# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^2, z^4 + z^2)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^2, z^4+z^2)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

---

*Date*: 9/04/2020 09:39:27 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

*Key words and phrases.* automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 3.5 minutes CPU time used.

**Theorem 1.1.** *When* $(a,b) = (z^2, z^4+z^2)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

*where, for* $\mathrm{CF}(a,b)$*,*

$$\begin{aligned} p_0(z) &= z^6+1,\\ p_1(z) &= z^{10}+z^8,\\ p_2(z) &= z^{12}+z^8,\\ p_3(z) &= z^{10}+z^8,\\ p_4(z) &= z^8+z^6+1, \end{aligned}$$

*and for* $\mathrm{CF}(b,a)$*,*

$$\begin{aligned} p_0(z) &= z^6+z^4+1,\\ p_1(z) &= z^{10}+z^8,\\ p_2(z) &= z^{12}+z^8,\\ p_3(z) &= z^{10}+z^8,\\ p_4(z) &= z^8+z^6+z^4+1. \end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) = {}& x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\ & \cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\ W_n(x) = {}& x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\ & \cdots \times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix}, \end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\ W_{n+1}(x) &= M_n(x)\cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k\ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:12u] &= M^e[:6u] + x^{2u}M^e[:4u] + x^{4u}M^e[:2u] + x^{2u}M^e[:6u] + x^{4u}M^e[:4u]\\ &\quad + x^{6u}M^e[:2u] + x^{8u}M^e[:4u] + (x^{6u}+x^{8u})R^e,\\ W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{4u}W^e[:2u] + x^{2u}W^e[:6u] + x^{4u}W^e[:4u]\\ &\quad + x^{6u}W^e[:2u] + x^{8u}W^e[:4u] + (x^{6u}+x^{8u})R^e.\end{aligned}$$

*For all integers $k\ge 2$ and $u = 2^{2k}$, the following identities hold.*

$$M^o[:12u] = M^o[:6u] + x^{2u}M^o[:4u] + x^{4u}M^o[:2u] + x^{2u}M^o[:6u] + x^{4u}M^o[:4u]$$

$$
\begin{aligned}
&+ x^{6u} M^o[:2u] + x^{8u} M^o[:4u] + (x^{6u} + x^{8u}) R^o, \\
W^o[:12u] &= W^o[:6u] + x^{2u} W^o[:4u] + x^{4u} W^o[:2u] + x^{2u} W^o[:6u] + x^{4u} W^o[:4u] \\
&+ x^{6u} W^o[:2u] + x^{8u} W^o[:4u] + (x^{6u} + x^{8u}) R^o.
\end{aligned}
$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] = &T[:6u] + x^{2u}T[:4u] + x^{4u}T[:2u] + x^{2u}T[:6u] + x^{4u}T[:4u] + x^{6u}T[:2u] \\
&+ x^{8u}T[:4u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
0 &= x^{6u}T[:u] + x^{4u}T[2u:3u] + x^{2u}T[4u:5u] + T[6u:7u], \\
0 &= x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{2u}T[5u:6u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + T[8u:9u], \\
0 &= x^{8u}T[u:2u] + T[9u:10u], \\
0 &= x^{8u}T[2u:3u] + T[10u:11u], \\
0 &= x^{8u}T[3u:4u] + T[11u:12u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[10w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[11w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[10w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 16 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.19}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.20}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1000)), \tag{2.21}$$

$$0 = \tau(A(s,1)) + \tau(A(s,1001)), \tag{2.22}$$

$$0 = \tau(A(s,10)) + \tau(A(s,1010)), \tag{2.23}$$

$$0 = \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.25}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.26}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1000)), \tag{2.27}$$

$$0 = \tau(A(s,1)) + \tau(A(s,1001)), \tag{2.28}$$

$$0 = \tau(A(s,10)) + \tau(A(s,1010)), \tag{2.29}$$

$$0 = \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^8), E_8) = (A(i,0^6), E_6)$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 4$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{2u}M^e[:4u] + x^{4u}M^e[:2u] + x^{6u}R^e,$$
$$W_{2k} = W^e[:6u] + x^{2u}W^e[:4u] + x^{4u}W^e[:2u] + x^{6u}R^e.$$

*For all integer $k \ge 2$, and $u = 2^{2k}$,*

$$M_{2k+1} = M^o[:6u] + x^{2u}M^o[:4u] + x^{4u}M^o[:2u] + x^{6u}R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{2u}W^o[:4u] + x^{4u}W^o[:2u] + x^{6u}R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(W^e[:6v] + x^{2v}W^e[:4v] + x^{4v}W^e[:2v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ + x^{2v}M^e[:4v] + x^{4v}M^e[:2v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is

$O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{12v}$, to

$$W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{8v}W^e[:4v]M^e[:4v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.31) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{12} + x^6, \\ q_1(x) &= x^4 + x^2, \\ q_2(x) &= x^4 + 1, \\ q_3(x) &= x^4 + x^2, \\ q_4(x) &= x^{12} + x^6 + x^4. \end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{72} + x^{60} + x^{56} + x^{48} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24},\\
p_3(x) &= x^{40} + x^{36} + x^{32} + x^{28},\\
p_6(x) &= x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^4 + 1,\\
p_9(x) &= x^{20} + x^4,\\
p_{12}(x) &= x^{16} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{66} + x^{64} + x^{54} + x^{48} + x^{46} + x^{40} + x^{38} + x^{36} + x^{30},\\
p_3(x) &= x^{22} + x^{18},\\
p_6(x) &= x^{18} + x^{16} + x^{14} + x^{12},\\
p_9(x) &= x^{14} + x^6,\\
p_{12}(x) &= x^{10} + x^8 + x^2 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e,1,0)$:

