PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^5 + z^2, z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^5 + z^2, z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b) = (z^5+z^2, z)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{14}+z^{12}+z^{11}+z^9+z^6+z^5+z^4+z^3+z^2+z+1,\\
p_1(z) &= z^{19}+z^{17}+z^{15}+z^{13}+z^{12}+z^{10}+z^6+z^4+z^3+z^2,\\
p_2(z) &= z^{20}+z^{18}+z^{10}+z^8+z^6+z^4,\\
p_3(z) &= z^{19}+z^{17}+z^{15}+z^{13}+z^{12}+z^{10}+z^6+z^4+z^3+z^2,\\
p_4(z) &= z^{18}+z^{16}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^5+z^4+z^3+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{16}+z^{14}+z^{12}+z^{11}+z^9+z^6+z^5+z^3+z^2+z,\\
p_1(z) &= z^{19}+z^{17}+z^{15}+z^{13}+z^{12}+z^{10}+z^6+z^4+z^3+z^2,\\
p_2(z) &= z^{20}+z^{18}+z^{10}+z^8+z^6+z^4,\\
p_3(z) &= z^{19}+z^{17}+z^{15}+z^{13}+z^{12}+z^{10}+z^6+z^4+z^3+z^2,\\
p_4(z) &= z^{18}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^5+z^3+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 540, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 660, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{5u} M^e[:u] \\ & + x^{2u} M^e[:6u] + x^{4u} M^e[:4u] + x^{5u} M^e[:3u] + x^{6u} M^e[:2u] \\ & + x^{7u} M^e[:u] + x^{3u} M^e[:6u] + x^{5u} M^e[:4u] + x^{6u} M^e[:3u] \\ & + x^{7u} M^e[:2u] + x^{8u} M^e[:u] + x^{5u} M^e[:6u] + x^{7u} M^e[:4u] \\ & + x^{8u} M^e[:3u] + x^{9u} M^e[:2u] + x^{10u} M^e[:u] + x^{6u} M^e[:6u] \\ & + x^{8u} M^e[:4u] + x^{10u} M^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u}) R^e, \end{aligned}$$

$$\begin{aligned}
W^e[:12u] = {} & W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u] \\
& + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u] \\
& + x^{7u}W^e[:u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u] \\
& + x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{5u}W^e[:6u] + x^{7u}W^e[:4u] \\
& + x^{8u}W^e[:3u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{6u}W^e[:6u] \\
& + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] = {} & M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u] \\
& + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u] \\
& + x^{7u}M^o[:u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u] \\
& + x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{5u}M^o[:6u] + x^{7u}M^o[:4u] \\
& + x^{8u}M^o[:3u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{6u}M^o[:6u] \\
& + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^o, \\
W^o[:12u] = {} & W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u] \\
& + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u] \\
& + x^{7u}W^o[:u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u] \\
& + x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{5u}W^o[:6u] + x^{7u}W^o[:4u] \\
& + x^{8u}W^o[:3u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{6u}W^o[:6u] \\
& + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] = {} & T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{5u}T[:u] + x^{2u}T[:6u] \\
& + x^{4u}T[:4u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{3u}T[:6u] \\
& + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{5u}T[:6u] \\
& + x^{7u}T[:4u] + x^{8u}T[:3u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{6u}T[:6u] \\
& + x^{8u}T[:4u] + x^{10u}T[:2u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
0 = {} & x^{6u}T[:u] + x^{5u}T[u:2u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] \\
& + x^{2u}T[4u:5u] + T[6u:7u], \\
0 = {} & x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] \\
& + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u], \\
0 = {} & x^{8u}T[:u] + x^{3u}T[5u:6u] + T[8u:9u], \\
0 = {} & x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u]
\end{aligned}$$

$$\begin{aligned}
&+T[9u:10u],\\
0&=x^{9u}T[u:2u]+x^{7u}T[3u:4u]+x^{6u}T[4u:5u]+x^{5u}T[5u:6u]\\
&+T[10u:11u],\\
0&=x^{10u}T[u:2u]+x^{8u}T[3u:4u]+x^{6u}T[5u:6u]+T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0&=T[[w]_2]+T[[1w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]\\
&+T[[110w]_2], \qquad (2.7)\\
0&=T[[w]_2]+T[[1w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]\\
&+T[[101w]_2]+T[[111w]_2], \qquad (2.8)\\
0&=T[[w]_2]+T[[101w]_2]+T[[1000w]_2], \qquad (2.9)\\
0&=T[[w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]+T[[1001w]_2], \qquad (2.10)\\
0&=T[[1w]_2]+T[[11w]_2]+T[[100w]_2]+T[[101w]_2]+T[[1010w]_2], \qquad (2.11)\\
0&=T[[1w]_2]+T[[11w]_2]+T[[101w]_2]+T[[1011w]_2], \qquad (2.12)
\end{aligned}$$

for all binary word w of length $2k$ and $w\neq 0^{2k}$, and

$$\begin{aligned}
0&=T[[w]_2]+T[[1w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]\\
&+T[[110w]_2], \qquad (2.13)\\
0&=T[[w]_2]+T[[1w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]\\
&+T[[101w]_2]+T[[111w]_2], \qquad (2.14)\\
0&=T[[w]_2]+T[[101w]_2]+T[[1000w]_2], \qquad (2.15)\\
0&=T[[w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]+T[[1001w]_2], \qquad (2.16)\\
0&=T[[1w]_2]+T[[11w]_2]+T[[100w]_2]+T[[101w]_2]+T[[1010w]_2], \qquad (2.17)\\
0&=T[[1w]_2]+T[[11w]_2]+T[[101w]_2]+T[[1011w]_2], \qquad (2.18)
\end{aligned}$$

for $w=0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1139 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k}=\{A(i,w):|w|=2k, w\neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$\begin{aligned}
0&=\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\
&+\tau(A(s,110)), \qquad (2.19)\\
0&=\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\
&+\tau(A(s,101))+\tau(A(s,111)), \qquad (2.20)\\
0&=\tau(A(s,\epsilon))+\tau(A(s,101))+\tau(A(s,1000)), \qquad (2.21)\\
0&=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\
&+\tau(A(s,1001)), \qquad (2.22)\\
0&=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,100))+\tau(A(s,101))
\end{aligned}$$

$$+\tau(A(s,1010)), \tag{2.23}$$

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100)) \\ &\quad +\tau(A(s,110)), \end{aligned} \tag{2.25}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100)) \\ &\quad +\tau(A(s,101))+\tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$0=\tau(A(s,\epsilon))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.27}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100)) \\ &\quad +\tau(A(s,1001)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1))+\tau(A(s,11))+\tau(A(s,100))+\tau(A(s,101)) \\ &\quad +\tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{32}), E_{32}) = (A(i,0^{20}), E_{20})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 16$, which turns out to be true. ☐

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u] \\ &\quad + x^{6u}R^e, \\ W_{2k} &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u] \\ &\quad + x^{6u}R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u] \\ &\quad + x^{6u}R^o, \\ W_{2k+1} &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u] \\ &\quad + x^{6u}R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \big(W^e[:6v] + x^{2v}W^e[:4v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{5v}W^e[:v]$$

$$+ x^{6u}R^e\big) \times \big(M^e[:6v] + x^{2v}M^e[:4v] + x^{3v}M^e[:3v] + x^{4v}M^e[:2v] + x^{5v}M^e[:v] + x^{6u}R^e\big); \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{6v}W^e[:6v]M^e[:6v] + x^{8v}W^e[:4v]M^e[:4v] + x^{10v}W^e[:2v]M^e[:2v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v : (n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v : (n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j,k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^9 + x^8 + x^6,$$

