777 | [891, 605] | 973 | [974, 975] |
| 190 | [190, 190] | 386 | [574, 575] | 582 | [500, 382] | 778 | [339, 224] | 974 | [448, 509] |
| 191 | [329, 293] | 387 | [516, 576] | 583 | [151, 772] | 779 | [892, 527] | 975 | [953, 392] |
| 192 | [330, 247] | 388 | [577, 578] | 584 | [441, 507] | 780 | [677, 893] | | |
| 193 | [331, 332] | 389 | [271, 579] | 585 | [170, 735] | 781 | [846, 555] | | |
| 194 | [333, 334] | 390 | [580, 581] | 586 | [437, 41] | 782 | [338, 223] | | |
| 195 | [335, 98] | 391 | [582, 583] | 587 | [505, 773] | 783 | [894, 895] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 140 | 1 | 280 | 0 | 420 | 1 | 560 | 1 | 700 | 0 | 840 | 1 |
| 1 | 0 | 141 | 0 | 281 | 1 | 421 | 0 | 561 | 1 | 701 | 0 | 841 | 0 |
| 2 | 0 | 142 | 1 | 282 | 0 | 422 | 0 | 562 | 0 | 702 | 1 | 842 | 1 |
| 3 | 0 | 143 | 1 | 283 | 1 | 423 | 1 | 563 | 0 | 703 | 1 | 843 | 0 |
| 4 | 0 | 144 | 0 | 284 | 1 | 424 | 1 | 564 | 0 | 704 | 0 | 844 | 1 |
| 5 | 0 | 145 | 1 | 285 | 1 | 425 | 0 | 565 | 1 | 705 | 0 | 845 | 0 |
| 6 | 1 | 146 | 1 | 286 | 1 | 426 | 1 | 566 | 1 | 706 | 0 | 846 | 0 |
| 7 | 0 | 147 | 0 | 287 | 0 | 427 | 0 | 567 | 0 | 707 | 0 | 847 | 1 |
| 8 | 0 | 148 | 1 | 288 | 1 | 428 | 0 | 568 | 0 | 708 | 1 | 848 | 0 |
| 9 | 0 | 149 | 1 | 289 | 0 | 429 | 0 | 569 | 0 | 709 | 1 | 849 | 1 |
| 10 | 1 | 150 | 1 | 290 | 1 | 430 | 0 | 570 | 1 | 710 | 0 | 850 | 1 |
| 11 | 0 | 151 | 1 | 291 | 0 | 431 | 0 | 571 | 0 | 711 | 1 | 851 | 0 |
| 12 | 1 | 152 | 1 | 292 | 0 | 432 | 0 | 572 | 1 | 712 | 1 | 852 | 0 |
| 13 | 1 | 153 | 1 | 293 | 0 | 433 | 0 | 573 | 0 | 713 | 0 | 853 | 0 |
| 14 | 1 | 154 | 0 | 294 | 1 | 434 | 0 | 574 | 0 | 714 | 1 | 854 | 0 |
| 15 | 0 | 155 | 0 | 295 | 0 | 435 | 1 | 575 | 0 | 715 | 1 | 855 | 1 |
| 16 | 0 | 156 | 1 | 296 | 1 | 436 | 0 | 576 | 1 | 716 | 1 | 856 | 0 |
| 17 | 0 | 157 | 1 | 297 | 1 | 437 | 1 | 577 | 0 | 717 | 0 | 857 | 0 |
| 18 | 1 | 158 | 1 | 298 | 1 | 438 | 0 | 578 | 0 | 718 | 0 | 858 | 1 |
| 19 | 0 | 159 | 1 | 299 | 0 | 439 | 0 | 579 | 1 | 719 | 1 | 859 | 1 |
| 20 | 1 | 160 | 0 | 300 | 1 | 440 | 0 | 580 | 0 | 720 | 0 | 860 | 0 |
| 21 | 1 | 161 | 0 | 301 | 1 | 441 | 0 | 581 | 0 | 721 | 1 | 861 | 1 |
