PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^2, z^4 + z^3 + z)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^2, z^4 + z^3 + z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 09:51:08 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 23.2 minutes CPU time used.

Theorem 1.1. *When* $(a,b)=(z^2,z^4+z^3+z)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{14}+z^{13}+z^{10}+z^7+z^5+z^4+z^2+z+1,\\
p_1(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^9+z^7+z^4+z^3+z^2,\\
p_2(z)&=z^{20}+z^{10}+z^8+z^6+z^4,\\
p_3(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^9+z^7+z^4+z^3+z^2,\\
p_4(z)&=z^{16}+z^{14}+z^{13}+z^{12}+z^8+z^7+z^6+z^5+z^4+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{14}+z^{13}+z^{12}+z^{10}+z^8+z^7+z^5+z^2+z,\\
p_1(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^9+z^7+z^4+z^3+z^2,\\
p_2(z)&=z^{20}+z^{10}+z^8+z^6+z^4,\\
p_3(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^9+z^7+z^4+z^3+z^2,\\
p_4(z)&=z^{16}+z^{14}+z^{13}+z^7+z^6+z^5+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)=x^{\deg(t_{2^n-1})}&\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)=x^{\deg(\bar t_{2^n-1})}&\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 540, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 660, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{5u} M^e[:u] \\ & + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ & + x^{6u} M^e[:u] + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{5u} M^e[:3u] \\ & + x^{6u} M^e[:2u] + x^{7u} M^e[:u] + x^{5u} M^e[:6u] + x^{6u} M^e[:5u] \\ & + x^{8u} M^e[:3u] + x^{9u} M^e[:2u] + x^{10u} M^e[:u] + x^{6u} M^e[:6u] \\ & + (x^{6u} + x^{7u} + x^{8u} + x^{11u}) R^e, \end{aligned}$$

$$\begin{aligned}W^e[:12u] &= W^e[:6u] + x^uW^e[:5u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u]\\ &+ x^uW^e[:6u] + x^{2u}W^e[:5u] + x^{4u}W^e[:3u] + x^{5u}W^e[:2u]\\ &+ x^{6u}W^e[:u] + x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{5u}W^e[:3u]\\ &+ x^{6u}W^e[:2u] + x^{7u}W^e[:u] + x^{5u}W^e[:6u] + x^{6u}W^e[:5u]\\ &+ x^{8u}W^e[:3u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{6u}W^e[:6u]\\ &+ (x^{6u} + x^{7u} + x^{8u} + x^{11u})R^e.\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}M^o[:12u] &= M^o[:6u] + x^uM^o[:5u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u]\\ &+ x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{4u}M^o[:3u] + x^{5u}M^o[:2u]\\ &+ x^{6u}M^o[:u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{5u}M^o[:3u]\\ &+ x^{6u}M^o[:2u] + x^{7u}M^o[:u] + x^{5u}M^o[:6u] + x^{6u}M^o[:5u]\\ &+ x^{8u}M^o[:3u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{6u}M^o[:6u]\\ &+ (x^{6u} + x^{7u} + x^{8u} + x^{11u})R^o,\\ W^o[:12u] &= W^o[:6u] + x^uW^o[:5u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u]\\ &+ x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{4u}W^o[:3u] + x^{5u}W^o[:2u]\\ &+ x^{6u}W^o[:u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{5u}W^o[:3u]\\ &+ x^{6u}W^o[:2u] + x^{7u}W^o[:u] + x^{5u}W^o[:6u] + x^{6u}W^o[:5u]\\ &+ x^{8u}W^o[:3u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{6u}W^o[:6u]\\ &+ (x^{6u} + x^{7u} + x^{8u} + x^{11u})R^o.\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}T[:12u] &= T[:6u] + x^uT[:5u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{5u}T[:u] + x^uT[:6u]\\ &+ x^{2u}T[:5u] + x^{4u}T[:3u] + x^{5u}T[:2u] + x^{6u}T[:u] + x^{2u}T[:6u]\\ &+ x^{3u}T[:5u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{5u}T[:6u]\\ &+ x^{6u}T[:5u] + x^{8u}T[:3u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{6u}T[:6u].\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}0 &= x^{5u}T[u:2u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] + x^uT[5u:6u]\\ &\quad + T[6u:7u],\\ 0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u],\\ 0 &= x^{8u}T[:u] + x^{5u}T[3u:4u] + T[8u:9u],\\ 0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{5u}T[4u:5u] + T[9u:10u],\\ 0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{5u}T[5u:6u]\\ &\quad + T[10u:11u],\end{aligned}$$

$$0 = x^{6u}T[5u:6u] + T[11u:12u],$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1196 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,110)), \end{aligned} \tag{2.19}$$
$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.20}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,1000)), \tag{2.21}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,1001)), \tag{2.22}$$
$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ & + \tau(A(s,1010)), \end{aligned} \tag{2.23}$$
$$0 = \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 = {} & \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,110)), \end{aligned} \tag{2.25}$$
$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.26}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,11)) + \tau(A(s,1000)), \tag{2.27}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,1001)), \tag{2.28}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.29}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{30}), E_{30}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 15$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{5u} M^e[:u] \\ &\quad + x^{6u} R^e, \\ W_{2k} &= W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{5u} W^e[:u] \\ &\quad + x^{6u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{4u} M^o[:2u] + x^{5u} M^o[:u] \\ &\quad + x^{6u} R^o, \\ W_{2k+1} &= W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{4u} W^o[:2u] + x^{5u} W^o[:u] \\ &\quad + x^{6u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(&W^e[:6v] + x^v W^e[:5v] + x^{3v} W^e[:3v] + x^{4v} W^e[:2v] + x^{5v} W^e[:v] \\ &+ x^{6u} R^e\big) \times \big(M^e[:6v] + x^v M^e[:5v] + x^{3v} M^e[:3v] + x^{4v} M^e[:2v] \\ &+ x^{5v} M^e[:v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} &W^e[:12v] M^e[:12v] + x^{2v} W^e[:10v] M^e[:10v] + x^{6v} W^e[:6v] M^e[:6v] \\ &\qquad + x^{8v} W^e[:4v] M^e[:4v] + x^{10v} W^e[:2v] M^e[:2v]. \end{aligned}$$

For all $n \leq 12$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12} \phi(M^e, 0, 1)(x, t/y) \right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{10} + x^7 + x^6, \\ q_1(x) &= x^{18} + x^{17} + x^{16} + x^{13} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^2, \\ q_2(x) &= x^{16} + x^{14} + x^{12} + x^{10} + 1, \\ q_3(x) &= x^{18} + x^{17} + x^{16} + x^{13} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^2, \\ q_4(x) &= x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^8 + x^7 + x^6 + x^4. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate

$f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{114}+x^{112}+x^{110}+x^{106}+x^{102}+x^{100}+x^{96}+x^{80}+x^{78}+x^{64}\\
&\quad +x^{62}+x^{60}+x^{54}+x^{52}+x^{50}+x^{48}+x^{44}+x^{38}+x^{30}+x^{28}+x^{24},\\
p_3(x) &= x^{106}+x^{104}+x^{102}+x^{100}+x^{96}+x^{94}+x^{90}+x^{86}+x^{84}+x^{80}+x^{74}\\
&\quad +x^{68}+x^{66}+x^{64}+x^{62}+x^{56}+x^{54}+x^{48}+x^{44}+x^{38}+x^{36}+x^{30}\\
&\quad +x^{24}+x^{18},\\
p_6(x) &= x^{106}+x^{104}+x^{94}+x^{92}+x^{90}+x^{88}+x^{84}+x^{82}+x^{80}+x^{76}+x^{74}\\
&\quad +x^{72}+x^{70}+x^{66}+x^{62}+x^{60}+x^{50}+x^{48}+x^{40}+x^{32}+x^{24}+x^{22}\\
&\quad +x^{20}+x^{16}+x^{14}+x^{12}+x^{10}+x^{8}+x^{4}+1,\\
p_9(x) &= x^{104}+x^{100}+x^{98}+x^{88}+x^{86}+x^{84}+x^{80}+x^{78}+x^{74}+x^{70}+x^{64}\\
&\quad +x^{62}+x^{60}+x^{58}+x^{50}+x^{44}+x^{40}+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}\\
&\quad +x^{24}+x^{22}+x^{20}+x^{12}+x^{10}+x^{4},\\
p_{12}(x) &= x^{104}+x^{80}+x^{72}+x^{64}+x^{56}+x^{32}+x^{16}+x^{8}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{98}+x^{97}+x^{96}\\
&\quad +x^{93}+x^{92}+x^{90}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{78}\\
&\quad +x^{77}+x^{74}+x^{72}+x^{71}+x^{69}+x^{68}+x^{66}+x^{64}+x^{63}+x^{61}+x^{59}\\
&\quad +x^{58}+x^{56}+x^{55}+x^{52}+x^{50}+x^{46}+x^{45}+x^{42}+x^{40}+x^{39}+x^{37}\\
&\quad +x^{36}+x^{35}+x^{32}+x^{30},\\
p_3(x) &= x^{101}+x^{100}+x^{93}+x^{92}+x^{85}+x^{84}+x^{83}+x^{82}+x^{77}+x^{76}+x^{61}\\
&\quad +x^{60}+x^{59}+x^{58}+x^{53}+x^{52}+x^{51}+x^{50}+x^{43}+x^{42}+x^{35}+x^{34}\\
&\quad +x^{29}+x^{28}+x^{21}+x^{20}+x^{19}+x^{18},\\
p_6(x) &= x^{99}+x^{96}+x^{95}+x^{94}+x^{93}+x^{90}+x^{88}+x^{87}+x^{86}+x^{83}+x^{81}\\
&\quad +x^{80}+x^{79}+x^{76}+x^{75}+x^{74}+x^{73}+x^{71}+x^{70}+x^{67}+x^{65}+x^{64}\\
&\quad +x^{55}+x^{52}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}+x^{43}+x^{40}+x^{38}\\
&\quad +x^{37}+x^{36}+x^{33}+x^{30}+x^{29}+x^{28}+x^{25}+x^{23}+x^{22}+x^{21}+x^{19}\\
&\quad +x^{18}+x^{15}+x^{13}+x^{12},
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79}\\
&\quad + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60}\\
&\quad + x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{42} + x^{39} + x^{38} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{17} + x^{16}\\
&\quad + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77}\\
&\quad + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} + x^{44} + x^{39}\\
&\quad + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21} + x^{20}\\
&\quad + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{96}\\
&\quad + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78}\\
&\quad + x^{77} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59}\\
&\quad + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{37}\\
&\quad + x^{36} + x^{35} + x^{32} + x^{30},\\
p_3(x) &= x^{101} + x^{100} + x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{61}\\
&\quad + x^{60} + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{50} + x^{43} + x^{42} + x^{35} + x^{34}\\
&\quad + x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{87} + x^{86} + x^{83} + x^{81}\\
&\quad + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64}\\
&\quad + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38}\\
&\quad + x^{37} + x^{36} + x^{33} + x^{30} + x^{29} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19}\\
&\quad + x^{18} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79}\\
&\quad + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60}\\
&\quad + x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{42} + x^{39} + x^{38} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{17} + x^{16}\\
&\quad + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77}\\
&\quad + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} + x^{44} + x^{39}\\
&\quad + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21} + x^{20}\\
&\quad + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88} + x^{86} + x^{76} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{66} + x^{58} + x^{56} + x^{54} + x^{42} + x^{36}, \\
p_3(x) &= x^{90} + x^{84} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} \\
&\quad + x^{54} + x^{52} + x^{48} + x^{44} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} \\
&\quad + x^{14} + x^{12}, \\
p_6(x) &= x^{90} + x^{86} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{56} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{38} + x^{34} + x^{28} + x^{22} + x^{18} + x^{14} + x^{10} + x^{8} + x^{4} + x^{2} \\
&\quad + 1, \\
p_9(x) &= x^{86} + x^{78} + x^{74} + x^{68} + x^{62} + x^{56} + x^{52} + x^{48} + x^{46} + x^{38} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{24} + x^{18} + x^{16} + x^{12} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{86} + x^{82} + x^{80} + x^{70} + x^{66} + x^{64} + x^{38} + x^{34} + x^{32} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{84} + x^{74} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{90} + x^{84} + x^{82} + x^{78} + x^{74} + x^{64} + x^{56} + x^{50} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18}, \\
p_6(x) &= x^{90} + x^{86} + x^{82} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} + x^{60} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{18} + x^{14} + x^{12} + x^{4} \\
&\quad + x^{2} + 1, \\
p_9(x) &= x^{86} + x^{78} + x^{74} + x^{68} + x^{62} + x^{56} + x^{52} + x^{48} + x^{46} + x^{38} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{24} + x^{18} + x^{16} + x^{12} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{86} + x^{82} + x^{80} + x^{70} + x^{66} + x^{64} + x^{38} + x^{34} + x^{32} + x^{22} + x^{18} \\
&\quad + x^{16} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} \\
&\quad + x^{74} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{59} + x^{58} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{54} + x^{52} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{39} \\
&\quad + x^{36} + x^{35} + x^{33} + x^{30}, \\
p_3(x) &= x^{101} + x^{100} + x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{50} + x^{43} + x^{42} + x^{35} + x^{34}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{87} + x^{86} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64}\\
&+ x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38}\\
&+ x^{37} + x^{36} + x^{33} + x^{30} + x^{29} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19}\\
&+ x^{18} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79}\\
&+ x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60}\\
&+ x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77}\\
&+ x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} + x^{44} + x^{39}\\
&+ x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21} + x^{20}\\
&+ x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94}\\
&+ x^{92} + x^{90} + x^{88} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76}\\
&+ x^{74} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{59} + x^{58} + x^{57}\\
&+ x^{56} + x^{55} + x^{54} + x^{52} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} + x^{39}\\
&+ x^{36} + x^{35} + x^{33} + x^{30},\\
p_3(x) &= x^{101} + x^{100} + x^{93} + x^{92} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{50} + x^{43} + x^{42} + x^{35} + x^{34}\\
&+ x^{29} + x^{28} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{87} + x^{86} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64}\\
&+ x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38}\\
&+ x^{37} + x^{36} + x^{33} + x^{30} + x^{29} + x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19}\\
&+ x^{18} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79}\\
&+ x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60}\\
&+ x^{59} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},
\end{aligned}
$$

$$\begin{aligned}p_{12}(x) = {} & x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} \\ & + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{47} + x^{45} + x^{44} + x^{39} \\ & + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{23} + x^{21} + x^{20} \\ & + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}p_0(x) = {} & x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{100} + x^{96} + x^{90} + x^{88} + x^{86} \\ & + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{66} + x^{62} + x^{60} + x^{54} \\ & + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}, \\ p_3(x) = {} & x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{88} + x^{86} + x^{80} + x^{74} + x^{62} + x^{58} \\ & + x^{54} + x^{52} + x^{48} + x^{44} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20} \\ & + x^{16} + x^{12}, \\ p_6(x) = {} & x^{106} + x^{104} + x^{100} + x^{98} + x^{94} + x^{88} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70} \\ & + x^{68} + x^{62} + x^{58} + x^{52} + x^{50} + x^{40} + x^{36} + x^{34} + x^{32} + x^{24} + x^{22} \\ & + x^{20} + x^{18} + x^{14} + x^{10} + x^{4} + 1, \\ p_9(x) = {} & x^{104} + x^{100} + x^{98} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{74} + x^{70} + x^{64} \\ & + x^{62} + x^{60} + x^{58} + x^{50} + x^{44} + x^{40} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} \\ & + x^{24} + x^{22} + x^{20} + x^{12} + x^{10} + x^{4}, \\ p_{12}(x) = {} & x^{104} + x^{80} + x^{72} + x^{64} + x^{56} + x^{32} + x^{16} + x^{8} + 1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}p_0(x) = {} & x^{108} + x^{106} + x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{89} \\ & + x^{88} + x^{86} + x^{85} + x^{83} + x^{81} + x^{75} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} \\ & + x^{64} + x^{61} + x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\ & + x^{48} + x^{47} + x^{43} + x^{41} + x^{38} + x^{35} + x^{32} + x^{29} + x^{28} + x^{27} + x^{25} \\ & + x^{24}, \\ p_3(x) = {} & x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{88} \\ & + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70} \\ & + x^{69} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{50} + x^{48} \\ & + x^{47} + x^{46} + x^{45} + x^{44} + x^{36} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} \\ & + x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16}, \\ p_6(x) = {} & x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{89} + x^{84} + x^{83} \\ & + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{63} \\ & + x^{62} + x^{61} + x^{60} + x^{59} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} \\ & + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{32} + x^{29} + x^{28}\end{aligned}$$