$$\begin{aligned}
q_1(x) &= x^{18} + x^{17} + x^{16} + x^{14} + x^{10} + x^8 + x^7 + x^5 + x^3 + x,\\
q_2(x) &= x^{16} + x^{14} + x^{12} + x^{10} + x^2 + 1,\\
q_3(x) &= x^{18} + x^{17} + x^{16} + x^{14} + x^{10} + x^8 + x^7 + x^5 + x^3 + x,\\
q_4(x) &= x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{11} + x^{10} + x^9 + x^8 + x^6 + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{94} + x^{90} + x^{84} + x^{82} + x^{74} + x^{72} + x^{68} + x^{56} + x^{52} + x^{48}\\
&\quad + x^{44} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{18} + x^{16} + x^{12},\\
p_3(x) &= x^{88} + x^{86} + x^{82} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60}\\
&\quad + x^{56} + x^{52} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18} + x^{10} + x^8,\\
p_6(x) &= x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60}\\
&\quad + x^{54} + x^{52} + x^{50} + x^{40} + x^{32} + x^{26} + x^{24} + x^{18} + x^{16} + x^6 + x^4 + x^2\\
&\quad + 1,\\
p_9(x) &= x^{86} + x^{82} + x^{76} + x^{72} + x^{68} + x^{64} + x^{58} + x^{56} + x^{50} + x^{42} + x^{40}\\
&\quad + x^{36} + x^{32} + x^{26} + x^{22} + x^{14} + x^{10} + x^6 + x^4 + x^2,\\
p_{12}(x) &= x^{86} + x^{80} + x^{70} + x^{64} + x^{54} + x^{48} + x^{22} + x^{16} + x^6 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83}\\
&\quad + x^{81} + x^{79} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{63} + x^{57}\\
&\quad + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40}\\
&\quad + x^{38} + x^{34} + x^{33} + x^{32} + x^{31} + x^{24},\\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85}\\
&\quad + x^{84} + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67}\\
&\quad + x^{63} + x^{62} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45}\\
&\quad + x^{44} + x^{43} + x^{41} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{73} + x^{70} + x^{69} + x^{66} + x^{51} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}\\
&+ x^{13} + x^{10} + x^{9} + x^{6},\\
p_9(x) &= x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80}\\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{62} + x^{59} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40}\\
&+ x^{39} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12} + x^{11} + x^{10} + x^{8} + x^{6} + x^{5} + x^{3},\\
p_{12}(x) &= x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63} + x^{60} + x^{51}\\
&+ x^{48} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83}\\
&+ x^{81} + x^{79} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{63} + x^{57}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40}\\
&+ x^{38} + x^{34} + x^{33} + x^{32} + x^{31} + x^{24},\\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67}\\
&+ x^{63} + x^{62} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{41} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26}\\
&+ x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{73} + x^{70} + x^{69} + x^{66} + x^{51} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}\\
&+ x^{13} + x^{10} + x^{9} + x^{6},\\
p_9(x) &= x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80}\\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{62} + x^{59} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40}\\
&+ x^{39} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12} + x^{11} + x^{10} + x^{8} + x^{6} + x^{5} + x^{3},\\
p_{12}(x) &= x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63} + x^{60} + x^{51}
\end{aligned}
$$

$$+ x^{48} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{112} + x^{110} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92} + x^{86} + x^{82} \\
&\quad + x^{80} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{54} + x^{50} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{36}, \\
p_3(x) &= x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{90} + x^{78} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{54} + x^{50} + x^{46} + x^{38} + x^{30} + x^{26} + x^{24} + x^{18} \\
&\quad + x^{16} + x^{14} + x^{8} + x^{6}, \\
p_6(x) &= x^{108} + x^{98} + x^{96} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} \\
&\quad + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{44} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{26} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{104} + x^{100} + x^{98} + x^{92} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} \\
&\quad + x^{68} + x^{60} + x^{52} + x^{46} + x^{36} + x^{34} + x^{32} + x^{24} + x^{22} + x^{18} + x^{12} \\
&\quad + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{104} + x^{88} + x^{80} + x^{72} + x^{64} + x^{48} + x^{40} + x^{24} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{110} + x^{106} + x^{104} + x^{102} + x^{96} + x^{94} + x^{92} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{50} + x^{38} + x^{30} + x^{26} + x^{24} + x^{20} + x^{12}, \\
p_3(x) &= x^{108} + x^{104} + x^{102} + x^{98} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{76} + x^{64} \\
&\quad + x^{62} + x^{60} + x^{54} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{28} \\
&\quad + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{8}, \\
p_6(x) &= x^{108} + x^{100} + x^{96} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} \\
&\quad + x^{72} + x^{70} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{20} + x^{18} + x^{16} + x^{14} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{104} + x^{100} + x^{98} + x^{92} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} \\
&\quad + x^{68} + x^{60} + x^{52} + x^{46} + x^{36} + x^{34} + x^{32} + x^{24} + x^{22} + x^{18} + x^{12} \\
&\quad + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{104} + x^{88} + x^{80} + x^{72} + x^{64} + x^{48} + x^{40} + x^{24} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{103} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{86} + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{64} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47}
\end{aligned}$$

$$\begin{aligned}
&+ x^{46} + x^{45} + x^{44} + x^{39} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30} + x^{27} + x^{25}\\
&+ x^{24} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67}\\
&+ x^{63} + x^{62} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{41} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26}\\
&+ x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{73} + x^{70} + x^{69} + x^{66} + x^{51} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}\\
&+ x^{13} + x^{10} + x^{9} + x^{6},\\
p_9(x) &= x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80}\\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{62} + x^{59} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40}\\
&+ x^{39} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{12} + x^{11} + x^{10} + x^{8} + x^{6} + x^{5} + x^{3},\\
p_{12}(x) &= x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63} + x^{60} + x^{51}\\
&+ x^{48} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{103} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92}\\
&+ x^{91} + x^{90} + x^{86} + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{69} + x^{68} + x^{67}\\
&+ x^{64} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47}\\
&+ x^{46} + x^{45} + x^{44} + x^{39} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30} + x^{27} + x^{25}\\
&+ x^{24} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67}\\
&+ x^{63} + x^{62} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{47} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{41} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26}\\
&+ x^{24} + x^{23} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{73} + x^{70} + x^{69} + x^{66} + x^{51} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{44} + x^{41} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}
\end{aligned}$$

