PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^4 + z^3, z^2 + z + 1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^4 + z^3, z^2 + z + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 09:39:27 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 33.8 minutes CPU time used.

Theorem 1.1. *When* $(a,b)=(z^4+z^3, z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{15}+z^{14}+z^{12}+z^{11}+z^{10}+z^7+z^5+z^4+z^3+1,\\
p_1(z)&=z^{22}+z^{21}+z^{20}+z^{18}+z^{15}+z^{14}+z^{12}+z^6+z^4+z^3,\\
p_2(z)&=z^{24}+z^{20}+z^{18}+z^{16}+z^{12}+z^{10}+z^8+z^6,\\
p_3(z)&=z^{22}+z^{21}+z^{20}+z^{18}+z^{15}+z^{14}+z^{12}+z^6+z^4+z^3,\\
p_4(z)&=z^{20}+z^{15}+z^{14}+z^{11}+z^7+z^6+z^5+z^3,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{11}+z^{10}+z^8+z^7+z^5+z^3,\\
p_1(z)&=z^{22}+z^{21}+z^{20}+z^{18}+z^{15}+z^{14}+z^{12}+z^6+z^4+z^3,\\
p_2(z)&=z^{24}+z^{20}+z^{18}+z^{16}+z^{12}+z^{10}+z^8+z^6,\\
p_3(z)&=z^{22}+z^{21}+z^{20}+z^{18}+z^{15}+z^{14}+z^{12}+z^6+z^4+z^3,\\
p_4(z)&=z^{20}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^6+z^5+z^4+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x,z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x, y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{3u} M^e[:6u] + x^{6u} M^e[:3u] \\ & + x^{7u} M^e[:2u] + x^{4u} M^e[:6u] + x^{7u} M^e[:3u] + x^{8u} M^e[:2u] \\ & + x^{7u} M^e[:5u] + x^{8u} M^e[:4u] + x^{11u} M^e[:u] \\ & + (x^{6u} + x^{9u} + x^{10u}) R^e, \\ W^e[:12u] = {} & W^e[:6u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{3u} W^e[:6u] + x^{6u} W^e[:3u] \\ & + x^{7u} W^e[:2u] + x^{4u} W^e[:6u] + x^{7u} W^e[:3u] + x^{8u} W^e[:2u] \end{aligned}$$

$$+ x^{7u}W^e[:5u] + x^{8u}W^e[:4u] + x^{11u}W^e[:u]$$
$$+ (x^{6u} + x^{9u} + x^{10u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u] \\
&\quad + x^{7u}M^o[:2u] + x^{4u}M^o[:6u] + x^{7u}M^o[:3u] + x^{8u}M^o[:2u] \\
&\quad + x^{7u}M^o[:5u] + x^{8u}M^o[:4u] + x^{11u}M^o[:u] \\
&\quad + (x^{6u} + x^{9u} + x^{10u})R^o, \\
W^o[:12u] &= W^o[:6u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u] \\
&\quad + x^{7u}W^o[:2u] + x^{4u}W^o[:6u] + x^{7u}W^o[:3u] + x^{8u}W^o[:2u] \\
&\quad + x^{7u}W^o[:5u] + x^{8u}W^o[:4u] + x^{11u}W^o[:u] \\
&\quad + (x^{6u} + x^{9u} + x^{10u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] &= T[:6u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{7u}T[:2u] \\
&\quad + x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{7u}T[:5u] + x^{8u}T[:4u] \\
&\quad + x^{11u}T[:u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
0 &= x^{6u}T[:u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] + T[6u:7u], \\
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + T[7u:8u], \\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] \\
&\quad + T[8u:9u], \\
0 &= x^{4u}T[5u:6u] + T[9u:10u], \\
0 &= x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[10w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1408 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.19}$$

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.20}$$

$$\begin{aligned} 0 = {} & \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1000)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.23}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.25}$$

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1000)), \end{aligned} \tag{2.27}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.29}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{46}), E_{46}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 23$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{6u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{6u}R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{6u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ &+ x^{3v}M^e[:3v] + x^{4v}M^e[:2v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v] + x^{8v}W^e[:4v]M^e[:4v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^6,\\
q_1(x) &= x^{21} + x^{20} + x^{18} + x^{12} + x^{10} + x^9 + x^6 + x^4 + x^3 + x^2,\\
q_2(x) &= x^{18} + x^{16} + x^{14} + x^{12} + x^8 + x^6 + x^4 + 1,\\
q_3(x) &= x^{21} + x^{20} + x^{18} + x^{12} + x^{10} + x^9 + x^6 + x^4 + x^3 + x^2,\\
q_4(x) &= x^{21} + x^{19} + x^{18} + x^{17} + x^{13} + x^{10} + x^9 + x^4.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{94} + x^{92}\\
&\quad + x^{90} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{56} + x^{54} + x^{50}\\
&\quad + x^{42} + x^{40} + x^{38} + x^{32} + x^{26} + x^{24},\\
p_3(x) &= x^{112} + x^{110} + x^{108} + x^{102} + x^{100} + x^{98} + x^{94} + x^{90} + x^{88} + x^{82} + x^{70}\\
&\quad + x^{68} + x^{58} + x^{56} + x^{48} + x^{46} + x^{34} + x^{32} + x^{30} + x^{28} + x^{20} + x^{18},\\
p_6(x) &= x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{90} + x^{88}\\
&\quad + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{64} + x^{62} + x^{56} + x^{52}\\
&\quad + x^{50} + x^{46} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} + x^{26} + x^{24} + x^{22}\\
&\quad + x^{14} + x^{12} + x^{10} + x^4 + x^2 + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) = {} & x^{114} + x^{112} + x^{108} + x^{106} + x^{98} + x^{96} + x^{94} + x^{90} + x^{86} + x^{82} + x^{78} \\
& + x^{76} + x^{74} + x^{70} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} \\
& + x^{32} + x^{26} + x^{22} + x^{18} + x^{16} + x^{12} + x^{8} + x^{4}, \\
p_{12}(x) = {} & x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{74} + x^{72} + x^{66} + x^{64} + x^{50} \\
& + x^{48} + x^{42} + x^{40} + x^{34} + x^{32} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} \\
& + x^{113} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} \\
& + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} \\
& + x^{85} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73} + x^{71} + x^{68} + x^{65} + x^{64} + x^{60} \\
& + x^{54} + x^{53} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{35} + x^{34} \\
& + x^{30}, \\
p_3(x) = {} & x^{120} + x^{119} + x^{116} + x^{114} + x^{111} + x^{109} + x^{106} + x^{104} + x^{101} + x^{99} \\
& + x^{96} + x^{93} + x^{92} + x^{84} + x^{83} + x^{80} + x^{77} + x^{75} + x^{72} + x^{69} + x^{68} \\
& + x^{60} + x^{59} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{42} \\
& + x^{40} + x^{37} + x^{36} + x^{28} + x^{27} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) = {} & x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{107} + x^{104} + x^{101} + x^{98} \\
& + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} \\
& + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} \\
& + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} \\
& + x^{40} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} \\
& + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{103} + x^{101} + x^{98} \\
& + x^{96} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} \\
& + x^{76} + x^{70} + x^{69} + x^{66} + x^{64} + x^{61} + x^{60} + x^{54} + x^{53} + x^{52} + x^{51} \\
& + x^{50} + x^{48} + x^{47} + x^{44} + x^{41} + x^{40} + x^{36} + x^{35} + x^{31} + x^{30} + x^{28} \\
& + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{12} + x^{11} \\
& + x^{7} + x^{6}, \\
p_{12}(x) = {} & x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} \\
& + x^{64} + x^{47} + x^{44} + x^{35} + x^{32} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} \\
& + x^{113} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101}
\end{aligned}$$