$$
\begin{aligned}
&+ x^{26} + x^{25} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{11} + x^9 + x^8 + x^7 + x^5 \\
&+ x^4 + x^3 + x + 1, \\
p_9(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} \\
&+ x^{73} + x^{72} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} \\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} \\
&+ x^{13} + x^{12} + x^{11} + x^9 + x^8, \\
p_{12}(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{75} + x^{73} \\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{55} + x^{53} + x^{52} + x^{51} \\
&+ x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{11} + x^9 + x^8 + x^7 \\
&+ x^5 + x^4 + x^3 + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} \\
&+ x^{89} + x^{85} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{68} \\
&+ x^{67} + x^{65} + x^{64} + x^{60} + x^{56} + x^{52} + x^{47} + x^{45} + x^{41} + x^{40} + x^{39} \\
&+ x^{37} + x^{34} + x^{33} + x^{31} + x^{30}, \\
p_3(x) &= x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{82} + x^{70} + x^{69} + x^{67} \\
&+ x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} + x^{52} + x^{51} + x^{49} + x^{46} \\
&+ x^{44} + x^{43} + x^{41} + x^{37} + x^{36} + x^{34} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} \\
&+ x^{21} + x^{20} + x^{18}, \\
p_6(x) &= x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76} + x^{74} + x^{70} + x^{64} + x^{52} \\
&+ x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{22} + x^{18} + x^{12}, \\
p_9(x) &= x^{98} + x^{97} + x^{96} + x^{94} + x^{90} + x^{89} + x^{88} + x^{86} + x^{82} + x^{81} + x^{80} \\
&+ x^{78} + x^{74} + x^{73} + x^{72} + x^{70} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} \\
&+ x^{40} + x^{38} + x^{34} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{22} + x^{18} \\
&+ x^{17} + x^{16} + x^{14} + x^{10} + x^9 + x^8 + x^6, \\
p_{12}(x) &= x^{100} + x^{96} + x^{88} + x^{72} + x^{68} + x^{64} + x^{52} + x^{48} + x^{40} + x^{24} + x^8 + x^4 \\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{111} + x^{109} + x^{108} + x^{105} + x^{101} + x^{97} + x^{94} + x^{88} \\
&+ x^{87} + x^{83} + x^{82} + x^{81} + x^{80} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} \\
&+ x^{67} + x^{66} + x^{65} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{50} + x^{49}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30},\\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90}\\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} + x^{69}\\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} + x^{52} + x^{51} + x^{50}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{32} + x^{31}\\
&+ x^{30} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{96} + x^{84} + x^{82} + x^{78} + x^{70} + x^{68} + x^{62} + x^{60}\\
&+ x^{54} + x^{52} + x^{48} + x^{40} + x^{38} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{18}\\
&+ x^{16} + x^{12},\\
p_9(x) &= x^{110} + x^{109} + x^{108} + x^{106} + x^{98} + x^{97} + x^{96} + x^{94} + x^{90} + x^{89} + x^{88}\\
&+ x^{86} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{62} + x^{61}\\
&+ x^{60} + x^{58} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{38} + x^{34}\\
&+ x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{22} + x^{18} + x^{17} + x^{16} + x^{13}\\
&+ x^{12} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{80} + x^{60} + x^{56}\\
&+ x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{28} + x^{24} + x^{20} + x^{16} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{102} + x^{99} + x^{97} + x^{95} + x^{93} + x^{90} + x^{86} + x^{84} + x^{82}\\
&+ x^{81} + x^{79} + x^{77} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{41}\\
&+ x^{37} + x^{36},\\
p_3(x) &= x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{71}\\
&+ x^{69} + x^{68} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{53} + x^{52} + x^{51} + x^{49}\\
&+ x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{40} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{26} + x^{24} + x^{21} + x^{20},\\
p_6(x) &= x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{84} + x^{81} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{52} + x^{48} + x^{46} + x^{43}\\
&+ x^{42} + x^{38} + x^{36} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{22} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{15} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}\\
&+ x^{13} + x^{12} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) = {} & x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{75} + x^{73}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{55} + x^{53} + x^{52} + x^{51}\\
&+ x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{11} + x^{9} + x^{8} + x^{7}\\
&+ x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{108} + x^{106} + x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{89}\\
&+ x^{88} + x^{86} + x^{85} + x^{83} + x^{81} + x^{75} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65}\\
&+ x^{64} + x^{61} + x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49}\\
&+ x^{48} + x^{47} + x^{43} + x^{41} + x^{38} + x^{35} + x^{32} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{24},\\
p_3(x) = {} & x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{88}\\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70}\\
&+ x^{69} + x^{66} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{50} + x^{48}\\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{36} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27}\\
&+ x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16},\\
p_6(x) = {} & x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{89} + x^{84} + x^{83}\\
&+ x^{80} + x^{79} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{67} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{61} + x^{60} + x^{59} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{32} + x^{29} + x^{28}\\
&+ x^{26} + x^{25} + x^{23} + x^{21} + x^{20} + x^{17} + x^{15} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5}\\
&+ x^{4} + x^{3} + x + 1,\\
p_9(x) = {} & x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}\\
&+ x^{13} + x^{12} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) = {} & x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{75} + x^{73}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{55} + x^{53} + x^{52} + x^{51}\\
&+ x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{11} + x^{9} + x^{8} + x^{7}\\
&+ x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{111} + x^{109} + x^{108} + x^{105} + x^{101} + x^{97} + x^{94} + x^{88} \\
&\quad + x^{87} + x^{83} + x^{82} + x^{81} + x^{80} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{50} + x^{49} \\
&\quad + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30}, \\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{73} + x^{72} + x^{69} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{96} + x^{84} + x^{82} + x^{78} + x^{70} + x^{68} + x^{62} + x^{60} \\
&\quad + x^{54} + x^{52} + x^{48} + x^{40} + x^{38} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{18} \\
&\quad + x^{16} + x^{12}, \\
p_9(x) &= x^{110} + x^{109} + x^{108} + x^{106} + x^{98} + x^{97} + x^{96} + x^{94} + x^{90} + x^{89} + x^{88} \\
&\quad + x^{86} + x^{82} + x^{81} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{58} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{38} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{22} + x^{18} + x^{17} + x^{16} + x^{13} \\
&\quad + x^{12} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{80} + x^{60} + x^{56} \\
&\quad + x^{52} + x^{48} + x^{44} + x^{40} + x^{36} + x^{28} + x^{24} + x^{20} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{89} + x^{85} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{68} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{60} + x^{56} + x^{52} + x^{47} + x^{45} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{34} + x^{33} + x^{31} + x^{30}, \\
p_3(x) &= x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{82} + x^{70} + x^{69} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} + x^{54} + x^{52} + x^{51} + x^{49} + x^{46} \\
&\quad + x^{44} + x^{43} + x^{41} + x^{37} + x^{36} + x^{34} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{21} + x^{20} + x^{18}, \\
p_6(x) &= x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{80} + x^{76} + x^{74} + x^{70} + x^{64} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{22} + x^{18} + x^{12}, \\
p_9(x) &= x^{98} + x^{97} + x^{96} + x^{94} + x^{90} + x^{89} + x^{88} + x^{86} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{78} + x^{74} + x^{73} + x^{72} + x^{70} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41}
\end{aligned}$$

$$+ x^{40} + x^{38} + x^{34} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{22} + x^{18} + x^{17} + x^{16} + x^{14} + x^{10} + x^{9} + x^{8} + x^{6},$$
$$p_{12}(x) = x^{100} + x^{96} + x^{88} + x^{72} + x^{68} + x^{64} + x^{52} + x^{48} + x^{40} + x^{24} + x^{8} + x^{4} + 1,$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$p_0(x) = x^{108} + x^{106} + x^{102} + x^{99} + x^{97} + x^{95} + x^{93} + x^{90} + x^{86} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{41} + x^{37} + x^{36},$$
$$p_3(x) = x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{71} + x^{69} + x^{68} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{40} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24} + x^{21} + x^{20},$$
$$p_6(x) = x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{81} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{52} + x^{48} + x^{46} + x^{43} + x^{42} + x^{38} + x^{36} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,$$
$$p_9(x) = x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8},$$
$$p_{12}(x) = x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	300	[172, 487]	600	[430, 410]	900	[1051, 1052]