$$
\begin{aligned}
&+x^{13}+x^{10}+x^{9}+x^{6},\\
p_9(x) &= x^{97}+x^{96}+x^{92}+x^{91}+x^{90}+x^{89}+x^{87}+x^{83}+x^{82}+x^{81}+x^{80}\\
&+x^{78}+x^{76}+x^{74}+x^{72}+x^{71}+x^{69}+x^{67}+x^{66}+x^{62}+x^{59}+x^{55}\\
&+x^{54}+x^{53}+x^{52}+x^{50}+x^{49}+x^{47}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}\\
&+x^{39}+x^{37}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{26}+x^{25}+x^{24}+x^{23}\\
&+x^{20}+x^{19}+x^{18}+x^{16}+x^{15}+x^{12}+x^{11}+x^{10}+x^{8}+x^{6}+x^{5}+x^{3},\\
p_{12}(x) &= x^{95}+x^{92}+x^{83}+x^{80}+x^{79}+x^{76}+x^{67}+x^{64}+x^{63}+x^{60}+x^{51}\\
&+x^{48}+x^{31}+x^{28}+x^{19}+x^{16}+x^{15}+x^{12}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{96}+x^{94}+x^{84}+x^{80}+x^{76}+x^{74}+x^{72}+x^{64}+x^{62}+x^{56}+x^{54}\\
&+x^{52}+x^{50}+x^{40}+x^{38}+x^{36}+x^{30}+x^{28}+x^{24},\\
p_3(x) &= x^{88}+x^{86}+x^{80}+x^{76}+x^{74}+x^{68}+x^{66}+x^{64}+x^{62}+x^{60}+x^{58}\\
&+x^{56}+x^{52}+x^{50}+x^{48}+x^{40}+x^{38}+x^{30}+x^{24}+x^{22}+x^{20}+x^{18}\\
&+x^{12}+x^{10}+x^{8}+x^{6},\\
p_6(x) &= x^{88}+x^{86}+x^{84}+x^{82}+x^{78}+x^{74}+x^{70}+x^{64}+x^{60}+x^{58}+x^{56}\\
&+x^{54}+x^{52}+x^{48}+x^{46}+x^{40}+x^{36}+x^{24}+x^{16}+x^{10}+x^{2}+1,\\
p_9(x) &= x^{86}+x^{82}+x^{76}+x^{72}+x^{68}+x^{64}+x^{58}+x^{56}+x^{50}+x^{42}+x^{40}\\
&+x^{36}+x^{32}+x^{26}+x^{22}+x^{14}+x^{10}+x^{6}+x^{4}+x^{2},\\
p_{12}(x) &= x^{86}+x^{80}+x^{70}+x^{64}+x^{54}+x^{48}+x^{22}+x^{16}+x^{6}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{101}+x^{97}+x^{96}+x^{94}+x^{92}\\
&+x^{88}+x^{87}+x^{86}+x^{85}+x^{83}+x^{76}+x^{75}+x^{70}+x^{66}+x^{65}+x^{60}\\
&+x^{58}+x^{56}+x^{54}+x^{53}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{42}+x^{41}\\
&+x^{39}+x^{38}+x^{35}+x^{34}+x^{31}+x^{27}+x^{25}+x^{23}+x^{21}+x^{19}+x^{17}\\
&+x^{16}+x^{15}+x^{14}+x^{12},\\
p_3(x) &= x^{103}+x^{102}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{92}+x^{89}+x^{88}+x^{78}\\
&+x^{77}+x^{76}+x^{73}+x^{72}+x^{70}+x^{69}+x^{66}+x^{64}+x^{60}+x^{59}+x^{58}\\
&+x^{57}+x^{54}+x^{49}+x^{48}+x^{45}+x^{42}+x^{41}+x^{39}+x^{35}+x^{34}+x^{30}\\
&+x^{29}+x^{28}+x^{27}+x^{24}+x^{22}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}\\
&+x^{10}+x^{8},\\
p_6(x) &= x^{103}+x^{102}+x^{98}+x^{97}+x^{96}+x^{95}+x^{91}+x^{86}+x^{85}+x^{79}+x^{78}\\
&+x^{74}+x^{73}+x^{72}+x^{55}+x^{54}+x^{53}+x^{50}+x^{46}+x^{44}+x^{43}+x^{42}\\
&+x^{40}+x^{38}+x^{36}+x^{35}+x^{34}+x^{32}+x^{29}+x^{28}+x^{24}+x^{19}+x^{18}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{14} + x^{13} + x^{12} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} + 1,\\
p_9(x) &= x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{67} + x^{64} + x^{55} + x^{52} + x^{39}\\
&\quad + x^{36} + x^{35} + x^{32} + x^{19} + x^{16} + x^{7} + x^{4},\\
p_{12}(x) &= x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{79} + x^{76} + x^{63} + x^{60} + x^{55}\\
&\quad + x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{15} + x^{12}\\
&\quad + x^{7} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{115} + x^{112} + x^{111} + x^{104} + x^{101} + x^{100} + x^{98}\\
&\quad + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{88} + x^{86} + x^{83} + x^{81} + x^{80}\\
&\quad + x^{78} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&\quad + x^{62} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{44} + x^{43}\\
&\quad + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{31} + x^{29} + x^{27} + x^{26}\\
&\quad + x^{24},\\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{100} + x^{97} + x^{94} + x^{91} + x^{88} + x^{85} + x^{82} + x^{79}\\
&\quad + x^{76} + x^{73} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{54}\\
&\quad + x^{51} + x^{48} + x^{45} + x^{42} + x^{39} + x^{36} + x^{34} + x^{31} + x^{30} + x^{28} + x^{27}\\
&\quad + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{13} + x^{9},\\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80}\\
&\quad + x^{78} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38}\\
&\quad + x^{34} + x^{28} + x^{24} + x^{20} + x^{10} + x^{6},\\
p_9(x) &= x^{110} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96}\\
&\quad + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79}\\
&\quad + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61}\\
&\quad + x^{59} + x^{58} + x^{55} + x^{51} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{35}\\
&\quad + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{18}\\
&\quad + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{7} + x^{3},\\
p_{12}(x) &= x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{64} + x^{52}\\
&\quad + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88}\\
&\quad + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{73} + x^{71} + x^{70}\\
&\quad + x^{69} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{51} + x^{49}\\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{31}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{29} + x^{25} + x^{24} + x^{23} + x^{22} + x^{18},\\
p_3(x) &= x^{94} + x^{93} + x^{91} + x^{88} + x^{85} + x^{81} + x^{70} + x^{69} + x^{67} + x^{64} + x^{61}\\
&+ x^{57} + x^{54} + x^{53} + x^{51} + x^{48} + x^{45} + x^{41} + x^{30} + x^{29} + x^{27} + x^{24}\\
&+ x^{22} + x^{19} + x^{17} + x^{16} + x^{13} + x^{9},\\
p_6(x) &= x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{70} + x^{66} + x^{48}\\
&+ x^{46} + x^{44} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18}\\
&+ x^{10} + x^{6},\\
p_9(x) &= x^{98} + x^{97} + x^{96} + x^{93} + x^{91} + x^{90} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80}\\
&+ x^{77} + x^{75} + x^{74} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58}\\
&+ x^{55} + x^{51} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27} + x^{26} + x^{23} + x^{19} + x^{18}\\
&+ x^{17} + x^{16} + x^{13} + x^{11} + x^{10} + x^{7} + x^{3},\\
p_{12}(x) &= x^{100} + x^{96} + x^{92} + x^{76} + x^{60} + x^{52} + x^{48} + x^{36} + x^{32} + x^{28} + x^{12}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{99} + x^{96} + x^{95} + x^{94} + x^{92} + x^{86} + x^{83} + x^{82}\\
&+ x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} + x^{63}\\
&+ x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{46} + x^{45} + x^{38} + x^{36}\\
&+ x^{35} + x^{30},\\
p_3(x) &= x^{103} + x^{102} + x^{99} + x^{98} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} + x^{82}\\
&+ x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{71} + x^{68} + x^{64} + x^{61} + x^{60} + x^{59}\\
&+ x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{26} + x^{25} + x^{23} + x^{20} + x^{19} + x^{15} + x^{14} + x^{12},\\
p_6(x) &= x^{103} + x^{102} + x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{83}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{64} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{42}\\
&+ x^{41} + x^{39} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{25} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{20} + x^{19} + x^{15} + x^{14} + x^{11} + x^{10} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{67} + x^{64} + x^{55} + x^{52} + x^{39}\\
&+ x^{36} + x^{35} + x^{32} + x^{19} + x^{16} + x^{7} + x^{4},\\
p_{12}(x) &= x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{79} + x^{76} + x^{63} + x^{60} + x^{55}\\
&+ x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{15} + x^{12}\\
&+ x^{7} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{97} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{76} + x^{75} + x^{70} + x^{66} + x^{65} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{41} \\
&\quad + x^{39} + x^{38} + x^{35} + x^{34} + x^{31} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{17} \\
&\quad + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_3(x) &= x^{103} + x^{102} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{89} + x^{88} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{69} + x^{66} + x^{64} + x^{60} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{54} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{39} + x^{35} + x^{34} + x^{30} \\
&\quad + x^{29} + x^{28} + x^{27} + x^{24} + x^{22} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} \\
&\quad + x^{10} + x^{8}, \\
p_6(x) &= x^{103} + x^{102} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{86} + x^{85} + x^{79} + x^{78} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{55} + x^{54} + x^{53} + x^{50} + x^{46} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{29} + x^{28} + x^{24} + x^{19} + x^{18} \\
&\quad + x^{14} + x^{13} + x^{12} + x^{9} + x^{7} + x^{6} + x^{4} + x^{3} + 1, \\
p_9(x) &= x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{67} + x^{64} + x^{55} + x^{52} + x^{39} \\
&\quad + x^{36} + x^{35} + x^{32} + x^{19} + x^{16} + x^{7} + x^{4}, \\
p_{12}(x) &= x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{79} + x^{76} + x^{63} + x^{60} + x^{55} \\
&\quad + x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{15} + x^{12} \\
&\quad + x^{7} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{57} + x^{55} + x^{51} + x^{49} \\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{31} \\
&\quad + x^{29} + x^{25} + x^{24} + x^{23} + x^{22} + x^{18}, \\
p_3(x) &= x^{94} + x^{93} + x^{91} + x^{88} + x^{85} + x^{81} + x^{70} + x^{69} + x^{67} + x^{64} + x^{61} \\
&\quad + x^{57} + x^{54} + x^{53} + x^{51} + x^{48} + x^{45} + x^{41} + x^{30} + x^{29} + x^{27} + x^{24} \\
&\quad + x^{22} + x^{19} + x^{17} + x^{16} + x^{13} + x^{9}, \\
p_6(x) &= x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{70} + x^{66} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{10} + x^{6}, \\
p_9(x) &= x^{98} + x^{97} + x^{96} + x^{93} + x^{91} + x^{90} + x^{87} + x^{83} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{77} + x^{75} + x^{74} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58}
\end{aligned}
$$

