PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^5+z^4+z^3,z+1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^5+z^4+z^3,z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^5+z^4+z^3,z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^8+z^5+z^4+z^3+1,\\
p_1(z)&=z^{23}+z^{22}+z^{17}+z^{15}+z^{14}+z^{11}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^{16}+z^{12}+z^8+z^6,\\
p_3(z)&=z^{23}+z^{22}+z^{17}+z^{15}+z^{14}+z^{11}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_4(z)&=z^{22}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^6+z^5+z^3,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{20}+z^{18}+z^{15}+z^{14}+z^{13}+z^{11}+z^8+z^5+z^3,\\
p_1(z)&=z^{23}+z^{22}+z^{17}+z^{15}+z^{14}+z^{11}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^{16}+z^{12}+z^8+z^6,\\
p_3(z)&=z^{23}+z^{22}+z^{17}+z^{15}+z^{14}+z^{11}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_4(z)&=z^{22}+z^{20}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^6+z^5+z^4+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 540, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 660, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^{2u}M^e[:6u] \\ & + x^{4u}M^e[:4u] + x^{5u}M^e[:3u] + x^{7u}M^e[:u] + x^{3u}M^e[:6u] \\ & + x^{5u}M^e[:4u] + x^{6u}M^e[:3u] + x^{8u}M^e[:u] + x^{4u}M^e[:6u] \\ & + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{9u}M^e[:u] + x^{5u}M^e[:6u] \\ & + x^{7u}M^e[:4u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] + x^{7u}M^e[:5u] \\ & + x^{9u}M^e[:3u] + x^{8u}M^e[:4u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^e,\\
W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{2u}W^e[:6u]\\
&+ x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{7u}W^e[:u] + x^{3u}W^e[:6u]\\
&+ x^{5u}W^e[:4u] + x^{6u}W^e[:3u] + x^{8u}W^e[:u] + x^{4u}W^e[:6u]\\
&+ x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{9u}W^e[:u] + x^{5u}W^e[:6u]\\
&+ x^{7u}W^e[:4u] + x^{8u}W^e[:3u] + x^{10u}W^e[:u] + x^{7u}W^e[:5u]\\
&+ x^{9u}W^e[:3u] + x^{8u}W^e[:4u]\\
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{2u}M^o[:6u]\\
&+ x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{7u}M^o[:u] + x^{3u}M^o[:6u]\\
&+ x^{5u}M^o[:4u] + x^{6u}M^o[:3u] + x^{8u}M^o[:u] + x^{4u}M^o[:6u]\\
&+ x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{9u}M^o[:u] + x^{5u}M^o[:6u]\\
&+ x^{7u}M^o[:4u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u] + x^{7u}M^o[:5u]\\
&+ x^{9u}M^o[:3u] + x^{8u}M^o[:4u]\\
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{2u}W^o[:6u]\\
&+ x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{7u}W^o[:u] + x^{3u}W^o[:6u]\\
&+ x^{5u}W^o[:4u] + x^{6u}W^o[:3u] + x^{8u}W^o[:u] + x^{4u}W^o[:6u]\\
&+ x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{9u}W^o[:u] + x^{5u}W^o[:6u]\\
&+ x^{7u}W^o[:4u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u] + x^{7u}W^o[:5u]\\
&+ x^{9u}W^o[:3u] + x^{8u}W^o[:4u]\\
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{5u}T[:u] + x^{2u}T[:6u] + x^{4u}T[:4u]\\
&+ x^{5u}T[:3u] + x^{7u}T[:u] + x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u]\\
&+ x^{8u}T[:u] + x^{4u}T[:6u] + x^{6u}T[:4u] + x^{7u}T[:3u] + x^{9u}T[:u]\\
&+ x^{5u}T[:6u] + x^{7u}T[:4u] + x^{8u}T[:3u] + x^{10u}T[:u] + x^{7u}T[:5u]\\
&+ x^{9u}T[:3u] + x^{8u}T[:4u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
0 &= x^{5u}T[u:2u] + x^{3u}T[3u:4u] + x^{2u}T[4u:5u] + T[6u:7u],\\
0 &= x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u],
\end{aligned}
$$

$$\begin{aligned}
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u],\\
0 &= x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u] + x^{4u}T[5u:6u]\\
&\quad + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{5u}T[5u:6u] + T[10u:11u],\\
0 &= x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1408 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.19}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.20}$$
$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1000)),\end{aligned} \tag{2.21}$$
$$\begin{aligned}0 &= \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1001)),\end{aligned} \tag{2.22}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.23}$$
$$0 = \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.25}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\qquad + \tau(A(s,1000)), \end{aligned} \tag{2.27}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\qquad + \tau(A(s,1001)), \end{aligned} \tag{2.28}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.29}$$

$$0 = \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{50}), E_{50}) = (A(i, 0^{26}), E_{26})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 25$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^{6u}R^e, \\ W_{2k} &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{6u}R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{6u}R^o, \\ W_{2k+1} &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{6u}R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(W^e[:6v] + x^{2v}W^e[:4v] + x^{3v}W^e[:3v] + x^{5v}W^e[:v] + x^{6u}R^e\big) \times \big(\\ M^e[:6v] + x^{2v}M^e[:4v] + x^{3v}M^e[:3v] + x^{5v}M^e[:v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{6v}W^e[:6v]M^e[:6v] \\ + x^{10v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \end{aligned}$$

$$\begin{aligned} b_n &:= M^e[n\cdot v:(n+1)\cdot v]/X^n,\\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1,...,a_{12},b_1,...,b_{12},c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0\le j,k\le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z=1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b)=\frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y)=\mathrm{Res}_t\left(\phi(M^e,0,1)(x,t),y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x)=M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y)=q_4(x)y^4+q_3(x)y^3+q_2(x)y^2+q_1(x)y+q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24}+x^{21}+x^{20}+x^{19}+x^{16}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^6,\\ q_1(x) &= x^{21}+x^{20}+x^{19}+x^{18}+x^{16}+x^{15}+x^{13}+x^{10}+x^9+x^7+x^2+x,\\ q_2(x) &= x^{18}+x^{16}+x^{12}+x^8+x^6+x^4+x^2+1,\\ q_3(x) &= x^{21}+x^{20}+x^{19}+x^{18}+x^{16}+x^{15}+x^{13}+x^{10}+x^9+x^7+x^2+x,\\ q_4(x) &= x^{21}+x^{19}+x^{18}+x^{13}+x^{11}+x^{10}+x^9+x^8+x^2. \end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y):=P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{104} + x^{102} + x^{100} + x^{94} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{70} + x^{68} + x^{66} + x^{58} + x^{56} + x^{48} + x^{42} + x^{40} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{22} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{106} + x^{102} + x^{100} + x^{90} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{54} + x^{50} \\
&\quad + x^{48} + x^{42} + x^{40} + x^{38} + x^{34} + x^{26} + x^{24} + x^{12} + x^{8}, \\
p_6(x) &= x^{106} + x^{92} + x^{82} + x^{78} + x^{72} + x^{68} + x^{60} + x^{56} + x^{54} + x^{48} + x^{46} \\
&\quad + x^{44} + x^{30} + x^{28} + x^{18} + x^{14} + x^{8} + 1, \\
p_9(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{70} + x^{64} + x^{60} + x^{56} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{12} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} \\
&\quad + x^{80} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{110} \\
&\quad + x^{109} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{55} + x^{50} + x^{48} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{33} \\
&\quad + x^{31} + x^{30} + x^{26} + x^{24}, \\
p_3(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{111} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101} \\
&\quad + x^{98} + x^{97} + x^{96} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} \\
&\quad + x^{77} + x^{74} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{55} \\
&\quad + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{39} + x^{36} + x^{33} \\
&\quad + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{20} + x^{17} + x^{15} + x^{13} + x^{12} \\
&\quad + x^{10} + x^{9}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{112} + x^{107} + x^{104} + x^{101} + x^{98} + x^{97} + x^{94} \\
&\quad + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} \\
&\quad + x^{80} + x^{79} + x^{76} + x^{73} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57}
\end{aligned}$$