$$
\begin{aligned}
&+x^{100}+x^{99}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}\\
&+x^{85}+x^{80}+x^{79}+x^{78}+x^{75}+x^{73}+x^{71}+x^{68}+x^{65}+x^{64}+x^{60}\\
&+x^{54}+x^{53}+x^{49}+x^{46}+x^{44}+x^{43}+x^{42}+x^{41}+x^{39}+x^{35}+x^{34}\\
&+x^{30},\\
p_3(x)&=x^{120}+x^{119}+x^{116}+x^{114}+x^{111}+x^{109}+x^{106}+x^{104}+x^{101}+x^{99}\\
&+x^{96}+x^{93}+x^{92}+x^{84}+x^{83}+x^{80}+x^{77}+x^{75}+x^{72}+x^{69}+x^{68}\\
&+x^{60}+x^{59}+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{44}+x^{42}\\
&+x^{40}+x^{37}+x^{36}+x^{28}+x^{27}+x^{23}+x^{21}+x^{20}+x^{19}+x^{18},\\
p_6(x)&=x^{117}+x^{115}+x^{114}+x^{113}+x^{112}+x^{110}+x^{107}+x^{104}+x^{101}+x^{98}\\
&+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{77}+x^{75}+x^{74}+x^{73}+x^{72}\\
&+x^{70}+x^{67}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{56}\\
&+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}+x^{43}+x^{42}\\
&+x^{40}+x^{33}+x^{31}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}\\
&+x^{20}+x^{18}+x^{17}+x^{15}+x^{14}+x^{12},\\
p_9(x)&=x^{114}+x^{113}+x^{112}+x^{111}+x^{110}+x^{108}+x^{107}+x^{103}+x^{101}+x^{98}\\
&+x^{96}+x^{93}+x^{91}+x^{90}+x^{89}+x^{87}+x^{84}+x^{81}+x^{79}+x^{78}+x^{77}\\
&+x^{76}+x^{70}+x^{69}+x^{66}+x^{64}+x^{61}+x^{60}+x^{54}+x^{53}+x^{52}+x^{51}\\
&+x^{50}+x^{48}+x^{47}+x^{44}+x^{41}+x^{40}+x^{36}+x^{35}+x^{31}+x^{30}+x^{28}\\
&+x^{27}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{12}+x^{11}\\
&+x^{7}+x^{6},\\
p_{12}(x)&=x^{111}+x^{108}+x^{99}+x^{96}+x^{95}+x^{92}+x^{83}+x^{80}+x^{79}+x^{76}+x^{67}\\
&+x^{64}+x^{47}+x^{44}+x^{35}+x^{32}+x^{15}+x^{12}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{132}+x^{130}+x^{128}+x^{126}+x^{124}+x^{116}+x^{102}+x^{98}+x^{86}+x^{78}\\
&+x^{72}+x^{68}+x^{58}+x^{56}+x^{52}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36},\\
p_3(x)&=x^{114}+x^{110}+x^{106}+x^{102}+x^{100}+x^{88}+x^{80}+x^{78}+x^{76}+x^{72}+x^{70}\\
&+x^{68}+x^{56}+x^{54}+x^{50}+x^{46}+x^{42}+x^{40}+x^{38}+x^{28}+x^{18}+x^{12},\\
p_6(x)&=x^{114}+x^{110}+x^{108}+x^{104}+x^{100}+x^{98}+x^{92}+x^{86}+x^{84}+x^{82}+x^{78}\\
&+x^{76}+x^{68}+x^{64}+x^{58}+x^{52}+x^{50}+x^{40}+x^{38}+x^{34}+x^{32}+x^{26}\\
&+x^{22}+x^{20}+x^{12}+x^{10}+x^{2}+1,\\
p_9(x)&=x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{88}+x^{86}+x^{82}+x^{78}+x^{74}\\
&+x^{72}+x^{70}+x^{66}+x^{56}+x^{52}+x^{50}+x^{48}+x^{42}+x^{38}+x^{30}+x^{24}\\
&+x^{20}+x^{12}+x^{10}+x^{4},\\
p_{12}(x)&=x^{108}+x^{106}+x^{104}+x^{100}+x^{98}+x^{96}+x^{92}+x^{90}+x^{88}+x^{84}+x^{82}
\end{aligned}
$$

$$+ x^{80} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} \\
&\quad + x^{108} + x^{106} + x^{102} + x^{98} + x^{94} + x^{92} + x^{86} + x^{84} + x^{80} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{60} + x^{54} + x^{52} + x^{50} + x^{44} + x^{42} + x^{34} + x^{28} + x^{26} \\
&\quad + x^{24}, \\
p_3(x) &= x^{114} + x^{110} + x^{106} + x^{100} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{78} \\
&\quad + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{34} \\
&\quad + x^{32} + x^{24} + x^{22} + x^{18}, \\
p_6(x) &= x^{114} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{16} + x^{14} + x^{8} \\
&\quad + x^{2} + 1, \\
p_9(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{88} + x^{86} + x^{82} + x^{78} + x^{74} \\
&\quad + x^{72} + x^{70} + x^{66} + x^{56} + x^{52} + x^{50} + x^{48} + x^{42} + x^{38} + x^{30} + x^{24} \\
&\quad + x^{20} + x^{12} + x^{10} + x^{4}, \\
p_{12}(x) &= x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} \\
&\quad + x^{80} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{44} + x^{42} + x^{40} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{117} + x^{115} + x^{111} + x^{110} + x^{108} \\
&\quad + x^{107} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{86} + x^{85} + x^{84} + x^{80} + x^{78} + x^{77} + x^{75} + x^{70} + x^{68} + x^{67} + x^{65} \\
&\quad + x^{64} + x^{62} + x^{61} + x^{50} + x^{48} + x^{47} + x^{46} + x^{42} + x^{39} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{30}, \\
p_3(x) &= x^{120} + x^{119} + x^{116} + x^{114} + x^{111} + x^{109} + x^{106} + x^{104} + x^{101} + x^{99} \\
&\quad + x^{96} + x^{93} + x^{92} + x^{84} + x^{83} + x^{80} + x^{77} + x^{75} + x^{72} + x^{69} + x^{68} \\
&\quad + x^{60} + x^{59} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{37} + x^{36} + x^{28} + x^{27} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{107} + x^{104} + x^{101} + x^{98} \\
&\quad + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}
\end{aligned}$$

$$
\begin{aligned}
&+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}+x^{43}+x^{42}\\
&+x^{40}+x^{33}+x^{31}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}\\
&+x^{20}+x^{18}+x^{17}+x^{15}+x^{14}+x^{12},\\
p_9(x)={}&x^{114}+x^{113}+x^{112}+x^{111}+x^{110}+x^{108}+x^{107}+x^{103}+x^{101}+x^{98}\\
&+x^{96}+x^{93}+x^{91}+x^{90}+x^{89}+x^{87}+x^{84}+x^{81}+x^{79}+x^{78}+x^{77}\\
&+x^{76}+x^{70}+x^{69}+x^{66}+x^{64}+x^{61}+x^{60}+x^{54}+x^{53}+x^{52}+x^{51}\\
&+x^{50}+x^{48}+x^{47}+x^{44}+x^{41}+x^{40}+x^{36}+x^{35}+x^{31}+x^{30}+x^{28}\\
&+x^{27}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{12}+x^{11}\\
&+x^{7}+x^{6},\\
p_{12}(x)={}&x^{111}+x^{108}+x^{99}+x^{96}+x^{95}+x^{92}+x^{83}+x^{80}+x^{79}+x^{76}+x^{67}\\
&+x^{64}+x^{47}+x^{44}+x^{35}+x^{32}+x^{15}+x^{12}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x)={}&x^{126}+x^{124}+x^{122}+x^{120}+x^{118}+x^{117}+x^{115}+x^{111}+x^{110}+x^{108}\\
&+x^{107}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{90}\\
&+x^{86}+x^{85}+x^{84}+x^{80}+x^{78}+x^{77}+x^{75}+x^{70}+x^{68}+x^{67}+x^{65}\\
&+x^{64}+x^{62}+x^{61}+x^{50}+x^{48}+x^{47}+x^{46}+x^{42}+x^{39}+x^{38}+x^{37}\\
&+x^{36}+x^{34}+x^{32}+x^{30},\\
p_3(x)={}&x^{120}+x^{119}+x^{116}+x^{114}+x^{111}+x^{109}+x^{106}+x^{104}+x^{101}+x^{99}\\
&+x^{96}+x^{93}+x^{92}+x^{84}+x^{83}+x^{80}+x^{77}+x^{75}+x^{72}+x^{69}+x^{68}\\
&+x^{60}+x^{59}+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{44}+x^{42}\\
&+x^{40}+x^{37}+x^{36}+x^{28}+x^{27}+x^{23}+x^{21}+x^{20}+x^{19}+x^{18},\\
p_6(x)={}&x^{117}+x^{115}+x^{114}+x^{113}+x^{112}+x^{110}+x^{107}+x^{104}+x^{101}+x^{98}\\
&+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{77}+x^{75}+x^{74}+x^{73}+x^{72}\\
&+x^{70}+x^{67}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{56}\\
&+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}+x^{43}+x^{42}\\
&+x^{40}+x^{33}+x^{31}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}\\
&+x^{20}+x^{18}+x^{17}+x^{15}+x^{14}+x^{12},\\
p_9(x)={}&x^{114}+x^{113}+x^{112}+x^{111}+x^{110}+x^{108}+x^{107}+x^{103}+x^{101}+x^{98}\\
&+x^{96}+x^{93}+x^{91}+x^{90}+x^{89}+x^{87}+x^{84}+x^{81}+x^{79}+x^{78}+x^{77}\\
&+x^{76}+x^{70}+x^{69}+x^{66}+x^{64}+x^{61}+x^{60}+x^{54}+x^{53}+x^{52}+x^{51}\\
&+x^{50}+x^{48}+x^{47}+x^{44}+x^{41}+x^{40}+x^{36}+x^{35}+x^{31}+x^{30}+x^{28}\\
&+x^{27}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{12}+x^{11}\\
&+x^{7}+x^{6},\\
p_{12}(x)={}&x^{111}+x^{108}+x^{99}+x^{96}+x^{95}+x^{92}+x^{83}+x^{80}+x^{79}+x^{76}+x^{67}
\end{aligned}
$$