$$+ x^{55} + x^{51} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27} + x^{26} + x^{23} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{11} + x^{10} + x^{7} + x^{3},$$

$$p_{12}(x) = x^{100} + x^{96} + x^{92} + x^{76} + x^{60} + x^{52} + x^{48} + x^{36} + x^{32} + x^{28} + x^{12} + x^{4} + 1,$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {}& x^{120} + x^{118} + x^{116} + x^{115} + x^{112} + x^{111} + x^{104} + x^{101} + x^{100} + x^{98} \\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{88} + x^{86} + x^{83} + x^{81} + x^{80} \\
&+ x^{78} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
&+ x^{62} + x^{61} + x^{60} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{44} + x^{43} \\
&+ x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{31} + x^{29} + x^{27} + x^{26} \\
&+ x^{24},
\end{aligned}$$

$$\begin{aligned}
p_3(x) = {}& x^{106} + x^{105} + x^{103} + x^{100} + x^{97} + x^{94} + x^{91} + x^{88} + x^{85} + x^{82} + x^{79} \\
&+ x^{76} + x^{73} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{54} \\
&+ x^{51} + x^{48} + x^{45} + x^{42} + x^{39} + x^{36} + x^{34} + x^{31} + x^{30} + x^{28} + x^{27} \\
&+ x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{13} + x^{9},
\end{aligned}$$

$$\begin{aligned}
p_6(x) = {}& x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} \\
&+ x^{78} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38} \\
&+ x^{34} + x^{28} + x^{24} + x^{20} + x^{10} + x^{6},
\end{aligned}$$

$$\begin{aligned}
p_9(x) = {}& x^{110} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} \\
&+ x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} \\
&+ x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} \\
&+ x^{59} + x^{58} + x^{55} + x^{51} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{35} \\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{18} \\
&+ x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{7} + x^{3},
\end{aligned}$$

$$\begin{aligned}
p_{12}(x) = {}& x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{64} + x^{52} \\
&+ x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{16} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {}& x^{108} + x^{106} + x^{104} + x^{99} + x^{96} + x^{95} + x^{94} + x^{92} + x^{86} + x^{83} + x^{82} \\
&+ x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} + x^{63} \\
&+ x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{46} + x^{45} + x^{38} + x^{36} \\
&+ x^{35} + x^{30},
\end{aligned}$$

$$\begin{aligned}
p_3(x) = {}& x^{103} + x^{102} + x^{99} + x^{98} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} + x^{82} \\
&+ x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{71} + x^{68} + x^{64} + x^{61} + x^{60} + x^{59}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{26} + x^{25} + x^{23} + x^{20} + x^{19} + x^{15} + x^{14} + x^{12},\\
p_6(x) = {}& x^{103} + x^{102} + x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{83}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{64} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{42}\\
&+ x^{41} + x^{39} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{25} + x^{24} + x^{23} + x^{22}\\
&+ x^{21} + x^{20} + x^{19} + x^{15} + x^{14} + x^{11} + x^{10} + x^{8} + x^{3} + 1,\\
p_9(x) = {}& x^{103} + x^{100} + x^{99} + x^{96} + x^{83} + x^{80} + x^{67} + x^{64} + x^{55} + x^{52} + x^{39}\\
&+ x^{36} + x^{35} + x^{32} + x^{19} + x^{16} + x^{7} + x^{4},\\
p_{12}(x) = {}& x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{79} + x^{76} + x^{63} + x^{60} + x^{55}\\
&+ x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{15} + x^{12}\\
&+ x^{7} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	285	[342, 450]	570	[167, 406]	855	[962, 784]
1	[3, 4]	286	[372, 450]	571	[748, 200]	856	[963, 388]
2	[5, 6]	287	[371, 342]	572	[713, 712]	857	[143, 770]
3	[7, 8]	288	[436, 61]	573	[749, 735]	858	[771, 14]
4	[9, 10]	289	[178, 48]	574	[411, 411]	859	[488, 749]
5	[11, 12]	290	[135, 186]	575	[750, 719]	860	[749, 736]
6	[13, 14]	291	[389, 451]	576	[751, 13]	861	[744, 766]
7	[15, 16]	292	[37, 452]	577	[732, 752]	862	[741, 791]
8	[17, 18]	293	[453, 454]	578	[52, 173]	863	[706, 386]
9	[19, 20]	294	[455, 456]	579	[753, 724]	864	[778, 964]
10	[21, 22]	295	[457, 135]	580	[754, 416]	865	[727, 709]
11	[23, 24]	296	[35, 37]	581	[747, 484]	866	[457, 394]
12	[25, 26]	297	[458, 459]	582	[493, 755]	867	[965, 133]
13	[27, 28]	298	[135, 355]	583	[747, 756]	868	[966, 389]
14	[29, 30]	299	[137, 460]	584	[757, 758]	869	[786, 717]
15	[31, 32]	300	[461, 138]	585	[488, 382]	870	[200, 138]
16	[33, 34]	301	[403, 462]	586	[453, 464]	871	[967, 784]
17	[35, 36]	302	[380, 158]	587	[759, 483]	872	[968, 421]
18	[37, 38]	303	[463, 131]	588	[198, 756]	873	[969, 122]
19	[39, 40]	304	[406, 464]	589	[178, 760]	874	[710, 970]
20	[41, 42]	305	[411, 465]	590	[761, 373]	875	[750, 408]
21	[43, 44]	306	[466, 365]	591	[147, 439]	876	[957, 424]
22	[45, 46]	307	[467, 348]	592	[762, 166]	877	[775, 442]

23	[47, 48]	308	[150, 468]	593	[763, 764]	878	[971, 370]
24	[49, 50]	309	[469, 379]	594	[354, 418]	879	[972, 175]
25	[51, 52]	310	[470, 144]	595	[765, 396]	880	[128, 133]
26	[53, 54]	311	[471, 401]	596	[396, 766]	881	[196, 353]
27	[55, 32]	312	[120, 472]	597	[767, 162]	882	[803, 57]
28	[56, 57]	313	[473, 124]	598	[768, 497]	883	[961, 748]
29	[58, 59]	314	[474, 150]	599	[763, 769]	884	[465, 411]
30	[60, 61]	315	[475, 127]	600	[714, 140]	885	[973, 347]
31	[62, 63]	316	[476, 128]	601	[431, 770]	886	[482, 974]
32	[64, 65]	317	[477, 123]	602	[725, 2]	887	[966, 491]
33	[66, 67]	318	[164, 478]	603	[396, 745]	888	[975, 40]
34	[68, 69]	319	[379, 472]	604	[771, 380]	889	[152, 421]
35	[70, 71]	320	[125, 469]	605	[158, 468]	890	[811, 490]
36	[72, 73]	321	[365, 45]	606	[722, 152]	891	[713, 713]
37	[69, 74]	322	[160, 479]	607	[772, 491]	892	[176, 434]
38	[75, 76]	323	[120, 357]	608	[202, 345]	893	[134, 394]
39	[77, 78]	324	[480, 36]	609	[481, 133]	894	[976, 46]
40	[79, 80]	325	[44, 189]	610	[773, 456]	895	[977, 459]
41	[81, 82]	326	[481, 129]	611	[439, 760]	896	[978, 357]
42	[83, 84]	327	[482, 483]	612	[43, 353]	897	[785, 387]
43	[85, 86]	328	[198, 484]	613	[735, 488]	898	[979, 755]
44	[87, 88]	329	[485, 168]	614	[737, 749]	899	[980, 459]
45	[89, 90]	330	[121, 486]	615	[774, 462]	900	[463, 476]
46	[91, 92]	331	[487, 488]	616	[51, 172]	901	[193, 378]
47	[93, 94]	332	[489, 490]	617	[775, 776]	902	[980, 964]
48	[95, 94]	333	[399, 175]	618	[777, 708]	903	[981, 353]
49	[96, 97]	334	[491, 390]	619	[778, 459]	904	[982, 755]
50	[98, 99]	335	[492, 60]	620	[143, 432]	905	[983, 135]
51	[100, 101]	336	[36, 38]	621	[427, 779]	906	[984, 183]
52	[102, 103]	337	[493, 494]	622	[766, 426]	907	[798, 343]
53	[101, 104]	338	[157, 495]	623	[192, 454]	908	[409, 496]
54	[105, 106]	339	[496, 421]	624	[140, 138]	909	[201, 460]
55	[107, 108]	340	[497, 163]	625	[154, 418]	910	[985, 342]
56	[109, 110]	341	[498, 499]	626	[44, 728]	911	[979, 494]
57	[6, 111]	342	[323, 500]	627	[141, 715]	912	[757, 947]
58	[112, 113]	343	[501, 64]	628	[422, 780]	913	[446, 468]
59	[92, 114]	344	[502, 503]	629	[141, 139]	914	[960, 12]
60	[115, 116]	345	[63, 312]	630	[707, 363]	915	[774, 404]
61	[117, 118]	346	[280, 504]	631	[781, 393]	916	[729, 202]
62	[119, 120]	347	[210, 274]	632	[782, 50]	917	[986, 198]
63	[121, 122]	348	[505, 506]	633	[751, 771]	918	[781, 34]
64	[123, 124]	349	[507, 508]	634	[131, 128]	919	[987, 791]
65	[125, 8]	350	[509, 510]	635	[783, 784]	920	[988, 168]
66	[126, 127]	351	[511, 251]	636	[149, 402]	921	[989, 780]