$$
\begin{aligned}
&+x^{56}+x^{54}+x^{53}+x^{51}+x^{50}+x^{48}+x^{47}+x^{45}+x^{44}+x^{42}+x^{41}\\
&+x^{39}+x^{38}+x^{37}+x^{36}+x^{34}+x^{31}+x^{29}+x^{28}+x^{26}+x^{25}+x^{23}\\
&+x^{22}+x^{20}+x^{15}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{6},\\
p_9(x)=&\,x^{114}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{103}+x^{101}\\
&+x^{99}+x^{98}+x^{94}+x^{92}+x^{87}+x^{83}+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}\\
&+x^{74}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{65}+x^{64}+x^{60}+x^{58}+x^{56}\\
&+x^{55}+x^{50}+x^{49}+x^{48}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}\\
&+x^{36}+x^{34}+x^{33}+x^{32}+x^{31}+x^{29}+x^{28}+x^{27}+x^{18}+x^{17}+x^{16}\\
&+x^{14}+x^{12}+x^{11}+x^{8}+x^{7}+x^{5}+x^{4}+x^{3},\\
p_{12}(x)=&\,x^{111}+x^{108}+x^{99}+x^{96}+x^{95}+x^{92}+x^{83}+x^{80}+x^{47}+x^{44}+x^{35}\\
&+x^{32}+x^{31}+x^{28}+x^{19}+x^{16}+x^{15}+x^{12}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{124}+x^{123}+x^{121}+x^{120}+x^{118}+x^{117}+x^{116}+x^{115}+x^{110}\\
&+x^{109}+x^{105}+x^{104}+x^{102}+x^{100}+x^{99}+x^{98}+x^{96}+x^{95}+x^{94}\\
&+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}+x^{85}+x^{83}+x^{82}+x^{79}+x^{78}+x^{77}\\
&+x^{75}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}+x^{62}+x^{60}+x^{59}\\
&+x^{58}+x^{57}+x^{55}+x^{50}+x^{48}+x^{45}+x^{44}+x^{41}+x^{40}+x^{39}+x^{33}\\
&+x^{31}+x^{30}+x^{26}+x^{24},\\
p_3(x)=&\,x^{120}+x^{119}+x^{118}+x^{116}+x^{111}+x^{108}+x^{105}+x^{104}+x^{102}+x^{101}\\
&+x^{98}+x^{97}+x^{96}+x^{94}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{82}\\
&+x^{77}+x^{74}+x^{72}+x^{70}+x^{69}+x^{66}+x^{65}+x^{63}+x^{60}+x^{57}+x^{55}\\
&+x^{53}+x^{52}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{44}+x^{39}+x^{36}+x^{33}\\
&+x^{32}+x^{30}+x^{29}+x^{26}+x^{25}+x^{23}+x^{20}+x^{17}+x^{15}+x^{13}+x^{12}\\
&+x^{10}+x^{9},\\
p_6(x)=&\,x^{117}+x^{115}+x^{114}+x^{112}+x^{107}+x^{104}+x^{101}+x^{98}+x^{97}+x^{94}\\
&+x^{93}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{82}\\
&+x^{80}+x^{79}+x^{76}+x^{73}+x^{70}+x^{65}+x^{63}+x^{62}+x^{60}+x^{59}+x^{57}\\
&+x^{56}+x^{54}+x^{53}+x^{51}+x^{50}+x^{48}+x^{47}+x^{45}+x^{44}+x^{42}+x^{41}\\
&+x^{39}+x^{38}+x^{37}+x^{36}+x^{34}+x^{31}+x^{29}+x^{28}+x^{26}+x^{25}+x^{23}\\
&+x^{22}+x^{20}+x^{15}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{6},\\
p_9(x)=&\,x^{114}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{103}+x^{101}\\
&+x^{99}+x^{98}+x^{94}+x^{92}+x^{87}+x^{83}+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}\\
&+x^{74}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{65}+x^{64}+x^{60}+x^{58}+x^{56}\\
&+x^{55}+x^{50}+x^{49}+x^{48}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{18} + x^{17} + x^{16} \\
&+ x^{14} + x^{12} + x^{11} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{47} + x^{44} + x^{35} \\
&+ x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{122} + x^{118} + x^{116} + x^{112} + x^{110} \\
&+ x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{88} + x^{86} + x^{84} + x^{82} + x^{72} + x^{70} \\
&+ x^{66} + x^{62} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{120} + x^{116} + x^{110} + x^{108} + x^{102} + x^{98} + x^{92} + x^{90} + x^{78} + x^{72} + x^{70} \\
&+ x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{36} + x^{34} + x^{28} \\
&+ x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{8} + x^{6}, \\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{92} + x^{90} + x^{86} + x^{82} + x^{80} + x^{72} \\
&+ x^{66} + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{32} + x^{26} + x^{20} + x^{18} \\
&+ x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + 1, \\
p_9(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} \\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{64} + x^{56} + x^{52} + x^{48} + x^{46} \\
&+ x^{36} + x^{30} + x^{24} + x^{22} + x^{20} + x^{16} + x^{12} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{50} + x^{48} + x^{42} \\
&+ x^{40} + x^{26} + x^{24} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{120} + x^{114} + x^{98} \\
&+ x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{66} + x^{62} + x^{60} \\
&+ x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} + x^{26} + x^{18} \\
&+ x^{16} + x^{14} + x^{12}, \\
p_3(x) &= x^{120} + x^{116} + x^{110} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{82} + x^{78} + x^{70} \\
&+ x^{68} + x^{60} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} + x^{34} \\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} + x^{10} + x^{8}, \\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{102} + x^{94} + x^{92} + x^{86} \\
&+ x^{82} + x^{68} + x^{62} + x^{60} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} \\
&+ x^{40} + x^{38} + x^{28} + x^{24} + x^{20} + x^{18} + x^{16} + x^{10} + x^{6} + x^{4} + 1, \\
p_9(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} \\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{64} + x^{56} + x^{52} + x^{48} + x^{46} \\
&+ x^{36} + x^{30} + x^{24} + x^{22} + x^{20} + x^{16} + x^{12} + x^{4} + x^{2},
\end{aligned}
$$

$$
\begin{aligned}
p_{12}(x) &= x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{50} + x^{48} + x^{42} \\
&\quad + x^{40} + x^{26} + x^{24} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} \\
&\quad + x^{94} + x^{93} + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{70} + x^{67} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{31} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_3(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{111} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101} \\
&\quad + x^{98} + x^{97} + x^{96} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} \\
&\quad + x^{77} + x^{74} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{55} \\
&\quad + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{39} + x^{36} + x^{33} \\
&\quad + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{20} + x^{17} + x^{15} + x^{13} + x^{12} \\
&\quad + x^{10} + x^{9}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{112} + x^{107} + x^{104} + x^{101} + x^{98} + x^{97} + x^{94} \\
&\quad + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} \\
&\quad + x^{80} + x^{79} + x^{76} + x^{73} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} \\
&\quad + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} \\
&\quad + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} \\
&\quad + x^{22} + x^{20} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) &= x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} \\
&\quad + x^{99} + x^{98} + x^{94} + x^{92} + x^{87} + x^{83} + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} \\
&\quad + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{55} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{18} + x^{17} + x^{16} \\
&\quad + x^{14} + x^{12} + x^{11} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{47} + x^{44} + x^{35} \\
&\quad + x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} \\
&\quad + x^{94} + x^{93} + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{81} + x^{77} + x^{76} + x^{75}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{70} + x^{67} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} \\
&+ x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} \\
&+ x^{36} + x^{31} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_3(x) = {}& x^{120} + x^{119} + x^{118} + x^{116} + x^{111} + x^{108} + x^{105} + x^{104} + x^{102} + x^{101} \\
&+ x^{98} + x^{97} + x^{96} + x^{94} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} \\
&+ x^{77} + x^{74} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{55} \\
&+ x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{39} + x^{36} + x^{33} \\
&+ x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{20} + x^{17} + x^{15} + x^{13} + x^{12} \\
&+ x^{10} + x^{9}, \\
p_6(x) = {}& x^{117} + x^{115} + x^{114} + x^{112} + x^{107} + x^{104} + x^{101} + x^{98} + x^{97} + x^{94} \\
&+ x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} \\
&+ x^{80} + x^{79} + x^{76} + x^{73} + x^{70} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} \\
&+ x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} \\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} \\
&+ x^{22} + x^{20} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) = {}& x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} \\
&+ x^{99} + x^{98} + x^{94} + x^{92} + x^{87} + x^{83} + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} \\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{60} + x^{58} + x^{56} \\
&+ x^{55} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} \\
&+ x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{18} + x^{17} + x^{16} \\
&+ x^{14} + x^{12} + x^{11} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) = {}& x^{111} + x^{108} + x^{99} + x^{96} + x^{95} + x^{92} + x^{83} + x^{80} + x^{47} + x^{44} + x^{35} \\
&+ x^{32} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{94} + x^{90} + x^{88} \\
&+ x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{62} + x^{54} + x^{50} + x^{46} + x^{42} + x^{40} \\
&+ x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24}, \\
p_3(x) = {}& x^{106} + x^{100} + x^{98} + x^{96} + x^{86} + x^{80} + x^{72} + x^{68} + x^{66} + x^{64} + x^{60} \\
&+ x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{30} + x^{24} + x^{22} \\
&+ x^{20} + x^{18} + x^{8} + x^{6}, \\
p_6(x) = {}& x^{106} + x^{102} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{78} + x^{66} + x^{52} \\
&+ x^{48} + x^{46} + x^{42} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{22} + x^{18} + x^{12} \\
&+ x^{10} + x^{8} + x^{4} + 1, \\
p_9(x) = {}& x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{80}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{78}+x^{76}+x^{70}+x^{64}+x^{60}+x^{56}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}\\
&+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{12}+x^{6}+x^{4}+x^{2},\\
p_{12}(x)=&\,x^{108}+x^{106}+x^{104}+x^{100}+x^{98}+x^{96}+x^{92}+x^{90}+x^{88}+x^{84}+x^{82}\\
&+x^{80}+x^{44}+x^{42}+x^{40}+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}+x^{24}+x^{20}\\
&+x^{18}+x^{16}+x^{12}+x^{10}+x^{8}+x^{4}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].
For $\phi(M^o,0,0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{124}+x^{123}+x^{121}+x^{118}+x^{117}+x^{116}+x^{113}+x^{112}+x^{109}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}\\
&+x^{86}+x^{81}+x^{79}+x^{77}+x^{76}+x^{73}+x^{72}+x^{70}+x^{68}+x^{67}+x^{65}\\
&+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{53}+x^{52}+x^{49}+x^{43}\\
&+x^{40}+x^{39}+x^{38}+x^{36}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{28}+x^{27}\\
&+x^{26}+x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12},\\
p_3(x)=&\,x^{123}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{112}+x^{110}+x^{108}+x^{107}\\
&+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{97}+x^{96}+x^{94}+x^{92}+x^{91}\\
&+x^{89}+x^{88}+x^{85}+x^{84}+x^{82}+x^{80}+x^{79}+x^{77}+x^{75}+x^{72}+x^{71}\\
&+x^{68}+x^{67}+x^{66}+x^{64}+x^{62}+x^{61}+x^{60}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{51}+x^{50}+x^{49}+x^{47}+x^{45}+x^{44}+x^{38}+x^{36}+x^{35}+x^{34}\\
&+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{25}+x^{23}+x^{19}+x^{18}+x^{17}\\
&+x^{13}+x^{11}+x^{10}+x^{8},\\
p_6(x)=&\,x^{123}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{112}+x^{110}+x^{107}+x^{105}\\
&+x^{102}+x^{100}+x^{96}+x^{95}+x^{93}+x^{91}+x^{90}+x^{89}+x^{85}+x^{82}+x^{81}\\
&+x^{79}+x^{75}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{61}+x^{60}\\
&+x^{58}+x^{56}+x^{54}+x^{50}+x^{49}+x^{48}+x^{46}+x^{44}+x^{43}+x^{42}+x^{41}\\
&+x^{38}+x^{35}+x^{30}+x^{29}+x^{28}+x^{23}+x^{22}+x^{21}+x^{20}+x^{16}+x^{14}\\
&+x^{13}+x^{12}+x^{9}+x^{7}+x^{6}+x^{4}+x^{3}+1,\\
p_9(x)=&\,x^{123}+x^{120}+x^{119}+x^{116}+x^{111}+x^{108}+x^{95}+x^{92}+x^{91}+x^{88}\\
&+x^{87}+x^{84}+x^{59}+x^{56}+x^{55}+x^{52}+x^{47}+x^{44}+x^{31}+x^{28}+x^{15}\\
&+x^{12}+x^{11}+x^{8}+x^{7}+x^{4},\\
p_{12}(x)=&\,x^{123}+x^{120}+x^{119}+x^{116}+x^{99}+x^{96}+x^{91}+x^{88}+x^{87}+x^{84}+x^{83}\\
&+x^{80}+x^{59}+x^{56}+x^{55}+x^{52}+x^{35}+x^{32}+x^{19}+x^{16}+x^{11}+x^{8}\\
&+x^{7}+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0].
For $\phi(M^o,0,1)$:

