PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^2,z^4+z^3+z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^2,z^4+z^3+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^2,z^4+z^3+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{12}+z^{11}+z^8+z^7+z^4+z^3+z^2+1,\\
p_1(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_2(z)&=z^{24}+z^{14}+z^{12}+z^{10}+z^8+z^4,\\
p_3(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_4(z)&=z^{20}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^7+z^4+z^3+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{16}+z^{11}+z^7+z^3+z^2,\\
p_1(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_2(z)&=z^{24}+z^{14}+z^{12}+z^{10}+z^8+z^4,\\
p_3(z)&=z^{22}+z^{20}+z^{19}+z^{18}+z^{14}+z^{13}+z^9+z^5+z^4+z^2,\\
p_4(z)&=z^{20}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^3+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge0}$ and $(Q_n(z))_{n\ge0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^u M^e[:6u] \\ & + x^{2u}M^e[:5u] + x^{4u}M^e[:3u] + x^{6u}M^e[:u] + x^{2u}M^e[:6u] \\ & + x^{3u}M^e[:5u] + x^{5u}M^e[:3u] + x^{7u}M^e[:u] + x^{4u}M^e[:6u] \\ & + x^{5u}M^e[:5u] + x^{7u}M^e[:3u] + x^{9u}M^e[:u] + x^{5u}M^e[:6u] \\ & + x^{6u}M^e[:5u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] + x^{7u}M^e[:5u] \\ & + x^{8u}M^e[:4u] + x^{9u}M^e[:3u] + x^{10u}M^e[:2u] + x^{11u}M^e[:u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^e,\\
W^e[:12u] &= W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^u W^e[:6u]\\
&+ x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{6u} W^e[:u] + x^{2u} W^e[:6u]\\
&+ x^{3u} W^e[:5u] + x^{5u} W^e[:3u] + x^{7u} W^e[:u] + x^{4u} W^e[:6u]\\
&+ x^{5u} W^e[:5u] + x^{7u} W^e[:3u] + x^{9u} W^e[:u] + x^{5u} W^e[:6u]\\
&+ x^{6u} W^e[:5u] + x^{8u} W^e[:3u] + x^{10u} W^e[:u] + x^{7u} W^e[:5u]\\
&+ x^{8u} W^e[:4u] + x^{9u} W^e[:3u] + x^{10u} W^e[:2u] + x^{11u} W^e[:u]\\
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{5u} M^o[:u] + x^u M^o[:6u]\\
&+ x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{6u} M^o[:u] + x^{2u} M^o[:6u]\\
&+ x^{3u} M^o[:5u] + x^{5u} M^o[:3u] + x^{7u} M^o[:u] + x^{4u} M^o[:6u]\\
&+ x^{5u} M^o[:5u] + x^{7u} M^o[:3u] + x^{9u} M^o[:u] + x^{5u} M^o[:6u]\\
&+ x^{6u} M^o[:5u] + x^{8u} M^o[:3u] + x^{10u} M^o[:u] + x^{7u} M^o[:5u]\\
&+ x^{8u} M^o[:4u] + x^{9u} M^o[:3u] + x^{10u} M^o[:2u] + x^{11u} M^o[:u]\\
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{5u} W^o[:u] + x^u W^o[:6u]\\
&+ x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{6u} W^o[:u] + x^{2u} W^o[:6u]\\
&+ x^{3u} W^o[:5u] + x^{5u} W^o[:3u] + x^{7u} W^o[:u] + x^{4u} W^o[:6u]\\
&+ x^{5u} W^o[:5u] + x^{7u} W^o[:3u] + x^{9u} W^o[:u] + x^{5u} W^o[:6u]\\
&+ x^{6u} W^o[:5u] + x^{8u} W^o[:3u] + x^{10u} W^o[:u] + x^{7u} W^o[:5u]\\
&+ x^{8u} W^o[:4u] + x^{9u} W^o[:3u] + x^{10u} W^o[:2u] + x^{11u} W^o[:u]\\
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] &= T[:6u] + x^u T[:5u] + x^{3u} T[:3u] + x^{5u} T[:u] + x^u T[:6u] + x^{2u} T[:5u]\\
&+ x^{4u} T[:3u] + x^{6u} T[:u] + x^{2u} T[:6u] + x^{3u} T[:5u] + x^{5u} T[:3u]\\
&+ x^{7u} T[:u] + x^{4u} T[:6u] + x^{5u} T[:5u] + x^{7u} T[:3u] + x^{9u} T[:u]\\
&+ x^{5u} T[:6u] + x^{6u} T[:5u] + x^{8u} T[:3u] + x^{10u} T[:u] + x^{7u} T[:5u]\\
&+ x^{8u} T[:4u] + x^{9u} T[:3u] + x^{10u} T[:2u] + x^{11u} T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
0 &= x^{5u} T[u:2u] + x^{3u} T[3u:4u] + x^u T[5u:6u] + T[6u:7u],\\
0 &= x^{7u} T[:u] + x^{6u} T[u:2u] + x^{5u} T[2u:3u] + x^{4u} T[3u:4u]
\end{aligned}
$$

$$\begin{aligned}
&+ x^{3u}T[4u\!:\!5u] + x^{2u}T[5u\!:\!6u] + T[7u\!:\!8u],\\
0 &= x^{6u}T[2u\!:\!3u] + x^{4u}T[4u\!:\!5u] + T[8u\!:\!9u],\\
0 &= x^{6u}T[3u\!:\!4u] + x^{4u}T[5u\!:\!6u] + T[9u\!:\!10u],\\
0 &= x^{9u}T[u\!:\!2u] + x^{7u}T[3u\!:\!4u] + x^{6u}T[4u\!:\!5u] + x^{5u}T[5u\!:\!6u]\\
&\quad + T[10u\!:\!11u],\\
0 &= x^{11u}T[\!:\!u] + x^{10u}T[u\!:\!2u] + x^{9u}T[2u\!:\!3u] + x^{8u}T[3u\!:\!4u]\\
&\quad + x^{7u}T[4u\!:\!5u] + T[11u\!:\!12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[101w]_2] + T[[111w]_2],\end{aligned} \tag{2.8}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[1011w]_2],\end{aligned} \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[101w]_2] + T[[111w]_2],\end{aligned} \tag{2.14}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[1011w]_2],\end{aligned} \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 522 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)), \tag{2.19}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,111)),\end{aligned} \tag{2.20}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1000)), \tag{2.21}$$

$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)), \tag{2.25}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1000)), \tag{2.27}$$

$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{6u} R^e, \\ W_{2k} &= W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^{6u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{5u} M^o[:u] + x^{6u} R^o, \\ W_{2k+1} &= W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{5u} W^o[:u] + x^{6u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} &= \big(W^e[:6v] + x^v W^e[:5v] + x^{3v} W^e[:3v] + x^{5v} W^e[:v] + x^{6u} R^e\big) \times \big(\\ &\qquad M^e[:6v] + x^v M^e[:5v] + x^{3v} M^e[:3v] + x^{5v} M^e[:v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is