67	[128, 129]	352	[512, 513]	637	[426, 744]	922	[990, 379]
68	[130, 131]	353	[114, 514]	638	[439, 48]	923	[991, 404]
69	[132, 133]	354	[515, 516]	639	[785, 187]	924	[487, 737]
70	[134, 135]	355	[230, 79]	640	[786, 125]	925	[992, 711]
71	[136, 137]	356	[28, 76]	641	[173, 54]	926	[989, 423]
72	[138, 139]	357	[18, 517]	642	[383, 787]	927	[993, 423]
73	[140, 141]	358	[518, 333]	643	[737, 382]	928	[994, 717]
74	[131, 132]	359	[519, 520]	644	[156, 443]	929	[995, 151]
75	[142, 143]	360	[521, 522]	645	[388, 145]	930	[996, 132]
76	[144, 145]	361	[523, 524]	646	[728, 417]	931	[170, 195]
77	[146, 147]	362	[301, 72]	647	[42, 375]	932	[410, 465]
78	[148, 149]	363	[525, 511]	648	[720, 363]	933	[997, 50]
79	[150, 151]	364	[65, 526]	649	[783, 788]	934	[981, 44]
80	[152, 153]	365	[527, 509]	650	[789, 463]	935	[988, 54]
81	[154, 155]	366	[528, 529]	651	[469, 120]	936	[991, 462]
82	[156, 157]	367	[530, 531]	652	[119, 379]	937	[998, 165]
83	[132, 129]	368	[532, 533]	653	[790, 791]	938	[999, 714]
84	[14, 158]	369	[264, 534]	654	[727, 349]	939	[270, 1000]
85	[159, 160]	370	[535, 536]	655	[382, 736]	940	[687, 655]
86	[161, 162]	371	[537, 538]	656	[772, 389]	941	[1001, 1002]
87	[8, 120]	372	[529, 539]	657	[385, 792]	942	[542, 1003]
88	[10, 163]	373	[526, 231]	658	[186, 387]	943	[294, 535]
89	[164, 165]	374	[540, 541]	659	[55, 202]	944	[1004, 838]
90	[54, 166]	375	[82, 542]	660	[366, 46]	945	[576, 1005]
91	[10, 167]	376	[543, 258]	661	[485, 54]	946	[1006, 1007]
92	[168, 169]	377	[544, 545]	662	[793, 712]	947	[1008, 1009]
93	[170, 162]	378	[235, 546]	663	[59, 437]	948	[1010, 1011]
94	[153, 171]	379	[214, 547]	664	[182, 373]	949	[1012, 1013]
95	[172, 173]	380	[548, 549]	665	[372, 392]	950	[1014, 659]
96	[174, 175]	381	[550, 551]	666	[794, 160]	951	[1015, 1016]
97	[176, 177]	382	[508, 552]	667	[795, 408]	952	[690, 339]
98	[147, 178]	383	[553, 554]	668	[796, 736]	953	[1017, 826]
99	[149, 179]	384	[555, 556]	669	[797, 384]	954	[1018, 1019]
100	[180, 181]	385	[557, 558]	670	[734, 491]	955	[667, 561]
101	[182, 183]	386	[559, 560]	671	[183, 374]	956	[1020, 295]
102	[184, 185]	387	[232, 561]	672	[798, 181]	957	[1021, 209]
103	[186, 187]	388	[562, 68]	673	[346, 479]	958	[1022, 890]
104	[181, 188]	389	[304, 302]	674	[799, 448]	959	[611, 503]
105	[189, 190]	390	[563, 564]	675	[41, 351]	960	[1023, 1024]
106	[144, 33]	391	[565, 502]	676	[800, 144]	961	[1025, 335]
107	[191, 192]	392	[566, 567]	677	[707, 433]	962	[1026, 1027]
108	[193, 194]	393	[568, 569]	678	[366, 350]	963	[1028, 66]
109	[161, 195]	394	[570, 571]	679	[436, 367]	964	[272, 217]
110	[158, 151]	395	[572, 560]	680	[801, 185]	965	[834, 98]

111	[54, 169]	396	[573, 574]	681	[802, 157]	966	[1029, 1030]
112	[196, 44]	397	[575, 576]	682	[803, 364]	967	[1031, 833]
113	[197, 198]	398	[577, 578]	683	[804, 398]	968	[888, 286]
114	[167, 192]	399	[579, 580]	684	[765, 744]	969	[1032, 1033]
115	[153, 199]	400	[547, 581]	685	[341, 372]	970	[1011, 1034]
116	[200, 141]	401	[582, 583]	686	[419, 438]	971	[873, 623]
117	[136, 201]	402	[584, 585]	687	[745, 426]	972	[1035, 1036]
118	[202, 56]	403	[586, 587]	688	[766, 765]	973	[103, 567]
119	[203, 204]	404	[588, 589]	689	[805, 194]	974	[822, 1037]
120	[16, 205]	405	[590, 228]	690	[9, 497]	975	[1038, 1039]
121	[206, 5]	406	[218, 591]	691	[799, 806]	976	[936, 605]
122	[207, 208]	407	[592, 593]	692	[497, 167]	977	[1040, 702]
123	[209, 210]	408	[506, 219]	693	[173, 168]	978	[686, 298]
124	[211, 212]	409	[594, 261]	694	[807, 365]	979	[1041, 102]
125	[213, 214]	410	[595, 585]	695	[808, 746]	980	[1042, 1043]
126	[215, 216]	411	[596, 550]	696	[809, 742]	981	[1044, 1045]
127	[217, 218]	412	[597, 598]	697	[810, 418]	982	[1046, 1047]
128	[219, 220]	413	[599, 600]	698	[706, 792]	983	[1048, 93]
129	[221, 222]	414	[220, 601]	699	[811, 812]	984	[1049, 1050]
130	[97, 223]	415	[602, 603]	700	[492, 436]	985	[1051, 1052]
131	[224, 225]	416	[539, 604]	701	[491, 451]	986	[1053, 683]
132	[223, 226]	417	[600, 605]	702	[800, 388]	987	[1054, 27]
133	[227, 228]	418	[73, 606]	703	[419, 370]	988	[1055, 517]
134	[229, 230]	419	[516, 83]	704	[813, 814]	989	[1056, 877]
135	[231, 232]	420	[607, 292]	705	[815, 816]	990	[1057, 334]
136	[233, 234]	421	[589, 608]	706	[817, 588]	991	[1058, 1059]
137	[24, 235]	422	[609, 610]	707	[814, 818]	992	[1060, 1061]
138	[236, 237]	423	[522, 611]	708	[591, 819]	993	[1062, 1063]
139	[238, 238]	424	[612, 211]	709	[820, 687]	994	[1064, 17]
140	[239, 240]	425	[613, 524]	710	[821, 822]	995	[546, 533]
141	[241, 242]	426	[603, 614]	711	[823, 91]	996	[1065, 663]
142	[243, 244]	427	[615, 616]	712	[712, 712]	997	[1066, 1067]
143	[245, 246]	428	[617, 95]	713	[637, 824]	998	[1068, 1069]
144	[26, 247]	429	[618, 612]	714	[825, 646]	999	[1070, 829]
145	[248, 115]	430	[321, 207]	715	[240, 826]	1000	[56, 364]
146	[249, 250]	431	[619, 620]	716	[827, 828]	1001	[998, 478]
147	[251, 252]	432	[621, 622]	717	[580, 638]	1002	[1071, 409]
148	[253, 254]	433	[556, 623]	718	[829, 221]	1003	[157, 444]
149	[255, 256]	434	[624, 525]	719	[830, 630]	1004	[1072, 199]
150	[208, 257]	435	[625, 530]	720	[831, 69]	1005	[14, 150]
151	[258, 221]	436	[536, 562]	721	[832, 324]	1006	[993, 780]
152	[259, 260]	437	[113, 626]	722	[833, 834]	1007	[1073, 172]
153	[261, 262]	438	[606, 627]	723	[835, 825]	1008	[759, 974]
154	[263, 264]	439	[628, 629]	724	[290, 657]	1009	[424, 473]