$$
p_0(x)=x^{140}+x^{136}+x^{132}+x^{131}+x^{126}+x^{124}+x^{123}+x^{121}+x^{120}+x^{119}
$$

$$
\begin{aligned}
&+ x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
&+ x^{107} + x^{106} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} + x^{92} \\
&+ x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{73} + x^{70} + x^{69} + x^{67} \\
&+ x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{55} + x^{54} + x^{52} + x^{49} + x^{44} + x^{43} \\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27} + x^{24}, \\
p_3(x) &= x^{119} + x^{118} + x^{115} + x^{108} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} \\
&+ x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} + x^{80} \\
&+ x^{78} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65} + x^{60} \\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{43} \\
&+ x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{20} + x^{17} + x^{16} \\
&+ x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96} \\
&+ x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{42} \\
&+ x^{40} + x^{32} + x^{24} + x^{22} + x^{18} + x^{8} + x^{6}, \\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{112} + x^{110} + x^{109} \\
&+ x^{106} + x^{104} + x^{100} + x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{84} \\
&+ x^{83} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{51} + x^{48} + x^{46} + x^{45} \\
&+ x^{42} + x^{40} + x^{36} + x^{32} + x^{30} + x^{29} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} \\
&+ x^{12} + x^{11} + x^{9} + x^{8} + x^{7} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{128} + x^{116} + x^{104} + x^{100} + x^{88} + x^{80} + x^{64} + x^{52} + x^{40} + x^{36} + x^{32} \\
&+ x^{24} + x^{20} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{118} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} + x^{105} + x^{102} + x^{101} + x^{100} \\
&+ x^{98} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} \\
&+ x^{79} + x^{77} + x^{76} + x^{73} + x^{71} + x^{69} + x^{67} + x^{64} + x^{61} + x^{59} + x^{58} \\
&+ x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{38} \\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{20} \\
&+ x^{18}, \\
p_3(x) &= x^{115} + x^{114} + x^{111} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{97} \\
&+ x^{95} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} \\
&+ x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{63} + x^{59} \\
&+ x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{43} + x^{42} + x^{39} \\
&+ x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{25} + x^{23} + x^{19} + x^{18} + x^{16} \\
&+ x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) = {} & x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{92} + x^{82} + x^{76} \\
& + x^{74} + x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{46} + x^{44} \\
& + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} \\
& + x^{8} + x^{6}, \\
p_9(x) = {} & x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} + x^{102} \\
& + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{57} \\
& + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{46} + x^{38} + x^{37} + x^{36} \\
& + x^{33} + x^{32} + x^{30} + x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{14} + x^{9} + x^{8} + x^{7} \\
& + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{124} + x^{116} + x^{100} + x^{96} + x^{92} + x^{80} + x^{60} + x^{52} + x^{36} + x^{32} + x^{20} \\
& + x^{16} + x^{12} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{118} + x^{116} + x^{112} + x^{111} + x^{108} + x^{107} + x^{104} + x^{103} \\
& + x^{98} + x^{92} + x^{90} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} \\
& + x^{69} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} \\
& + x^{46} + x^{45} + x^{42} + x^{40} + x^{36} + x^{35} + x^{30}, \\
p_3(x) = {} & x^{123} + x^{119} + x^{118} + x^{116} + x^{114} + x^{111} + x^{110} + x^{106} + x^{105} + x^{102} \\
& + x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} \\
& + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} \\
& + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{49} + x^{48} + x^{47} \\
& + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{34} + x^{29} + x^{25} + x^{20} + x^{18} \\
& + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_6(x) = {} & x^{123} + x^{119} + x^{118} + x^{116} + x^{114} + x^{111} + x^{110} + x^{108} + x^{105} + x^{104} \\
& + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{86} + x^{83} \\
& + x^{81} + x^{80} + x^{79} + x^{73} + x^{69} + x^{68} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} \\
& + x^{57} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{39} + x^{37} + x^{36} \\
& + x^{35} + x^{34} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{21} + x^{19} + x^{15} \\
& + x^{14} + x^{11} + x^{10} + x^{8} + x^{3} + 1, \\
p_9(x) = {} & x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{95} + x^{92} + x^{91} + x^{88} \\
& + x^{87} + x^{84} + x^{59} + x^{56} + x^{55} + x^{52} + x^{47} + x^{44} + x^{31} + x^{28} + x^{15} \\
& + x^{12} + x^{11} + x^{8} + x^{7} + x^{4}, \\
p_{12}(x) = {} & x^{123} + x^{120} + x^{119} + x^{116} + x^{99} + x^{96} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} \\
& + x^{80} + x^{59} + x^{56} + x^{55} + x^{52} + x^{35} + x^{32} + x^{19} + x^{16} + x^{11} + x^{8} \\
& + x^{7} + x^{4} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{117} + x^{116} + x^{113} + x^{112} + x^{109} \\
&\quad + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{86} + x^{81} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{65} \\
&\quad + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{49} + x^{43} \\
&\quad + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_3(x) &= x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{112} + x^{110} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} \\
&\quad + x^{89} + x^{88} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{75} + x^{72} + x^{71} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{56} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{38} + x^{36} + x^{35} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} + x^{23} + x^{19} + x^{18} + x^{17} \\
&\quad + x^{13} + x^{11} + x^{10} + x^{8}, \\
p_6(x) &= x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{112} + x^{110} + x^{107} + x^{105} \\
&\quad + x^{102} + x^{100} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{89} + x^{85} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{75} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{38} + x^{35} + x^{30} + x^{29} + x^{28} + x^{23} + x^{22} + x^{21} + x^{20} + x^{16} + x^{14} \\
&\quad + x^{13} + x^{12} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} + 1, \\
p_9(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{111} + x^{108} + x^{95} + x^{92} + x^{91} + x^{88} \\
&\quad + x^{87} + x^{84} + x^{59} + x^{56} + x^{55} + x^{52} + x^{47} + x^{44} + x^{31} + x^{28} + x^{15} \\
&\quad + x^{12} + x^{11} + x^{8} + x^{7} + x^{4}, \\
p_{12}(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{99} + x^{96} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} \\
&\quad + x^{80} + x^{59} + x^{56} + x^{55} + x^{52} + x^{35} + x^{32} + x^{19} + x^{16} + x^{11} + x^{8} \\
&\quad + x^{7} + x^{4} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{118} + x^{115} + x^{112} + x^{110} + x^{109} + x^{107} + x^{105} + x^{102} + x^{101} + x^{100} \\
&\quad + x^{98} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{77} + x^{76} + x^{73} + x^{71} + x^{69} + x^{67} + x^{64} + x^{61} + x^{59} + x^{58} \\
&\quad + x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{20} \\
&\quad + x^{18},
\end{aligned}$$

$$
\begin{aligned}
p_3(x) = {} & x^{115} + x^{114} + x^{111} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{97} \\
& + x^{95} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{63} + x^{59} \\
& + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{43} + x^{42} + x^{39} \\
& + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{25} + x^{23} + x^{19} + x^{18} + x^{16} \\
& + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{92} + x^{82} + x^{76} \\
& + x^{74} + x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{46} + x^{44} \\
& + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} \\
& + x^{8} + x^{6}, \\
p_9(x) = {} & x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{110} + x^{102} \\
& + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{57} \\
& + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{46} + x^{38} + x^{37} + x^{36} \\
& + x^{33} + x^{32} + x^{30} + x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{14} + x^{9} + x^{8} + x^{7} \\
& + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{124} + x^{116} + x^{100} + x^{96} + x^{92} + x^{80} + x^{60} + x^{52} + x^{36} + x^{32} + x^{20} \\
& + x^{16} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{140} + x^{136} + x^{132} + x^{131} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} \\
& + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
& + x^{107} + x^{106} + x^{105} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} + x^{92} \\
& + x^{91} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{73} + x^{70} + x^{69} + x^{67} \\
& + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{55} + x^{54} + x^{52} + x^{49} + x^{44} + x^{43} \\
& + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27} + x^{24}, \\
p_3(x) = {} & x^{119} + x^{118} + x^{115} + x^{108} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} + x^{80} \\
& + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65} + x^{60} \\
& + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{43} \\
& + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{20} + x^{17} + x^{16} \\
& + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96} \\
& + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{42} \\
& + x^{40} + x^{32} + x^{24} + x^{22} + x^{18} + x^{8} + x^{6}, \\
p_9(x) = {} & x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{112} + x^{110} + x^{109}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{106}+x^{104}+x^{100}+x^{96}+x^{94}+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}+x^{84}\\
&+x^{83}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{55}+x^{51}+x^{48}+x^{46}+x^{45}\\
&+x^{42}+x^{40}+x^{36}+x^{32}+x^{30}+x^{29}+x^{26}+x^{24}+x^{20}+x^{16}+x^{14}\\
&+x^{12}+x^{11}+x^{9}+x^{8}+x^{7}+x^{4}+x^{3},\\
p_{12}(x) ={}& x^{128}+x^{116}+x^{104}+x^{100}+x^{88}+x^{80}+x^{64}+x^{52}+x^{40}+x^{36}+x^{32}\\
&+x^{24}+x^{20}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) ={}& x^{126}+x^{124}+x^{118}+x^{116}+x^{112}+x^{111}+x^{108}+x^{107}+x^{104}+x^{103}\\
&+x^{98}+x^{92}+x^{90}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{76}+x^{75}\\
&+x^{69}+x^{64}+x^{62}+x^{59}+x^{58}+x^{57}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}\\
&+x^{46}+x^{45}+x^{42}+x^{40}+x^{36}+x^{35}+x^{30},\\
p_3(x) ={}& x^{123}+x^{119}+x^{118}+x^{116}+x^{114}+x^{111}+x^{110}+x^{106}+x^{105}+x^{102}\\
&+x^{101}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}+x^{88}+x^{86}+x^{85}\\
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{79}+x^{78}+x^{75}+x^{74}+x^{73}+x^{72}+x^{70}\\
&+x^{67}+x^{66}+x^{64}+x^{63}+x^{62}+x^{61}+x^{59}+x^{57}+x^{49}+x^{48}+x^{47}\\
&+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{38}+x^{34}+x^{29}+x^{25}+x^{20}+x^{18}\\
&+x^{16}+x^{15}+x^{14}+x^{12},\\
p_6(x) ={}& x^{123}+x^{119}+x^{118}+x^{116}+x^{114}+x^{111}+x^{110}+x^{108}+x^{105}+x^{104}\\
&+x^{102}+x^{101}+x^{100}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{86}+x^{83}\\
&+x^{81}+x^{80}+x^{79}+x^{73}+x^{69}+x^{68}+x^{63}+x^{62}+x^{61}+x^{59}+x^{58}\\
&+x^{57}+x^{53}+x^{51}+x^{49}+x^{47}+x^{46}+x^{45}+x^{43}+x^{39}+x^{37}+x^{36}\\
&+x^{35}+x^{34}+x^{30}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{21}+x^{19}+x^{15}\\
&+x^{14}+x^{11}+x^{10}+x^{8}+x^{3}+1,\\
p_9(x) ={}& x^{123}+x^{120}+x^{119}+x^{116}+x^{111}+x^{108}+x^{95}+x^{92}+x^{91}+x^{88}\\
&+x^{87}+x^{84}+x^{59}+x^{56}+x^{55}+x^{52}+x^{47}+x^{44}+x^{31}+x^{28}+x^{15}\\
&+x^{12}+x^{11}+x^{8}+x^{7}+x^{4},\\
p_{12}(x) ={}& x^{123}+x^{120}+x^{119}+x^{116}+x^{99}+x^{96}+x^{91}+x^{88}+x^{87}+x^{84}+x^{83}\\
&+x^{80}+x^{59}+x^{56}+x^{55}+x^{52}+x^{35}+x^{32}+x^{19}+x^{16}+x^{11}+x^{8}\\
&+x^{7}+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$