$O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{2v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:6v]M^e[:6v] \\ + x^{10v}W^e[:2v]M^e[:2v].$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^7 + x^6, \\ q_1(x) &= x^{22} + x^{20} + x^{19} + x^{15} + x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2, \\ q_2(x) &= x^{20} + x^{16} + x^{14} + x^{12} + x^{10} + 1, \\ q_3(x) &= x^{22} + x^{20} + x^{19} + x^{15} + x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2, \\ q_4(x) &= x^{22} + x^{21} + x^{20} + x^{17} + x^{14} + x^{13} + x^{10} + x^8 + x^7 + x^6 + x^4. \end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned} p_0(x) &= x^{132} + x^{130} + x^{126} + x^{120} + x^{118} + x^{116} + x^{108} + x^{106} + x^{94} + x^{92} \\ &\quad + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{68} + x^{64} + x^{62} + x^{58} \\ &\quad + x^{56} + x^{52} + x^{48} + x^{46} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{24}, \\ p_3(x) &= x^{126} + x^{124} + x^{120} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{98} \\ &\quad + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{70} + x^{64} + x^{58} \\ &\quad + x^{56} + x^{54} + x^{52} + x^{46} + x^{42} + x^{34} + x^{30} + x^{28} + x^{18}, \\ p_6(x) &= x^{126} + x^{122} + x^{116} + x^{112} + x^{108} + x^{100} + x^{94} + x^{92} + x^{90} + x^{88} \\ &\quad + x^{82} + x^{80} + x^{78} + x^{72} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} \\ &\quad + x^{46} + x^{36} + x^{26} + x^{22} + x^{20} + x^{14} + x^{12} + x^{10} + x^{8} + x^{4} + 1, \\ p_9(x) &= x^{128} + x^{124} + x^{120} + x^{118} + x^{110} + x^{104} + x^{100} + x^{92} + x^{90} + x^{86} \\ &\quad + x^{84} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{62} + x^{58} + x^{56} + x^{50} + x^{48} \\ &\quad + x^{40} + x^{38} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{16} + x^{12} + x^{10} + x^{4}, \\ p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{96} + x^{88} + x^{80} + x^{64} + x^{56} + x^{48} + x^{16} + x^{8} + 1 \\ &\quad , \end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{126} + x^{123} + x^{122} + x^{120} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} \\ &\quad + x^{105} + x^{104} + x^{103} + x^{98} + x^{95} + x^{92} + x^{91} + x^{89} + x^{85} + x^{82} + x^{81} \\ &\quad + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} + x^{63} \\ &\quad + x^{61} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} \\ &\quad + x^{43} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30}, \\ p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{112} + x^{109} + x^{108} + x^{102} \\ &\quad + x^{101} + x^{100} + x^{99} + x^{98} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{81} + x^{79} \\ &\quad + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} \end{aligned}$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{46} + x^{43} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{108} + x^{107} + x^{105}\\
&+ x^{103} + x^{101} + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82}\\
&+ x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}\\
&+ x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{38} + x^{37}\\
&+ x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21}\\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{110} + x^{109} + x^{105} + x^{103} + x^{99}\\
&+ x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&+ x^{81} + x^{80} + x^{72} + x^{71} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58}\\
&+ x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{34} + x^{33} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19}\\
&+ x^{18} + x^{17} + x^{15} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100}\\
&+ x^{96} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{52}\\
&+ x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{32} + x^{31}\\
&+ x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8}\\
&+ x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{120} + x^{113} + x^{112} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{105} + x^{104} + x^{103} + x^{98} + x^{95} + x^{92} + x^{91} + x^{89} + x^{85} + x^{82} + x^{81}\\
&+ x^{79} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} + x^{63}\\
&+ x^{61} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45}\\
&+ x^{43} + x^{41} + x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30},\\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{112} + x^{109} + x^{108} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{98} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{81} + x^{79}\\
&+ x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{46} + x^{43} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{108} + x^{107} + x^{105}\\
&+ x^{103} + x^{101} + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{81}+x^{80}+x^{78}+x^{77}+x^{76}+x^{70}+x^{69}+x^{67}+x^{65}+x^{64}+x^{62}\\
&+x^{61}+x^{60}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{38}+x^{37}\\
&+x^{36}+x^{34}+x^{33}+x^{31}+x^{30}+x^{28}+x^{27}+x^{26}+x^{24}+x^{23}+x^{21}\\
&+x^{19}+x^{18}+x^{16}+x^{15}+x^{13}+x^{12},\\
p_9(x)&=x^{118}+x^{117}+x^{116}+x^{115}+x^{114}+x^{110}+x^{109}+x^{105}+x^{103}+x^{99}\\
&+x^{98}+x^{96}+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}+x^{89}+x^{88}+x^{86}+x^{85}\\
&+x^{81}+x^{80}+x^{72}+x^{71}+x^{68}+x^{66}+x^{63}+x^{62}+x^{60}+x^{59}+x^{58}\\
&+x^{57}+x^{56}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{44}+x^{43}+x^{42}+x^{41}\\
&+x^{40}+x^{34}+x^{33}+x^{29}+x^{27}+x^{26}+x^{25}+x^{23}+x^{22}+x^{21}+x^{19}\\
&+x^{18}+x^{17}+x^{15}+x^{11}+x^{9}+x^{8}+x^{7}+x^{6},\\
p_{12}(x)&=x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}+x^{104}+x^{103}+x^{101}+x^{100}\\
&+x^{96}+x^{95}+x^{93}+x^{92}+x^{88}+x^{87}+x^{85}+x^{83}+x^{81}+x^{80}+x^{52}\\
&+x^{51}+x^{49}+x^{47}+x^{45}+x^{44}+x^{40}+x^{39}+x^{37}+x^{36}+x^{32}+x^{31}\\
&+x^{29}+x^{28}+x^{24}+x^{23}+x^{21}+x^{20}+x^{16}+x^{15}+x^{13}+x^{12}+x^{8}\\
&+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1].