155	[265, 266]	440	[630, 564]	725	[836, 837]	1010	[762, 169]
156	[267, 268]	441	[631, 632]	726	[838, 839]	1011	[951, 359]
157	[269, 270]	442	[633, 634]	727	[649, 840]	1012	[1074, 177]
158	[271, 272]	443	[635, 636]	728	[330, 501]	1013	[802, 443]
159	[273, 274]	444	[268, 637]	729	[841, 831]	1014	[790, 742]
160	[275, 276]	445	[78, 98]	730	[842, 843]	1015	[976, 350]
161	[277, 278]	446	[638, 541]	731	[703, 118]	1016	[1075, 430]
162	[279, 280]	447	[639, 582]	732	[844, 845]	1017	[975, 59]
163	[20, 281]	448	[640, 641]	733	[846, 316]	1018	[738, 939]
164	[282, 283]	449	[642, 643]	734	[847, 848]	1019	[1076, 424]
165	[284, 275]	450	[499, 504]	735	[688, 849]	1020	[753, 942]
166	[285, 286]	451	[644, 645]	736	[551, 850]	1021	[1077, 351]
167	[287, 288]	452	[71, 298]	737	[552, 572]	1022	[793, 713]
168	[289, 290]	453	[646, 321]	738	[851, 852]	1023	[955, 190]
169	[291, 292]	454	[281, 647]	739	[853, 548]	1024	[1078, 942]
170	[293, 294]	455	[648, 649]	740	[854, 527]	1025	[1079, 36]
171	[260, 204]	456	[650, 651]	741	[855, 856]	1026	[1080, 708]
172	[295, 289]	457	[652, 318]	742	[212, 857]	1027	[1081, 185]
173	[296, 291]	458	[653, 21]	743	[674, 858]	1028	[718, 128]
174	[297, 213]	459	[533, 654]	744	[524, 859]	1029	[1082, 44]
175	[298, 299]	460	[234, 655]	745	[860, 861]	1030	[1083, 201]
176	[300, 301]	461	[656, 238]	746	[862, 863]	1031	[967, 788]
177	[302, 303]	462	[657, 658]	747	[864, 865]	1032	[356, 472]
178	[250, 304]	463	[659, 660]	748	[866, 241]	1033	[1084, 764]
179	[254, 305]	464	[228, 258]	749	[849, 595]	1034	[717, 469]
180	[306, 307]	465	[585, 596]	750	[867, 691]	1035	[809, 791]
181	[308, 309]	466	[661, 89]	751	[868, 685]	1036	[1081, 147]
182	[310, 311]	467	[662, 507]	752	[869, 870]	1037	[473, 160]
183	[312, 313]	468	[30, 663]	753	[871, 866]	1038	[470, 388]
184	[314, 315]	469	[664, 665]	754	[872, 873]	1039	[1085, 706]
185	[316, 317]	470	[666, 667]	755	[645, 279]	1040	[174, 400]
186	[318, 319]	471	[668, 669]	756	[874, 551]	1041	[458, 964]
187	[320, 308]	472	[204, 285]	757	[875, 876]	1042	[1086, 724]
188	[307, 321]	473	[670, 671]	758	[877, 568]	1043	[1087, 434]
189	[86, 322]	474	[672, 647]	759	[878, 879]	1044	[968, 153]
190	[88, 323]	475	[673, 674]	760	[317, 880]	1045	[948, 478]
191	[324, 222]	476	[675, 227]	761	[881, 532]	1046	[1088, 175]
192	[22, 325]	477	[676, 677]	762	[678, 542]	1047	[1089, 365]
193	[326, 327]	478	[601, 678]	763	[882, 619]	1048	[191, 406]
194	[328, 16]	479	[274, 679]	764	[883, 570]	1049	[1090, 135]
195	[256, 105]	480	[680, 681]	765	[859, 884]	1050	[986, 747]
196	[329, 330]	481	[658, 626]	766	[884, 573]	1051	[1091, 155]
197	[331, 332]	482	[682, 269]	767	[885, 886]	1052	[1085, 385]
198	[333, 284]	483	[4, 87]	768	[887, 888]	1053	[425, 765]

199	[292, 334]	484	[683, 684]	769	[837, 889]	1054	[777, 430]
200	[335, 336]	485	[685, 285]	770	[890, 891]	1055	[1092, 351]
201	[337, 338]	486	[246, 686]	771	[892, 271]	1056	[731, 139]
202	[339, 340]	487	[574, 687]	772	[893, 563]	1057	[1092, 42]
203	[341, 342]	488	[622, 688]	773	[894, 895]	1058	[440, 171]
204	[343, 344]	489	[689, 690]	774	[896, 617]	1059	[1093, 733]
205	[32, 345]	490	[691, 692]	775	[897, 817]	1060	[767, 195]
206	[346, 347]	491	[94, 693]	776	[616, 624]	1061	[1094, 130]
207	[348, 349]	492	[694, 695]	777	[898, 676]	1062	[1095, 166]
208	[45, 350]	493	[696, 697]	778	[899, 15]	1063	[1093, 752]
209	[58, 40]	494	[608, 698]	779	[587, 900]	1064	[1079, 37]
210	[351, 352]	495	[699, 613]	780	[299, 236]	1065	[480, 37]
211	[345, 57]	496	[700, 701]	781	[901, 627]	1066	[1086, 942]
212	[353, 189]	497	[702, 703]	782	[902, 903]	1067	[963, 144]
213	[354, 155]	498	[704, 434]	783	[904, 905]	1068	[1096, 715]
214	[355, 187]	499	[148, 401]	784	[237, 328]	1069	[1084, 769]
215	[356, 357]	500	[163, 192]	785	[315, 92]	1070	[984, 373]
216	[358, 359]	501	[168, 166]	786	[906, 907]	1071	[1097, 259]
217	[360, 361]	502	[705, 706]	787	[545, 908]	1072	[660, 645]
218	[362, 363]	503	[177, 707]	788	[118, 909]	1073	[1098, 543]
219	[345, 364]	504	[434, 707]	789	[910, 656]	1074	[1099, 926]
220	[365, 366]	505	[429, 708]	790	[911, 892]	1075	[1100, 1101]
221	[60, 367]	506	[348, 709]	791	[504, 830]	1076	[1102, 1057]
222	[183, 368]	507	[710, 711]	792	[912, 860]	1077	[1103, 512]
223	[369, 370]	508	[712, 713]	793	[560, 637]	1078	[1104, 1105]
224	[371, 372]	509	[714, 200]	794	[907, 679]	1079	[1106, 75]
225	[42, 352]	510	[160, 347]	795	[913, 914]	1080	[1107, 1108]
226	[177, 362]	511	[138, 715]	796	[643, 891]	1081	[1109, 628]
227	[373, 374]	512	[716, 360]	797	[915, 869]	1082	[1110, 901]
228	[351, 375]	513	[32, 56]	798	[916, 917]	1083	[1111, 1112]
229	[376, 150]	514	[717, 8]	799	[918, 919]	1084	[1113, 1114]
230	[377, 378]	515	[718, 132]	800	[920, 921]	1085	[1115, 912]
231	[8, 379]	516	[407, 719]	801	[922, 923]	1086	[1116, 1117]
232	[13, 380]	517	[36, 452]	802	[924, 925]	1087	[1118, 1119]
233	[381, 382]	518	[720, 433]	803	[926, 526]	1088	[1120, 1121]
234	[383, 384]	519	[721, 722]	804	[927, 928]	1089	[1122, 528]
235	[385, 386]	520	[163, 406]	805	[929, 555]	1090	[1123, 1124]
236	[355, 387]	521	[723, 724]	806	[876, 265]	1091	[1125, 1126]
237	[388, 33]	522	[443, 495]	807	[930, 931]	1092	[1127, 1128]
238	[389, 390]	523	[725, 726]	808	[932, 523]	1093	[1129, 862]
239	[391, 181]	524	[465, 465]	809	[933, 934]	1094	[1130, 224]
240	[342, 392]	525	[172, 53]	810	[935, 936]	1095	[677, 513]
241	[33, 393]	526	[124, 347]	811	[937, 938]	1096	[1000, 531]
242	[394, 186]	527	[123, 160]	812	[593, 320]	1097	[180, 343]