$$+ x^{64} + x^{47} + x^{44} + x^{35} + x^{32} + x^{15} + x^{12} + x^{3} + 1,$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96}\\
&\quad + x^{90} + x^{88} + x^{84} + x^{82} + x^{74} + x^{70} + x^{60} + x^{56} + x^{48} + x^{46} + x^{40}\\
&\quad + x^{38} + x^{36},\\
p_3(x) &= x^{112} + x^{110} + x^{104} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{76}\\
&\quad + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{52} + x^{48} + x^{44} + x^{42} + x^{40}\\
&\quad + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{16} + x^{12},\\
p_6(x) &= x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88}\\
&\quad + x^{78} + x^{70} + x^{66} + x^{54} + x^{52} + x^{50} + x^{46} + x^{38} + x^{34} + x^{26} + x^{20}\\
&\quad + x^{18} + x^{16} + x^{12} + x^{8} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{98} + x^{96} + x^{94} + x^{90} + x^{86} + x^{82} + x^{78}\\
&\quad + x^{76} + x^{74} + x^{70} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36}\\
&\quad + x^{32} + x^{26} + x^{22} + x^{18} + x^{16} + x^{12} + x^{8} + x^{4},\\
p_{12}(x) &= x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{74} + x^{72} + x^{66} + x^{64} + x^{50}\\
&\quad + x^{48} + x^{42} + x^{40} + x^{34} + x^{32} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{113} + x^{112}\\
&\quad + x^{110} + x^{109} + x^{107} + x^{105} + x^{103} + x^{102} + x^{99} + x^{97} + x^{95} + x^{93}\\
&\quad + x^{92} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{75} + x^{72} + x^{70} + x^{63} + x^{59}\\
&\quad + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45}\\
&\quad + x^{43} + x^{41} + x^{38} + x^{36} + x^{33} + x^{31} + x^{28} + x^{27} + x^{24},\\
p_3(x) &= x^{123} + x^{119} + x^{118} + x^{117} + x^{113} + x^{108} + x^{107} + x^{106} + x^{100} + x^{99}\\
&\quad + x^{98} + x^{97} + x^{94} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{73}\\
&\quad + x^{72} + x^{71} + x^{70} + x^{68} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{51} + x^{47}\\
&\quad + x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{33} + x^{30} + x^{29} + x^{28} + x^{25} + x^{19}\\
&\quad + x^{16},\\
p_6(x) &= x^{123} + x^{119} + x^{118} + x^{117} + x^{113} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100}\\
&\quad + x^{99} + x^{98} + x^{94} + x^{93} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{80}\\
&\quad + x^{79} + x^{78} + x^{74} + x^{73} + x^{71} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60}\\
&\quad + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{36}\\
&\quad + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{19}\\
&\quad + x^{17} + x^{16} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{108} + x^{103} + x^{100} \\
&\quad + x^{99} + x^{96} + x^{95} + x^{92} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} \\
&\quad + x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} \\
&\quad + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8}, \\
p_{12}(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{87} + x^{84} \\
&\quad + x^{75} + x^{72} + x^{67} + x^{64} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43} \\
&\quad + x^{40} + x^{35} + x^{32} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8} \\
&\quad + x^{3} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{132} + x^{127} + x^{126} + x^{122} + x^{119} + x^{116} + x^{115} + x^{114} + x^{111} \\
&\quad + x^{109} + x^{106} + x^{104} + x^{101} + x^{100} + x^{99} + x^{95} + x^{94} + x^{89} + x^{88} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{65} \\
&\quad + x^{64} + x^{62} + x^{61} + x^{57} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30}, \\
p_3(x) &= x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102} + x^{100} \\
&\quad + x^{97} + x^{95} + x^{93} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{73} + x^{71} + x^{69} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{51} + x^{50} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{33} + x^{32} \\
&\quad + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{90} + x^{88} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} \\
&\quad + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{57} + x^{56} \\
&\quad + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} \\
&\quad + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{124} + x^{112} + x^{104} + x^{100} + x^{88} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} \\
&\quad + x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1].

For $\phi(M^o, 1, 0)$:

$$p_0(x) = x^{122} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{108} + x^{105} + x^{104}$$

$$
\begin{aligned}
&+ x^{103} + x^{100} + x^{99} + x^{97} + x^{94} + x^{93} + x^{89} + x^{88} + x^{85} + x^{81} + x^{78}\\
&+ x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50}\\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30},\\
p_3(x) = &\, x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101}\\
&+ x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68}\\
&+ x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}\\
&+ x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18},\\
p_6(x) = &\, x^{122} + x^{120} + x^{116} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{94}\\
&+ x^{90} + x^{88} + x^{82} + x^{80} + x^{76} + x^{64} + x^{60} + x^{56} + x^{50} + x^{44} + x^{42}\\
&+ x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{22} + x^{16} + x^{14} + x^{12},\\
p_9(x) = &\, x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{112} + x^{109} + x^{107}\\
&+ x^{104} + x^{102} + x^{99} + x^{96} + x^{93} + x^{91} + x^{88} + x^{86} + x^{83} + x^{80} + x^{76}\\
&+ x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{51} + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{29} + x^{28} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{19} + x^{16} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_{12}(x) = &\, x^{128} + x^{120} + x^{112} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80}\\
&+ x^{76} + x^{68} + x^{56} + x^{44} + x^{36} + x^{24} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = &\, x^{126} + x^{124} + x^{122} + x^{118} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{87} + x^{84} + x^{81}\\
&+ x^{79} + x^{78} + x^{76} + x^{70} + x^{69} + x^{68} + x^{64} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{51} + x^{50} + x^{46} + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_3(x) = &\, x^{123} + x^{119} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{106} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{95} + x^{93} + x^{92} + x^{90} + x^{84}\\
&+ x^{82} + x^{80} + x^{79} + x^{74} + x^{72} + x^{69} + x^{66} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{55} + x^{52} + x^{48} + x^{47} + x^{46} + x^{45} + x^{40} + x^{39} + x^{37} + x^{36}\\
&+ x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{23} + x^{22} + x^{20},\\
p_6(x) = &\, x^{123} + x^{119} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{94} + x^{92} + x^{90}\\
&+ x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{61} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{47}\\
&+ x^{44} + x^{42} + x^{39} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{26} + x^{23} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} + x^{3} + 1,\\
p_9(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{108} + x^{103} + x^{100}\\
&+ x^{99} + x^{96} + x^{95} + x^{92} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75}\\
&+ x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40}\\
&+ x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{87} + x^{84}\\
&+ x^{75} + x^{72} + x^{67} + x^{64} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43}\\
&+ x^{40} + x^{35} + x^{32} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{113} + x^{112}\\
&+ x^{110} + x^{109} + x^{107} + x^{105} + x^{103} + x^{102} + x^{99} + x^{97} + x^{95} + x^{93}\\
&+ x^{92} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{75} + x^{72} + x^{70} + x^{63} + x^{59}\\
&+ x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45}\\
&+ x^{43} + x^{41} + x^{38} + x^{36} + x^{33} + x^{31} + x^{28} + x^{27} + x^{24},\\
p_3(x) &= x^{123} + x^{119} + x^{118} + x^{117} + x^{113} + x^{108} + x^{107} + x^{106} + x^{100} + x^{99}\\
&+ x^{98} + x^{97} + x^{94} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{68} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{51} + x^{47}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{33} + x^{30} + x^{29} + x^{28} + x^{25} + x^{19}\\
&+ x^{16},\\
p_6(x) &= x^{123} + x^{119} + x^{118} + x^{117} + x^{113} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100}\\
&+ x^{99} + x^{98} + x^{94} + x^{93} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{80}\\
&+ x^{79} + x^{78} + x^{74} + x^{73} + x^{71} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60}\\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{36}\\
&+ x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{19}\\
&+ x^{17} + x^{16} + x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{108} + x^{103} + x^{100}\\
&+ x^{99} + x^{96} + x^{95} + x^{92} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75}\\
&+ x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40}\\
&+ x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{87} + x^{84}\\
&+ x^{75} + x^{72} + x^{67} + x^{64} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43}\\
&+ x^{40} + x^{35} + x^{32} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{122} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{108} + x^{105} + x^{104} \\
&\quad + x^{103} + x^{100} + x^{99} + x^{97} + x^{94} + x^{93} + x^{89} + x^{88} + x^{85} + x^{81} + x^{78} \\
&\quad + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50} \\
&\quad + x^{48} + x^{47} + x^{46} + x^{45} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} \\
&\quad + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} \\
&\quad + x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} \\
&\quad + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{122} + x^{120} + x^{116} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{94} \\
&\quad + x^{90} + x^{88} + x^{82} + x^{80} + x^{76} + x^{64} + x^{60} + x^{56} + x^{50} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{112} + x^{109} + x^{107} \\
&\quad + x^{104} + x^{102} + x^{99} + x^{96} + x^{93} + x^{91} + x^{88} + x^{86} + x^{83} + x^{80} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{51} + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{29} + x^{28} + x^{26} \\
&\quad + x^{25} + x^{24} + x^{23} + x^{19} + x^{16} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} \\
&\quad + x^{76} + x^{68} + x^{56} + x^{44} + x^{36} + x^{24} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{132} + x^{127} + x^{126} + x^{122} + x^{119} + x^{116} + x^{115} + x^{114} + x^{111} \\
&\quad + x^{109} + x^{106} + x^{104} + x^{101} + x^{100} + x^{99} + x^{95} + x^{94} + x^{89} + x^{88} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{65} \\
&\quad + x^{64} + x^{62} + x^{61} + x^{57} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30}, \\
p_3(x) &= x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102} + x^{100} \\
&\quad + x^{97} + x^{95} + x^{93} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{73} + x^{71} + x^{69} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{51} + x^{50} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{33} + x^{32} \\
&\quad + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{98} + x^{96} + x^{94}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{90} + x^{88} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54} + x^{48}\\
&+ x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{30} + x^{26} + x^{20} + x^{16} + x^{14} + x^{12},\\
p_9(x) &= x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110}\\
&+ x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96}\\
&+ x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{57} + x^{56}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{6},\\
p_{12}(x) &= x^{124} + x^{112} + x^{104} + x^{100} + x^{88} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60}\\
&+ x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{87} + x^{84} + x^{81}\\
&+ x^{79} + x^{78} + x^{76} + x^{70} + x^{69} + x^{68} + x^{64} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{51} + x^{50} + x^{46} + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{123} + x^{119} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{106} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{95} + x^{93} + x^{92} + x^{90} + x^{84}\\
&+ x^{82} + x^{80} + x^{79} + x^{74} + x^{72} + x^{69} + x^{66} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{55} + x^{52} + x^{48} + x^{47} + x^{46} + x^{45} + x^{40} + x^{39} + x^{37} + x^{36}\\
&+ x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{23} + x^{22} + x^{20},\\
p_6(x) &= x^{123} + x^{119} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{94} + x^{92} + x^{90}\\
&+ x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{61} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{47}\\
&+ x^{44} + x^{42} + x^{39} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{23} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} + x^{3} + 1,\\
p_9(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{111} + x^{108} + x^{103} + x^{100}\\
&+ x^{99} + x^{96} + x^{95} + x^{92} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75}\\
&+ x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40}\\
&+ x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} + x^{103} + x^{100} + x^{87} + x^{84}\\
&+ x^{75} + x^{72} + x^{67} + x^{64} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{43}\\
&+ x^{40} + x^{35} + x^{32} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8}
\end{aligned}
$$