0	[1, 2]	353	[447, 527]	706	[895, 46]	1059	[1182, 488]
1	[3, 4]	354	[528, 529]	707	[205, 450]	1060	[1183, 215]
2	[5, 6]	355	[530, 493]	708	[896, 442]	1061	[1184, 1140]
3	[7, 8]	356	[220, 515]	709	[897, 159]	1062	[1185, 855]
4	[9, 10]	357	[440, 518]	710	[514, 495]	1063	[1186, 829]
5	[11, 12]	358	[531, 8]	711	[787, 493]	1064	[1187, 429]
6	[13, 14]	359	[183, 495]	712	[898, 850]	1065	[1188, 454]
7	[15, 16]	360	[532, 188]	713	[899, 478]	1066	[1189, 412]
8	[17, 18]	361	[505, 515]	714	[890, 487]	1067	[1158, 807]
9	[19, 20]	362	[482, 410]	715	[900, 207]	1068	[393, 397]
10	[21, 22]	363	[533, 155]	716	[414, 838]	1069	[1190, 459]
11	[23, 24]	364	[427, 50]	717	[901, 829]	1070	[496, 124]
12	[25, 26]	365	[462, 526]	718	[13, 471]	1071	[797, 482]
13	[27, 28]	366	[58, 534]	719	[902, 408]	1072	[1124, 842]
14	[29, 30]	367	[467, 535]	720	[903, 424]	1073	[136, 56]
15	[31, 32]	368	[409, 536]	721	[904, 821]	1074	[1191, 821]
16	[33, 34]	369	[537, 46]	722	[905, 168]	1075	[1171, 53]
17	[35, 36]	370	[488, 190]	723	[502, 422]	1076	[200, 539]
18	[37, 38]	371	[538, 397]	724	[906, 482]	1077	[494, 184]
19	[39, 40]	372	[210, 539]	725	[907, 415]	1078	[1192, 812]
20	[41, 42]	373	[540, 415]	726	[811, 447]	1079	[1193, 909]
21	[43, 44]	374	[541, 542]	727	[839, 486]	1080	[1194, 454]
22	[45, 46]	375	[411, 543]	728	[451, 515]	1081	[1195, 388]
23	[47, 48]	376	[544, 545]	729	[908, 909]	1082	[828, 868]
24	[49, 50]	377	[546, 469]	730	[144, 168]	1083	[1196, 221]
25	[51, 52]	378	[138, 547]	731	[910, 166]	1084	[881, 1197]
26	[53, 54]	379	[548, 549]	732	[161, 455]	1085	[1196, 515]
27	[55, 56]	380	[457, 397]	733	[911, 415]	1086	[1198, 440]
28	[57, 58]	381	[550, 493]	734	[449, 146]	1087	[831, 842]
29	[31, 59]	382	[551, 552]	735	[912, 422]	1088	[1199, 126]
30	[60, 61]	383	[182, 442]	736	[913, 523]	1089	[1200, 469]
31	[62, 63]	384	[553, 447]	737	[405, 914]	1090	[842, 389]
32	[64, 65]	385	[554, 555]	738	[915, 204]	1091	[1201, 1132]
33	[66, 67]	386	[556, 557]	739	[916, 166]	1092	[1202, 422]
34	[68, 69]	387	[558, 559]	740	[917, 58]	1093	[43, 386]
35	[70, 71]	388	[560, 561]	741	[918, 189]	1094	[1203, 399]
36	[72, 73]	389	[75, 562]	742	[532, 835]	1095	[1167, 431]
37	[74, 75]	390	[563, 564]	743	[919, 147]	1096	[837, 36]
38	[76, 77]	391	[565, 566]	744	[130, 879]	1097	[401, 826]
39	[78, 79]	392	[362, 567]	745	[920, 921]	1098	[1204, 219]
40	[80, 81]	393	[24, 568]	746	[51, 523]	1099	[1165, 162]
41	[82, 83]	394	[569, 570]	747	[922, 455]	1100	[1205, 844]
42	[84, 85]	395	[571, 572]	748	[923, 50]	1101	[472, 448]

43	[86, 87]	396	[573, 574]	749	[130, 924]	1102	[1206, 41]
44	[88, 89]	397	[575, 576]	750	[925, 549]	1103	[1207, 535]
45	[90, 91]	398	[577, 16]	751	[926, 845]	1104	[1208, 1147]
46	[92, 93]	399	[578, 579]	752	[927, 476]	1105	[1200, 468]
47	[94, 85]	400	[580, 581]	753	[928, 536]	1106	[215, 1209]
48	[95, 96]	401	[582, 583]	754	[419, 929]	1107	[935, 164]
49	[97, 98]	402	[584, 585]	755	[839, 799]	1108	[1210, 789]
50	[99, 100]	403	[586, 249]	756	[930, 542]	1109	[1211, 830]
51	[101, 102]	404	[587, 588]	757	[931, 406]	1110	[1212, 217]
52	[103, 104]	405	[589, 590]	758	[932, 787]	1111	[876, 1213]
53	[105, 106]	406	[591, 592]	759	[880, 188]	1112	[899, 155]
54	[107, 108]	407	[593, 594]	760	[151, 418]	1113	[847, 543]
55	[109, 110]	408	[243, 569]	761	[143, 192]	1114	[1177, 1208]
56	[111, 112]	409	[595, 370]	762	[826, 131]	1115	[1214, 529]
57	[113, 114]	410	[596, 276]	763	[1, 420]	1116	[1215, 431]
58	[115, 116]	411	[597, 598]	764	[920, 933]	1117	[1216, 561]
59	[117, 118]	412	[598, 231]	765	[58, 934]	1118	[1217, 97]
60	[119, 120]	413	[599, 600]	766	[11, 441]	1119	[1218, 1219]
61	[121, 122]	414	[601, 602]	767	[935, 41]	1120	[1220, 242]
62	[123, 124]	415	[603, 604]	768	[927, 2]	1121	[1221, 980]
63	[125, 126]	416	[605, 360]	769	[936, 807]	1122	[1222, 352]
64	[127, 128]	417	[606, 607]	770	[937, 529]	1123	[1223, 222]
65	[129, 130]	418	[608, 609]	771	[932, 160]	1124	[1224, 274]
66	[131, 132]	419	[610, 611]	772	[938, 512]	1125	[1225, 224]
67	[133, 134]	420	[612, 613]	773	[939, 872]	1126	[1226, 666]
68	[135, 136]	421	[4, 334]	774	[940, 525]	1127	[1227, 1228]
69	[128, 137]	422	[614, 363]	775	[941, 942]	1128	[1229, 284]
70	[138, 139]	423	[366, 615]	776	[546, 468]	1129	[1230, 329]
71	[140, 41]	424	[616, 617]	777	[943, 185]	1130	[1231, 1038]
72	[141, 44]	425	[618, 619]	778	[843, 944]	1131	[1232, 68]
73	[142, 143]	426	[620, 621]	779	[945, 399]	1132	[1233, 1234]
74	[144, 145]	427	[622, 623]	780	[434, 59]	1133	[1235, 102]
75	[146, 147]	428	[624, 596]	781	[803, 946]	1134	[1236, 633]
76	[148, 149]	429	[625, 626]	782	[947, 42]	1135	[1237, 702]
77	[150, 151]	430	[627, 359]	783	[948, 909]	1136	[104, 965]
78	[152, 153]	431	[431, 431]	784	[825, 823]	1137	[1238, 963]
79	[154, 155]	432	[628, 629]	785	[949, 839]	1138	[1239, 646]
80	[156, 157]	433	[630, 335]	786	[950, 74]	1139	[1240, 332]
81	[158, 159]	434	[631, 24]	787	[602, 256]	1140	[1241, 1242]
82	[160, 161]	435	[632, 25]	788	[951, 659]	1141	[1243, 612]
83	[42, 162]	436	[93, 633]	789	[952, 953]	1142	[744, 118]
84	[161, 163]	437	[634, 635]	790	[954, 955]	1143	[1244, 1245]
85	[162, 164]	438	[636, 340]	791	[956, 269]	1144	[1246, 1247]
86	[165, 166]	439	[637, 638]	792	[957, 958]	1145	[1248, 1249]