243	[395, 396]	528	[377, 194]	813	[405, 192]	1098	[949, 60]
244	[397, 398]	529	[124, 479]	814	[126, 939]	1099	[990, 120]
245	[399, 400]	530	[142, 431]	815	[940, 745]	1100	[962, 788]
246	[401, 402]	531	[418, 369]	816	[941, 726]	1101	[1087, 177]
247	[52, 53]	532	[467, 727]	817	[723, 942]	1102	[985, 372]
248	[403, 404]	533	[353, 728]	818	[192, 464]	1103	[1131, 32]
249	[405, 406]	534	[421, 199]	819	[746, 361]	1104	[982, 494]
250	[407, 408]	535	[130, 476]	820	[804, 943]	1105	[1089, 416]
251	[53, 168]	536	[406, 454]	821	[944, 122]	1106	[1132, 144]
252	[409, 152]	537	[729, 32]	822	[721, 409]	1107	[997, 414]
253	[410, 411]	538	[730, 727]	823	[11, 945]	1108	[31, 202]
254	[412, 413]	539	[496, 153]	824	[745, 765]	1109	[461, 141]
255	[49, 414]	540	[705, 385]	825	[184, 147]	1110	[996, 128]
256	[137, 415]	541	[185, 178]	826	[181, 344]	1111	[940, 766]
257	[416, 366]	542	[155, 369]	827	[796, 735]	1112	[1133, 413]
258	[373, 368]	543	[435, 60]	828	[946, 947]	1113	[971, 438]
259	[391, 343]	544	[731, 715]	829	[761, 183]	1114	[1134, 788]
260	[343, 188]	545	[732, 733]	830	[401, 179]	1115	[381, 749]
261	[189, 417]	546	[728, 190]	831	[948, 165]	1116	[1080, 430]
262	[418, 419]	547	[155, 419]	832	[949, 436]	1117	[810, 155]
263	[420, 421]	548	[734, 389]	833	[754, 365]	1118	[474, 158]
264	[422, 423]	549	[59, 445]	834	[145, 393]	1119	[795, 719]
265	[380, 150]	550	[382, 735]	835	[950, 50]	1120	[1088, 400]
266	[424, 123]	551	[736, 488]	836	[805, 378]	1121	[983, 394]
267	[425, 426]	552	[736, 737]	837	[789, 130]	1122	[159, 124]
268	[427, 428]	553	[738, 127]	838	[951, 952]	1123	[953, 141]
269	[429, 430]	554	[739, 13]	839	[748, 140]	1124	[969, 486]
270	[431, 432]	555	[455, 740]	840	[746, 449]	1125	[794, 124]
271	[362, 433]	556	[477, 473]	841	[953, 138]	1126	[475, 939]
272	[434, 362]	557	[741, 742]	842	[395, 744]	1127	[807, 416]
273	[435, 436]	558	[201, 415]	843	[954, 428]	1128	[1083, 137]
274	[40, 437]	559	[412, 743]	844	[955, 417]	1129	[47, 760]
275	[369, 438]	560	[744, 745]	845	[956, 755]	1130	[1077, 42]
276	[185, 439]	561	[379, 357]	846	[957, 477]	1131	[1135, 109]
277	[440, 199]	562	[476, 132]	847	[801, 147]	1132	[1136, 248]
278	[441, 442]	563	[716, 746]	848	[958, 431]	1133	[1137, 854]
279	[443, 444]	564	[394, 355]	849	[426, 396]	1134	[1138, 594]
280	[145, 34]	565	[704, 177]	850	[735, 737]	1135	[376, 158]
281	[40, 445]	566	[197, 747]	851	[959, 464]	1136	[420, 153]
282	[446, 151]	567	[416, 45]	852	[960, 945]	1137	[944, 486]
283	[447, 448]	568	[722, 496]	853	[39, 59]	1138	[950, 414]
284	[360, 449]	569	[421, 171]	854	[961, 714]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	190	0	380	1	570	1	760	1	950	1
1	0	191	1	381	0	571	0	761	0	951	0
2	1	192	0	382	0	572	1	762	1	952	0
3	0	193	0	383	0	573	1	763	1	953	1
4	0	194	1	384	1	574	1	764	0	954	0
5	1	195	1	385	0	575	1	765	0	955	0
6	1	196	0	386	0	576	1	766	0	956	1
7	0	197	1	387	1	577	0	767	0	957	0
8	0	198	1	388	0	578	1	768	1	958	0
9	0	199	0	389	1	579	1	769	1	959	0
10	0	200	1	390	1	580	1	770	0	960	0
11	1	201	0	391	0	581	0	771	0	961	0
12	0	202	1	392	1	582	0	772	0	962	0
13	1	203	0	393	1	583	0	773	0	963	0
14	0	204	1	394	1	584	1	774	1	964	0
15	0	205	0	395	1	585	0	775	1	965	1
16	0	206	1	396	1	586	1	776	0	966	1
17	0	207	1	397	1	587	0	777	1	967	1
18	0	208	0	398	0	588	1	778	0	968	1
19	0	209	0	399	1	589	1	779	0	969	0
20	0	210	1	400	1	590	0	780	1	970	0
21	0	211	1	401	0	591	1	781	0	971	0
22	0	212	1	402	1	592	1	782	0	972	0
23	1	213	0	403	1	593	1	783	0	973	0
24	1	214	1	404	1	594	0	784	0	974	1
25	0	215	0	405	0	595	0	785	1	975	1
26	1	216	1	406	1	596	1	786	1	976	0
27	1	217	1	407	1	597	0	787	0	977	1
28	0	218	1	408	1	598	1	788	1	978	1
29	0	219	1	409	0	599	1	789	1	979	1
30	0	220	0	410	0	600	1	790	1	980	0
31	0	221	0	411	1	601	0	791	0	981	1
32	0	222	0	412	0	602	1	792	1	982	0
33	0	223	0	413	1	603	1	793	0	983	1
34	0	224	1	414	0	604	0	794	1	984	1
35	0	225	0	415	1	605	1	795	0	985	1
36	1	226	1	416	1	606	1	796	1	986	0
37	0	227	1	417	1	607	0	797	1	987	1
38	1	228	1	418	0	608	1	798	1	988	1
39	0	229	0	419	1	609	0	799	0	989	1
40	0	230	1	420	0	610	0	800	1	990	1
41	0	231	0	421	1	611	0	801	1	991	0
42	0	232	1	422	0	612	0	802	1	992	0