For $\phi(M^e,1,1)$:

$$
\begin{aligned}
p_0(x)&=x^{120}+x^{118}+x^{116}+x^{114}+x^{112}+x^{108}+x^{104}+x^{102}+x^{98}+x^{96}\\
&+x^{94}+x^{92}+x^{88}+x^{82}+x^{74}+x^{68}+x^{64}+x^{62}+x^{56}+x^{54}+x^{48}\\
&+x^{42}+x^{36},\\
p_3(x)&=x^{108}+x^{106}+x^{104}+x^{96}+x^{94}+x^{92}+x^{88}+x^{82}+x^{80}+x^{78}+x^{76}\\
&+x^{70}+x^{68}+x^{66}+x^{64}+x^{62}+x^{58}+x^{56}+x^{54}+x^{48}+x^{36}+x^{34}\\
&+x^{26}+x^{16}+x^{14}+x^{12},\\
p_6(x)&=x^{108}+x^{106}+x^{102}+x^{100}+x^{98}+x^{96}+x^{92}+x^{90}+x^{88}+x^{86}+x^{78}\\
&+x^{72}+x^{68}+x^{66}+x^{62}+x^{54}+x^{52}+x^{50}+x^{48}+x^{44}+x^{42}+x^{38}\\
&+x^{36}+x^{34}+x^{32}+x^{22}+x^{20}+x^{18}+x^{14}+x^{12}+x^{10}+x^{4}+x^{2}+1,\\
p_9(x)&=x^{104}+x^{102}+x^{100}+x^{96}+x^{92}+x^{90}+x^{88}+x^{82}+x^{78}+x^{74}+x^{72}\\
&+x^{70}+x^{62}+x^{60}+x^{54}+x^{52}+x^{50}+x^{46}+x^{44}+x^{40}+x^{38}+x^{36}\\
&+x^{30}+x^{24}+x^{20}+x^{18}+x^{6}+x^{4},\\
p_{12}(x)&=x^{104}+x^{102}+x^{98}+x^{96}+x^{88}+x^{86}+x^{82}+x^{80}+x^{40}+x^{38}+x^{34}\\
&+x^{32}+x^{24}+x^{22}+x^{18}+x^{16}+x^{8}+x^{6}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e,0,0)$:

$$
\begin{aligned}
p_0(x)&=x^{120}+x^{118}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}+x^{104}+x^{100}+x^{96}\\
&+x^{78}+x^{76}+x^{70}+x^{68}+x^{60}+x^{58}+x^{56}+x^{50}+x^{48}+x^{38}+x^{34}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{30} + x^{28} + x^{26} + x^{24},\\
p_3(x) &= x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{92} + x^{90} + x^{88} + x^{78} + x^{76} + x^{72}\\
&\quad + x^{62} + x^{58} + x^{48} + x^{46} + x^{44} + x^{28} + x^{26} + x^{22} + x^{18},\\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{76} + x^{70}\\
&\quad + x^{68} + x^{66} + x^{64} + x^{60} + x^{56} + x^{46} + x^{44} + x^{40} + x^{38} + x^{30} + x^{18}\\
&\quad + x^{16} + x^{14} + x^{8} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{90} + x^{88} + x^{82} + x^{78} + x^{74} + x^{72}\\
&\quad + x^{70} + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36}\\
&\quad + x^{30} + x^{24} + x^{20} + x^{18} + x^{6} + x^{4},\\
p_{12}(x) &= x^{104} + x^{102} + x^{98} + x^{96} + x^{88} + x^{86} + x^{82} + x^{80} + x^{40} + x^{38} + x^{34}\\
&\quad + x^{32} + x^{24} + x^{22} + x^{18} + x^{16} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].
For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{119} + x^{117} + x^{116} + x^{115} + x^{113} + x^{110} + x^{106}\\
&\quad + x^{105} + x^{104} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{81} + x^{80}\\
&\quad + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61}\\
&\quad + x^{59} + x^{58} + x^{54} + x^{44} + x^{39} + x^{38} + x^{35} + x^{33} + x^{30},\\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{112} + x^{109} + x^{108} + x^{102}\\
&\quad + x^{101} + x^{100} + x^{99} + x^{98} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{81} + x^{79}\\
&\quad + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61}\\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{46} + x^{43} + x^{41}\\
&\quad + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20}\\
&\quad + x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{108} + x^{107} + x^{105}\\
&\quad + x^{103} + x^{101} + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82}\\
&\quad + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}\\
&\quad + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{38} + x^{37}\\
&\quad + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21}\\
&\quad + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{110} + x^{109} + x^{105} + x^{103} + x^{99}\\
&\quad + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85}\\
&\quad + x^{81} + x^{80} + x^{72} + x^{71} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58}\\
&\quad + x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41}\\
&\quad + x^{40} + x^{34} + x^{33} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19}\\
&\quad + x^{18} + x^{17} + x^{15} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6},
\end{aligned}
$$

$$\begin{aligned}
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100} \\
&\quad + x^{96} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{52} \\
&\quad + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{32} + x^{31} \\
&\quad + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} \\
&\quad + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{119} + x^{117} + x^{116} + x^{115} + x^{113} + x^{110} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{81} + x^{80} \\
&\quad + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} \\
&\quad + x^{59} + x^{58} + x^{54} + x^{44} + x^{39} + x^{38} + x^{35} + x^{33} + x^{30}, \\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{112} + x^{109} + x^{108} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{99} + x^{98} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{81} + x^{79} \\
&\quad + x^{76} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{46} + x^{43} + x^{41} \\
&\quad + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{18}, \\
p_6(x) &= x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{108} + x^{107} + x^{105} \\
&\quad + x^{103} + x^{101} + x^{100} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} \\
&\quad + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12}, \\