87	[164, 42]	440	[227, 639]	793	[723, 249]	1146	[1250, 668]
88	[167, 168]	441	[225, 640]	794	[959, 960]	1147	[1251, 745]
89	[169, 170]	442	[641, 284]	795	[961, 223]	1148	[345, 74]
90	[171, 172]	443	[642, 643]	796	[962, 963]	1149	[1252, 697]
91	[173, 174]	444	[102, 597]	797	[964, 965]	1150	[609, 312]
92	[175, 146]	445	[644, 568]	798	[966, 967]	1151	[1253, 363]
93	[176, 177]	446	[645, 646]	799	[968, 76]	1152	[1254, 84]
94	[178, 40]	447	[647, 648]	800	[969, 666]	1153	[1255, 332]
95	[179, 180]	448	[16, 649]	801	[970, 971]	1154	[1256, 109]
96	[181, 182]	449	[650, 651]	802	[707, 727]	1155	[231, 1011]
97	[183, 184]	450	[304, 595]	803	[972, 242]	1156	[1257, 639]
98	[125, 185]	451	[652, 18]	804	[973, 619]	1157	[613, 654]
99	[186, 187]	452	[653, 654]	805	[974, 975]	1158	[1258, 239]
100	[188, 189]	453	[655, 656]	806	[976, 365]	1159	[1259, 1260]
101	[190, 191]	454	[657, 658]	807	[977, 270]	1160	[1261, 996]
102	[192, 193]	455	[555, 659]	808	[978, 368]	1161	[67, 238]
103	[194, 195]	456	[660, 661]	809	[348, 967]	1162	[1262, 976]
104	[128, 196]	457	[63, 662]	810	[979, 101]	1163	[1263, 268]
105	[197, 198]	458	[118, 663]	811	[22, 980]	1164	[1264, 1265]
106	[194, 128]	459	[664, 665]	812	[646, 309]	1165	[1266, 82]
107	[37, 50]	460	[590, 120]	813	[981, 982]	1166	[312, 575]
108	[133, 131]	461	[666, 558]	814	[983, 71]	1167	[1267, 628]
109	[186, 199]	462	[667, 668]	815	[984, 985]	1168	[1268, 1269]
110	[200, 201]	463	[669, 670]	816	[986, 718]	1169	[1270, 266]
111	[202, 203]	464	[671, 245]	817	[987, 988]	1170	[1015, 1271]
112	[204, 205]	465	[672, 305]	818	[989, 762]	1171	[1272, 955]
113	[169, 204]	466	[673, 674]	819	[275, 990]	1172	[1273, 587]
114	[206, 207]	467	[675, 676]	820	[991, 238]	1173	[1274, 641]
115	[208, 209]	468	[20, 661]	821	[992, 993]	1174	[106, 28]
116	[210, 201]	469	[661, 321]	822	[994, 995]	1175	[1275, 1276]
117	[211, 212]	470	[677, 563]	823	[996, 997]	1176	[1277, 122]
118	[213, 136]	471	[678, 679]	824	[696, 68]	1177	[1278, 1251]
119	[214, 215]	472	[680, 307]	825	[998, 999]	1178	[1279, 76]
120	[216, 217]	473	[681, 682]	826	[581, 1000]	1179	[1260, 91]
121	[218, 219]	474	[683, 684]	827	[1001, 1002]	1180	[568, 1026]
122	[220, 221]	475	[685, 686]	828	[1003, 1004]	1181	[1280, 648]
123	[222, 223]	476	[687, 688]	829	[639, 1005]	1182	[697, 340]
124	[224, 225]	477	[689, 629]	830	[971, 1006]	1183	[1281, 988]
125	[226, 227]	478	[690, 691]	831	[960, 623]	1184	[1282, 1219]
126	[228, 229]	479	[692, 693]	832	[1007, 99]	1185	[1283, 243]
127	[230, 231]	480	[694, 695]	833	[714, 607]	1186	[1068, 1260]
128	[232, 233]	481	[615, 696]	834	[1008, 338]	1187	[1284, 696]
129	[234, 235]	482	[559, 697]	835	[1009, 976]	1188	[1285, 716]
130	[236, 237]	483	[698, 699]	836	[1010, 1011]	1189	[28, 1019]

131	[238, 239]	484	[700, 240]	837	[1012, 290]	1190	[1286, 343]
132	[240, 241]	485	[701, 82]	838	[1013, 113]	1191	[688, 244]
133	[242, 243]	486	[693, 702]	839	[1014, 97]	1192	[651, 1287]
134	[244, 245]	487	[703, 704]	840	[746, 1015]	1193	[1288, 630]
135	[246, 111]	488	[705, 380]	841	[1016, 98]	1194	[254, 93]
136	[247, 248]	489	[706, 67]	842	[1017, 608]	1195	[1289, 265]
137	[249, 250]	490	[621, 707]	843	[1018, 1019]	1196	[1290, 334]
138	[251, 252]	491	[708, 75]	844	[1020, 1021]	1197	[1291, 1292]
139	[253, 254]	492	[709, 710]	845	[566, 732]	1198	[965, 1293]
140	[83, 84]	493	[659, 711]	846	[1022, 603]	1199	[1294, 310]
141	[255, 256]	494	[712, 293]	847	[1023, 107]	1200	[1295, 20]
142	[257, 258]	495	[713, 714]	848	[314, 1024]	1201	[1296, 665]
143	[259, 260]	496	[715, 716]	849	[1025, 1026]	1202	[1024, 1297]
144	[261, 262]	497	[717, 575]	850	[1027, 1028]	1203	[1298, 613]
145	[263, 264]	498	[718, 695]	851	[1029, 608]	1204	[1299, 637]
146	[79, 265]	499	[719, 720]	852	[1030, 66]	1205	[663, 599]
147	[266, 267]	500	[721, 292]	853	[1031, 1002]	1206	[1300, 566]
148	[268, 269]	501	[722, 723]	854	[1032, 240]	1207	[110, 581]
149	[270, 271]	502	[724, 361]	855	[1033, 70]	1208	[1301, 356]
150	[272, 273]	503	[725, 686]	856	[567, 375]	1209	[988, 354]
151	[274, 275]	504	[726, 630]	857	[1034, 705]	1210	[640, 584]
152	[276, 103]	505	[727, 307]	858	[1035, 762]	1211	[233, 693]
153	[277, 278]	506	[728, 109]	859	[1036, 593]	1212	[1302, 651]
154	[279, 280]	507	[729, 730]	860	[1037, 1038]	1213	[1058, 1303]
155	[281, 234]	508	[731, 732]	861	[339, 730]	1214	[1304, 1297]
156	[282, 283]	509	[733, 244]	862	[1039, 958]	1215	[1305, 280]
157	[284, 285]	510	[245, 734]	863	[1040, 584]	1216	[817, 534]
158	[286, 287]	511	[735, 574]	864	[1041, 341]	1217	[1306, 424]
159	[288, 289]	512	[736, 737]	865	[336, 273]	1218	[841, 150]
160	[290, 87]	513	[738, 108]	866	[1042, 1039]	1219	[54, 404]
161	[291, 290]	514	[739, 291]	867	[1043, 4]	1220	[543, 412]
162	[262, 292]	515	[740, 615]	868	[370, 1044]	1221	[902, 132]
163	[87, 293]	516	[741, 742]	869	[1045, 662]	1222	[195, 196]
164	[85, 294]	517	[743, 707]	870	[1046, 1047]	1223	[1307, 789]
165	[295, 260]	518	[665, 744]	871	[1048, 345]	1224	[170, 449]
166	[296, 297]	519	[745, 746]	872	[1049, 1050]	1225	[1308, 389]
167	[298, 294]	520	[747, 554]	873	[373, 79]	1226	[1163, 132]
168	[299, 300]	521	[649, 676]	874	[1051, 562]	1227	[1215, 432]
169	[301, 302]	522	[748, 111]	875	[1052, 971]	1228	[786, 1309]
170	[303, 304]	523	[682, 749]	876	[1053, 968]	1229	[1310, 921]
171	[305, 306]	524	[750, 25]	877	[1054, 1000]	1230	[814, 163]
172	[307, 308]	525	[751, 752]	878	[380, 674]	1231	[477, 155]
173	[309, 310]	526	[753, 754]	879	[662, 1055]	1232	[1311, 56]
174	[311, 312]	527	[308, 755]	880	[1056, 604]	1233	[491, 842]