1	[3, 4]	301	[472, 377]	601	[389, 786]	901	[1053, 1054]
2	[5, 6]	302	[406, 127]	602	[51, 204]	902	[1055, 1056]
3	[7, 8]	303	[488, 489]	603	[122, 160]	903	[923, 1031]
4	[9, 10]	304	[490, 472]	604	[410, 835]	904	[1057, 934]
5	[11, 12]	305	[150, 130]	605	[836, 837]	905	[1058, 585]
6	[13, 14]	306	[491, 165]	606	[178, 829]	906	[1059, 593]
7	[15, 16]	307	[492, 493]	607	[838, 154]	907	[709, 735]
8	[17, 18]	308	[201, 470]	608	[839, 192]	908	[1060, 1061]
9	[19, 20]	309	[494, 167]	609	[840, 357]	909	[1062, 742]
10	[21, 22]	310	[377, 429]	610	[137, 841]	910	[253, 1063]
11	[23, 24]	311	[495, 496]	611	[522, 842]	911	[1064, 103]
12	[25, 26]	312	[137, 38]	612	[440, 802]	912	[1065, 699]
13	[27, 28]	313	[489, 497]	613	[843, 495]	913	[1008, 220]
14	[29, 30]	314	[454, 164]	614	[44, 844]	914	[1066, 1067]
15	[31, 32]	315	[166, 179]	615	[506, 175]	915	[1068, 1069]
16	[33, 34]	316	[33, 59]	616	[472, 445]	916	[1023, 81]
17	[35, 36]	317	[14, 176]	617	[845, 791]	917	[1070, 1071]
18	[37, 38]	318	[498, 499]	618	[431, 411]	918	[98, 699]
19	[39, 2]	319	[500, 501]	619	[357, 468]	919	[1072, 609]
20	[40, 41]	320	[502, 503]	620	[841, 531]	920	[1073, 621]
21	[42, 43]	321	[504, 184]	621	[794, 163]	921	[182, 375]
22	[44, 45]	322	[505, 506]	622	[489, 459]	922	[1074, 12]
23	[46, 47]	323	[507, 388]	623	[846, 454]	923	[358, 200]
24	[48, 49]	324	[508, 509]	624	[847, 449]	924	[1075, 138]
25	[12, 50]	325	[406, 419]	625	[848, 837]	925	[1076, 58]
26	[14, 51]	326	[510, 511]	626	[849, 373]	926	[1077, 167]
27	[52, 53]	327	[385, 169]	627	[393, 850]	927	[1078, 389]
28	[54, 55]	328	[512, 513]	628	[847, 851]	928	[1079, 188]
29	[56, 57]	329	[514, 396]	629	[407, 391]	929	[203, 844]
30	[58, 59]	330	[120, 515]	630	[852, 433]	930	[875, 152]
31	[60, 61]	331	[445, 378]	631	[853, 396]	931	[450, 183]
32	[62, 63]	332	[463, 516]	632	[443, 121]	932	[854, 812]
33	[64, 65]	333	[54, 517]	633	[164, 401]	933	[827, 870]
34	[66, 67]	334	[518, 429]	634	[141, 361]	934	[1080, 820]
35	[68, 69]	335	[519, 469]	635	[35, 374]	935	[150, 386]
36	[70, 71]	336	[520, 521]	636	[854, 149]	936	[837, 816]
37	[72, 73]	337	[522, 523]	637	[423, 486]	937	[1081, 399]
38	[74, 75]	338	[524, 172]	638	[855, 371]	938	[1082, 786]
39	[76, 77]	339	[385, 503]	639	[440, 522]	939	[1083, 134]
40	[78, 62]	340	[525, 59]	640	[856, 419]	940	[390, 455]
41	[79, 80]	341	[526, 12]	641	[414, 493]	941	[861, 522]
42	[81, 82]	342	[527, 528]	642	[161, 383]	942	[404, 865]
43	[83, 84]	343	[529, 521]	643	[536, 124]	943	[842, 404]
44	[85, 86]	344	[530, 41]	644	[442, 125]	944	[439, 861]

45	[87, 88]	345	[531, 532]	645	[822, 497]	945	[1084, 844]
46	[89, 90]	346	[503, 399]	646	[822, 459]	946	[11, 536]
47	[91, 92]	347	[57, 533]	647	[531, 43]	947	[370, 1085]
48	[93, 94]	348	[478, 390]	648	[784, 376]	948	[1086, 365]
49	[95, 96]	349	[408, 534]	649	[490, 499]	949	[1087, 2]
50	[24, 97]	350	[140, 533]	650	[857, 825]	950	[360, 142]
51	[28, 98]	351	[535, 460]	651	[858, 859]	951	[402, 812]
52	[99, 100]	352	[536, 50]	652	[860, 161]	952	[1088, 841]
53	[101, 102]	353	[47, 8]	653	[37, 841]	953	[384, 192]
54	[103, 104]	354	[537, 407]	654	[528, 528]	954	[1089, 1085]
55	[105, 106]	355	[357, 475]	655	[861, 802]	955	[1090, 398]
56	[107, 108]	356	[538, 539]	656	[862, 453]	956	[155, 401]
57	[109, 110]	357	[86, 540]	657	[431, 835]	957	[1081, 170]
58	[111, 112]	358	[541, 542]	658	[863, 405]	958	[175, 51]
59	[113, 114]	359	[543, 285]	659	[864, 462]	959	[1091, 421]
60	[115, 49]	360	[544, 221]	660	[832, 133]	960	[1092, 385]
61	[116, 117]	361	[545, 546]	661	[466, 489]	961	[189, 147]
62	[118, 119]	362	[547, 548]	662	[800, 865]	962	[443, 515]
63	[120, 121]	363	[549, 550]	663	[434, 861]	963	[356, 467]
64	[122, 123]	364	[551, 268]	664	[477, 389]	964	[916, 841]
65	[12, 124]	365	[552, 553]	665	[866, 410]	965	[1093, 121]
66	[125, 58]	366	[554, 555]	666	[485, 424]	966	[505, 13]
67	[126, 127]	367	[556, 557]	667	[466, 822]	967	[786, 197]
68	[128, 129]	368	[558, 559]	668	[867, 427]	968	[1094, 392]
69	[130, 131]	369	[260, 560]	669	[868, 501]	969	[436, 410]
70	[132, 133]	370	[561, 109]	670	[869, 870]	970	[1087, 513]
71	[134, 135]	371	[562, 563]	671	[363, 487]	971	[801, 434]
72	[136, 137]	372	[16, 564]	672	[871, 41]	972	[1095, 177]
73	[138, 55]	373	[565, 566]	673	[872, 142]	973	[510, 144]
74	[139, 140]	374	[567, 301]	674	[840, 467]	974	[1096, 532]
75	[141, 142]	375	[283, 568]	675	[800, 366]	975	[799, 884]
76	[143, 144]	376	[321, 569]	676	[873, 135]	976	[1097, 371]
77	[128, 145]	377	[570, 571]	677	[444, 507]	977	[518, 378]
78	[146, 147]	378	[572, 330]	678	[483, 804]	978	[526, 536]
79	[148, 149]	379	[573, 574]	679	[421, 203]	979	[1098, 170]
80	[53, 150]	380	[575, 74]	680	[874, 131]	980	[807, 357]
81	[151, 152]	381	[576, 577]	681	[796, 178]	981	[1099, 1100]
82	[153, 154]	382	[300, 578]	682	[875, 479]	982	[1101, 365]
83	[155, 156]	383	[579, 313]	683	[13, 175]	983	[462, 837]
84	[157, 158]	384	[580, 581]	684	[135, 142]	984	[862, 918]
85	[159, 160]	385	[26, 582]	685	[876, 877]	985	[891, 506]
86	[161, 162]	386	[583, 584]	686	[391, 534]	986	[902, 357]
87	[163, 164]	387	[585, 586]	687	[32, 850]	987	[1102, 175]
88	[130, 165]	388	[587, 588]	688	[789, 154]	988	[1077, 157]