| 22 | 1 | 162 | 1 | 302 | 1 | 442 | 1 | 582 | 1 | 722 | 1 | 862 | 1 |
| 23 | 0 | 163 | 1 | 303 | 0 | 443 | 0 | 583 | 1 | 723 | 1 | 863 | 1 |
| 24 | 1 | 164 | 1 | 304 | 0 | 444 | 1 | 584 | 0 | 724 | 1 | 864 | 0 |
| 25 | 0 | 165 | 0 | 305 | 0 | 445 | 0 | 585 | 1 | 725 | 0 | 865 | 1 |
| 26 | 1 | 166 | 1 | 306 | 0 | 446 | 1 | 586 | 1 | 726 | 1 | 866 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 27 | 0 | 167 | 0 | 307 | 0 | 447 | 0 | 587 | 0 | 727 | 0 | 867 | 0 |
| 28 | 1 | 168 | 0 | 308 | 0 | 448 | 1 | 588 | 0 | 728 | 1 | 868 | 0 |
| 29 | 0 | 169 | 0 | 309 | 1 | 449 | 1 | 589 | 0 | 729 | 1 | 869 | 0 |
| 30 | 0 | 170 | 1 | 310 | 1 | 450 | 1 | 590 | 1 | 730 | 1 | 870 | 1 |
| 31 | 1 | 171 | 1 | 311 | 1 | 451 | 1 | 591 | 1 | 731 | 0 | 871 | 0 |
| 32 | 0 | 172 | 1 | 312 | 0 | 452 | 0 | 592 | 1 | 732 | 1 | 872 | 1 |
| 33 | 0 | 173 | 1 | 313 | 0 | 453 | 1 | 593 | 0 | 733 | 0 | 873 | 1 |
| 34 | 0 | 174 | 1 | 314 | 1 | 454 | 0 | 594 | 0 | 734 | 1 | 874 | 0 |
| 35 | 1 | 175 | 0 | 315 | 1 | 455 | 1 | 595 | 0 | 735 | 0 | 875 | 1 |
| 36 | 1 | 176 | 0 | 316 | 0 | 456 | 0 | 596 | 0 | 736 | 0 | 876 | 1 |
| 37 | 0 | 177 | 0 | 317 | 1 | 457 | 0 | 597 | 0 | 737 | 1 | 877 | 1 |
| 38 | 0 | 178 | 1 | 318 | 0 | 458 | 1 | 598 | 1 | 738 | 1 | 878 | 0 |
| 39 | 0 | 179 | 0 | 319 | 1 | 459 | 0 | 599 | 1 | 739 | 0 | 879 | 0 |
| 40 | 1 | 180 | 1 | 320 | 1 | 460 | 1 | 600 | 0 | 740 | 1 | 880 | 0 |
| 41 | 1 | 181 | 1 | 321 | 0 | 461 | 0 | 601 | 0 | 741 | 0 | 881 | 0 |
| 42 | 1 | 182 | 1 | 322 | 0 | 462 | 1 | 602 | 0 | 742 | 0 | 882 | 1 |
| 43 | 0 | 183 | 1 | 323 | 0 | 463 | 0 | 603 | 0 | 743 | 0 | 883 | 1 |
| 44 | 1 | 184 | 0 | 324 | 1 | 464 | 0 | 604 | 1 | 744 | 0 | 884 | 1 |
| 45 | 0 | 185 | 0 | 325 | 1 | 465 | 1 | 605 | 0 | 745 | 0 | 885 | 0 |
| 46 | 0 | 186 | 0 | 326 | 1 | 466 | 1 | 606 | 0 | 746 | 1 | 886 | 1 |
| 47 | 1 | 187 | 0 | 327 | 0 | 467 | 1 | 607 | 1 | 747 | 0 | 887 | 1 |
| 48 | 1 | 188 | 1 | 328 | 1 | 468 | 0 | 608 | 0 | 748 | 1 | 888 | 1 |
| 49 | 0 | 189 | 0 | 329 | 1 | 469 | 1 | 609 | 1 | 749 | 0 | 889 | 1 |
| 50 | 0 | 190 | 0 | 330 | 0 | 470 | 0 | 610 | 0 | 750 | 1 | 890 | 0 |
| 51 | 1 | 191 | 1 | 331 | 1 | 471 | 1 | 611 | 0 | 751 | 1 | 891 | 0 |