175	[313, 314]	528	[756, 571]	881	[1057, 1015]	1234	[1159, 925]
176	[315, 316]	529	[757, 577]	882	[572, 1058]	1235	[1312, 422]
177	[317, 318]	530	[758, 555]	883	[557, 959]	1236	[1313, 395]
178	[319, 304]	531	[759, 760]	884	[1059, 595]	1237	[1314, 921]
179	[320, 321]	532	[761, 101]	885	[1060, 1061]	1238	[500, 177]
180	[322, 323]	533	[752, 628]	886	[1062, 641]	1239	[1315, 392]
181	[324, 325]	534	[762, 763]	887	[1063, 726]	1240	[886, 921]
182	[326, 327]	535	[237, 764]	888	[1064, 704]	1241	[947, 845]
183	[328, 329]	536	[561, 765]	889	[1065, 711]	1242	[1144, 1316]
184	[330, 331]	537	[333, 356]	890	[1066, 780]	1243	[1317, 189]
185	[332, 333]	538	[766, 235]	891	[273, 587]	1244	[1134, 59]
186	[334, 222]	539	[767, 768]	892	[1067, 1068]	1245	[885, 920]
187	[335, 336]	540	[769, 637]	893	[1069, 746]	1246	[1318, 844]
188	[337, 103]	541	[770, 585]	894	[1070, 999]	1247	[385, 44]
189	[338, 339]	542	[771, 701]	895	[1071, 1072]	1248	[1319, 457]
190	[340, 338]	543	[772, 599]	896	[574, 17]	1249	[387, 797]
191	[341, 342]	544	[773, 650]	897	[1073, 65]	1250	[156, 448]
192	[258, 228]	545	[774, 775]	898	[81, 327]	1251	[864, 929]
193	[260, 343]	546	[776, 594]	899	[1074, 431]	1252	[1320, 487]
194	[344, 345]	547	[777, 778]	900	[1075, 590]	1253	[1321, 909]
195	[346, 347]	548	[779, 780]	901	[357, 1076]	1254	[1322, 139]
196	[108, 348]	549	[754, 781]	902	[1077, 699]	1255	[1323, 459]
197	[349, 339]	550	[782, 635]	903	[1078, 306]	1256	[1197, 1170]
198	[350, 351]	551	[783, 17]	904	[1079, 583]	1257	[1324, 487]
199	[352, 353]	552	[784, 627]	905	[1080, 83]	1258	[1325, 192]
200	[354, 99]	553	[785, 308]	906	[306, 596]	1259	[1326, 876]
201	[355, 328]	554	[786, 499]	907	[1081, 611]	1260	[1184, 542]
202	[356, 357]	555	[455, 787]	908	[1082, 305]	1261	[464, 807]
203	[223, 63]	556	[788, 495]	909	[1083, 1084]	1262	[443, 879]
204	[358, 359]	557	[142, 789]	910	[720, 953]	1263	[905, 145]
205	[360, 266]	558	[46, 124]	911	[1085, 341]	1264	[1323, 174]
206	[361, 362]	559	[790, 392]	912	[1086, 1087]	1265	[1168, 1214]
207	[363, 107]	560	[791, 58]	913	[1088, 373]	1266	[926, 42]
208	[364, 114]	561	[446, 792]	914	[1089, 565]	1267	[1206, 164]
209	[365, 366]	562	[442, 8]	915	[1090, 650]	1268	[1327, 920]
210	[367, 368]	563	[793, 195]	916	[1091, 226]	1269	[1154, 855]
211	[369, 370]	564	[212, 794]	917	[1092, 734]	1270	[1328, 529]
212	[371, 372]	565	[795, 454]	918	[287, 726]	1271	[863, 1329]
213	[264, 373]	566	[469, 468]	919	[1093, 559]	1272	[1330, 488]
214	[278, 116]	567	[515, 172]	920	[1094, 570]	1273	[1331, 196]
215	[374, 375]	568	[488, 415]	921	[351, 1041]	1274	[59, 457]
216	[376, 314]	569	[53, 412]	922	[1095, 602]	1275	[919, 848]
217	[377, 378]	570	[796, 797]	923	[1096, 274]	1276	[1131, 1332]
218	[379, 380]	571	[796, 388]	924	[235, 1097]	1277	[835, 189]

219	[381, 382]	572	[172, 199]	925	[1098, 769]	1278	[1333, 920]
220	[383, 283]	573	[798, 191]	926	[1099, 262]	1279	[1334, 545]
221	[384, 352]	574	[799, 37]	927	[1100, 954]	1280	[1335, 215]
222	[192, 125]	575	[800, 801]	928	[1101, 959]	1281	[1336, 528]
223	[385, 386]	576	[463, 802]	929	[1102, 994]	1282	[215, 545]
224	[387, 388]	577	[803, 535]	930	[1103, 228]	1283	[1337, 549]
225	[389, 390]	578	[804, 140]	931	[1104, 996]	1284	[1338, 868]
226	[391, 177]	579	[795, 805]	932	[1105, 291]	1285	[1339, 1132]
227	[207, 392]	580	[806, 148]	933	[1061, 1106]	1286	[1340, 124]
228	[172, 187]	581	[483, 807]	934	[269, 1037]	1287	[931, 914]
229	[32, 393]	582	[808, 37]	935	[1107, 316]	1288	[1341, 196]
230	[394, 1]	583	[176, 501]	936	[1055, 268]	1289	[1342, 133]
231	[395, 207]	584	[809, 188]	937	[737, 1087]	1290	[1343, 192]
232	[396, 397]	585	[482, 536]	938	[1108, 275]	1291	[1344, 914]
233	[398, 399]	586	[810, 191]	939	[1109, 225]	1292	[936, 465]
234	[400, 401]	587	[811, 792]	940	[1006, 1068]	1293	[511, 130]
235	[402, 193]	588	[447, 812]	941	[1110, 1072]	1294	[508, 48]
236	[403, 404]	589	[813, 549]	942	[372, 1111]	1295	[922, 163]
237	[405, 406]	590	[510, 209]	943	[1112, 66]	1296	[801, 461]
238	[407, 10]	591	[814, 455]	944	[1113, 1114]	1297	[852, 174]
239	[134, 408]	592	[815, 816]	945	[1115, 760]	1298	[1345, 545]
240	[409, 410]	593	[815, 166]	946	[674, 1032]	1299	[1346, 1147]
241	[388, 409]	594	[432, 432]	947	[1116, 995]	1300	[530, 161]
242	[411, 53]	595	[134, 132]	948	[588, 270]	1301	[1347, 914]
243	[412, 413]	596	[57, 817]	949	[638, 968]	1302	[1348, 535]
244	[414, 36]	597	[422, 395]	950	[1117, 429]	1303	[528, 1349]
245	[415, 191]	598	[193, 126]	951	[1118, 499]	1304	[1350, 872]
246	[416, 204]	599	[818, 817]	952	[1119, 125]	1305	[550, 161]
247	[417, 418]	600	[512, 819]	953	[9, 180]	1306	[1351, 335]
248	[419, 219]	601	[820, 821]	954	[425, 48]	1307	[1352, 1292]
249	[420, 421]	602	[493, 163]	955	[8, 157]	1308	[1019, 563]
250	[422, 206]	603	[822, 145]	956	[1120, 401]	1309	[1353, 1354]
251	[423, 424]	604	[207, 823]	957	[1121, 199]	1310	[654, 612]
252	[425, 426]	605	[824, 440]	958	[519, 217]	1311	[1000, 1061]
253	[427, 38]	606	[825, 392]	959	[799, 49]	1312	[1355, 571]
254	[428, 429]	607	[826, 134]	960	[789, 192]	1313	[607, 710]
255	[430, 40]	608	[827, 828]	961	[1122, 424]	1314	[116, 325]
256	[431, 432]	609	[463, 829]	962	[503, 395]	1315	[1356, 598]
257	[433, 172]	610	[61, 830]	963	[203, 12]	1316	[1357, 1253]
258	[434, 32]	611	[33, 518]	964	[1123, 447]	1317	[65, 682]
259	[435, 436]	612	[831, 150]	965	[827, 212]	1318	[1358, 106]
260	[437, 386]	613	[189, 523]	966	[889, 36]	1319	[1293, 287]
261	[438, 139]	614	[832, 37]	967	[131, 408]	1320	[1359, 982]
262	[432, 431]	615	[833, 512]	968	[1124, 150]	1321	[1360, 365]

263	[47, 426]	616	[834, 835]	969	[1125, 46]	1322	[975, 603]
264	[181, 1]	617	[452, 544]	970	[1126, 418]	1323	[1361, 227]
265	[415, 439]	618	[836, 499]	971	[854, 1127]	1324	[100, 769]
266	[440, 34]	619	[469, 469]	972	[1128, 399]	1325	[734, 985]
267	[188, 51]	620	[837, 838]	973	[1129, 787]	1326	[1362, 737]
268	[203, 441]	621	[479, 839]	974	[1130, 12]	1327	[1363, 248]
269	[442, 421]	622	[179, 10]	975	[1131, 1132]	1328	[120, 361]
270	[129, 443]	623	[49, 38]	976	[818, 58]	1329	[648, 1282]
271	[198, 444]	624	[840, 812]	977	[1133, 198]	1330	[1002, 252]
272	[445, 130]	625	[841, 842]	978	[1134, 32]	1331	[368, 757]
273	[446, 447]	626	[466, 803]	979	[456, 10]	1332	[1364, 1052]
274	[421, 448]	627	[843, 844]	980	[202, 436]	1333	[679, 351]
275	[449, 450]	628	[845, 162]	981	[1135, 201]	1334	[1287, 360]
276	[198, 56]	629	[164, 845]	982	[809, 835]	1335	[1365, 774]
277	[451, 221]	630	[800, 512]	983	[804, 162]	1336	[1366, 757]
278	[452, 215]	631	[846, 488]	984	[1136, 429]	1337	[643, 705]
279	[453, 454]	632	[847, 53]	985	[35, 838]	1338	[585, 1367]
280	[455, 160]	633	[146, 848]	986	[1137, 848]	1339	[1368, 1293]
281	[456, 180]	634	[849, 850]	987	[1138, 199]	1340	[980, 656]
282	[167, 145]	635	[845, 140]	988	[1139, 1140]	1341	[702, 1269]
283	[193, 185]	636	[851, 471]	989	[1141, 1]	1342	[668, 1247]
284	[457, 458]	637	[189, 52]	990	[789, 402]	1343	[114, 226]
285	[459, 33]	638	[807, 148]	991	[1142, 159]	1344	[1369, 572]
286	[460, 461]	639	[395, 790]	992	[923, 38]	1345	[958, 302]
287	[462, 463]	640	[388, 482]	993	[903, 1143]	1346	[670, 224]
288	[464, 465]	641	[852, 459]	994	[1144, 821]	1347	[1370, 362]
289	[466, 467]	642	[853, 536]	995	[41, 845]	1348	[248, 742]
290	[468, 468]	643	[854, 855]	996	[54, 413]	1349	[347, 1371]
291	[140, 164]	644	[856, 204]	997	[807, 510]	1350	[1372, 357]
292	[162, 41]	645	[857, 487]	998	[141, 386]	1351	[1373, 868]
293	[468, 469]	646	[540, 190]	999	[450, 848]	1352	[174, 440]
294	[163, 160]	647	[858, 209]	1000	[148, 209]	1353	[1315, 823]
295	[470, 471]	648	[398, 859]	1001	[1145, 839]	1354	[1321, 1374]
296	[472, 157]	649	[32, 457]	1002	[402, 125]	1355	[221, 172]
297	[473, 474]	650	[860, 32]	1003	[1146, 518]	1356	[1152, 177]
298	[475, 476]	651	[389, 418]	1004	[1139, 542]	1357	[1199, 185]
299	[477, 478]	652	[861, 449]	1005	[487, 540]	1358	[792, 812]
300	[479, 480]	653	[862, 863]	1006	[941, 1147]	1359	[465, 148]
301	[481, 482]	654	[864, 219]	1007	[834, 188]	1360	[1375, 534]
302	[483, 465]	655	[865, 471]	1008	[1148, 486]	1361	[1376, 395]
303	[484, 134]	656	[407, 180]	1009	[1149, 212]	1362	[1377, 863]
304	[485, 48]	657	[810, 439]	1010	[1150, 1132]	1363	[1378, 941]
305	[213, 198]	658	[158, 866]	1011	[154, 478]	1364	[1190, 174]
306	[135, 198]	659	[787, 161]	1012	[1151, 476]	1365	[1379, 941]