89	[52, 166]	389	[589, 590]	689	[435, 844]	989	[367, 439]
90	[167, 158]	390	[591, 592]	690	[841, 42]	990	[1103, 836]
91	[168, 151]	391	[593, 594]	691	[421, 44]	991	[824, 117]
92	[169, 170]	392	[327, 595]	692	[878, 879]	992	[1104, 829]
93	[171, 172]	393	[596, 345]	693	[469, 202]	993	[171, 363]
94	[173, 174]	394	[61, 113]	694	[184, 496]	994	[845, 811]
95	[175, 176]	395	[597, 565]	695	[880, 203]	995	[1105, 534]
96	[177, 58]	396	[335, 543]	696	[881, 875]	996	[1106, 472]
97	[47, 178]	397	[598, 599]	697	[882, 789]	997	[500, 1107]
98	[53, 179]	398	[600, 601]	698	[883, 177]	998	[836, 463]
99	[180, 33]	399	[582, 602]	699	[425, 884]	999	[853, 369]
100	[181, 54]	400	[603, 604]	700	[179, 130]	1000	[833, 133]
101	[182, 183]	401	[605, 606]	701	[885, 462]	1001	[195, 481]
102	[184, 185]	402	[607, 608]	702	[38, 42]	1002	[1108, 1107]
103	[186, 187]	403	[609, 610]	703	[886, 382]	1003	[1109, 175]
104	[176, 188]	404	[611, 612]	704	[878, 474]	1004	[1083, 427]
105	[189, 190]	405	[613, 614]	705	[427, 141]	1005	[1110, 851]
106	[177, 33]	406	[228, 615]	706	[887, 127]	1006	[508, 119]
107	[191, 192]	407	[578, 616]	707	[888, 166]	1007	[1111, 816]
108	[193, 194]	408	[617, 618]	708	[149, 889]	1008	[1112, 382]
109	[195, 196]	409	[6, 25]	709	[381, 120]	1009	[1113, 403]
110	[197, 198]	410	[619, 620]	710	[890, 463]	1010	[520, 433]
111	[199, 200]	411	[621, 317]	711	[843, 184]	1011	[837, 516]
112	[201, 202]	412	[622, 71]	712	[891, 13]	1012	[1114, 796]
113	[49, 203]	413	[623, 624]	713	[892, 893]	1013	[537, 820]
114	[176, 204]	414	[625, 626]	714	[894, 438]	1014	[802, 842]
115	[205, 206]	415	[307, 627]	715	[895, 438]	1015	[1115, 515]
116	[207, 208]	416	[628, 629]	716	[497, 792]	1016	[1116, 536]
117	[209, 210]	417	[630, 631]	717	[896, 201]	1017	[366, 801]
118	[211, 212]	418	[632, 633]	718	[159, 123]	1018	[1088, 38]
119	[213, 214]	419	[634, 635]	719	[428, 378]	1019	[1080, 407]
120	[215, 216]	420	[636, 637]	720	[897, 799]	1020	[920, 410]
121	[217, 81]	421	[305, 87]	721	[842, 800]	1021	[1117, 34]
122	[218, 219]	422	[638, 639]	722	[898, 456]	1022	[1118, 792]
123	[220, 221]	423	[640, 641]	723	[864, 836]	1023	[882, 153]
124	[222, 223]	424	[615, 642]	724	[136, 37]	1024	[1119, 851]
125	[224, 66]	425	[643, 622]	725	[413, 35]	1025	[869, 828]
126	[225, 226]	426	[644, 645]	726	[408, 392]	1026	[834, 145]
127	[227, 228]	427	[646, 545]	727	[524, 363]	1027	[455, 198]
128	[229, 230]	428	[637, 308]	728	[899, 155]	1028	[806, 893]
129	[110, 217]	429	[629, 647]	729	[56, 140]	1029	[153, 405]
130	[102, 231]	430	[648, 649]	730	[380, 37]	1030	[1120, 198]
131	[232, 90]	431	[594, 224]	731	[900, 407]	1031	[172, 795]
132	[233, 234]	432	[650, 651]	732	[892, 194]	1032	[1121, 125]

133	[235, 236]	433	[652, 653]	733	[51, 188]	1033	[1122, 391]
134	[237, 238]	434	[654, 655]	734	[901, 147]	1034	[1123, 201]
135	[239, 240]	435	[656, 657]	735	[820, 391]	1035	[1124, 799]
136	[241, 242]	436	[658, 659]	736	[507, 476]	1036	[1125, 150]
137	[243, 244]	437	[660, 278]	737	[493, 796]	1037	[885, 836]
138	[245, 246]	438	[208, 661]	738	[902, 467]	1038	[1126, 44]
139	[247, 248]	439	[555, 662]	739	[376, 120]	1039	[815, 393]
140	[4, 249]	440	[210, 663]	740	[903, 816]	1040	[1127, 377]
141	[250, 251]	441	[664, 564]	741	[904, 363]	1041	[1111, 516]
142	[252, 253]	442	[665, 666]	742	[905, 2]	1042	[1090, 1100]
143	[254, 255]	443	[568, 667]	743	[154, 406]	1043	[914, 825]
144	[256, 257]	444	[668, 669]	744	[149, 813]	1044	[118, 509]
145	[67, 258]	445	[670, 671]	745	[906, 820]	1045	[803, 484]
146	[259, 260]	446	[672, 209]	746	[907, 408]	1046	[830, 859]
147	[261, 262]	447	[325, 634]	747	[168, 875]	1047	[441, 376]
148	[263, 264]	448	[673, 674]	748	[908, 153]	1048	[817, 163]
149	[221, 265]	449	[230, 675]	749	[57, 416]	1049	[1125, 179]
150	[266, 267]	450	[676, 677]	750	[909, 910]	1050	[1091, 49]
151	[268, 213]	451	[678, 679]	751	[911, 398]	1051	[1074, 536]
152	[269, 270]	452	[680, 89]	752	[912, 156]	1052	[473, 879]
153	[271, 272]	453	[601, 604]	753	[895, 913]	1053	[1098, 399]
154	[273, 227]	454	[681, 605]	754	[865, 367]	1054	[1112, 161]
155	[274, 275]	455	[682, 683]	755	[146, 190]	1055	[863, 154]
156	[276, 277]	456	[684, 583]	756	[897, 425]	1056	[48, 421]
157	[278, 279]	457	[685, 261]	757	[478, 786]	1057	[1128, 150]
158	[280, 281]	458	[279, 686]	758	[914, 117]	1058	[1119, 449]
159	[282, 283]	459	[687, 688]	759	[909, 418]	1059	[1129, 799]
160	[82, 284]	460	[633, 311]	760	[915, 144]	1060	[1130, 445]
161	[285, 286]	461	[689, 552]	761	[163, 155]	1061	[1075, 54]
162	[287, 288]	462	[690, 691]	762	[916, 38]	1062	[1131, 913]
163	[265, 289]	463	[692, 693]	763	[917, 501]	1063	[135, 361]
164	[290, 291]	464	[694, 6]	764	[9, 458]	1064	[857, 117]
165	[267, 292]	465	[695, 696]	765	[904, 172]	1065	[199, 359]
166	[293, 294]	466	[697, 687]	766	[166, 150]	1066	[525, 34]
167	[295, 296]	467	[657, 646]	767	[865, 801]	1067	[519, 201]
168	[297, 298]	468	[698, 95]	768	[46, 796]	1068	[1084, 45]
169	[299, 300]	469	[699, 700]	769	[452, 918]	1069	[1132, 469]
170	[301, 302]	470	[701, 702]	770	[812, 889]	1070	[872, 361]
171	[303, 304]	471	[703, 704]	771	[488, 822]	1071	[860, 382]
172	[112, 305]	472	[353, 705]	772	[785, 134]	1072	[912, 401]
173	[306, 307]	473	[706, 707]	773	[523, 404]	1073	[1102, 14]
174	[18, 308]	474	[348, 708]	774	[919, 484]	1074	[1133, 1134]
175	[309, 310]	475	[539, 709]	775	[143, 511]	1075	[1135, 760]
176	[311, 312]	476	[669, 252]	776	[480, 196]	1076	[700, 998]