$$\begin{aligned}
p_0(x) &= x^{66} + x^{64} + x^{54} + x^{48} + x^{46} + x^{40} + x^{38} + x^{36} + x^{30},\\
p_3(x) &= x^{22} + x^{18},\\
p_6(x) &= x^{18} + x^{16} + x^{14} + x^{12},\\
p_9(x) &= x^{14} + x^6,\\
p_{12}(x) &= x^{10} + x^8 + x^2 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e,1,1)$:

$$\begin{aligned}
p_0(x) &= x^{60} + x^{56} + x^{48} + x^{40} + x^{36},\\
p_3(x) &= x^{28} + x^{12},\\
p_6(x) &= x^{24} + 1,
\end{aligned}$$

$$\begin{aligned} p_9(x) &= x^8 + x^4, \\ p_{12}(x) &= x^4 + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned} p_0(x) &= x^{60} + x^{56} + x^{52} + x^{40} + x^{36} + x^{32} + x^{24}, \\ p_3(x) &= x^{28} + x^{20}, \\ p_6(x) &= x^{24} + x^{16} + x^8 + 1, \\ p_9(x) &= x^8 + x^4, \\ p_{12}(x) &= x^4 + 1, \end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{66} + x^{64} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{38} + x^{32} + x^{30}, \\ p_3(x) &= x^{22} + x^{18}, \\ p_6(x) &= x^{18} + x^{16} + x^{14} + x^{12}, \\ p_9(x) &= x^{14} + x^6, \\ p_{12}(x) &= x^{10} + x^8 + x^2 + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned} p_0(x) &= x^{66} + x^{64} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{38} + x^{32} + x^{30}, \\ p_3(x) &= x^{22} + x^{18}, \\ p_6(x) &= x^{18} + x^{16} + x^{14} + x^{12}, \\ p_9(x) &= x^{14} + x^6, \\ p_{12}(x) &= x^{10} + x^8 + x^2 + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned} p_0(x) &= x^{72} + x^{64} + x^{60} + x^{56} + x^{44} + x^{40} + x^{36}, \\ p_3(x) &= x^{40} + x^{36} + x^{16} + x^{12}, \\ p_6(x) &= x^{36} + x^{32} + x^{20} + x^{16} + x^{12} + x^8 + x^4 + 1, \\ p_9(x) &= x^{20} + x^4, \\ p_{12}(x) &= x^{16} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned} p_0(x) &= x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} \\ &\quad + x^{40} + x^{38} + x^{34} + x^{30} + x^{26} + x^{24}, \\ p_3(x) &= x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18} + x^{16}, \\ p_6(x) &= x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^2 + 1, \end{aligned}$$

$$
\begin{aligned}
p_9(x) &= x^{10} + x^8,\\
p_{12}(x) &= x^2 + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{64} + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{38} + x^{32} + x^{30},\\
p_3(x) &= x^{24} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{16} + x^{12},\\
p_9(x) &= x^8 + x^6,\\
p_{12}(x) &= 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{50} + x^{46} + x^{40} + x^{38} + x^{36}\\
&\quad + x^{34} + x^{30},\\
p_3(x) &= x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{24} + x^{20} + x^{16} + x^{12},\\
p_9(x) &= x^{16} + x^{14} + x^8 + x^6,\\
p_{12}(x) &= x^8 + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{52} + x^{46} + x^{44} + x^{36},\\
p_3(x) &= x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{20},\\
p_6(x) &= x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12} + x^{10} + x^8 + x^2 + 1,\\
p_9(x) &= x^{10} + x^8,\\
p_{12}(x) &= x^2 + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}\\
&\quad + x^{40} + x^{38} + x^{34} + x^{30} + x^{26} + x^{24},\\
p_3(x) &= x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18} + x^{16},\\
p_6(x) &= x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^2 + 1,\\
p_9(x) &= x^{10} + x^8,\\
p_{12}(x) &= x^2 + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{50} + x^{46} + x^{40} + x^{38} + x^{36}\\
&\quad + x^{34} + x^{30},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{24} + x^{20} + x^{16} + x^{12},\\
p_9(x) &= x^{16} + x^{14} + x^{8} + x^{6},\\
p_{12}(x) &= x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{64} + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{38} + x^{32} + x^{30},\\
p_3(x) &= x^{24} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{16} + x^{12},\\
p_9(x) &= x^{8} + x^{6},\\
p_{12}(x) &= 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{52} + x^{46} + x^{44} + x^{36},\\
p_3(x) &= x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{20},\\
p_6(x) &= x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12} + x^{10} + x^{8} + x^{2} + 1,\\
p_9(x) &= x^{10} + x^{8},\\
p_{12}(x) &= x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 4 | [7, 8] | 8 | [13, 2] | 12 | [7, 15] |
| 1 | [3, 4] | 5 | [9, 4] | 9 | [2, 14] | 13 | [6, 8] |
| 2 | [2, 2] | 6 | [10, 11] | 10 | [6, 15] | 14 | [11, 11] |
| 3 | [5, 6] | 7 | [12, 2] | 11 | [14, 2] | 15 | [4, 11] |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 3 | 0 | 6 | 1 | 9 | 0 | 12 | 0 | 15 | 0 |
| 1 | 0 | 4 | 0 | 7 | 0 | 10 | 1 | 13 | 1 | | |
| 2 | 0 | 5 | 0 | 8 | 1 | 11 | 1 | 14 | 1 | | |

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
*Email address*: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
*Email address*: huyining@hust.edu.cn