| 52 | 0 | 192 | 0 | 332 | 1 | 472 | 1 | 612 | 0 | 752 | 1 | 892 | 1 |
| 53 | 0 | 193 | 1 | 333 | 0 | 473 | 1 | 613 | 1 | 753 | 0 | 893 | 0 |
| 54 | 0 | 194 | 0 | 334 | 0 | 474 | 0 | 614 | 1 | 754 | 0 | 894 | 1 |
| 55 | 1 | 195 | 1 | 335 | 1 | 475 | 0 | 615 | 0 | 755 | 0 | 895 | 0 |
| 56 | 0 | 196 | 1 | 336 | 1 | 476 | 1 | 616 | 0 | 756 | 0 | 896 | 0 |
| 57 | 0 | 197 | 0 | 337 | 1 | 477 | 0 | 617 | 0 | 757 | 0 | 897 | 0 |
| 58 | 0 | 198 | 1 | 338 | 0 | 478 | 1 | 618 | 0 | 758 | 0 | 898 | 1 |
| 59 | 1 | 199 | 0 | 339 | 1 | 479 | 1 | 619 | 1 | 759 | 0 | 899 | 1 |
| 60 | 1 | 200 | 1 | 340 | 1 | 480 | 0 | 620 | 1 | 760 | 1 | 900 | 0 |
| 61 | 1 | 201 | 1 | 341 | 0 | 481 | 0 | 621 | 0 | 761 | 1 | 901 | 0 |
| 62 | 0 | 202 | 0 | 342 | 0 | 482 | 1 | 622 | 0 | 762 | 0 | 902 | 0 |
| 63 | 1 | 203 | 0 | 343 | 1 | 483 | 1 | 623 | 0 | 763 | 1 | 903 | 0 |
| 64 | 0 | 204 | 1 | 344 | 1 | 484 | 0 | 624 | 0 | 764 | 0 | 904 | 1 |
| 65 | 1 | 205 | 1 | 345 | 1 | 485 | 1 | 625 | 0 | 765 | 0 | 905 | 1 |
| 66 | 0 | 206 | 0 | 346 | 1 | 486 | 0 | 626 | 0 | 766 | 0 | 906 | 0 |
| 67 | 0 | 207 | 1 | 347 | 1 | 487 | 1 | 627 | 0 | 767 | 0 | 907 | 0 |
| 68 | 1 | 208 | 1 | 348 | 0 | 488 | 1 | 628 | 0 | 768 | 1 | 908 | 1 |
| 69 | 1 | 209 | 0 | 349 | 0 | 489 | 1 | 629 | 0 | 769 | 1 | 909 | 1 |
| 70 | 1 | 210 | 0 | 350 | 0 | 490 | 1 | 630 | 0 | 770 | 0 | 910 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 71 | 1 | 211 | 0 | 351 | 1 | 491 | 1 | 631 | 0 | 771 | 0 | 911 | 0 |
| 72 | 0 | 212 | 1 | 352 | 1 | 492 | 1 | 632 | 0 | 772 | 0 | 912 | 0 |
| 73 | 0 | 213 | 0 | 353 | 1 | 493 | 0 | 633 | 1 | 773 | 1 | 913 | 0 |
| 74 | 0 | 214 | 0 | 354 | 0 | 494 | 1 | 634 | 0 | 774 | 1 | 914 | 0 |
| 75 | 1 | 215 | 1 | 355 | 0 | 495 | 1 | 635 | 1 | 775 | 1 | 915 | 0 |
| 76 | 0 | 216 | 0 | 356 | 0 | 496 | 1 | 636 | 0 | 776 | 1 | 916 | 1 |
| 77 | 1 | 217 | 1 | 357 | 1 | 497 | 0 | 637 | 0 | 777 | 0 | 917 | 0 |
| 78 | 1 | 218 | 1 | 358 | 1 | 498 | 1 | 638 | 0 | 778 | 1 | 918 | 1 |
| 79 | 1 | 219 | 0 | 359 | 0 | 499 | 0 | 639 | 0 | 779 | 1 | 919 | 0 |
| 80 | 1 | 220 | 0 | 360 | 1 | 500 | 1 | 640 | 0 | 780 | 1 | 920 | 0 |
| 81 | 0 | 221 | 0 | 361 | 0 | 501 | 1 | 641 | 1 | 781 | 0 | 921 | 1 |