177	[313, 314]	477	[710, 711]	777	[415, 796]	1077	[1136, 280]
178	[90, 315]	478	[712, 682]	778	[192, 169]	1078	[1137, 591]
179	[100, 316]	479	[713, 276]	779	[134, 141]	1079	[84, 601]
180	[317, 113]	480	[714, 715]	780	[197, 456]	1080	[1138, 769]
181	[318, 319]	481	[716, 332]	781	[821, 489]	1081	[933, 285]
182	[320, 321]	482	[717, 718]	782	[376, 443]	1082	[1139, 347]
183	[322, 323]	483	[719, 720]	783	[920, 431]	1083	[1140, 250]
184	[324, 325]	484	[344, 721]	784	[921, 215]	1084	[704, 975]
185	[326, 327]	485	[722, 712]	785	[922, 923]	1085	[951, 781]
186	[328, 329]	486	[610, 619]	786	[924, 684]	1086	[1141, 1142]
187	[330, 331]	487	[723, 681]	787	[292, 661]	1087	[1143, 1144]
188	[332, 333]	488	[724, 84]	788	[925, 926]	1088	[1145, 694]
189	[334, 335]	489	[725, 726]	789	[927, 225]	1089	[1146, 1147]
190	[336, 337]	490	[727, 670]	790	[928, 748]	1090	[1148, 245]
191	[338, 299]	491	[635, 543]	791	[737, 549]	1091	[1149, 85]
192	[223, 339]	492	[728, 729]	792	[929, 930]	1092	[1150, 1065]
193	[340, 341]	493	[730, 17]	793	[931, 333]	1093	[590, 1029]
194	[329, 342]	494	[731, 732]	794	[932, 933]	1094	[1151, 642]
195	[343, 344]	495	[651, 733]	795	[934, 935]	1095	[1152, 1151]
196	[345, 346]	496	[734, 735]	796	[626, 936]	1096	[627, 220]
197	[347, 348]	497	[736, 737]	797	[937, 765]	1097	[1153, 4]
198	[349, 350]	498	[738, 656]	798	[938, 939]	1098	[994, 352]
199	[351, 72]	499	[739, 572]	799	[708, 940]	1099	[1154, 1155]
200	[231, 352]	500	[740, 741]	800	[941, 942]	1100	[1156, 574]
201	[291, 353]	501	[742, 556]	801	[612, 943]	1101	[1157, 1158]
202	[354, 237]	502	[743, 301]	802	[662, 944]	1102	[219, 311]
203	[206, 355]	503	[581, 744]	803	[945, 946]	1103	[1159, 1025]
204	[30, 347]	504	[745, 734]	804	[947, 654]	1104	[73, 757]
205	[356, 357]	505	[746, 747]	805	[948, 949]	1105	[666, 619]
206	[358, 359]	506	[748, 749]	806	[950, 951]	1106	[1160, 570]
207	[360, 361]	507	[750, 632]	807	[952, 953]	1107	[1161, 1162]
208	[362, 363]	508	[751, 256]	808	[954, 955]	1108	[1163, 1156]
209	[364, 365]	509	[602, 349]	809	[277, 956]	1109	[1164, 930]
210	[366, 367]	510	[752, 652]	810	[957, 79]	1110	[1165, 1166]
211	[368, 369]	511	[753, 754]	811	[781, 770]	1111	[718, 708]
212	[370, 371]	512	[755, 750]	812	[240, 958]	1112	[1167, 579]
213	[178, 372]	513	[756, 757]	813	[959, 690]	1113	[1168, 1169]
214	[203, 45]	514	[758, 759]	814	[935, 252]	1114	[1170, 292]
215	[373, 374]	515	[569, 694]	815	[960, 961]	1115	[333, 553]
216	[375, 376]	516	[691, 21]	816	[106, 962]	1116	[1171, 222]
217	[377, 378]	517	[760, 761]	817	[963, 290]	1117	[1052, 549]
218	[379, 204]	518	[94, 25]	818	[964, 959]	1118	[930, 563]
219	[380, 137]	519	[762, 701]	819	[965, 966]	1119	[1172, 756]
220	[375, 381]	520	[763, 764]	820	[967, 779]	1120	[564, 653]

221	[382, 383]	521	[765, 766]	821	[707, 573]	1121	[1173, 64]
222	[384, 385]	522	[262, 611]	822	[696, 929]	1122	[550, 327]
223	[179, 386]	523	[236, 767]	823	[69, 16]	1123	[1174, 354]
224	[154, 126]	524	[768, 723]	824	[968, 969]	1124	[1175, 1176]
225	[387, 388]	525	[769, 770]	825	[970, 971]	1125	[966, 350]
226	[389, 390]	526	[771, 772]	826	[972, 973]	1126	[286, 679]
227	[391, 392]	527	[527, 527]	827	[974, 927]	1127	[620, 629]
228	[393, 394]	528	[257, 773]	828	[553, 975]	1128	[1177, 583]
229	[395, 396]	529	[774, 324]	829	[936, 215]	1129	[1178, 644]
230	[397, 398]	530	[775, 776]	830	[976, 638]	1130	[288, 778]
231	[183, 376]	531	[244, 777]	831	[595, 232]	1131	[1179, 295]
232	[169, 399]	532	[22, 736]	832	[977, 978]	1132	[1180, 1181]
233	[400, 401]	533	[108, 778]	833	[979, 980]	1133	[793, 786]
234	[402, 149]	534	[779, 780]	834	[981, 982]	1134	[883, 125]
235	[40, 403]	535	[749, 781]	835	[649, 739]	1135	[1182, 163]
236	[404, 366]	536	[571, 782]	836	[940, 983]	1136	[1183, 462]
237	[405, 406]	537	[783, 617]	837	[984, 643]	1137	[1184, 164]
238	[407, 408]	538	[784, 381]	838	[985, 228]	1138	[896, 469]
239	[173, 409]	539	[785, 427]	839	[986, 603]	1139	[1078, 478]
240	[410, 411]	540	[786, 455]	840	[987, 698]	1140	[1129, 425]
241	[412, 169]	541	[787, 43]	841	[988, 83]	1141	[1110, 449]
242	[413, 373]	542	[788, 466]	842	[663, 989]	1142	[417, 910]
243	[414, 415]	543	[789, 405]	843	[990, 991]	1143	[1185, 893]
244	[140, 416]	544	[790, 791]	844	[281, 716]	1144	[808, 447]
245	[417, 418]	545	[192, 503]	845	[992, 664]	1145	[888, 53]
246	[126, 419]	546	[459, 792]	846	[993, 994]	1146	[1186, 147]
247	[420, 421]	547	[793, 390]	847	[995, 5]	1147	[858, 831]
248	[132, 422]	548	[794, 454]	848	[782, 106]	1148	[919, 804]
249	[8, 372]	549	[363, 795]	849	[996, 567]	1149	[1187, 161]
250	[423, 424]	550	[796, 8]	850	[997, 998]	1150	[886, 161]
251	[425, 426]	551	[797, 144]	851	[999, 941]	1151	[819, 409]
252	[427, 135]	552	[798, 799]	852	[1000, 1001]	1152	[1124, 425]
253	[386, 165]	553	[415, 47]	853	[1002, 68]	1153	[797, 511]
254	[428, 429]	554	[523, 800]	854	[1003, 1004]	1154	[917, 1107]
255	[430, 431]	555	[801, 439]	855	[1005, 1006]	1155	[446, 809]
256	[432, 433]	556	[802, 523]	856	[350, 80]	1156	[1132, 201]
257	[367, 434]	557	[528, 527]	857	[1007, 1008]	1157	[1186, 190]
258	[58, 34]	558	[803, 804]	858	[1009, 1010]	1158	[186, 823]
259	[435, 45]	559	[805, 187]	859	[104, 1011]	1159	[866, 431]
260	[436, 431]	560	[493, 47]	860	[1012, 1013]	1160	[818, 812]
261	[437, 438]	561	[806, 194]	861	[943, 1014]	1161	[1101, 877]
262	[439, 440]	562	[807, 467]	862	[1015, 101]	1162	[434, 440]
263	[441, 381]	563	[183, 381]	863	[542, 635]	1163	[1105, 392]
264	[442, 177]	564	[32, 394]	864	[1016, 984]	1164	[492, 415]