$+ x^3 + 1,$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	353	[147, 417]	706	[193, 212]	1059	[1220, 49]
1	[3, 4]	354	[427, 541]	707	[496, 864]	1060	[1221, 516]
2	[5, 6]	355	[175, 542]	708	[128, 174]	1061	[496, 569]
3	[7, 8]	356	[543, 446]	709	[121, 484]	1062	[974, 49]
4	[9, 10]	357	[544, 43]	710	[934, 159]	1063	[550, 199]
5	[11, 12]	358	[418, 545]	711	[935, 936]	1064	[212, 1216]
6	[13, 14]	359	[51, 216]	712	[937, 838]	1065	[509, 144]
7	[15, 16]	360	[546, 547]	713	[938, 488]	1066	[414, 409]
8	[17, 18]	361	[49, 548]	714	[939, 940]	1067	[994, 160]
9	[19, 20]	362	[549, 429]	715	[941, 509]	1068	[1222, 177]
10	[21, 22]	363	[550, 209]	716	[35, 406]	1069	[1223, 148]
11	[23, 24]	364	[551, 46]	717	[942, 574]	1070	[880, 563]
12	[25, 26]	365	[552, 553]	718	[943, 566]	1071	[484, 1224]
13	[27, 28]	366	[476, 477]	719	[884, 944]	1072	[852, 933]
14	[29, 30]	367	[159, 158]	720	[945, 419]	1073	[1225, 918]
15	[31, 32]	368	[157, 477]	721	[208, 212]	1074	[1226, 847]
16	[33, 34]	369	[554, 463]	722	[215, 561]	1075	[48, 397]
17	[35, 36]	370	[555, 182]	723	[946, 536]	1076	[427, 1227]
18	[37, 38]	371	[556, 8]	724	[947, 895]	1077	[986, 503]
19	[39, 40]	372	[557, 558]	725	[443, 948]	1078	[442, 881]
20	[41, 39]	373	[559, 464]	726	[949, 569]	1079	[1228, 482]
21	[42, 43]	374	[560, 561]	727	[146, 133]	1080	[1229, 1230]
22	[44, 45]	375	[452, 467]	728	[950, 951]	1081	[1231, 10]
23	[46, 47]	376	[184, 562]	729	[930, 558]	1082	[12, 574]
24	[48, 49]	377	[516, 195]	730	[952, 463]	1083	[393, 543]
25	[50, 51]	378	[442, 563]	731	[953, 854]	1084	[885, 477]
26	[52, 53]	379	[446, 564]	732	[954, 563]	1085	[1232, 918]
27	[54, 55]	380	[565, 566]	733	[955, 39]	1086	[937, 459]
28	[56, 57]	381	[567, 532]	734	[535, 847]	1087	[1233, 461]
29	[58, 59]	382	[568, 569]	735	[466, 37]	1088	[1234, 571]
30	[60, 61]	383	[511, 523]	736	[56, 956]	1089	[1235, 567]
31	[62, 63]	384	[570, 147]	737	[957, 174]	1090	[1236, 998]
32	[64, 65]	385	[142, 571]	738	[958, 151]	1091	[1237, 895]
33	[66, 67]	386	[572, 573]	739	[862, 956]	1092	[895, 1215]
34	[68, 69]	387	[574, 39]	740	[959, 426]	1093	[906, 157]
35	[70, 71]	388	[575, 204]	741	[960, 409]	1094	[1238, 936]
36	[72, 73]	389	[403, 576]	742	[502, 488]	1095	[919, 926]

37	[74, 75]	390	[577, 37]	743	[471, 191]	1096	[1239, 883]
38	[76, 77]	391	[510, 191]	744	[961, 828]	1097	[524, 491]
39	[78, 79]	392	[578, 579]	745	[392, 46]	1098	[1240, 881]
40	[80, 81]	393	[580, 581]	746	[962, 561]	1099	[394, 573]
41	[82, 80]	394	[582, 583]	747	[475, 43]	1100	[1000, 882]
42	[83, 84]	395	[263, 584]	748	[963, 184]	1101	[154, 574]
43	[85, 86]	396	[585, 236]	749	[964, 168]	1102	[1241, 444]
44	[87, 88]	397	[586, 238]	750	[965, 136]	1103	[60, 516]
45	[89, 90]	398	[587, 588]	751	[966, 530]	1104	[1242, 442]
46	[91, 92]	399	[589, 590]	752	[954, 881]	1105	[1243, 530]
47	[93, 94]	400	[591, 592]	753	[440, 901]	1106	[920, 173]
48	[95, 96]	401	[593, 594]	754	[967, 847]	1107	[934, 158]
49	[97, 98]	402	[595, 596]	755	[886, 503]	1108	[44, 1244]
50	[99, 100]	403	[597, 598]	756	[938, 503]	1109	[1245, 435]
51	[101, 102]	404	[599, 600]	757	[905, 175]	1110	[1246, 136]
52	[22, 84]	405	[601, 286]	758	[547, 968]	1111	[1247, 200]
53	[103, 104]	406	[602, 603]	759	[33, 8]	1112	[1205, 218]
54	[105, 106]	407	[604, 605]	760	[969, 158]	1113	[534, 1230]
55	[107, 108]	408	[606, 68]	761	[970, 895]	1114	[1000, 200]
56	[109, 110]	409	[607, 387]	762	[516, 480]	1115	[1248, 475]
57	[111, 112]	410	[608, 383]	763	[971, 134]	1116	[915, 1249]
58	[113, 114]	411	[609, 610]	764	[854, 948]	1117	[503, 546]
59	[115, 116]	412	[611, 612]	765	[972, 461]	1118	[1193, 138]
60	[117, 118]	413	[613, 614]	766	[973, 951]	1119	[571, 55]
61	[119, 120]	414	[615, 616]	767	[974, 397]	1120	[1250, 891]
62	[32, 121]	415	[617, 618]	768	[552, 936]	1121	[548, 534]
63	[122, 32]	416	[619, 620]	769	[975, 419]	1122	[433, 409]
64	[123, 124]	417	[227, 621]	770	[523, 211]	1123	[1251, 478]
65	[125, 126]	418	[622, 623]	771	[976, 448]	1124	[492, 140]
66	[127, 128]	419	[624, 625]	772	[172, 179]	1125	[1252, 59]
67	[129, 130]	420	[626, 138]	773	[977, 838]	1126	[1253, 500]
68	[131, 132]	421	[627, 628]	774	[505, 46]	1127	[1254, 951]
69	[133, 134]	422	[629, 107]	775	[11, 154]	1128	[1255, 40]
70	[135, 136]	423	[630, 335]	776	[543, 394]	1129	[129, 534]
71	[137, 138]	424	[631, 338]	777	[884, 968]	1130	[859, 998]
72	[139, 140]	425	[632, 326]	778	[978, 476]	1131	[400, 1190]
73	[141, 142]	426	[633, 349]	779	[462, 979]	1132	[425, 997]
74	[143, 144]	427	[634, 635]	780	[446, 573]	1133	[1211, 1256]
75	[145, 124]	428	[636, 235]	781	[980, 968]	1134	[1257, 896]
76	[146, 147]	429	[637, 638]	782	[177, 884]	1135	[481, 954]
77	[148, 149]	430	[639, 640]	783	[981, 464]	1136	[1258, 163]
78	[150, 36]	431	[641, 642]	784	[907, 2]	1137	[422, 571]
79	[151, 152]	432	[643, 644]	785	[982, 574]	1138	[1259, 838]
80	[121, 153]	433	[645, 17]	786	[983, 984]	1139	[432, 414]