| 82 | 1 | 222 | 1 | 362 | 0 | 502 | 0 | 642 | 0 | 782 | 0 | 922 | 1 |
| 83 | 1 | 223 | 0 | 363 | 0 | 503 | 1 | 643 | 1 | 783 | 1 | 923 | 1 |
| 84 | 0 | 224 | 0 | 364 | 0 | 504 | 1 | 644 | 1 | 784 | 0 | 924 | 1 |
| 85 | 0 | 225 | 0 | 365 | 0 | 505 | 0 | 645 | 0 | 785 | 0 | 925 | 1 |
| 86 | 1 | 226 | 1 | 366 | 1 | 506 | 0 | 646 | 1 | 786 | 0 | 926 | 1 |
| 87 | 1 | 227 | 1 | 367 | 1 | 507 | 0 | 647 | 1 | 787 | 1 | 927 | 0 |
| 88 | 0 | 228 | 1 | 368 | 1 | 508 | 1 | 648 | 0 | 788 | 0 | 928 | 1 |
| 89 | 0 | 229 | 1 | 369 | 1 | 509 | 1 | 649 | 1 | 789 | 0 | 929 | 0 |
| 90 | 0 | 230 | 0 | 370 | 0 | 510 | 0 | 650 | 1 | 790 | 0 | 930 | 1 |
| 91 | 1 | 231 | 0 | 371 | 0 | 511 | 0 | 651 | 1 | 791 | 1 | 931 | 1 |
| 92 | 1 | 232 | 1 | 372 | 0 | 512 | 1 | 652 | 0 | 792 | 0 | 932 | 1 |
| 93 | 1 | 233 | 1 | 373 | 0 | 513 | 0 | 653 | 1 | 793 | 0 | 933 | 1 |
| 94 | 1 | 234 | 1 | 374 | 0 | 514 | 0 | 654 | 1 | 794 | 0 | 934 | 0 |
| 95 | 0 | 235 | 0 | 375 | 0 | 515 | 1 | 655 | 1 | 795 | 0 | 935 | 1 |
| 96 | 0 | 236 | 0 | 376 | 1 | 516 | 1 | 656 | 0 | 796 | 1 | 936 | 0 |
| 97 | 0 | 237 | 1 | 377 | 1 | 517 | 0 | 657 | 1 | 797 | 1 | 937 | 0 |
| 98 | 1 | 238 | 1 | 378 | 0 | 518 | 1 | 658 | 1 | 798 | 1 | 938 | 0 |
| 99 | 1 | 239 | 0 | 379 | 1 | 519 | 1 | 659 | 0 | 799 | 1 | 939 | 0 |
| 100 | 0 | 240 | 1 | 380 | 0 | 520 | 0 | 660 | 1 | 800 | 1 | 940 | 1 |
| 101 | 0 | 241 | 0 | 381 | 1 | 521 | 0 | 661 | 1 | 801 | 1 | 941 | 0 |
| 102 | 1 | 242 | 1 | 382 | 0 | 522 | 0 | 662 | 1 | 802 | 0 | 942 | 0 |
| 103 | 0 | 243 | 1 | 383 | 0 | 523 | 1 | 663 | 1 | 803 | 1 | 943 | 0 |
| 104 | 1 | 244 | 0 | 384 | 1 | 524 | 0 | 664 | 0 | 804 | 0 | 944 | 0 |
| 105 | 0 | 245 | 0 | 385 | 1 | 525 | 1 | 665 | 0 | 805 | 0 | 945 | 0 |
| 106 | 1 | 246 | 1 | 386 | 0 | 526 | 1 | 666 | 1 | 806 | 1 | 946 | 1 |
| 107 | 1 | 247 | 1 | 387 | 1 | 527 | 1 | 667 | 1 | 807 | 0 | 947 | 1 |
| 108 | 0 | 248 | 0 | 388 | 0 | 528 | 0 | 668 | 0 | 808 | 0 | 948 | 1 |
| 109 | 0 | 249 | 1 | 389 | 0 | 529 | 1 | 669 | 1 | 809 | 1 | 949 | 1 |
| 110 | 1 | 250 | 0 | 390 | 0 | 530 | 0 | 670 | 1 | 810 | 1 | 950 | 1 |
| 111 | 0 | 251 | 1 | 391 | 1 | 531 | 0 | 671 | 0 | 811 | 0 | 951 | 1 |
| 112 | 1 | 252 | 1 | 392 | 0 | 532 | 1 | 672 | 0 | 812 | 1 | 952 | 0 |