p_9(x) &= x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{110} + x^{109} + x^{105} + x^{103} + x^{99} \\
&\quad + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} \\
&\quad + x^{81} + x^{80} + x^{72} + x^{71} + x^{68} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{56} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{34} + x^{33} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} \\
&\quad + x^{18} + x^{17} + x^{15} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100} \\
&\quad + x^{96} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{52} \\
&\quad + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{32} + x^{31} \\
&\quad + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} \\
&\quad + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{122} + x^{118} + x^{114} + x^{112} + x^{104} + x^{98} \\
&\quad + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{38} + x^{36}, \\
p_3(x) &= x^{126} + x^{118} + x^{116} + x^{114} + x^{110} + x^{104} + x^{100} + x^{90} + x^{86} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{64} + x^{54} + x^{46} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12}, \\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{92} \\
&\quad + x^{90} + x^{86} + x^{84} + x^{76} + x^{74} + x^{72} + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} \\
&\quad + x^{38} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{10} \\
&\quad + x^{4} + 1, \\
p_9(x) &= x^{128} + x^{124} + x^{120} + x^{118} + x^{110} + x^{104} + x^{100} + x^{92} + x^{90} + x^{86} \\
&\quad + x^{84} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{62} + x^{58} + x^{56} + x^{50} + x^{48} \\
&\quad + x^{40} + x^{38} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{16} + x^{12} + x^{10} + x^{4}, \\
p_{12}(x) &= x^{128} + x^{120} + x^{112} + x^{96} + x^{88} + x^{80} + x^{64} + x^{56} + x^{48} + x^{16} + x^{8} + 1 \\
&\quad ,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{117} + x^{113} + x^{112} + x^{111} + x^{110} + x^{106} + x^{105} + x^{103} \\
&\quad + x^{100} + x^{99} + x^{87} + x^{86} + x^{82} + x^{79} + x^{76} + x^{73} + x^{70} + x^{69} + x^{67} \\
&\quad + x^{64} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{29} + x^{27} \\
&\quad + x^{25} + x^{24}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{110} + x^{109} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{91} + x^{86} + x^{84} + x^{80} + x^{78} + x^{75} + x^{74} + x^{71} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{45} + x^{44} + x^{42} + x^{36} + x^{34} + x^{32} + x^{28} + x^{27} + x^{26} + x^{21} \\
&\quad + x^{19} + x^{17} + x^{16}, \\
p_6(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{108} + x^{103} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{91} + x^{90} \\
&\quad + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} \\
&\quad + x^{74} + x^{71} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} \\
&\quad + x^{37} + x^{34} + x^{30} + x^{29} + x^{26} + x^{25} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108}\\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{60}\\
&+ x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17}\\
&+ x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{100} + x^{99} + x^{97} + x^{96}\\
&+ x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57}\\
&+ x^{55} + x^{53} + x^{52} + x^{36} + x^{35} + x^{33} + x^{32} + x^{20} + x^{19} + x^{17} + x^{16}\\
&+ x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{128} + x^{124} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{104} + x^{101} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&+ x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{76}\\
&+ x^{75} + x^{72} + x^{68} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{52} + x^{50}\\
&+ x^{47} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{33} + x^{31} + x^{30},\\
p_3(x) &= x^{114} + x^{112} + x^{111} + x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{100} + x^{98}\\
&+ x^{82} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65}\\
&+ x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{43}\\
&+ x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{32} + x^{31} + x^{28} + x^{27} + x^{25} + x^{24}\\
&+ x^{21} + x^{20} + x^{18},\\
p_6(x) &= x^{116} + x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{86} + x^{78}\\
&+ x^{76} + x^{66} + x^{62} + x^{60} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38}\\
&+ x^{36} + x^{30} + x^{24} + x^{22} + x^{18} + x^{12},\\
p_9(x) &= x^{118} + x^{116} + x^{115} + x^{114} + x^{112} + x^{108} + x^{106} + x^{105} + x^{104} + x^{100}\\
&+ x^{99} + x^{98} + x^{96} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{48} + x^{44} + x^{42} + x^{41} + x^{40} + x^{36} + x^{35} + x^{34} + x^{32} + x^{28}\\
&+ x^{26} + x^{25} + x^{24} + x^{20} + x^{19} + x^{18} + x^{16} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{120} + x^{116} + x^{96} + x^{88} + x^{84} + x^{80} + x^{56} + x^{52} + x^{32} + x^{16} + x^{8}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117}\\
&+ x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{108} + x^{107} + x^{105} + x^{103}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{100} + x^{99} + x^{97} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} + x^{80}\\
&+ x^{78} + x^{77} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{57}\\