81	[154, 41]	434	[646, 647]	787	[8, 985]	1140	[1260, 421]
82	[155, 154]	435	[648, 649]	788	[548, 130]	1141	[866, 184]
83	[42, 156]	436	[650, 368]	789	[414, 33]	1142	[1261, 567]
84	[43, 157]	437	[251, 590]	790	[828, 200]	1143	[1214, 59]
85	[158, 159]	438	[651, 288]	791	[981, 436]	1144	[988, 53]
86	[160, 161]	439	[652, 653]	792	[205, 464]	1145	[863, 421]
87	[162, 163]	440	[654, 113]	793	[986, 488]	1146	[1262, 530]
88	[164, 165]	441	[655, 630]	794	[987, 846]	1147	[868, 55]
89	[166, 46]	442	[656, 644]	795	[983, 32]	1148	[1263, 846]
90	[167, 168]	443	[657, 658]	796	[177, 547]	1149	[929, 214]
91	[169, 51]	444	[618, 659]	797	[988, 416]	1150	[1264, 40]
92	[170, 171]	445	[660, 356]	798	[828, 882]	1151	[128, 905]
93	[172, 173]	446	[92, 661]	799	[901, 889]	1152	[914, 161]
94	[174, 175]	447	[662, 663]	800	[200, 883]	1153	[1265, 898]
95	[176, 177]	448	[664, 665]	801	[989, 491]	1154	[1266, 898]
96	[178, 179]	449	[666, 667]	802	[990, 444]	1155	[1267, 1227]
97	[180, 124]	450	[346, 668]	803	[966, 835]	1156	[13, 918]
98	[181, 182]	451	[669, 670]	804	[991, 951]	1157	[1268, 542]
99	[183, 184]	452	[671, 672]	805	[992, 993]	1158	[1269, 883]
100	[185, 186]	453	[673, 674]	806	[166, 393]	1159	[575, 160]
101	[187, 188]	454	[675, 594]	807	[913, 40]	1160	[1192, 219]
102	[189, 151]	455	[676, 384]	808	[994, 204]	1161	[939, 429]
103	[190, 191]	456	[383, 677]	809	[995, 993]	1162	[513, 412]
104	[192, 193]	457	[678, 679]	810	[969, 159]	1163	[978, 157]
105	[194, 195]	458	[680, 681]	811	[147, 134]	1164	[1270, 47]
106	[196, 197]	459	[682, 683]	812	[426, 996]	1165	[1271, 133]
107	[198, 199]	460	[359, 684]	813	[186, 997]	1166	[170, 421]
108	[200, 201]	461	[685, 63]	814	[998, 999]	1167	[32, 483]
109	[202, 203]	462	[686, 687]	815	[173, 1000]	1168	[173, 828]
110	[204, 205]	463	[688, 689]	816	[1001, 209]	1169	[1272, 478]
111	[206, 207]	464	[291, 255]	817	[841, 831]	1170	[1273, 993]
112	[192, 208]	465	[690, 279]	818	[898, 1002]	1171	[1274, 460]
113	[198, 209]	466	[691, 692]	819	[1003, 558]	1172	[1275, 55]
114	[155, 12]	467	[693, 264]	820	[399, 933]	1173	[1220, 397]
115	[210, 211]	468	[694, 263]	821	[1004, 426]	1174	[167, 1276]
116	[212, 213]	469	[695, 696]	822	[498, 956]	1175	[1200, 138]
117	[148, 214]	470	[381, 697]	823	[47, 446]	1176	[1277, 846]
118	[215, 216]	471	[698, 85]	824	[485, 161]	1177	[870, 519]
119	[217, 218]	472	[699, 700]	825	[1005, 219]	1178	[500, 998]
120	[127, 219]	473	[312, 278]	826	[987, 1006]	1179	[415, 1278]
121	[220, 221]	474	[326, 281]	827	[596, 304]	1180	[1279, 573]
122	[222, 223]	475	[86, 701]	828	[1007, 1008]	1181	[39, 1280]
123	[224, 225]	476	[701, 342]	829	[1009, 592]	1182	[1281, 968]
124	[226, 227]	477	[71, 702]	830	[683, 650]	1183	[1282, 854]

125	[228, 229]	478	[703, 704]	831	[1010, 117]	1184	[503, 177]
126	[230, 231]	479	[363, 236]	832	[269, 617]	1185	[1274, 197]
127	[232, 233]	480	[239, 705]	833	[1011, 227]	1186	[1283, 196]
128	[234, 235]	481	[706, 707]	834	[1012, 817]	1187	[1027, 106]
129	[236, 237]	482	[67, 708]	835	[1013, 1007]	1188	[1103, 1051]
130	[238, 239]	483	[63, 709]	836	[1014, 1015]	1189	[1284, 317]
131	[240, 241]	484	[65, 690]	837	[1016, 663]	1190	[1022, 1012]
132	[242, 243]	485	[710, 610]	838	[1017, 1018]	1191	[1177, 1053]
133	[244, 245]	486	[711, 712]	839	[1019, 606]	1192	[1285, 1036]
134	[246, 247]	487	[713, 714]	840	[1020, 1010]	1193	[1286, 310]
135	[248, 249]	488	[715, 716]	841	[1021, 1022]	1194	[1287, 804]
136	[250, 251]	489	[717, 607]	842	[1023, 606]	1195	[1288, 380]
137	[252, 253]	490	[718, 273]	843	[1024, 274]	1196	[1289, 692]
138	[138, 138]	491	[88, 719]	844	[1025, 384]	1197	[1290, 1291]
139	[254, 255]	492	[720, 292]	845	[684, 616]	1198	[1292, 318]
140	[256, 257]	493	[721, 722]	846	[1026, 1027]	1199	[1293, 378]
141	[258, 259]	494	[723, 302]	847	[1028, 228]	1200	[1294, 252]
142	[260, 261]	495	[724, 725]	848	[1029, 772]	1201	[1295, 1048]
143	[262, 263]	496	[726, 727]	849	[1030, 736]	1202	[1296, 1145]
144	[264, 265]	497	[728, 729]	850	[1031, 349]	1203	[794, 4]
145	[266, 267]	498	[730, 293]	851	[1032, 285]	1204	[1297, 788]
146	[268, 269]	499	[731, 732]	852	[1033, 1034]	1205	[1298, 16]
147	[270, 76]	500	[733, 734]	853	[820, 1019]	1206	[1299, 790]
148	[271, 272]	501	[735, 736]	854	[1035, 1036]	1207	[1034, 1298]
149	[273, 274]	502	[737, 114]	855	[1037, 1038]	1208	[1300, 1166]
150	[275, 276]	503	[738, 739]	856	[1039, 1040]	1209	[600, 1045]
151	[277, 278]	504	[740, 735]	857	[1041, 1042]	1210	[1301, 681]
152	[279, 280]	505	[741, 742]	858	[1043, 86]	1211	[1092, 1291]
153	[281, 115]	506	[335, 743]	859	[1044, 1045]	1212	[1302, 107]
154	[282, 283]	507	[744, 745]	860	[267, 647]	1213	[1303, 1121]
155	[284, 82]	508	[104, 264]	861	[73, 102]	1214	[1304, 320]
156	[285, 286]	509	[746, 735]	862	[665, 366]	1215	[1131, 1305]
157	[84, 285]	510	[642, 747]	863	[809, 252]	1216	[348, 1113]
158	[287, 288]	511	[748, 375]	864	[594, 1046]	1217	[1050, 786]
159	[288, 289]	512	[749, 703]	865	[1047, 368]	1218	[1306, 281]
160	[290, 291]	513	[750, 367]	866	[296, 1048]	1219	[1307, 272]
161	[292, 293]	514	[751, 752]	867	[1049, 1050]	1220	[1308, 755]
162	[294, 238]	515	[753, 593]	868	[1051, 377]	1221	[1309, 239]
163	[295, 296]	516	[754, 755]	869	[1052, 98]	1222	[672, 797]
164	[297, 298]	517	[756, 757]	870	[1053, 360]	1223	[1310, 273]
165	[299, 300]	518	[758, 759]	871	[1054, 66]	1224	[1100, 1168]
166	[301, 302]	519	[760, 240]	872	[1055, 1056]	1225	[1311, 78]
167	[303, 304]	520	[761, 762]	873	[1057, 696]	1226	[1312, 237]
168	[305, 306]	521	[79, 338]	874	[1058, 701]	1227	[1095, 800]

169	[307, 308]	522	[692, 706]	875	[1059, 360]	1228	[1313, 722]
170	[309, 310]	523	[763, 100]	876	[1060, 764]	1229	[1314, 118]
171	[311, 312]	524	[764, 762]	877	[1061, 596]	1230	[776, 1101]
172	[313, 314]	525	[765, 381]	878	[1062, 298]	1231	[1315, 788]
173	[315, 316]	526	[766, 734]	879	[1063, 662]	1232	[1305, 283]
174	[233, 317]	527	[767, 768]	880	[261, 614]	1233	[1316, 1065]
175	[235, 318]	528	[769, 702]	881	[707, 1064]	1234	[1129, 1317]
176	[245, 319]	529	[677, 770]	882	[325, 1065]	1235	[1318, 661]
177	[320, 321]	530	[771, 772]	883	[1008, 1066]	1236	[298, 1042]
178	[322, 323]	531	[773, 774]	884	[739, 1014]	1237	[1319, 115]
179	[324, 325]	532	[583, 294]	885	[1067, 85]	1238	[1320, 739]
180	[231, 326]	533	[775, 776]	886	[1068, 795]	1239	[1321, 353]
181	[327, 287]	534	[736, 337]	887	[1069, 76]	1240	[1317, 607]
182	[328, 329]	535	[777, 776]	888	[1070, 331]	1241	[1112, 233]
183	[330, 331]	536	[778, 779]	889	[696, 1071]	1242	[1322, 813]
184	[332, 333]	537	[272, 780]	890	[729, 323]	1243	[1323, 676]
185	[334, 335]	538	[781, 782]	891	[770, 799]	1244	[1324, 1325]
186	[336, 225]	539	[783, 784]	892	[1072, 1037]	1245	[1064, 65]
187	[337, 267]	540	[785, 786]	893	[1073, 1034]	1246	[90, 348]
188	[338, 339]	541	[4, 787]	894	[247, 262]	1247	[1326, 759]
189	[340, 279]	542	[788, 789]	895	[1074, 1075]	1248	[1327, 747]
190	[341, 342]	543	[579, 790]	896	[18, 1076]	1249	[1328, 1156]
191	[343, 344]	544	[791, 792]	897	[1077, 20]	1250	[590, 314]
192	[345, 346]	545	[793, 794]	898	[1048, 1078]	1251	[1329, 653]
193	[347, 348]	546	[795, 796]	899	[1079, 823]	1252	[623, 795]
194	[349, 96]	547	[797, 676]	900	[1080, 1008]	1253	[1330, 1331]
195	[118, 350]	548	[755, 798]	901	[1040, 674]	1254	[705, 92]
196	[351, 352]	549	[799, 391]	902	[1081, 19]	1255	[1332, 630]
197	[353, 354]	550	[800, 679]	903	[1082, 352]	1256	[1045, 777]
198	[355, 356]	551	[708, 93]	904	[221, 1083]	1257	[1042, 742]
199	[357, 358]	552	[801, 319]	905	[658, 782]	1258	[1333, 584]
200	[304, 359]	553	[802, 803]	906	[1084, 71]	1259	[1334, 796]
201	[360, 361]	554	[804, 333]	907	[1085, 1086]	1260	[1335, 253]
202	[362, 363]	555	[805, 289]	908	[1087, 1078]	1261	[385, 811]
203	[364, 365]	556	[806, 640]	909	[1088, 337]	1262	[1336, 1014]
204	[366, 367]	557	[807, 757]	910	[1089, 635]	1263	[1337, 1144]
205	[368, 290]	558	[808, 809]	911	[1090, 69]	1264	[1338, 707]
206	[369, 291]	559	[810, 366]	912	[1091, 1092]	1265	[663, 687]
207	[370, 307]	560	[811, 375]	913	[283, 813]	1266	[709, 1063]
208	[371, 372]	561	[308, 812]	914	[1093, 650]	1267	[1339, 1331]
209	[373, 275]	562	[697, 634]	915	[1094, 1095]	1268	[1340, 1019]
210	[374, 308]	563	[813, 814]	916	[1096, 221]	1269	[1341, 1289]
211	[375, 376]	564	[94, 815]	917	[1097, 26]	1270	[592, 582]
212	[377, 378]	565	[816, 93]	918	[1098, 1039]	1271	[344, 727]