| 113 | 0 | 253 | 0 | 393 | 1 | 533 | 0 | 673 | 1 | 813 | 0 | 953 | 0 |
| 114 | 1 | 254 | 1 | 394 | 1 | 534 | 1 | 674 | 1 | 814 | 1 | 954 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 115 | 0 | 255 | 1 | 395 | 0 | 535 | 0 | 675 | 0 | 815 | 1 | 955 | 0 |
| 116 | 1 | 256 | 1 | 396 | 0 | 536 | 1 | 676 | 0 | 816 | 1 | 956 | 0 |
| 117 | 1 | 257 | 0 | 397 | 0 | 537 | 0 | 677 | 1 | 817 | 1 | 957 | 0 |
| 118 | 1 | 258 | 1 | 398 | 1 | 538 | 0 | 678 | 0 | 818 | 1 | 958 | 1 |
| 119 | 0 | 259 | 1 | 399 | 1 | 539 | 0 | 679 | 1 | 819 | 1 | 959 | 1 |
| 120 | 0 | 260 | 0 | 400 | 0 | 540 | 0 | 680 | 1 | 820 | 1 | 960 | 1 |
| 121 | 1 | 261 | 1 | 401 | 1 | 541 | 0 | 681 | 0 | 821 | 0 | 961 | 0 |
| 122 | 0 | 262 | 0 | 402 | 1 | 542 | 0 | 682 | 0 | 822 | 0 | 962 | 0 |
| 123 | 0 | 263 | 1 | 403 | 0 | 543 | 0 | 683 | 0 | 823 | 1 | 963 | 0 |
| 124 | 1 | 264 | 1 | 404 | 0 | 544 | 1 | 684 | 0 | 824 | 0 | 964 | 0 |
| 125 | 0 | 265 | 1 | 405 | 1 | 545 | 0 | 685 | 1 | 825 | 1 | 965 | 1 |
| 126 | 0 | 266 | 1 | 406 | 1 | 546 | 0 | 686 | 1 | 826 | 1 | 966 | 1 |
| 127 | 0 | 267 | 1 | 407 | 0 | 547 | 1 | 687 | 1 | 827 | 1 | 967 | 1 |
| 128 | 0 | 268 | 1 | 408 | 0 | 548 | 1 | 688 | 1 | 828 | 0 | 968 | 1 |
| 129 | 1 | 269 | 0 | 409 | 1 | 549 | 0 | 689 | 1 | 829 | 0 | 969 | 1 |
| 130 | 1 | 270 | 0 | 410 | 0 | 550 | 1 | 690 | 0 | 830 | 1 | 970 | 0 |
| 131 | 0 | 271 | 0 | 411 | 1 | 551 | 1 | 691 | 0 | 831 | 0 | 971 | 0 |
| 132 | 1 | 272 | 0 | 412 | 0 | 552 | 1 | 692 | 1 | 832 | 1 | 972 | 1 |
| 133 | 1 | 273 | 1 | 413 | 0 | 553 | 1 | 693 | 1 | 833 | 1 | 973 | 1 |
| 134 | 0 | 274 | 1 | 414 | 1 | 554 | 0 | 694 | 0 | 834 | 0 | 974 | 1 |
| 135 | 1 | 275 | 0 | 415 | 1 | 555 | 0 | 695 | 1 | 835 | 1 | 975 | 0 |
| 136 | 0 | 276 | 1 | 416 | 1 | 556 | 0 | 696 | 1 | 836 | 1 | | |
| 137 | 0 | 277 | 0 | 417 | 0 | 557 | 0 | 697 | 1 | 837 | 0 | | |
| 138 | 1 | 278 | 1 | 418 | 1 | 558 | 0 | 698 | 1 | 838 | 1 | | |
| 139 | 0 | 279 | 0 | 419 | 1 | 559 | 1 | 699 | 1 | 839 | 1 | | |

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
*Email address*: `guoniu.han@unistra.fr`

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
*Email address*: `huyining@hust.edu.cn`