&+ x^{56} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{39} + x^{36}\\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{30},\\
p_3(x) &= x^{130} + x^{128} + x^{127} + x^{126} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113}\\
&+ x^{112} + x^{110} + x^{109} + x^{107} + x^{102} + x^{101} + x^{100} + x^{96} + x^{95} + x^{94}\\
&+ x^{90} + x^{89} + x^{88} + x^{84} + x^{83} + x^{82} + x^{78} + x^{77} + x^{76} + x^{74} + x^{58}\\
&+ x^{56} + x^{55} + x^{54} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{22} + x^{21} + x^{20}\\
&+ x^{18},\\
p_6(x) &= x^{132} + x^{128} + x^{126} + x^{124} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{102}\\
&+ x^{100} + x^{98} + x^{88} + x^{86} + x^{80} + x^{78} + x^{74} + x^{72} + x^{66} + x^{60} + x^{58}\\
&+ x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{34} + x^{26} + x^{18} + x^{16} + x^{12},\\
p_9(x) &= x^{134} + x^{132} + x^{131} + x^{127} + x^{126} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117}\\
&+ x^{111} + x^{109} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{94} + x^{93} + x^{89}\\
&+ x^{88} + x^{86} + x^{70} + x^{68} + x^{67} + x^{63} + x^{62} + x^{57} + x^{56} + x^{55} + x^{54}\\
&+ x^{53} + x^{47} + x^{45} + x^{39} + x^{37} + x^{31} + x^{29} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{14} + x^{13} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{136} + x^{128} + x^{124} + x^{116} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{80}\\
&+ x^{72} + x^{64} + x^{60} + x^{52} + x^{48} + x^{44} + x^{36} + x^{28} + x^{24} + x^{20} + x^{16} + 1\\
&,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} + x^{106}\\
&+ x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{60} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46}\\
&+ x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{113} + x^{110} + x^{109} + x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100}\\
&+ x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&+ x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{68} + x^{66}\\
&+ x^{65} + x^{62} + x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{50} + x^{48} + x^{45} + x^{44}\\
&+ x^{41} + x^{39} + x^{37} + x^{33} + x^{21} + x^{20},\\
p_6(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{109} + x^{104} + x^{101} + x^{98} + x^{97} + x^{96} + x^{91} + x^{88} + x^{87} + x^{84} + x^{82} \\
&+ x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} \\
&+ x^{63} + x^{62} + x^{59} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} \\
&+ x^{40} + x^{39} + x^{37} + x^{33} + x^{32} + x^{28} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} \\
&+ x^{8} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} \\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{60} \\
&+ x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} \\
&+ x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} \\
&+ x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{100} + x^{99} + x^{97} + x^{96} \\
&+ x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57} \\
&+ x^{55} + x^{53} + x^{52} + x^{36} + x^{35} + x^{33} + x^{32} + x^{20} + x^{19} + x^{17} + x^{16} \\
&+ x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{117} + x^{113} + x^{112} + x^{111} + x^{110} + x^{106} + x^{105} + x^{103} \\
&+ x^{100} + x^{99} + x^{87} + x^{86} + x^{82} + x^{79} + x^{76} + x^{73} + x^{70} + x^{69} + x^{67} \\
&+ x^{64} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} \\
&+ x^{44} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{29} + x^{27} \\
&+ x^{25} + x^{24}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{113} + x^{110} + x^{109} \\
&+ x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} \\
&+ x^{94} + x^{92} + x^{91} + x^{86} + x^{84} + x^{80} + x^{78} + x^{75} + x^{74} + x^{71} + x^{69} \\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} \\
&+ x^{49} + x^{45} + x^{44} + x^{42} + x^{36} + x^{34} + x^{32} + x^{28} + x^{27} + x^{26} + x^{21} \\
&+ x^{19} + x^{17} + x^{16}, \\
p_6(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} \\
&+ x^{111} + x^{108} + x^{103} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{91} + x^{90} \\
&+ x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} \\
&+ x^{74} + x^{71} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} \\
&+ x^{37} + x^{34} + x^{30} + x^{29} + x^{26} + x^{25} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} \\
&+ x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{60}\\
&+ x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17}\\
&+ x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) = {}& x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{100} + x^{99} + x^{97} + x^{96}\\
&+ x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57}\\
&+ x^{55} + x^{53} + x^{52} + x^{36} + x^{35} + x^{33} + x^{32} + x^{20} + x^{19} + x^{17} + x^{16}\\
&+ x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{132} + x^{129} + x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117}\\
&+ x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{108} + x^{107} + x^{105} + x^{103}\\
&+ x^{100} + x^{99} + x^{97} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} + x^{80}\\
&+ x^{78} + x^{77} + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{57}\\
&+ x^{56} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{39} + x^{36}\\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{30},\\