213	[249, 379]	566	[817, 818]	919	[1099, 1100]	1272	[1342, 1040]
214	[380, 381]	567	[819, 24]	920	[1101, 1007]	1273	[1343, 1317]
215	[382, 383]	568	[722, 246]	921	[1102, 1103]	1274	[712, 223]
216	[384, 385]	569	[225, 820]	922	[1104, 377]	1275	[1344, 346]
217	[386, 387]	570	[821, 246]	923	[318, 1010]	1276	[1345, 1330]
218	[388, 389]	571	[378, 670]	924	[1105, 668]	1277	[1346, 17]
219	[390, 391]	572	[822, 811]	925	[1106, 1100]	1278	[317, 1347]
220	[392, 393]	573	[679, 643]	926	[1107, 1108]	1279	[1348, 1037]
221	[47, 394]	574	[24, 823]	927	[1109, 1063]	1280	[757, 1184]
222	[395, 129]	575	[824, 229]	928	[1110, 290]	1281	[1349, 770]
223	[396, 397]	576	[825, 826]	929	[1111, 1112]	1282	[1350, 618]
224	[398, 399]	577	[329, 245]	930	[1113, 1114]	1283	[1178, 1289]
225	[189, 400]	578	[827, 828]	931	[1115, 1116]	1284	[1351, 482]
226	[401, 188]	579	[507, 128]	932	[798, 1117]	1285	[219, 174]
227	[402, 215]	580	[829, 186]	933	[588, 380]	1286	[544, 156]
228	[403, 404]	581	[125, 140]	934	[1118, 241]	1287	[1352, 12]
229	[137, 137]	582	[830, 412]	935	[1119, 716]	1288	[1353, 567]
230	[405, 406]	583	[509, 831]	936	[1120, 604]	1289	[467, 522]
231	[407, 148]	584	[150, 406]	937	[780, 1121]	1290	[1354, 442]
232	[408, 409]	585	[832, 174]	938	[1122, 320]	1291	[895, 59]
233	[396, 49]	586	[833, 133]	939	[1123, 1124]	1292	[1355, 214]
234	[410, 210]	587	[834, 565]	940	[1125, 751]	1293	[1356, 565]
235	[411, 412]	588	[835, 836]	941	[1126, 75]	1294	[559, 436]
236	[413, 414]	589	[837, 484]	942	[102, 78]	1295	[1252, 1215]
237	[174, 415]	590	[458, 838]	943	[734, 579]	1296	[1357, 1256]
238	[52, 416]	591	[839, 522]	944	[796, 62]	1297	[871, 414]
239	[133, 417]	592	[840, 509]	945	[1127, 1081]	1298	[415, 542]
240	[418, 419]	593	[841, 144]	946	[1128, 1129]	1299	[1358, 904]
241	[158, 158]	594	[842, 841]	947	[1130, 1131]	1300	[149, 478]
242	[420, 421]	595	[843, 469]	948	[1132, 1119]	1301	[1005, 128]
243	[141, 422]	596	[844, 51]	949	[69, 269]	1302	[1359, 936]
244	[423, 152]	597	[845, 846]	950	[1133, 581]	1303	[848, 173]
245	[402, 51]	598	[164, 847]	951	[1134, 1135]	1304	[1360, 218]
246	[424, 425]	599	[848, 179]	952	[1136, 1050]	1305	[998, 1256]
247	[426, 427]	600	[849, 163]	953	[1137, 1132]	1306	[1361, 879]
248	[428, 429]	601	[850, 203]	954	[1138, 66]	1307	[1362, 429]
249	[430, 431]	602	[851, 191]	955	[1139, 1114]	1308	[179, 828]
250	[432, 433]	603	[514, 852]	956	[1140, 1141]	1309	[1363, 933]
251	[434, 435]	604	[853, 854]	957	[727, 1124]	1310	[1364, 461]
252	[204, 161]	605	[531, 855]	958	[1141, 1022]	1311	[1365, 1227]
253	[436, 204]	606	[570, 133]	959	[1142, 1143]	1312	[1366, 1230]
254	[437, 45]	607	[178, 173]	960	[75, 1144]	1313	[190, 472]
255	[160, 205]	608	[856, 469]	961	[100, 1145]	1314	[206, 412]
256	[438, 182]	609	[857, 527]	962	[661, 667]	1315	[130, 1230]

257	[439, 440]	610	[43, 476]	963	[1146, 697]	1316	[1367, 212]
258	[441, 442]	611	[858, 406]	964	[1147, 325]	1317	[1231, 539]
259	[443, 444]	612	[499, 859]	965	[1148, 372]	1318	[1368, 901]
260	[445, 446]	613	[860, 121]	966	[1149, 1061]	1319	[1369, 212]
261	[447, 448]	614	[413, 433]	967	[1150, 798]	1320	[996, 1227]
262	[425, 188]	615	[861, 210]	968	[681, 1151]	1321	[858, 36]
263	[449, 210]	616	[862, 57]	969	[1152, 287]	1322	[1370, 998]
264	[185, 425]	617	[863, 171]	970	[1153, 690]	1323	[1371, 933]
265	[193, 450]	618	[124, 864]	971	[1065, 672]	1324	[31, 984]
266	[451, 193]	619	[865, 140]	972	[1154, 359]	1325	[991, 1372]
267	[452, 37]	620	[866, 867]	973	[1155, 743]	1326	[1352, 154]
268	[453, 184]	621	[142, 868]	974	[1066, 120]	1327	[1248, 477]
269	[454, 124]	622	[869, 163]	975	[1156, 113]	1328	[903, 984]
270	[455, 456]	623	[870, 536]	976	[1157, 709]	1329	[990, 948]
271	[457, 446]	624	[871, 433]	977	[1158, 321]	1330	[1283, 461]
272	[458, 459]	625	[849, 872]	978	[1159, 276]	1331	[835, 891]
273	[196, 460]	626	[873, 419]	979	[1160, 1161]	1332	[139, 126]
274	[461, 197]	627	[874, 53]	980	[1162, 1061]	1333	[1373, 879]
275	[462, 463]	628	[495, 1]	981	[1163, 110]	1334	[1374, 542]
276	[464, 204]	629	[875, 200]	982	[667, 28]	1335	[1375, 404]
277	[465, 144]	630	[523, 456]	983	[1164, 220]	1336	[1376, 571]
278	[466, 467]	631	[876, 399]	984	[1165, 1166]	1337	[1377, 1216]
279	[468, 400]	632	[877, 456]	985	[16, 1167]	1338	[851, 472]
280	[469, 470]	633	[878, 534]	986	[1168, 1075]	1339	[1257, 566]
281	[471, 472]	634	[530, 836]	987	[1169, 68]	1340	[528, 416]
282	[473, 186]	635	[854, 444]	988	[1170, 792]	1341	[555, 132]
283	[181, 132]	636	[562, 879]	989	[1171, 605]	1342	[1250, 836]
284	[474, 121]	637	[880, 881]	990	[598, 745]	1343	[1378, 168]
285	[156, 157]	638	[853, 443]	991	[1172, 582]	1344	[1379, 532]
286	[157, 475]	639	[882, 883]	992	[1161, 261]	1345	[1379, 855]
287	[475, 156]	640	[546, 884]	993	[1173, 1174]	1346	[470, 889]
288	[161, 436]	641	[885, 475]	994	[1175, 292]	1347	[49, 129]
289	[156, 476]	642	[486, 136]	995	[1176, 1177]	1348	[874, 416]
290	[436, 160]	643	[393, 47]	996	[725, 1092]	1349	[420, 171]
291	[477, 156]	644	[886, 488]	997	[278, 1178]	1350	[1380, 901]
292	[464, 160]	645	[887, 37]	998	[732, 764]	1351	[1381, 817]
293	[138, 137]	646	[888, 889]	999	[1086, 1179]	1352	[1347, 774]
294	[214, 478]	647	[830, 207]	1000	[314, 1129]	1353	[1382, 583]
295	[479, 480]	648	[890, 891]	1001	[1180, 94]	1354	[1383, 79]
296	[481, 442]	649	[892, 531]	1002	[333, 1181]	1355	[321, 598]
297	[34, 482]	650	[205, 436]	1003	[1182, 80]	1356	[1384, 18]
298	[483, 484]	651	[893, 45]	1004	[1183, 634]	1357	[1385, 752]
299	[485, 205]	652	[894, 895]	1005	[1184, 658]	1358	[1386, 262]
300	[486, 487]	653	[494, 896]	1006	[1185, 1186]	1359	[1387, 797]