43	0	233	0	423	0	613	0	803	1	993	0
44	0	234	0	424	0	614	1	804	0	994	0
45	0	235	0	425	0	615	1	805	1	995	1
46	0	236	1	426	1	616	0	806	0	996	1
47	1	237	0	427	1	617	1	807	1	997	0
48	0	238	1	428	1	618	1	808	0	998	0
49	1	239	0	429	1	619	0	809	0	999	1
50	1	240	0	430	0	620	1	810	1	1000	0
51	0	241	0	431	0	621	1	811	0	1001	0
52	1	242	1	432	1	622	0	812	1	1002	0
53	1	243	1	433	1	623	0	813	0	1003	1
54	0	244	1	434	0	624	0	814	0	1004	1
55	1	245	1	435	0	625	0	815	1	1005	0
56	0	246	0	436	1	626	0	816	0	1006	0
57	1	247	1	437	1	627	0	817	1	1007	1
58	0	248	1	438	1	628	0	818	0	1008	0
59	1	249	0	439	0	629	0	819	0	1009	0
60	0	250	1	440	0	630	0	820	0	1010	1
61	0	251	1	441	0	631	0	821	1	1011	0
62	0	252	0	442	1	632	0	822	1	1012	1
63	1	253	0	443	0	633	1	823	1	1013	1
64	0	254	0	444	1	634	1	824	1	1014	1
65	0	255	1	445	0	635	0	825	1	1015	0
66	0	256	1	446	0	636	1	826	0	1016	0
67	1	257	1	447	1	637	1	827	1	1017	1
68	0	258	1	448	1	638	0	828	0	1018	0
69	0	259	0	449	0	639	1	829	0	1019	1
70	0	260	1	450	0	640	1	830	0	1020	1
71	0	261	0	451	0	641	0	831	1	1021	0
72	1	262	0	452	0	642	0	832	1	1022	0
73	0	263	0	453	1	643	1	833	1	1023	0
74	1	264	0	454	0	644	0	834	1	1024	0
75	1	265	1	455	1	645	0	835	1	1025	0
76	1	266	0	456	1	646	1	836	1	1026	0
77	0	267	0	457	0	647	0	837	1	1027	0
78	0	268	1	458	1	648	1	838	0	1028	0
79	0	269	1	459	1	649	0	839	0	1029	1
80	0	270	0	460	0	650	1	840	0	1030	1
81	0	271	1	461	0	651	1	841	1	1031	1
82	0	272	0	462	0	652	0	842	1	1032	0
83	0	273	0	463	1	653	1	843	0	1033	0
84	0	274	0	464	1	654	0	844	0	1034	1
85	0	275	0	465	0	655	0	845	1	1035	0
86	0	276	0	466	0	656	0	846	0	1036	0

87	0	277	0	467	0	657	0	847	1	1037	1
88	0	278	0	468	0	658	0	848	0	1038	1
89	0	279	0	469	1	659	1	849	1	1039	0
90	0	280	1	470	1	660	1	850	0	1040	1
91	0	281	0	471	1	661	0	851	0	1041	1
92	1	282	0	472	1	662	0	852	0	1042	0
93	1	283	1	473	1	663	1	853	0	1043	1
94	0	284	1	474	1	664	1	854	0	1044	1
95	0	285	0	475	1	665	1	855	0	1045	1
96	1	286	1	476	0	666	1	856	0	1046	0
97	0	287	1	477	1	667	0	857	1	1047	0
98	1	288	1	478	0	668	1	858	0	1048	1
99	1	289	1	479	0	669	1	859	0	1049	1
100	0	290	0	480	1	670	1	860	1	1050	0
101	1	291	1	481	0	671	0	861	0	1051	1
102	1	292	0	482	1	672	1	862	0	1052	0
103	0	293	1	483	0	673	1	863	1	1053	0
104	0	294	1	484	0	674	0	864	0	1054	1
105	0	295	0	485	0	675	0	865	0	1055	1
106	1	296	0	486	0	676	1	866	0	1056	1
107	1	297	1	487	1	677	0	867	1	1057	1
108	0	298	0	488	0	678	1	868	1	1058	0
109	0	299	1	489	1	679	1	869	1	1059	1
110	1	300	0	490	0	680	1	870	1	1060	0
111	0	301	1	491	0	681	1	871	1	1061	0
112	0	302	1	492	1	682	1	872	1	1062	0
113	1	303	1	493	0	683	0	873	0	1063	1
114	1	304	1	494	1	684	0	874	1	1064	0
115	0	305	1	495	0	685	0	875	1	1065	1
116	1	306	0	496	1	686	1	876	0	1066	0
117	0	307	0	497	1	687	1	877	1	1067	0
118	1	308	0	498	0	688	0	878	0	1068	0
119	0	309	1	499	0	689	1	879	0	1069	0
120	0	310	1	500	0	690	0	880	1	1070	1
121	1	311	1	501	1	691	0	881	0	1071	0
122	1	312	0	502	1	692	1	882	1	1072	1
123	0	313	1	503	1	693	0	883	0	1073	1
124	1	314	1	504	0	694	1	884	0	1074	1
125	0	315	1	505	1	695	0	885	0	1075	0
126	0	316	0	506	1	696	0	886	1	1076	1
127	1	317	1	507	1	697	1	887	1	1077	0
128	1	318	0	508	0	698	1	888	1	1078	0
129	0	319	1	509	1	699	0	889	0	1079	0
130	0	320	0	510	0	700	1	890	0	1080	0

131	1	321	0	511	1	701	0	891	1	1081	0
132	0	322	0	512	1	702	1	892	0	1082	1
133	1	323	0	513	0	703	1	893	0	1083	1
134	0	324	1	514	1	704	0	894	0	1084	0
135	0	325	0	515	0	705	1	895	1	1085	0
136	0	326	0	516	1	706	1	896	1	1086	0
137	1	327	1	517	1	707	0	897	1	1087	1
138	1	328	1	518	1	708	1	898	1	1088	0
139	1	329	0	519	1	709	0	899	0	1089	0
140	0	330	1	520	0	710	1	900	1	1090	1
141	0	331	1	521	1	711	1	901	0	1091	1
142	1	332	1	522	0	712	0	902	0	1092	1
143	1	333	1	523	1	713	1	903	1	1093	1
144	1	334	0	524	0	714	1	904	0	1094	0
145	1	335	1	525	0	715	0	905	1	1095	0
146	0	336	1	526	1	716	1	906	1	1096	0
147	1	337	0	527	0	717	1	907	1	1097	0
148	0	338	1	528	1	718	0	908	0	1098	1
149	1	339	1	529	1	719	0	909	0	1099	1
150	0	340	1	530	1	720	1	910	1	1100	0
151	1	341	0	531	0	721	1	911	1	1101	1
152	0	342	0	532	0	722	1	912	1	1102	1
153	0	343	1	533	1	723	1	913	0	1103	0
154	0	344	1	534	1	724	0	914	0	1104	0
155	1	345	1	535	0	725	1	915	1	1105	0
156	0	346	1	536	1	726	0	916	1	1106	0
157	1	347	1	537	1	727	0	917	0	1107	0
158	1	348	1	538	1	728	1	918	0	1108	0
159	0	349	1	539	1	729	1	919	1	1109	0
160	0	350	1	540	1	730	1	920	1	1110	1
161	0	351	1	541	0	731	1	921	1	1111	1
162	0	352	1	542	1	732	0	922	1	1112	1
163	0	353	1	543	0	733	0	923	0	1113	0
164	0	354	0	544	1	734	1	924	1	1114	1
165	1	355	1	545	0	735	0	925	0	1115	0
166	0	356	0	546	1	736	1	926	1	1116	0
167	1	357	0	547	1	737	1	927	0	1117	1
168	1	358	1	548	1	738	0	928	0	1118	1
169	1	359	1	549	1	739	0	929	1	1119	0
170	1	360	1	550	0	740	0	930	1	1120	0
171	1	361	1	551	1	741	0	931	1	1121	1
172	0	362	1	552	1	742	1	932	0	1122	0
173	0	363	0	553	0	743	0	933	0	1123	1
174	1	364	0	554	0	744	0	934	1	1124	0

175	0	365	0	555	1	745	1	935	1	1125	1
176	0	366	1	556	1	746	0	936	0	1126	1
177	1	367	1	557	0	747	0	937	0	1127	1
178	1	368	0	558	0	748	0	938	1	1128	1
179	0	369	0	559	0	749	1	939	0	1129	1
180	0	370	0	560	0	750	1	940	1	1130	0
181	0	371	1	561	1	751	1	941	0	1131	0
182	1	372	1	562	0	752	1	942	1	1132	0
183	0	373	1	563	1	753	1	943	1	1133	1
184	1	374	1	564	1	754	1	944	1	1134	1
185	0	375	0	565	0	755	0	945	1	1135	0
186	0	376	0	566	1	756	1	946	0	1136	0
187	0	377	1	567	1	757	1	947	0	1137	1
188	0	378	0	568	1	758	1	948	1	1138	1
189	0	379	1	569	1	759	0	949	1		

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