p_3(x) = {}& x^{130} + x^{128} + x^{127} + x^{126} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113}\\
&+ x^{112} + x^{110} + x^{109} + x^{107} + x^{102} + x^{101} + x^{100} + x^{96} + x^{95} + x^{94}\\
&+ x^{90} + x^{89} + x^{88} + x^{84} + x^{83} + x^{82} + x^{78} + x^{77} + x^{76} + x^{74} + x^{58}\\
&+ x^{56} + x^{55} + x^{54} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{27} + x^{22} + x^{21} + x^{20}\\
&+ x^{18},\\
p_6(x) = {}& x^{132} + x^{128} + x^{126} + x^{124} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{102}\\
&+ x^{100} + x^{98} + x^{88} + x^{86} + x^{80} + x^{78} + x^{74} + x^{72} + x^{66} + x^{60} + x^{58}\\
&+ x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{34} + x^{26} + x^{18} + x^{16} + x^{12},\\
p_9(x) = {}& x^{134} + x^{132} + x^{131} + x^{127} + x^{126} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117}\\
&+ x^{111} + x^{109} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{94} + x^{93} + x^{89}\\
&+ x^{88} + x^{86} + x^{70} + x^{68} + x^{67} + x^{63} + x^{62} + x^{57} + x^{56} + x^{55} + x^{54}\\
&+ x^{53} + x^{47} + x^{45} + x^{39} + x^{37} + x^{31} + x^{29} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{14} + x^{13} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{136} + x^{128} + x^{124} + x^{116} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{80}\\
&+ x^{72} + x^{64} + x^{60} + x^{52} + x^{48} + x^{44} + x^{36} + x^{28} + x^{24} + x^{20} + x^{16} + 1\\
&,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o,1,0)$:

$$\begin{aligned}
p_0(x) &= x^{128}+x^{124}+x^{121}+x^{120}+x^{118}+x^{115}+x^{114}+x^{113}+x^{112}+x^{111}\\
&\quad+x^{110}+x^{108}+x^{107}+x^{104}+x^{101}+x^{98}+x^{97}+x^{95}+x^{94}+x^{93}\\
&\quad+x^{92}+x^{91}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{81}+x^{80}+x^{79}+x^{76}\\
&\quad+x^{75}+x^{72}+x^{68}+x^{64}+x^{61}+x^{60}+x^{58}+x^{57}+x^{56}+x^{52}+x^{50}\\
&\quad+x^{47}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}+x^{33}+x^{31}+x^{30},\\
p_3(x) &= x^{114}+x^{112}+x^{111}+x^{108}+x^{107}+x^{105}+x^{104}+x^{101}+x^{100}+x^{98}\\
&\quad+x^{82}+x^{80}+x^{79}+x^{76}+x^{75}+x^{74}+x^{73}+x^{71}+x^{69}+x^{67}+x^{65}\\
&\quad+x^{63}+x^{61}+x^{59}+x^{57}+x^{55}+x^{53}+x^{51}+x^{49}+x^{47}+x^{45}+x^{43}\\
&\quad+x^{42}+x^{41}+x^{40}+x^{37}+x^{36}+x^{32}+x^{31}+x^{28}+x^{27}+x^{25}+x^{24}\\
&\quad+x^{21}+x^{20}+x^{18},\\
p_6(x) &= x^{116}+x^{110}+x^{108}+x^{106}+x^{102}+x^{98}+x^{96}+x^{94}+x^{92}+x^{86}+x^{78}\\
&\quad+x^{76}+x^{66}+x^{62}+x^{60}+x^{54}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}\\
&\quad+x^{36}+x^{30}+x^{24}+x^{22}+x^{18}+x^{12},\\
p_9(x) &= x^{118}+x^{116}+x^{115}+x^{114}+x^{112}+x^{108}+x^{106}+x^{105}+x^{104}+x^{100}\\
&\quad+x^{99}+x^{98}+x^{96}+x^{92}+x^{90}+x^{89}+x^{88}+x^{86}+x^{54}+x^{52}+x^{51}\\
&\quad+x^{50}+x^{48}+x^{44}+x^{42}+x^{41}+x^{40}+x^{36}+x^{35}+x^{34}+x^{32}+x^{28}\\
&\quad+x^{26}+x^{25}+x^{24}+x^{20}+x^{19}+x^{18}+x^{16}+x^{12}+x^{10}+x^{9}+x^{8}+x^{6},\\
p_{12}(x) &= x^{120}+x^{116}+x^{96}+x^{88}+x^{84}+x^{80}+x^{56}+x^{52}+x^{32}+x^{16}+x^{8}\\
&\quad+x^{4}+1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^o,1,1)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}+x^{110}+x^{108}+x^{106}\\
&\quad+x^{103}+x^{102}+x^{101}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{92}+x^{91}+x^{82}\\
&\quad+x^{81}+x^{80}+x^{79}+x^{77}+x^{76}+x^{75}+x^{71}+x^{70}+x^{68}+x^{67}+x^{65}\\
&\quad+x^{64}+x^{63}+x^{62}+x^{60}+x^{57}+x^{55}+x^{53}+x^{52}+x^{51}+x^{48}+x^{46}\\
&\quad+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36},\\
p_3(x) &= x^{124}+x^{123}+x^{122}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}\\
&\quad+x^{113}+x^{110}+x^{109}+x^{108}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}\\
&\quad+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}+x^{91}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}\\
&\quad+x^{83}+x^{82}+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{72}+x^{68}+x^{66}\\
&\quad+x^{65}+x^{62}+x^{61}+x^{58}+x^{57}+x^{54}+x^{53}+x^{50}+x^{48}+x^{45}+x^{44}\\
&\quad+x^{41}+x^{39}+x^{37}+x^{33}+x^{21}+x^{20},\\
p_6(x) &= x^{124}+x^{123}+x^{122}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{113}\\
&\quad+x^{109}+x^{104}+x^{101}+x^{98}+x^{97}+x^{96}+x^{91}+x^{88}+x^{87}+x^{84}+x^{82}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} \\
&+ x^{63} + x^{62} + x^{59} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} \\
&+ x^{40} + x^{39} + x^{37} + x^{33} + x^{32} + x^{28} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} \\
&+ x^{8} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) = {}& x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} \\
&+ x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{60} \\
&+ x^{59} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} \\
&+ x^{37} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} \\
&+ x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) = {}& x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{100} + x^{99} + x^{97} + x^{96} \\
&+ x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{60} + x^{59} + x^{57} \\
&+ x^{55} + x^{53} + x^{52} + x^{36} + x^{35} + x^{33} + x^{32} + x^{20} + x^{19} + x^{17} + x^{16} \\