265	[381, 443]	565	[808, 809]	865	[767, 1017]	1165	[903, 516]
266	[444, 387]	566	[497, 460]	866	[1018, 1019]	1166	[148, 812]
267	[445, 429]	567	[810, 484]	867	[1020, 239]	1167	[1188, 489]
268	[446, 447]	568	[157, 451]	868	[1021, 600]	1168	[1185, 194]
269	[448, 449]	569	[503, 170]	869	[1022, 1023]	1169	[457, 10]
270	[125, 33]	570	[790, 811]	870	[667, 657]	1170	[908, 789]
271	[450, 375]	571	[812, 813]	871	[1024, 596]	1171	[1128, 179]
272	[167, 451]	572	[499, 445]	872	[1025, 284]	1172	[1117, 59]
273	[31, 393]	573	[373, 36]	873	[958, 588]	1173	[1116, 12]
274	[452, 453]	574	[506, 14]	874	[272, 560]	1174	[838, 405]
275	[382, 162]	575	[814, 141]	875	[1026, 1027]	1175	[1114, 47]
276	[454, 155]	576	[815, 32]	876	[1028, 1026]	1176	[839, 385]
277	[455, 456]	577	[463, 816]	877	[978, 1029]	1177	[465, 821]
278	[457, 458]	578	[38, 531]	878	[1030, 931]	1178	[1188, 822]
279	[459, 460]	579	[817, 454]	879	[766, 1031]	1179	[1108, 501]
280	[461, 422]	580	[818, 149]	880	[645, 281]	1180	[1109, 14]
281	[462, 463]	581	[819, 174]	881	[1032, 269]	1181	[471, 499]
282	[464, 372]	582	[820, 408]	882	[1033, 273]	1182	[1189, 274]
283	[465, 466]	583	[138, 517]	883	[1034, 111]	1183	[1190, 692]
284	[467, 468]	584	[821, 822]	884	[939, 967]	1184	[540, 276]
285	[469, 470]	585	[805, 823]	885	[1035, 544]	1185	[1191, 1192]
286	[405, 126]	586	[42, 532]	886	[1036, 1037]	1186	[1193, 1194]
287	[471, 472]	587	[824, 825]	887	[294, 610]	1187	[1195, 287]
288	[390, 197]	588	[499, 377]	888	[1038, 1039]	1188	[641, 736]
289	[49, 44]	589	[788, 821]	889	[608, 743]	1189	[1187, 382]
290	[473, 474]	590	[495, 185]	890	[97, 761]	1190	[1123, 469]
291	[467, 475]	591	[826, 507]	891	[1040, 309]	1191	[1094, 534]
292	[387, 476]	592	[164, 156]	892	[1041, 562]	1192	[798, 425]
293	[477, 478]	593	[827, 828]	893	[1042, 554]	1193	[400, 156]
294	[151, 479]	594	[799, 426]	894	[1043, 1044]	1194	[362, 172]
295	[480, 481]	595	[8, 829]	895	[1045, 1046]	1195	[1082, 390]
296	[386, 131]	596	[830, 831]	896	[1047, 1048]		
297	[482, 134]	597	[832, 422]	897	[1049, 1050]		
298	[483, 484]	598	[833, 422]	898	[323, 766]		
299	[485, 486]	599	[834, 129]	899	[702, 270]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	200	1	400	0	600	0	800	1	1000	0
1	0	201	1	401	1	601	1	801	0	1001	1
2	0	202	0	402	0	602	1	802	1	1002	1
3	0	203	0	403	0	603	0	803	0	1003	1
4	0	204	1	404	0	604	0	804	1	1004	0
5	0	205	0	405	1	605	1	805	0	1005	0

6	0	206	0	406	1	606	0	806	1	1006	1
7	0	207	1	407	1	607	0	807	1	1007	1
8	1	208	1	408	0	608	1	808	0	1008	0
9	0	209	1	409	0	609	0	809	0	1009	0
10	1	210	0	410	0	610	0	810	1	1010	0
11	0	211	0	411	1	611	0	811	1	1011	1
12	1	212	1	412	1	612	0	812	0	1012	1
13	0	213	0	413	1	613	1	813	0	1013	0
14	0	214	0	414	0	614	1	814	0	1014	1
15	0	215	0	415	1	615	1	815	1	1015	1
16	0	216	0	416	1	616	1	816	1	1016	0
17	1	217	1	417	0	617	0	817	0	1017	0
18	1	218	0	418	1	618	1	818	0	1018	1
19	0	219	0	419	0	619	0	819	0	1019	1
20	1	220	0	420	1	620	0	820	0	1020	0
21	1	221	1	421	0	621	1	821	1	1021	1
22	1	222	0	422	0	622	1	822	0	1022	0
23	0	223	1	423	0	623	1	823	1	1023	0
24	1	224	0	424	1	624	1	824	0	1024	1
25	1	225	0	425	0	625	0	825	0	1025	0
26	0	226	1	426	1	626	0	826	1	1026	0
27	0	227	1	427	0	627	1	827	1	1027	0
28	1	228	1	428	0	628	1	828	1	1028	1
29	0	229	1	429	1	629	1	829	1	1029	1
30	1	230	0	430	0	630	0	830	1	1030	0
31	0	231	1	431	1	631	1	831	1	1031	1
32	0	232	1	432	1	632	1	832	1	1032	0
33	0	233	0	433	1	633	1	833	0	1033	0
34	0	234	0	434	1	634	0	834	0	1034	0
35	1	235	1	435	1	635	1	835	0	1035	1
36	0	236	0	436	1	636	1	836	1	1036	1
37	1	237	1	437	1	637	0	837	1	1037	1
38	1	238	1	438	1	638	0	838	0	1038	1
39	0	239	1	439	0	639	0	839	1	1039	1
40	1	240	0	440	0	640	0	840	0	1040	0
41	1	241	1	441	1	641	0	841	0	1041	1
42	1	242	1	442	1	642	0	842	1	1042	1
43	0	243	0	443	1	643	0	843	1	1043	0
44	1	244	0	444	0	644	1	844	0	1044	0
45	1	245	0	445	0	645	0	845	0	1045	0
46	0	246	0	446	1	646	0	846	1	1046	1
47	1	247	1	447	1	647	0	847	1	1047	1
48	1	248	0	448	0	648	0	848	0	1048	0
49	1	249	1	449	0	649	0	849	0	1049	1