301	[488, 177]	654	[897, 415]	1007	[489, 414]	1360	[1388, 67]
302	[489, 433]	655	[867, 898]	1008	[574, 913]	1361	[1389, 725]
303	[490, 491]	656	[899, 558]	1009	[1187, 193]	1362	[1390, 617]
304	[492, 126]	657	[900, 536]	1010	[844, 215]	1363	[640, 88]
305	[194, 480]	658	[540, 154]	1011	[1188, 142]	1364	[1391, 353]
306	[493, 494]	659	[901, 902]	1012	[184, 898]	1365	[1392, 732]
307	[495, 426]	660	[903, 32]	1013	[1189, 539]	1366	[1393, 593]
308	[145, 496]	661	[51, 561]	1014	[151, 1190]	1367	[1114, 712]
309	[497, 45]	662	[484, 904]	1015	[571, 1191]	1368	[782, 623]
310	[476, 475]	663	[905, 415]	1016	[1192, 128]	1369	[823, 249]
311	[498, 57]	664	[906, 476]	1017	[1193, 137]	1370	[237, 1086]
312	[499, 500]	665	[907, 527]	1018	[995, 1194]	1371	[387, 729]
313	[501, 129]	666	[908, 426]	1019	[400, 152]	1372	[1394, 724]
314	[502, 503]	667	[842, 509]	1020	[1195, 148]	1373	[1395, 1143]
315	[504, 151]	668	[530, 891]	1021	[1196, 134]	1374	[1396, 1132]
316	[411, 207]	669	[909, 516]	1022	[840, 841]	1375	[768, 1112]
317	[409, 34]	670	[565, 896]	1023	[1197, 399]	1376	[759, 1177]
318	[210, 456]	671	[910, 500]	1024	[1198, 854]	1377	[1397, 259]
319	[131, 182]	672	[424, 186]	1025	[1199, 142]	1378	[1076, 316]
320	[505, 393]	673	[911, 565]	1026	[943, 896]	1379	[687, 82]
321	[409, 8]	674	[567, 855]	1027	[493, 565]	1380	[20, 1081]
322	[506, 47]	675	[912, 500]	1028	[1200, 137]	1381	[865, 126]
323	[507, 219]	676	[186, 188]	1029	[984, 121]	1382	[1398, 214]
324	[508, 509]	677	[469, 901]	1030	[1201, 889]	1383	[1399, 427]
325	[510, 472]	678	[551, 393]	1031	[1202, 168]	1384	[1400, 427]
326	[511, 210]	679	[483, 153]	1032	[1203, 203]	1385	[1217, 532]
327	[512, 404]	680	[153, 904]	1033	[1204, 415]	1386	[405, 36]
328	[513, 207]	681	[41, 913]	1034	[1205, 431]	1387	[1401, 1256]
329	[514, 399]	682	[914, 205]	1035	[1206, 209]	1388	[1402, 904]
330	[515, 516]	683	[915, 203]	1036	[122, 984]	1389	[1241, 948]
331	[61, 480]	684	[898, 879]	1037	[467, 38]	1390	[1403, 1216]
332	[517, 39]	685	[916, 448]	1038	[933, 1207]	1391	[1, 427]
333	[518, 519]	686	[917, 918]	1039	[892, 567]	1392	[1404, 460]
334	[520, 148]	687	[919, 448]	1040	[61, 195]	1393	[1260, 171]
335	[468, 151]	688	[920, 179]	1041	[1208, 435]	1394	[1404, 197]
336	[521, 152]	689	[434, 921]	1042	[430, 218]	1395	[890, 836]
337	[37, 522]	690	[124, 569]	1043	[1209, 527]	1396	[928, 956]
338	[449, 523]	691	[922, 193]	1044	[1210, 884]	1397	[58, 1215]
339	[399, 524]	692	[454, 496]	1045	[557, 931]	1398	[1121, 703]
340	[525, 469]	693	[923, 522]	1046	[500, 1211]	1399	[356, 363]
341	[526, 527]	694	[924, 142]	1047	[1212, 993]	1400	[790, 1095]
342	[161, 464]	695	[925, 484]	1048	[494, 566]	1401	[1405, 588]
343	[528, 53]	696	[447, 926]	1049	[1213, 884]	1402	[1406, 1131]
344	[439, 469]	697	[461, 460]	1050	[9, 539]	1403	[1407, 1051]

345	[529, 530]	698	[927, 463]	1051	[1214, 1215]	1404	[372, 220]
346	[531, 532]	699	[928, 57]	1052	[450, 1216]	1405	[479, 195]
347	[533, 534]	700	[407, 929]	1053	[8, 482]	1406	[438, 132]
348	[535, 165]	701	[159, 159]	1054	[397, 129]	1407	[1240, 563]
349	[518, 536]	702	[477, 43]	1055	[1217, 855]		
350	[214, 537]	703	[930, 931]	1056	[947, 1214]		
351	[538, 539]	704	[534, 932]	1057	[1218, 435]		
352	[540, 12]	705	[933, 491]	1058	[1219, 404]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	235	1	470	0	705	0	940	0	1175	0
1	0	236	1	471	0	706	0	941	1	1176	0
2	0	237	0	472	0	707	1	942	0	1177	0
3	0	238	0	473	0	708	0	943	0	1178	1
4	1	239	1	474	0	709	0	944	1	1179	0
5	0	240	1	475	0	710	1	945	0	1180	0
6	1	241	0	476	1	711	1	946	1	1181	1
7	0	242	0	477	1	712	1	947	0	1182	0
8	0	243	1	478	1	713	0	948	1	1183	0
9	1	244	1	479	0	714	1	949	1	1184	0
10	0	245	1	480	1	715	1	950	1	1185	1
11	0	246	0	481	0	716	0	951	1	1186	1
12	1	247	1	482	1	717	0	952	1	1187	1
13	1	248	0	483	1	718	0	953	0	1188	0
14	1	249	1	484	1	719	0	954	0	1189	0
15	0	250	1	485	1	720	0	955	1	1190	0
16	0	251	1	486	1	721	1	956	1	1191	0
17	0	252	1	487	0	722	0	957	1	1192	1
18	1	253	0	488	1	723	1	958	0	1193	1
19	1	254	1	489	0	724	0	959	1	1194	1
20	0	255	0	490	0	725	0	960	1	1195	0
21	0	256	0	491	1	726	1	961	1	1196	0
22	0	257	1	492	0	727	1	962	1	1197	1
23	0	258	1	493	1	728	1	963	0	1198	1
24	1	259	0	494	1	729	1	964	0	1199	0
25	1	260	1	495	0	730	1	965	1	1200	0
26	0	261	1	496	1	731	0	966	1	1201	0
27	1	262	1	497	1	732	0	967	0	1202	1
28	1	263	1	498	1	733	1	968	0	1203	0
29	1	264	1	499	0	734	0	969	0	1204	0
30	0	265	0	500	1	735	0	970	1	1205	0
31	0	266	1	501	0	736	1	971	1	1206	1
32	0	267	1	502	1	737	1	972	0	1207	0

33	0	268	1	503	0	738	0	973	1	1208	1
34	1	269	0	504	1	739	0	974	1	1209	0
35	0	270	0	505	1	740	1	975	1	1210	0
36	1	271	0	506	1	741	1	976	1	1211	1
37	1	272	0	507	1	742	1	977	1	1212	0
38	1	273	1	508	0	743	0	978	1	1213	1
39	1	274	0	509	1	744	1	979	1	1214	0
40	0	275	1	510	1	745	0	980	1	1215	1
41	0	276	1	511	0	746	1	981	1	1216	0
42	0	277	0	512	0	747	0	982	1	1217	1
43	0	278	0	513	1	748	0	983	0	1218	0
44	0	279	1	514	1	749	0	984	1	1219	0
45	1	280	0	515	1	750	1	985	0	1220	0
46	0	281	0	516	0	751	1	986	1	1221	0
47	0	282	0	517	0	752	0	987	0	1222	0
48	1	283	0	518	1	753	1	988	1	1223	0
49	1	284	0	519	0	754	0	989	1	1224	1
50	1	285	1	520	1	755	0	990	1	1225	0
51	1	286	0	521	0	756	0	991	0	1226	1
52	0	287	0	522	0	757	1	992	1	1227	0
53	1	288	1	523	1	758	1	993	0	1228	1
54	1	289	1	524	1	759	0	994	0	1229	0
55	1	290	0	525	0	760	0	995	0	1230	1
56	1	291	1	526	1	761	1	996	0	1231	0
57	0	292	1	527	1	762	0	997	0	1232	0
58	1	293	0	528	1	763	1	998	0	1233	1
59	0	294	0	529	0	764	1	999	1	1234	1
60	0	295	0	530	1	765	0	1000	1	1235	1
61	1	296	0	531	1	766	1	1001	0	1236	1
62	0	297	1	532	1	767	1	1002	1	1237	0
63	1	298	1	533	0	768	1	1003	0	1238	0
64	0	299	1	534	1	769	1	1004	0	1239	0
65	1	300	1	535	0	770	1	1005	0	1240	0
66	0	301	1	536	1	771	1	1006	1	1241	0
67	1	302	0	537	0	772	0	1007	0	1242	0
68	1	303	0	538	1	773	1	1008	1	1243	1
69	1	304	0	539	1	774	1	1009	1	1244	0
70	0	305	1	540	1	775	0	1010	0	1245	0
71	1	306	1	541	1	776	1	1011	0	1246	0
72	1	307	0	542	0	777	0	1012	0	1247	1
73	1	308	1	543	1	778	1	1013	0	1248	1
74	1	309	1	544	1	779	1	1014	0	1249	1
75	1	310	1	545	1	780	1	1015	1	1250	0
76	1	311	1	546	0	781	1	1016	1	1251	1