&+ x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	105	[186, 142]	210	[178, 329]	315	[388, 168]	420	[478, 337]
1	[3, 4]	106	[48, 187]	211	[330, 53]	316	[376, 389]	421	[318, 482]
2	[5, 6]	107	[188, 144]	212	[331, 176]	317	[390, 391]	422	[324, 483]
3	[7, 8]	108	[55, 189]	213	[332, 194]	318	[392, 67]	423	[38, 36]
4	[9, 10]	109	[122, 190]	214	[159, 57]	319	[83, 393]	424	[484, 485]
5	[11, 12]	110	[154, 191]	215	[333, 334]	320	[201, 394]	425	[349, 179]
6	[13, 14]	111	[56, 160]	216	[335, 325]	321	[250, 63]	426	[363, 367]
7	[15, 16]	112	[192, 188]	217	[336, 337]	322	[395, 94]	427	[379, 486]
8	[17, 18]	113	[193, 194]	218	[41, 338]	323	[396, 25]	428	[9, 366]
9	[19, 20]	114	[195, 123]	219	[339, 340]	324	[397, 239]	429	[42, 487]
10	[21, 22]	115	[165, 196]	220	[195, 190]	325	[398, 119]	430	[474, 485]
11	[23, 24]	116	[197, 12]	221	[38, 341]	326	[399, 66]	431	[345, 166]
12	[25, 26]	117	[141, 186]	222	[339, 342]	327	[4, 400]	432	[8, 480]
13	[27, 28]	118	[49, 178]	223	[343, 191]	328	[401, 402]	433	[473, 34]
14	[29, 30]	119	[60, 49]	224	[48, 344]	329	[226, 75]	434	[389, 138]
15	[31, 32]	120	[198, 199]	225	[141, 141]	330	[403, 97]	435	[354, 10]
16	[33, 34]	121	[200, 201]	226	[178, 60]	331	[404, 270]	436	[167, 488]
17	[35, 36]	122	[202, 203]	227	[345, 196]	332	[405, 232]	437	[12, 135]
18	[37, 38]	123	[204, 205]	228	[153, 132]	333	[406, 279]	438	[58, 489]
19	[39, 40]	124	[206, 203]	229	[155, 346]	334	[407, 260]	439	[389, 490]
20	[41, 42]	125	[207, 208]	230	[121, 319]	335	[408, 291]	440	[56, 491]
21	[43, 44]	126	[209, 210]	231	[148, 194]	336	[409, 410]	441	[121, 487]

22	[45, 46]	127	[211, 212]	232	[347, 346]	337	[411, 412]	442	[483, 53]
23	[47, 48]	128	[213, 214]	233	[348, 125]	338	[413, 107]	443	[128, 188]
24	[49, 50]	129	[26, 20]	234	[325, 145]	339	[414, 71]	444	[476, 158]
25	[51, 52]	130	[215, 216]	235	[179, 177]	340	[85, 415]	445	[348, 150]
26	[53, 54]	131	[217, 218]	236	[349, 60]	341	[210, 416]	446	[388, 488]
27	[11, 55]	132	[219, 100]	237	[177, 50]	342	[417, 283]	447	[318, 192]
28	[56, 57]	133	[220, 221]	238	[175, 350]	343	[418, 294]	448	[492, 36]
29	[58, 59]	134	[222, 223]	239	[188, 181]	344	[94, 419]	449	[363, 482]
30	[60, 61]	135	[24, 224]	240	[192, 351]	345	[420, 284]	450	[13, 386]
31	[62, 63]	136	[225, 226]	241	[349, 329]	346	[421, 422]	451	[336, 125]
32	[64, 65]	137	[227, 228]	242	[352, 353]	347	[423, 87]	452	[148, 372]
33	[66, 67]	138	[229, 230]	243	[354, 355]	348	[424, 313]	453	[363, 192]
34	[68, 69]	139	[231, 232]	244	[328, 356]	349	[425, 241]	454	[188, 378]
35	[70, 71]	140	[233, 234]	245	[14, 154]	350	[426, 427]	455	[178, 493]
36	[72, 73]	141	[235, 236]	246	[333, 194]	351	[67, 223]	456	[137, 490]
37	[74, 75]	142	[236, 237]	247	[124, 357]	352	[428, 82]	457	[163, 158]
38	[76, 77]	143	[238, 26]	248	[358, 359]	353	[279, 429]	458	[348, 337]
39	[78, 79]	144	[239, 240]	249	[360, 135]	354	[430, 391]	459	[339, 382]
40	[80, 81]	145	[214, 202]	250	[170, 55]	355	[431, 432]	460	[486, 55]
41	[82, 83]	146	[241, 242]	251	[32, 361]	356	[433, 434]	461	[476, 385]
42	[84, 85]	147	[243, 244]	252	[171, 140]	357	[435, 436]	462	[335, 483]
43	[86, 87]	148	[245, 109]	253	[55, 135]	358	[437, 245]	463	[140, 353]
44	[88, 89]	149	[246, 76]	254	[332, 334]	359	[438, 439]	464	[143, 494]
45	[90, 91]	150	[247, 248]	255	[362, 14]	360	[440, 302]	465	[194, 56]
46	[92, 93]	151	[249, 250]	256	[363, 128]	361	[63, 441]	466	[333, 149]
47	[20, 94]	152	[16, 251]	257	[133, 364]	362	[442, 219]	467	[475, 137]
48	[95, 96]	153	[252, 237]	258	[317, 180]	363	[18, 301]	468	[131, 44]
49	[97, 24]	154	[28, 253]	259	[134, 189]	364	[257, 443]	469	[358, 495]
50	[98, 97]	155	[71, 218]	260	[339, 320]	365	[444, 101]	470	[375, 54]
51	[99, 100]	156	[254, 255]	261	[365, 182]	366	[445, 446]	471	[43, 147]
52	[101, 102]	157	[65, 220]	262	[161, 366]	367	[447, 239]	472	[325, 375]
53	[103, 104]	158	[256, 257]	263	[127, 367]	368	[448, 405]	473	[496, 301]
54	[105, 106]	159	[258, 259]	264	[9, 355]	369	[449, 450]	474	[497, 278]
55	[107, 108]	160	[260, 261]	265	[368, 359]	370	[234, 114]	475	[203, 293]
56	[109, 110]	161	[262, 263]	266	[325, 53]	371	[451, 452]	476	[498, 293]
57	[111, 112]	162	[264, 265]	267	[317, 369]	372	[453, 111]	477	[499, 118]
58	[113, 114]	163	[266, 267]	268	[370, 121]	373	[454, 221]	478	[500, 501]
59	[115, 116]	164	[268, 28]	269	[371, 162]	374	[455, 456]	479	[282, 229]
60	[117, 118]	165	[269, 270]	270	[193, 372]	375	[308, 259]	480	[502, 456]
61	[119, 117]	166	[271, 272]	271	[348, 357]	376	[457, 220]	481	[503, 230]
62	[120, 121]	167	[273, 274]	272	[45, 356]	377	[458, 250]	482	[504, 291]
63	[122, 123]	168	[275, 73]	273	[373, 374]	378	[459, 429]	483	[505, 241]
64	[124, 125]	169	[276, 110]	274	[330, 375]	379	[460, 107]	484	[506, 507]
65	[126, 38]	170	[277, 226]	275	[376, 377]	380	[461, 462]	485	[508, 509]

66	[127, 128]	171	[278, 279]	276	[143, 378]	381	[237, 463]	486	[510, 287]
67	[129, 130]	172	[280, 91]	277	[33, 173]	382	[464, 441]	487	[199, 298]
68	[131, 40]	173	[281, 282]	278	[120, 42]	383	[310, 104]	488	[511, 512]
69	[14, 132]	174	[212, 283]	279	[129, 369]	384	[232, 465]	489	[513, 245]
70	[133, 52]	175	[253, 284]	280	[43, 40]	385	[466, 219]	490	[267, 296]
71	[134, 135]	176	[285, 6]	281	[354, 366]	386	[467, 236]	491	[514, 419]
72	[60, 136]	177	[286, 287]	282	[197, 55]	387	[468, 469]	492	[515, 393]
73	[137, 138]	178	[118, 105]	283	[149, 56]	388	[470, 69]	493	[516, 210]
74	[139, 140]	179	[288, 98]	284	[321, 367]	389	[471, 472]	494	[517, 465]