50	1	250	0	450	1	650	1	850	0	1050	0
51	1	251	0	451	0	651	0	851	1	1051	1
52	0	252	0	452	0	652	1	852	0	1052	1
53	0	253	0	453	1	653	1	853	1	1053	0
54	1	254	0	454	0	654	1	854	1	1054	0
55	1	255	0	455	0	655	1	855	0	1055	1
56	0	256	1	456	1	656	1	856	0	1056	1
57	1	257	1	457	1	657	1	857	1	1057	0
58	1	258	1	458	0	658	1	858	0	1058	1
59	1	259	1	459	0	659	0	859	0	1059	0
60	0	260	1	460	1	660	1	860	1	1060	1
61	1	261	1	461	1	661	0	861	1	1061	0
62	0	262	0	462	0	662	1	862	1	1062	1
63	0	263	1	463	0	663	1	863	1	1063	1
64	0	264	1	464	1	664	1	864	0	1064	1
65	1	265	1	465	0	665	1	865	1	1065	1
66	0	266	0	466	0	666	1	866	1	1066	0
67	0	267	0	467	1	667	0	867	0	1067	0
68	1	268	1	468	0	668	0	868	1	1068	0
69	1	269	0	469	0	669	1	869	0	1069	1
70	0	270	0	470	1	670	0	870	0	1070	0
71	1	271	1	471	1	671	0	871	1	1071	1
72	1	272	0	472	1	672	1	872	0	1072	1
73	0	273	0	473	1	673	0	873	1	1073	0
74	1	274	0	474	0	674	0	874	0	1074	1
75	0	275	1	475	1	675	1	875	0	1075	0
76	0	276	0	476	1	676	1	876	1	1076	1
77	1	277	0	477	1	677	0	877	1	1077	0
78	1	278	1	478	0	678	0	878	0	1078	0
79	1	279	0	479	1	679	0	879	1	1079	1
80	0	280	1	480	0	680	0	880	0	1080	1
81	1	281	0	481	1	681	0	881	0	1081	1
82	1	282	1	482	1	682	0	882	0	1082	0
83	0	283	0	483	0	683	0	883	0	1083	0
84	1	284	1	484	0	684	1	884	0	1084	0
85	1	285	0	485	1	685	1	885	1	1085	0
86	0	286	1	486	0	686	1	886	1	1086	0
87	1	287	1	487	0	687	0	887	1	1087	0
88	1	288	1	488	1	688	0	888	1	1088	1
89	0	289	1	489	1	689	1	889	1	1089	0
90	0	290	1	490	0	690	0	890	1	1090	1
91	1	291	1	491	1	691	0	891	0	1091	0
92	1	292	0	492	1	692	0	892	1	1092	1
93	1	293	1	493	0	693	0	893	1	1093	0

94	1	294	1	494	1	694	1	894	0	1094	0
95	1	295	0	495	0	695	0	895	0	1095	1
96	1	296	0	496	0	696	0	896	1	1096	1
97	1	297	1	497	1	697	0	897	1	1097	1
98	0	298	0	498	1	698	0	898	1	1098	0
99	0	299	1	499	0	699	0	899	1	1099	0
100	1	300	1	500	0	700	1	900	1	1100	1
101	0	301	1	501	1	701	1	901	0	1101	0
102	1	302	1	502	0	702	1	902	1	1102	0
103	1	303	1	503	0	703	1	903	0	1103	1
104	0	304	0	504	0	704	0	904	0	1104	0
105	1	305	0	505	1	705	0	905	1	1105	1
106	1	306	1	506	1	706	1	906	0	1106	0
107	0	307	1	507	1	707	1	907	1	1107	0
108	0	308	1	508	1	708	1	908	1	1108	1
109	1	309	1	509	1	709	1	909	1	1109	1
110	1	310	1	510	1	710	1	910	0	1110	0
111	1	311	0	511	0	711	1	911	1	1111	1
112	1	312	0	512	1	712	0	912	1	1112	0
113	1	313	1	513	1	713	1	913	0	1113	0
114	0	314	0	514	0	714	0	914	0	1114	1
115	0	315	1	515	0	715	0	915	0	1115	1
116	1	316	0	516	0	716	1	916	0	1116	0
117	1	317	0	517	0	717	1	917	0	1117	1
118	0	318	1	518	1	718	1	918	0	1118	0
119	0	319	0	519	0	719	0	919	1	1119	1
120	0	320	0	520	0	720	1	920	0	1120	0
121	1	321	0	521	0	721	1	921	0	1121	0
122	0	322	1	522	0	722	1	922	1	1122	0
123	0	323	1	523	0	723	0	923	0	1123	0
124	0	324	1	524	0	724	1	924	0	1124	1
125	0	325	1	525	0	725	1	925	1	1125	1
126	0	326	1	526	1	726	0	926	0	1126	1
127	1	327	0	527	0	727	0	927	0	1127	0
128	1	328	1	528	1	728	1	928	1	1128	0
129	1	329	0	529	1	729	0	929	0	1129	0
130	1	330	0	530	0	730	0	930	0	1130	1
131	1	331	0	531	0	731	1	931	1	1131	1
132	0	332	0	532	1	732	1	932	1	1132	1
133	1	333	1	533	0	733	1	933	1	1133	1
134	1	334	1	534	1	734	0	934	1	1134	0
135	1	335	0	535	1	735	0	935	0	1135	0
136	1	336	0	536	0	736	1	936	1	1136	0
137	0	337	0	537	0	737	0	937	1	1137	0

138	0	338	0	538	0	738	1	938	0	1138	1
139	1	339	0	539	1	739	0	939	0	1139	0
140	0	340	0	540	0	740	0	940	1	1140	0
141	0	341	1	541	0	741	0	941	1	1141	0
142	0	342	0	542	1	742	1	942	0	1142	0
143	0	343	1	543	0	743	0	943	1	1143	0
144	1	344	0	544	1	744	1	944	0	1144	0
145	0	345	0	545	1	745	0	945	0	1145	1
146	1	346	0	546	0	746	1	946	0	1146	0
147	1	347	1	547	1	747	1	947	1	1147	0
148	1	348	0	548	1	748	1	948	0	1148	1
149	1	349	0	549	0	749	1	949	0	1149	0
150	0	350	0	550	0	750	1	950	1	1150	1
151	1	351	1	551	1	751	1	951	0	1151	0
152	0	352	0	552	0	752	1	952	1	1152	1
153	1	353	1	553	1	753	0	953	0	1153	1
154	0	354	0	554	0	754	1	954	0	1154	0
155	0	355	0	555	0	755	1	955	1	1155	1
156	0	356	0	556	1	756	1	956	0	1156	1
157	1	357	0	557	1	757	0	957	1	1157	0
158	1	358	0	558	0	758	0	958	1	1158	1
159	1	359	0	559	0	759	1	959	0	1159	1
160	1	360	1	560	0	760	0	960	1	1160	0
161	0	361	1	561	1	761	1	961	1	1161	0
162	1	362	1	562	1	762	0	962	1	1162	1
163	1	363	0	563	1	763	0	963	0	1163	1
164	1	364	1	564	0	764	0	964	0	1164	1
165	0	365	0	565	0	765	0	965	0	1165	0
166	1	366	0	566	1	766	1	966	1	1166	1
167	0	367	1	567	1	767	1	967	0	1167	0
168	1	368	0	568	1	768	0	968	0	1168	0
169	1	369	1	569	0	769	0	969	1	1169	1
170	1	370	1	570	1	770	0	970	0	1170	1
171	1	371	1	571	0	771	1	971	0	1171	0
172	1	372	0	572	0	772	1	972	1	1172	1
173	1	373	0	573	0	773	0	973	1	1173	0
174	1	374	1	574	1	774	1	974	1	1174	0
175	1	375	0	575	0	775	0	975	1	1175	1
176	0	376	0	576	1	776	0	976	1	1176	1
177	1	377	1	577	0	777	1	977	1	1177	0
178	0	378	0	578	1	778	1	978	1	1178	0
179	1	379	0	579	0	779	1	979	0	1179	1
180	0	380	0	580	0	780	1	980	1	1180	1
181	1	381	1	581	0	781	1	981	0	1181	1

182	0	382	1	582	0	782	0	982	0	1182	0
183	1	383	0	583	0	783	0	983	0	1183	0
184	1	384	0	584	1	784	0	984	1	1184	0
185	1	385	0	585	0	785	1	985	0	1185	0
186	1	386	0	586	1	786	0	986	1	1186	0
187	0	387	0	587	0	787	0	987	0	1187	0
188	0	388	0	588	0	788	1	988	0	1188	0
189	1	389	1	589	1	789	0	989	1	1189	0
190	0	390	1	590	0	790	1	990	1	1190	0
191	0	391	1	591	1	791	0	991	0	1191	0
192	1	392	0	592	1	792	0	992	0	1192	0
193	0	393	1	593	1	793	1	993	1	1193	0
194	0	394	1	594	1	794	1	994	0	1194	1
195	1	395	1	595	1	795	1	995	1	1195	0
196	0	396	0	596	1	796	0	996	0		
197	1	397	0	597	1	797	1	997	0		
198	0	398	0	598	0	798	0	998	1		
199	1	399	0	599	0	799	1	999	1		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: `guoniu.han@unistra.fr`

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: `huyining@hust.edu.cn`