77	0	312	0	547	1	782	1	1017	1	1252	0
78	1	313	0	548	0	783	1	1018	0	1253	1
79	0	314	1	549	1	784	0	1019	1	1254	0
80	0	315	1	550	0	785	1	1020	0	1255	1
81	0	316	1	551	0	786	0	1021	0	1256	0
82	0	317	1	552	1	787	0	1022	0	1257	1
83	0	318	0	553	1	788	0	1023	1	1258	1
84	0	319	1	554	0	789	1	1024	1	1259	0
85	0	320	1	555	1	790	0	1025	0	1260	1
86	0	321	1	556	1	791	1	1026	0	1261	1
87	0	322	1	557	0	792	0	1027	1	1262	0
88	1	323	1	558	0	793	1	1028	0	1263	1
89	1	324	0	559	0	794	0	1029	1	1264	0
90	0	325	1	560	0	795	0	1030	0	1265	1
91	0	326	0	561	1	796	1	1031	1	1266	0
92	1	327	0	562	0	797	1	1032	0	1267	1
93	0	328	1	563	0	798	0	1033	0	1268	1
94	0	329	1	564	0	799	1	1034	0	1269	1
95	1	330	1	565	0	800	0	1035	1	1270	0
96	1	331	1	566	0	801	1	1036	1	1271	1
97	1	332	0	567	0	802	1	1037	0	1272	0
98	0	333	1	568	0	803	1	1038	0	1273	1
99	1	334	1	569	0	804	0	1039	0	1274	1
100	1	335	1	570	0	805	1	1040	1	1275	0
101	1	336	0	571	1	806	1	1041	1	1276	0
102	0	337	1	572	1	807	0	1042	1	1277	0
103	1	338	1	573	1	808	0	1043	0	1278	1
104	0	339	1	574	1	809	0	1044	0	1279	0
105	1	340	0	575	1	810	0	1045	0	1280	1
106	1	341	1	576	0	811	0	1046	1	1281	0
107	1	342	1	577	1	812	1	1047	0	1282	0
108	0	343	1	578	0	813	0	1048	1	1283	1
109	1	344	1	579	1	814	0	1049	1	1284	0
110	1	345	0	580	1	815	1	1050	1	1285	1
111	0	346	1	581	1	816	0	1051	0	1286	1
112	0	347	0	582	0	817	0	1052	1	1287	1
113	1	348	0	583	1	818	1	1053	0	1288	0
114	0	349	1	584	1	819	0	1054	0	1289	0
115	0	350	0	585	0	820	1	1055	1	1290	1
116	0	351	1	586	0	821	0	1056	0	1291	1
117	0	352	1	587	0	822	1	1057	0	1292	1
118	0	353	0	588	0	823	0	1058	0	1293	0
119	1	354	1	589	1	824	1	1059	0	1294	0
120	0	355	1	590	0	825	0	1060	0	1295	0

121	0	356	1	591	1	826	0	1061	1	1296	1
122	1	357	1	592	0	827	0	1062	1	1297	0
123	0	358	1	593	0	828	0	1063	0	1298	0
124	0	359	1	594	1	829	1	1064	0	1299	1
125	1	360	0	595	1	830	0	1065	1	1300	1
126	1	361	1	596	0	831	0	1066	1	1301	0
127	0	362	1	597	1	832	0	1067	0	1302	0
128	0	363	0	598	1	833	0	1068	0	1303	1
129	1	364	0	599	1	834	0	1069	0	1304	0
130	0	365	1	600	0	835	0	1070	1	1305	0
131	1	366	1	601	1	836	0	1071	1	1306	0
132	0	367	1	602	0	837	1	1072	0	1307	0
133	1	368	0	603	1	838	1	1073	0	1308	0
134	0	369	0	604	1	839	1	1074	1	1309	0
135	0	370	1	605	1	840	0	1075	1	1310	0
136	1	371	1	606	0	841	0	1076	1	1311	0
137	1	372	0	607	1	842	1	1077	1	1312	1
138	0	373	0	608	0	843	1	1078	1	1313	1
139	1	374	0	609	1	844	0	1079	1	1314	0
140	0	375	1	610	0	845	1	1080	0	1315	0
141	1	376	0	611	0	846	0	1081	0	1316	1
142	1	377	0	612	0	847	0	1082	1	1317	0
143	1	378	1	613	1	848	1	1083	1	1318	1
144	1	379	1	614	1	849	0	1084	0	1319	0
145	1	380	0	615	1	850	1	1085	0	1320	0
146	1	381	0	616	0	851	0	1086	1	1321	0
147	0	382	0	617	0	852	0	1087	1	1322	0
148	0	383	0	618	0	853	1	1088	1	1323	1
149	1	384	0	619	0	854	1	1089	1	1324	0
150	1	385	1	620	0	855	0	1090	1	1325	0
151	0	386	1	621	1	856	0	1091	0	1326	1
152	1	387	1	622	1	857	1	1092	1	1327	1
153	0	388	1	623	0	858	0	1093	0	1328	1
154	0	389	1	624	0	859	0	1094	0	1329	1
155	0	390	1	625	0	860	1	1095	0	1330	1
156	1	391	1	626	0	861	1	1096	0	1331	0
157	0	392	0	627	0	862	0	1097	1	1332	1
158	0	393	1	628	0	863	0	1098	0	1333	1
159	1	394	0	629	0	864	1	1099	0	1334	0
160	0	395	1	630	1	865	0	1100	1	1335	1
161	1	396	0	631	0	866	0	1101	0	1336	0
162	0	397	0	632	1	867	1	1102	0	1337	1
163	0	398	0	633	1	868	0	1103	0	1338	0
164	1	399	1	634	1	869	1	1104	0	1339	1

165	1	400	1	635	1	870	0	1105	1	1340	1
166	1	401	0	636	0	871	0	1106	0	1341	1
167	0	402	1	637	1	872	1	1107	1	1342	0
168	1	403	1	638	1	873	0	1108	0	1343	1
169	0	404	1	639	1	874	0	1109	0	1344	0
170	1	405	1	640	0	875	0	1110	0	1345	0
171	1	406	0	641	0	876	0	1111	1	1346	0
172	0	407	1	642	1	877	1	1112	0	1347	1
173	1	408	0	643	1	878	1	1113	1	1348	0
174	0	409	1	644	0	879	0	1114	1	1349	0
175	1	410	0	645	0	880	1	1115	1	1350	0
176	1	411	1	646	1	881	1	1116	0	1351	0
177	1	412	0	647	0	882	1	1117	0	1352	1
178	1	413	1	648	1	883	1	1118	1	1353	0
179	0	414	1	649	0	884	0	1119	1	1354	1
180	1	415	0	650	0	885	0	1120	0	1355	1
181	0	416	0	651	0	886	0	1121	0	1356	0
182	1	417	1	652	1	887	0	1122	0	1357	1
183	1	418	1	653	1	888	1	1123	1	1358	1
184	0	419	0	654	1	889	1	1124	0	1359	0
185	1	420	0	655	1	890	1	1125	0	1360	0
186	0	421	0	656	1	891	1	1126	1	1361	0
187	1	422	0	657	0	892	0	1127	0	1362	0
188	1	423	1	658	1	893	0	1128	1	1363	0
189	0	424	0	659	1	894	1	1129	1	1364	0
190	1	425	1	660	1	895	1	1130	0	1365	0
191	1	426	1	661	1	896	1	1131	1	1366	1
192	0	427	1	662	1	897	1	1132	1	1367	1
193	0	428	0	663	1	898	1	1133	1	1368	1
194	1	429	1	664	0	899	1	1134	1	1369	0
195	0	430	1	665	0	900	0	1135	0	1370	0
196	1	431	0	666	1	901	1	1136	1	1371	1
197	0	432	1	667	1	902	0	1137	0	1372	0
198	1	433	0	668	1	903	1	1138	0	1373	1
199	1	434	1	669	1	904	0	1139	1	1374	0
200	0	435	1	670	0	905	1	1140	1	1375	1
201	0	436	0	671	1	906	0	1141	0	1376	0
202	1	437	1	672	0	907	0	1142	1	1377	1
203	0	438	0	673	1	908	1	1143	0	1378	1
204	1	439	1	674	0	909	1	1144	1	1379	0
205	0	440	1	675	0	910	1	1145	0	1380	0
206	0	441	1	676	0	911	1	1146	0	1381	0
207	1	442	1	677	0	912	0	1147	0	1382	0
208	1	443	0	678	0	913	0	1148	1	1383	1

209	0	444	0	679	1	914	0	1149	1	1384	0
210	0	445	1	680	0	915	0	1150	0	1385	1
211	1	446	1	681	0	916	0	1151	0	1386	1
212	0	447	1	682	0	917	1	1152	0	1387	0
213	1	448	0	683	0	918	0	1153	1	1388	0
214	0	449	1	684	1	919	0	1154	0	1389	0
215	0	450	1	685	0	920	0	1155	1	1390	0
216	0	451	1	686	1	921	0	1156	1	1391	0
217	1	452	1	687	0	922	0	1157	1	1392	0
218	1	453	1	688	0	923	0	1158	1	1393	1
219	1	454	0	689	1	924	1	1159	1	1394	0
220	0	455	0	690	0	925	0	1160	1	1395	1
221	0	456	0	691	0	926	1	1161	1	1396	0
222	1	457	0	692	0	927	0	1162	1	1397	1
223	0	458	0	693	0	928	0	1163	1	1398	0
224	0	459	0	694	1	929	1	1164	0	1399	1
225	0	460	1	695	0	930	1	1165	1	1400	0
226	0	461	0	696	1	931	1	1166	1	1401	0
227	1	462	1	697	0	932	0	1167	0	1402	0
228	1	463	0	698	0	933	0	1168	1	1403	0
229	1	464	1	699	0	934	1	1169	0	1404	0
230	1	465	0	700	1	935	1	1170	1	1405	0
231	1	466	0	701	1	936	0	1171	1	1406	0
232	0	467	0	702	1	937	1	1172	0	1407	0
233	0	468	1	703	1	938	0	1173	0		
234	0	469	0	704	1	939	1	1174	0		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: `guoniu.han@unistra.fr`

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: `huyining@hust.edu.cn`