75	[141, 142]	180	[289, 290]	285	[370, 323]	390	[154, 374]	495	[518, 519]
76	[143, 144]	181	[291, 292]	286	[178, 179]	391	[163, 365]	496	[479, 158]
77	[145, 146]	182	[293, 294]	287	[179, 61]	392	[14, 343]	497	[484, 196]
78	[39, 147]	183	[295, 296]	288	[60, 177]	393	[154, 381]	498	[153, 343]
79	[148, 149]	184	[297, 298]	289	[370, 338]	394	[128, 351]	499	[33, 32]
80	[124, 150]	185	[75, 299]	290	[379, 170]	395	[197, 134]	500	[371, 366]
81	[151, 152]	186	[186, 186]	291	[51, 364]	396	[332, 149]	501	[120, 338]
82	[153, 154]	187	[114, 300]	292	[365, 133]	397	[386, 154]	502	[139, 172]
83	[155, 156]	188	[301, 302]	293	[175, 380]	398	[473, 173]	503	[159, 520]
84	[157, 158]	189	[287, 303]	294	[343, 381]	399	[474, 196]	504	[157, 365]
85	[159, 160]	190	[304, 305]	295	[319, 382]	400	[8, 152]	505	[171, 352]
86	[161, 162]	191	[30, 306]	296	[149, 383]	401	[132, 374]	506	[326, 147]
87	[163, 164]	192	[307, 308]	297	[319, 342]	402	[37, 35]	507	[479, 365]
88	[165, 166]	193	[309, 310]	298	[184, 341]	403	[475, 389]	508	[345, 327]
89	[167, 168]	194	[311, 70]	299	[140, 384]	404	[184, 36]	509	[368, 495]
90	[169, 54]	195	[312, 313]	300	[194, 383]	405	[330, 145]	510	[139, 352]
91	[170, 12]	196	[314, 315]	301	[347, 156]	406	[170, 134]	511	[47, 489]
92	[171, 172]	197	[316, 117]	302	[38, 185]	407	[476, 365]	512	[32, 174]
93	[173, 174]	198	[126, 184]	303	[352, 384]	408	[477, 184]	513	[124, 337]
94	[175, 176]	199	[317, 130]	304	[157, 385]	409	[478, 125]	514	[51, 481]
95	[131, 147]	200	[318, 128]	305	[362, 386]	410	[479, 164]	515	[159, 491]
96	[126, 35]	201	[319, 320]	306	[59, 187]	411	[131, 387]	516	[177, 349]
97	[177, 178]	202	[145, 54]	307	[322, 42]	412	[151, 480]	517	[56, 520]
98	[179, 49]	203	[321, 128]	308	[319, 340]	413	[322, 121]	518	[475, 377]
99	[129, 180]	204	[322, 323]	309	[37, 184]	414	[331, 350]	519	[173, 361]
100	[53, 146]	205	[324, 325]	310	[331, 380]	415	[158, 133]	520	[521, 300]
101	[143, 181]	206	[326, 40]	311	[370, 42]	416	[59, 344]	521	[188, 494]
102	[158, 182]	207	[165, 327]	312	[343, 374]	417	[133, 481]		
103	[133, 183]	208	[328, 46]	313	[193, 149]	418	[51, 183]		
104	[184, 185]	209	[47, 59]	314	[43, 387]	419	[42, 319]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	75	0	150	0	225	0	300	1	375	0	450	0
1	0	76	0	151	1	226	1	301	0	376	1	451	1

2	0	77	1	152	1	227	1	302	0	377	0	452	0
3	0	78	0	153	0	228	0	303	0	378	0	453	0
4	0	79	0	154	1	229	0	304	0	379	0	454	0
5	0	80	0	155	0	230	0	305	0	380	0	455	1
6	0	81	1	156	1	231	0	306	1	381	1	456	1
7	0	82	0	157	0	232	0	307	1	382	0	457	1
8	1	83	0	158	0	233	0	308	0	383	0	458	0
9	0	84	0	159	1	234	1	309	0	384	0	459	0
10	1	85	1	160	0	235	0	310	0	385	1	460	0
11	0	86	1	161	1	236	1	311	1	386	1	461	0
12	1	87	1	162	0	237	1	312	1	387	1	462	1
13	0	88	1	163	1	238	0	313	0	388	0	463	0
14	0	89	0	164	1	239	0	314	1	389	1	464	0
15	0	90	0	165	1	240	1	315	0	390	1	465	1
16	1	91	1	166	0	241	1	316	1	391	1	466	1
17	1	92	0	167	0	242	0	317	1	392	0	467	1
18	0	93	0	168	1	243	0	318	0	393	1	468	1
19	0	94	0	169	0	244	0	319	0	394	1	469	1
20	0	95	1	170	1	245	0	320	0	395	1	470	0
21	1	96	0	171	0	246	1	321	1	396	1	471	1
22	0	97	1	172	1	247	0	322	1	397	1	472	1
23	0	98	0	173	0	248	1	323	1	398	1	473	1
24	1	99	1	174	0	249	1	324	1	399	0	474	0
25	1	100	1	175	0	250	1	325	1	400	1	475	1
26	1	101	0	176	1	251	0	326	0	401	0	476	0
27	0	102	0	177	1	252	0	327	0	402	0	477	1
28	1	103	1	178	1	253	0	328	0	403	1	478	1
29	0	104	0	179	0	254	1	329	1	404	0	479	1
30	0	105	0	180	1	255	0	330	1	405	1	480	0
31	0	106	1	181	1	256	0	331	0	406	1	481	1
32	0	107	0	182	0	257	1	332	1	407	0	482	0
33	1	108	0	183	0	258	1	333	1	408	1	483	0
34	1	109	1	184	0	259	0	334	0	409	1	484	0
35	1	110	1	185	0	260	0	335	1	410	1	485	1
36	0	111	1	186	0	261	0	336	1	411	1	486	0
37	0	112	1	187	1	262	1	337	1	412	1	487	1
38	0	113	0	188	0	263	1	338	1	413	1	488	0
39	0	114	1	189	0	264	0	339	0	414	0	489	0
40	0	115	1	190	0	265	1	340	1	415	0	490	1
41	0	116	1	191	0	266	1	341	1	416	1	491	1
42	0	117	0	192	1	267	1	342	1	417	1	492	1
43	1	118	1	193	0	268	1	343	1	418	1	493	1
44	1	119	0	194	1	269	1	344	0	419	0	494	1
45	0	120	0	195	1	270	0	345	1	420	1	495	1

46	0	121	0	196	1	271	0	346	0	421	0	496	1
47	0	122	1	197	1	272	0	347	0	422	1	497	0
48	1	123	1	198	0	273	0	348	0	423	0	498	0
49	1	124	0	199	1	274	1	349	1	424	0	499	1
50	0	125	1	200	0	275	1	350	0	425	1	500	1
51	1	126	0	201	0	276	0	351	1	426	0	501	0
52	0	127	1	202	1	277	1	352	0	427	0	502	0
53	1	128	1	203	1	278	0	353	1	428	0	503	1
54	0	129	1	204	1	279	1	354	0	429	0	504	0
55	0	130	1	205	1	280	1	355	1	430	0	505	0
56	1	131	1	206	0	281	0	356	1	431	1	506	0
57	1	132	0	207	1	282	1	357	0	432	1	507	1
58	0	133	1	208	0	283	1	358	1	433	1	508	1
59	1	134	0	209	0	284	1	359	0	434	1	509	1
60	0	135	1	210	1	285	1	360	1	435	0	510	0
61	0	136	0	211	1	286	1	361	1	436	0	511	0
62	0	137	1	212	0	287	0	362	0	437	1	512	0
63	1	138	0	213	1	288	0	363	0	438	0	513	0
64	0	139	0	214	1	289	1	364	1	439	1	514	1
65	0	140	0	215	1	290	0	365	0	440	1	515	1
66	1	141	0	216	1	291	1	366	0	441	0	516	1
67	1	142	1	217	1	292	0	367	0	442	0	517	1
68	1	143	0	218	0	293	0	368	1	443	1	518	1
69	0	144	0	219	0	294	1	369	0	444	0	519	0
70	1	145	1	220	1	295	0	370	1	445	0	520	0
71	0	146	1	221	0	296	1	371	1	446	0	521	0
72	0	147	0	222	0	297	0	372	0	447	0		
73	1	148	0	223	1	298	0	373	0	448	1		
74	0	149	1	224	1	299	0	374	1	449	0		

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