307	[450, 147]	660	[867, 850]	1013	[1152, 501]	1366	[1380, 803]
308	[486, 49]	661	[163, 787]	1014	[1153, 125]	1367	[940, 872]
309	[487, 488]	662	[124, 203]	1015	[1154, 1127]	1368	[1381, 879]
310	[190, 439]	663	[212, 868]	1016	[1155, 486]	1369	[1382, 830]
311	[489, 463]	664	[869, 130]	1017	[1156, 463]	1370	[1383, 1170]
312	[490, 463]	665	[833, 801]	1018	[1157, 534]	1371	[813, 1384]
313	[491, 150]	666	[206, 790]	1019	[883, 212]	1372	[1385, 1170]
314	[393, 458]	667	[870, 789]	1020	[1158, 465]	1373	[1219, 1251]
315	[492, 476]	668	[420, 8]	1021	[1159, 813]	1374	[1386, 1387]
316	[160, 493]	669	[871, 404]	1022	[1160, 207]	1375	[760, 774]
317	[494, 495]	670	[524, 872]	1023	[1161, 133]	1376	[562, 954]
318	[45, 496]	671	[873, 449]	1024	[150, 389]	1377	[1388, 1287]
319	[497, 159]	672	[874, 512]	1025	[1162, 153]	1378	[1389, 1367]
320	[498, 499]	673	[875, 876]	1026	[391, 501]	1379	[1390, 372]
321	[42, 140]	674	[216, 520]	1027	[1163, 408]	1380	[1245, 237]
322	[500, 501]	675	[877, 855]	1028	[428, 1164]	1381	[611, 779]
323	[502, 503]	676	[860, 59]	1029	[526, 829]	1382	[1269, 1249]
324	[504, 148]	677	[878, 879]	1030	[509, 133]	1383	[1004, 258]
325	[505, 221]	678	[880, 835]	1031	[1137, 147]	1384	[1076, 589]
326	[506, 134]	679	[881, 863]	1032	[876, 830]	1385	[1367, 1047]
327	[437, 44]	680	[788, 184]	1033	[1165, 140]	1386	[1195, 797]
328	[507, 40]	681	[882, 461]	1034	[1166, 457]	1387	[60, 876]
329	[493, 455]	682	[883, 828]	1035	[943, 126]	1388	[1391, 941]
330	[508, 426]	683	[884, 797]	1036	[1167, 432]	1389	[1392, 803]
331	[400, 509]	684	[885, 886]	1037	[401, 133]	1390	[1393, 876]
332	[510, 149]	685	[887, 153]	1038	[421, 157]	1391	[1021, 1004]
333	[459, 440]	686	[822, 168]	1039	[1168, 528]	1392	[1394, 643]
334	[8, 448]	687	[798, 439]	1040	[1169, 217]	1393	[1395, 1058]
335	[511, 443]	688	[473, 153]	1041	[863, 1170]	1394	[1396, 803]
336	[512, 461]	689	[888, 139]	1042	[1171, 543]	1395	[1397, 528]
337	[513, 128]	690	[889, 838]	1043	[1172, 844]	1396	[1398, 779]
338	[436, 441]	691	[890, 846]	1044	[46, 202]	1397	[1399, 347]
339	[486, 37]	692	[891, 412]	1045	[1173, 1]	1398	[1400, 813]
340	[514, 184]	693	[173, 459]	1046	[1174, 389]	1399	[1401, 813]
341	[151, 390]	694	[892, 436]	1047	[894, 543]	1400	[1402, 670]
342	[515, 186]	695	[485, 426]	1048	[1130, 441]	1401	[1403, 1076]
343	[436, 12]	696	[793, 128]	1049	[1129, 160]	1402	[1404, 920]
344	[516, 401]	697	[124, 436]	1050	[507, 1175]	1403	[1405, 813]
345	[205, 146]	698	[480, 486]	1051	[1176, 839]	1404	[1406, 110]
346	[517, 518]	699	[790, 823]	1052	[1177, 941]	1405	[626, 754]
347	[519, 520]	700	[884, 388]	1053	[1178, 201]	1406	[1407, 528]
348	[204, 449]	701	[551, 850]	1054	[1179, 1147]	1407	[1387, 233]
349	[521, 401]	702	[412, 404]	1055	[1, 442]		
350	[522, 523]	703	[893, 146]	1056	[1180, 442]		

351	[524, 525]	704	[533, 478]	1057	[1181, 863]		
352	[490, 526]	705	[894, 53]	1058	[945, 859]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	235	1	470	1	705	1	940	0	1175	1
1	0	236	1	471	1	706	1	941	0	1176	1
2	1	237	0	472	1	707	0	942	0	1177	0
3	0	238	0	473	1	708	0	943	1	1178	1
4	1	239	1	474	0	709	0	944	0	1179	0
5	1	240	1	475	0	710	1	945	0	1180	1
6	0	241	0	476	0	711	1	946	1	1181	0
7	0	242	0	477	0	712	0	947	1	1182	0
8	0	243	0	478	0	713	1	948	1	1183	0
9	1	244	1	479	0	714	1	949	0	1184	0
10	1	245	0	480	1	715	1	950	1	1185	1
11	1	246	0	481	1	716	1	951	1	1186	0
12	1	247	0	482	0	717	1	952	1	1187	0
13	0	248	0	483	1	718	0	953	1	1188	0
14	0	249	1	484	0	719	0	954	0	1189	1
15	0	250	0	485	1	720	0	955	0	1190	1
16	0	251	0	486	1	721	0	956	0	1191	1
17	0	252	0	487	1	722	1	957	0	1192	0
18	1	253	0	488	1	723	0	958	0	1193	0
19	1	254	1	489	1	724	0	959	0	1194	1
20	0	255	0	490	0	725	1	960	1	1195	1
21	1	256	0	491	0	726	1	961	1	1196	0
22	1	257	0	492	0	727	0	962	1	1197	0
23	1	258	0	493	1	728	1	963	1	1198	1
24	0	259	0	494	0	729	0	964	1	1199	0
25	1	260	0	495	1	730	1	965	1	1200	0
26	1	261	1	496	1	731	0	966	0	1201	1
27	0	262	1	497	1	732	0	967	0	1202	1
28	1	263	1	498	0	733	0	968	0	1203	0
29	0	264	1	499	0	734	1	969	0	1204	1
30	1	265	0	500	0	735	1	970	1	1205	1
31	0	266	1	501	1	736	0	971	1	1206	0
32	1	267	0	502	0	737	0	972	1	1207	1
33	0	268	1	503	1	738	0	973	1	1208	1
34	0	269	0	504	1	739	1	974	0	1209	1
35	0	270	0	505	0	740	1	975	1	1210	0
36	0	271	1	506	1	741	1	976	0	1211	1
37	1	272	1	507	0	742	0	977	0	1212	0
38	1	273	0	508	0	743	1	978	1	1213	0

39	1	274	1	509	0	744	1	979	1	1214	0
40	0	275	1	510	0	745	0	980	1	1215	1
41	0	276	1	511	1	746	1	981	0	1216	1
42	0	277	1	512	0	747	0	982	1	1217	1
43	1	278	1	513	0	748	1	983	1	1218	1
44	0	279	0	514	1	749	1	984	0	1219	1
45	1	280	0	515	1	750	1	985	0	1220	0
46	0	281	1	516	1	751	0	986	0	1221	0
47	1	282	0	517	1	752	1	987	1	1222	1
48	0	283	0	518	1	753	1	988	1	1223	1
49	0	284	1	519	0	754	0	989	0	1224	0
50	0	285	0	520	0	755	0	990	1	1225	0
51	1	286	0	521	1	756	1	991	1	1226	1
52	1	287	1	522	1	757	1	992	1	1227	1
53	1	288	0	523	0	758	0	993	0	1228	1
54	1	289	0	524	1	759	1	994	0	1229	1
55	0	290	0	525	0	760	1	995	0	1230	1
56	1	291	0	526	1	761	0	996	1	1231	0
57	1	292	1	527	1	762	1	997	0	1232	1
58	0	293	0	528	1	763	0	998	0	1233	0
59	0	294	1	529	1	764	0	999	1	1234	1
60	1	295	1	530	0	765	0	1000	1	1235	0
61	0	296	1	531	1	766	1	1001	1	1236	1
62	0	297	1	532	0	767	1	1002	1	1237	0
63	1	298	0	533	1	768	1	1003	0	1238	0
64	1	299	0	534	1	769	0	1004	1	1239	1
65	0	300	0	535	0	770	0	1005	1	1240	1
66	0	301	1	536	0	771	0	1006	0	1241	1
67	0	302	1	537	0	772	0	1007	0	1242	0
68	0	303	0	538	1	773	1	1008	0	1243	0
69	0	304	1	539	1	774	0	1009	1	1244	1
70	0	305	1	540	0	775	0	1010	0	1245	0
71	0	306	0	541	0	776	1	1011	0	1246	0
72	0	307	1	542	0	777	1	1012	1	1247	1
73	0	308	1	543	0	778	0	1013	1	1248	1
74	1	309	1	544	1	779	0	1014	0	1249	0
75	0	310	1	545	0	780	0	1015	0	1250	0
76	1	311	1	546	1	781	1	1016	1	1251	1
77	1	312	0	547	1	782	1	1017	0	1252	1
78	1	313	0	548	0	783	1	1018	0	1253	1
79	0	314	0	549	0	784	0	1019	0	1254	1
80	0	315	0	550	1	785	0	1020	1	1255	0
81	0	316	0	551	1	786	1	1021	1	1256	0
82	0	317	0	552	0	787	1	1022	0	1257	0

83	0	318	1	553	0	788	1	1023	0	1258	1
84	0	319	1	554	1	789	1	1024	1	1259	1
85	1	320	0	555	0	790	0	1025	0	1260	0
86	1	321	0	556	1	791	0	1026	1	1261	0
87	1	322	0	557	0	792	0	1027	1	1262	0
88	0	323	0	558	0	793	0	1028	1	1263	1
89	1	324	1	559	0	794	0	1029	1	1264	0
90	1	325	0	560	0	795	1	1030	0	1265	1
91	1	326	1	561	0	796	1	1031	0	1266	0
92	0	327	0	562	0	797	1	1032	1	1267	0
93	0	328	0	563	0	798	0	1033	0	1268	1
94	1	329	1	564	1	799	0	1034	0	1269	0
95	0	330	0	565	1	800	0	1035	1	1270	1
96	1	331	0	566	1	801	1	1036	0	1271	1
97	0	332	0	567	1	802	0	1037	1	1272	1
98	1	333	0	568	1	803	1	1038	1	1273	1
99	0	334	0	569	1	804	1	1039	1	1274	0
100	0	335	1	570	1	805	0	1040	1	1275	1
101	1	336	0	571	1	806	0	1041	1	1276	1
102	0	337	0	572	1	807	0	1042	1	1277	1
103	1	338	0	573	0	808	1	1043	1	1278	0
104	0	339	1	574	0	809	1	1044	0	1279	1
105	1	340	1	575	0	810	1	1045	0	1280	0
106	1	341	1	576	0	811	1	1046	1	1281	0
107	1	342	1	577	1	812	0	1047	1	1282	0
108	0	343	0	578	1	813	0	1048	0	1283	1
109	0	344	1	579	1	814	1	1049	1	1284	0
110	1	345	0	580	0	815	0	1050	0	1285	0
111	1	346	1	581	1	816	0	1051	1	1286	1
112	1	347	0	582	1	817	1	1052	0	1287	1
113	1	348	1	583	0	818	0	1053	1	1288	0
114	0	349	1	584	1	819	1	1054	0	1289	1
115	0	350	1	585	0	820	1	1055	0	1290	0
116	0	351	1	586	1	821	1	1056	1	1291	0
117	0	352	0	587	1	822	0	1057	0	1292	0
118	1	353	1	588	1	823	1	1058	0	1293	1
119	1	354	1	589	0	824	0	1059	0	1294	0
120	1	355	0	590	0	825	0	1060	0	1295	0
121	0	356	1	591	1	826	1	1061	0	1296	1
122	1	357	1	592	0	827	1	1062	1	1297	0
123	0	358	1	593	0	828	0	1063	0	1298	0
124	0	359	0	594	1	829	1	1064	0	1299	1
125	1	360	0	595	1	830	1	1065	0	1300	0
126	1	361	0	596	1	831	1	1066	1	1301	1

127	1	362	0	597	0	832	0	1067	1	1302	0
128	0	363	1	598	0	833	1	1068	0	1303	1
129	0	364	0	599	0	834	0	1069	1	1304	0
130	1	365	1	600	0	835	1	1070	1	1305	1
131	0	366	0	601	1	836	0	1071	1	1306	1
132	1	367	0	602	1	837	1	1072	0	1307	1
133	0	368	1	603	0	838	1	1073	0	1308	0
134	1	369	0	604	1	839	0	1074	1	1309	1
135	0	370	1	605	0	840	1	1075	1	1310	1
136	0	371	1	606	0	841	1	1076	1	1311	1
137	1	372	0	607	1	842	0	1077	0	1312	0
138	0	373	0	608	1	843	0	1078	0	1313	1
139	0	374	0	609	0	844	1	1079	0	1314	0
140	0	375	0	610	0	845	1	1080	1	1315	1
141	0	376	1	611	0	846	0	1081	1	1316	0
142	0	377	1	612	1	847	0	1082	0	1317	0
143	0	378	0	613	0	848	0	1083	0	1318	0
144	1	379	0	614	0	849	0	1084	0	1319	1
145	1	380	1	615	1	850	1	1085	0	1320	1
146	0	381	1	616	0	851	1	1086	1	1321	1
147	1	382	1	617	1	852	0	1087	1	1322	1
148	1	383	1	618	0	853	0	1088	0	1323	0
149	0	384	0	619	1	854	1	1089	0	1324	0
150	1	385	1	620	1	855	0	1090	0	1325	1
151	1	386	1	621	0	856	1	1091	1	1326	1
152	1	387	0	622	0	857	0	1092	1	1327	1
153	1	388	0	623	0	858	1	1093	1	1328	1
154	0	389	0	624	1	859	0	1094	0	1329	1
155	1	390	0	625	1	860	1	1095	0	1330	1
156	0	391	1	626	0	861	1	1096	1	1331	1
157	1	392	0	627	0	862	1	1097	1	1332	1
158	0	393	0	628	1	863	1	1098	1	1333	0
159	0	394	1	629	1	864	1	1099	0	1334	1
160	0	395	1	630	0	865	0	1100	1	1335	0
161	0	396	0	631	0	866	1	1101	1	1336	0
162	1	397	0	632	0	867	1	1102	0	1337	1
163	1	398	1	633	0	868	1	1103	1	1338	0
164	1	399	1	634	0	869	0	1104	1	1339	0
165	1	400	0	635	1	870	1	1105	0	1340	1
166	1	401	1	636	1	871	0	1106	0	1341	0
167	0	402	1	637	0	872	1	1107	1	1342	1
168	0	403	1	638	0	873	0	1108	0	1343	0
169	1	404	1	639	1	874	1	1109	1	1344	0
170	0	405	0	640	0	875	0	1110	0	1345	0

171	1	406	1	641	0	876	1	1111	1	1346	1
172	1	407	0	642	0	877	0	1112	1	1347	1
173	1	408	0	643	1	878	1	1113	0	1348	0
174	1	409	1	644	1	879	0	1114	0	1349	0
175	0	410	1	645	0	880	1	1115	0	1350	0
176	0	411	0	646	0	881	0	1116	1	1351	1
177	0	412	0	647	1	882	1	1117	1	1352	1
178	1	413	0	648	1	883	0	1118	1	1353	1
179	0	414	1	649	1	884	0	1119	1	1354	1
180	0	415	0	650	1	885	0	1120	0	1355	0
181	1	416	0	651	0	886	1	1121	0	1356	1
182	1	417	0	652	1	887	0	1122	1	1357	0
183	0	418	1	653	1	888	0	1123	1	1358	0
184	0	419	0	654	1	889	0	1124	0	1359	1
185	0	420	1	655	0	890	1	1125	0	1360	1
186	0	421	1	656	0	891	0	1126	1	1361	0
187	1	422	0	657	1	892	1	1127	1	1362	1
188	0	423	0	658	0	893	1	1128	1	1363	1
189	0	424	0	659	1	894	1	1129	1	1364	1
190	1	425	0	660	1	895	1	1130	0	1365	0
191	1	426	1	661	1	896	0	1131	1	1366	0
192	0	427	0	662	0	897	0	1132	0	1367	0
193	0	428	1	663	1	898	0	1133	0	1368	0
194	1	429	1	664	0	899	1	1134	1	1369	0
195	1	430	0	665	1	900	1	1135	0	1370	1
196	0	431	0	666	0	901	1	1136	0	1371	0
197	1	432	1	667	1	902	0	1137	0	1372	0
198	1	433	0	668	1	903	0	1138	1	1373	1
199	0	434	0	669	0	904	0	1139	1	1374	1
200	1	435	0	670	1	905	1	1140	1	1375	1
201	0	436	0	671	0	906	0	1141	0	1376	0
202	1	437	0	672	1	907	1	1142	1	1377	1
203	1	438	1	673	0	908	0	1143	1	1378	1
204	1	439	0	674	1	909	0	1144	0	1379	0
205	0	440	1	675	0	910	0	1145	1	1380	0
206	0	441	0	676	1	911	0	1146	0	1381	0
207	1	442	0	677	1	912	1	1147	1	1382	0
208	0	443	0	678	1	913	0	1148	0	1383	1
209	1	444	0	679	0	914	0	1149	1	1384	1
210	0	445	1	680	1	915	0	1150	0	1385	0
211	0	446	0	681	1	916	1	1151	1	1386	1
212	1	447	1	682	0	917	1	1152	1	1387	1
213	1	448	0	683	0	918	1	1153	0	1388	1
214	1	449	1	684	0	919	1	1154	0	1389	1

215	0	450	1	685	0	920	0	1155	1	1390	0
216	1	451	1	686	0	921	1	1156	0	1391	1
217	1	452	1	687	0	922	0	1157	0	1392	1
218	0	453	0	688	1	923	1	1158	1	1393	0
219	1	454	1	689	0	924	1	1159	1	1394	1
220	1	455	0	690	0	925	1	1160	0	1395	0
221	0	456	1	691	1	926	0	1161	0	1396	1
222	0	457	1	692	0	927	1	1162	0	1397	0
223	1	458	1	693	1	928	1	1163	1	1398	1
224	0	459	0	694	1	929	0	1164	0	1399	0
225	0	460	0	695	1	930	1	1165	0	1400	1
226	1	461	0	696	0	931	1	1166	0	1401	0
227	1	462	1	697	0	932	0	1167	0	1402	1
228	1	463	0	698	1	933	0	1168	1	1403	0
229	1	464	0	699	0	934	0	1169	1	1404	1
230	1	465	1	700	0	935	1	1170	0	1405	0
231	1	466	0	701	1	936	0	1171	1	1406	1
232	0	467	0	702	0	937	0	1172	1	1407	1
233	1	468	0	703	1	938	0	1173	0		
234	0	469	1	704	1	939	1	1174	1		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn