PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^5+z^4+z^2,z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^5+z^4+z^2,z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^5+z^4+z^2,z)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{13}+z^{11}+z^{10}+z^9+z^4+z^2+z+1,\\
p_1(z)&=z^{19}+z^{17}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^3+z^2,\\
p_2(z)&=z^{20}+z^{18}+z^{16}+z^{14}+z^{10}+z^6+z^4,\\
p_3(z)&=z^{19}+z^{17}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^3+z^2,\\
p_4(z)&=z^{18}+z^{16}+z^{13}+z^{12}+z^{11}+z^9+z^8+z^6+z^4+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{16}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^2+z,\\
p_1(z)&=z^{19}+z^{17}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^3+z^2,\\
p_2(z)&=z^{20}+z^{18}+z^{16}+z^{14}+z^{10}+z^6+z^4,\\
p_3(z)&=z^{19}+z^{17}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^3+z^2,\\
p_4(z)&=z^{18}+z^{13}+z^{11}+z^9+z^8+z^6+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n}\tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n}\tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] &= M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{4u} M^e[:2u] + x^u M^e[:6u] \\
&\quad + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{3u} M^e[:6u] \\
&\quad + x^{4u} M^e[:5u] + x^{5u} M^e[:4u] + x^{7u} M^e[:2u] + x^{6u} M^e[:6u] \\
&\quad + (x^{6u} + x^{7u} + x^{9u}) R^e, \\
W^e[:12u] &= W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{4u} W^e[:2u] + x^u W^e[:6u] \\
&\quad + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{3u} W^e[:6u]
\end{aligned}$$

$$+ x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u] + x^{6u}W^e[:6u] + (x^{6u} + x^{7u} + x^{9u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^uM^o[:5u] + x^{2u}M^o[:4u] + x^{4u}M^o[:2u] + x^uM^o[:6u] \\ &+ x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{3u}M^o[:6u] \\ &+ x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u] + x^{6u}M^o[:6u] \\ &+ (x^{6u} + x^{7u} + x^{9u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^uW^o[:5u] + x^{2u}W^o[:4u] + x^{4u}W^o[:2u] + x^uW^o[:6u] \\ &+ x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{3u}W^o[:6u] \\ &+ x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u] + x^{6u}W^o[:6u] \\ &+ (x^{6u} + x^{7u} + x^{9u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] = T[:6u] &+ x^uT[:5u] + x^{2u}T[:4u] + x^{4u}T[:2u] + x^uT[:6u] + x^{2u}T[:5u] \\ &+ x^{3u}T[:4u] + x^{5u}T[:2u] + x^{3u}T[:6u] + x^{4u}T[:5u] + x^{5u}T[:4u] \\ &+ x^{7u}T[:2u] + x^{6u}T[:6u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{6u}T[:u] + x^{4u}T[2u:3u] + x^{2u}T[4u:5u] + x^uT[5u:6u] + T[6u:7u], \\ 0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] \\ &\quad + x^{3u}T[4u:5u] + T[7u:8u], \\ 0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u] \\ &\quad + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{6u}T[3u:4u] + T[9u:10u], \\ 0 &= x^{6u}T[4u:5u] + T[10u:11u], \\ 0 &= x^{6u}T[5u:6u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[11w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[100w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[11w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1044 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.19}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.20}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.21}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 1001)), \tag{2.22}$$

$$0 = \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.23}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.25}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.26}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.27}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 1001)), \tag{2.28}$$

$$0 = \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.29}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{20}), E_{20}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 10$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{4u} M^e[:2u] + x^{6u} R^e, \\ W_{2k} &= W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{4u} W^e[:2u] + x^{6u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^u M^o[:5u] + x^{2u} M^o[:4u] + x^{4u} M^o[:2u] + x^{6u} R^o, \\ W_{2k+1} &= W^o[:6u] + x^u W^o[:5u] + x^{2u} W^o[:4u] + x^{4u} W^o[:2u] + x^{6u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(&W^e[:6v] + x^v W^e[:5v] + x^{2v} W^e[:4v] + x^{4v} W^e[:2v] + x^{6u} R^e\big) \times \big(\\ &M^e[:6v] + x^v M^e[:5v] + x^{2v} M^e[:4v] + x^{4v} M^e[:2v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} W^e[:12v]M^e[:12v] + x^{2v} W^e[:10v]M^e[:10v] + x^{4v} W^e[:8v]M^e[:8v] \\ + x^{8v} W^e[:4v]M^e[:4v]. \end{aligned}$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned}
\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\
\lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\
\lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\
\lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.
\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{20} + x^{19} + x^{18} + x^{16} + x^{11} + x^{10} + x^9 + x^7 + x^6,\\
q_1(x) &= x^{18} + x^{17} + x^{13} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^3 + x,\\
q_2(x) &= x^{16} + x^{14} + x^{10} + x^6 + x^4 + x^2 + 1,\\
q_3(x) &= x^{18} + x^{17} + x^{13} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^3 + x,\\
q_4(x) &= x^{19} + x^{18} + x^{16} + x^{14} + x^{12} + x^{11} + x^9 + x^8 + x^7 + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$p_0(x) = x^{96} + x^{94} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66}$$

$$
\begin{aligned}
&\quad + x^{64} + x^{60} + x^{56} + x^{46} + x^{44} + x^{40} + x^{36} + x^{16} + x^{14} + x^{12},\\
p_3(x) &= x^{88} + x^{86} + x^{84} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{56} + x^{54} + x^{44}\\
&\quad + x^{42} + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{12} + x^{8},\\
p_6(x) &= x^{88} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60}\\
&\quad + x^{56} + x^{52} + x^{42} + x^{38} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14}\\
&\quad + x^{10} + 1,\\
p_9(x) &= x^{86} + x^{72} + x^{64} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38}\\
&\quad + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{16} + x^{12} + x^{8} + x^{2},\\
p_{12}(x) &= x^{86} + x^{82} + x^{80} + x^{70} + x^{66} + x^{64} + x^{54} + x^{50} + x^{48} + x^{38} + x^{34}\\
&\quad + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94}\\
&\quad + x^{93} + x^{87} + x^{86} + x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70}\\
&\quad + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} + x^{49}\\
&\quad + x^{46} + x^{44} + x^{42} + x^{39} + x^{38} + x^{33} + x^{32} + x^{30} + x^{29} + x^{26} + x^{24},\\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82}\\
&\quad + x^{81} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{49} + x^{48} + x^{46} + x^{45} + x^{41}\\
&\quad + x^{40} + x^{38} + x^{37} + x^{33} + x^{32} + x^{30} + x^{29} + x^{25} + x^{24} + x^{22} + x^{20}\\
&\quad + x^{19} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{99} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{80}\\
&\quad + x^{75} + x^{73} + x^{72} + x^{65} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55} + x^{53} + x^{52}\\
&\quad + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} + x^{34} + x^{33}\\
&\quad + x^{32} + x^{25} + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{83} + x^{82} + x^{80}\\
&\quad + x^{78} + x^{77} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{59} + x^{58} + x^{57} + x^{53}\\
&\quad + x^{52} + x^{51} + x^{47} + x^{46} + x^{45} + x^{39} + x^{38} + x^{37} + x^{29} + x^{28} + x^{27}\\
&\quad + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11}\\
&\quad + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77}\\
&\quad + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{55}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{39} + x^{37} + x^{36}\\
&\quad + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{87} + x^{86} + x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} \\
&\quad + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{57} + x^{55} + x^{54} + x^{49} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{39} + x^{38} + x^{33} + x^{32} + x^{30} + x^{29} + x^{26} + x^{24}, \\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82} \\
&\quad + x^{81} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{49} + x^{48} + x^{46} + x^{45} + x^{41} \\
&\quad + x^{40} + x^{38} + x^{37} + x^{33} + x^{32} + x^{30} + x^{29} + x^{25} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{19} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{99} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{75} + x^{73} + x^{72} + x^{65} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{25} + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{83} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{77} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{59} + x^{58} + x^{57} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{47} + x^{46} + x^{45} + x^{39} + x^{38} + x^{37} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} \\
&\quad + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} \\
&\quad + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{55} \\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{39} + x^{37} + x^{36} \\
&\quad + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{110} + x^{94} + x^{88} + x^{84} + x^{74} + x^{64} + x^{62} \\
&\quad + x^{60} + x^{58} + x^{56} + x^{52} + x^{44} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{108} + x^{102} + x^{96} + x^{94} + x^{90} + x^{84} + x^{80} + x^{76} + x^{72} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{32} + x^{26} + x^{22} + x^{20} + x^{18} + x^{8} + x^{6}, \\
p_6(x) &= x^{108} + x^{104} + x^{100} + x^{96} + x^{88} + x^{78} + x^{74} + x^{72} + x^{66} + x^{60} + x^{58} \\
&\quad + x^{34} + x^{32} + x^{28} + x^{22} + x^{20} + x^{18} + x^{10} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{104} + x^{100} + x^{98} + x^{96} + x^{88} + x^{84} + x^{78} + x^{76} + x^{72} + x^{64} + x^{62} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{38} + x^{34} + x^{26} + x^{20} + x^{16} \\
&\quad + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2},
\end{aligned}$$

$$p_{12}(x) = x^{104} + x^{80} + x^{64} + x^{48} + x^{40} + x^{32} + x^{24} + x^{8} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{114} + x^{110} + x^{102} + x^{92} + x^{88} + x^{82} + x^{80} + x^{78} + x^{76} \\
&\quad + x^{72} + x^{66} + x^{56} + x^{54} + x^{44} + x^{42} + x^{40} + x^{36} + x^{30} + x^{26} + x^{22} \\
&\quad + x^{12}, \\
p_3(x) &= x^{108} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{46} + x^{42} + x^{40} + x^{34} + x^{32} \\
&\quad + x^{28} + x^{22} + x^{20} + x^{16} + x^{14} + x^{8}, \\
p_6(x) &= x^{108} + x^{104} + x^{98} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} \\
&\quad + x^{72} + x^{70} + x^{66} + x^{58} + x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{34} + x^{32} \\
&\quad + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{10} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{104} + x^{100} + x^{98} + x^{96} + x^{88} + x^{84} + x^{78} + x^{76} + x^{72} + x^{64} + x^{62} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{38} + x^{34} + x^{26} + x^{20} + x^{16} \\
&\quad + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{104} + x^{80} + x^{64} + x^{48} + x^{40} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{94} + x^{91} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{37} + x^{35} + x^{33} + x^{30} \\
&\quad + x^{29} + x^{28} + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82} \\
&\quad + x^{81} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{61} + x^{57} + x^{56} + x^{54} + x^{53} + x^{49} + x^{48} + x^{46} + x^{45} + x^{41} \\
&\quad + x^{40} + x^{38} + x^{37} + x^{33} + x^{32} + x^{30} + x^{29} + x^{25} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{19} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{99} + x^{96} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{75} + x^{73} + x^{72} + x^{65} + x^{62} + x^{60} + x^{59} + x^{58} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{25} + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{83} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{77} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{59} + x^{58} + x^{57} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{47} + x^{46} + x^{45} + x^{39} + x^{38} + x^{37} + x^{29} + x^{28} + x^{27}
\end{aligned}$$

$$\begin{aligned}
&+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{16}+x^{15}+x^{13}+x^{12}+x^{11}\\
&+x^{9}+x^{8}+x^{7}+x^{5}+x^{4}+x^{3},\\
p_{12}(x)=\,&x^{95}+x^{93}+x^{92}+x^{87}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{79}+x^{77}\\
&+x^{76}+x^{71}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{55}\\
&+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}+x^{39}+x^{37}+x^{36}\\
&+x^{35}+x^{33}+x^{32}+x^{15}+x^{13}+x^{12}+x^{7}+x^{5}+x^{4}+x^{3}+x+1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0].

For $\phi(W^e,1,0)$:

$$\begin{aligned}
p_0(x)=\,&x^{108}+x^{106}+x^{103}+x^{102}+x^{101}+x^{100}+x^{97}+x^{94}+x^{91}+x^{90}\\
&+x^{88}+x^{86}+x^{85}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{74}+x^{73}+x^{71}\\
&+x^{70}+x^{69}+x^{64}+x^{63}+x^{62}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}\\
&+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{37}+x^{35}+x^{33}+x^{30}\\
&+x^{29}+x^{28}+x^{27}+x^{24}+x^{21}+x^{20}+x^{19}+x^{18},\\
p_3(x)=\,&x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{94}+x^{92}+x^{90}+x^{89}+x^{83}+x^{82}\\
&+x^{81}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{70}+x^{69}+x^{67}+x^{66}+x^{64}\\
&+x^{62}+x^{61}+x^{57}+x^{56}+x^{54}+x^{53}+x^{49}+x^{48}+x^{46}+x^{45}+x^{41}\\
&+x^{40}+x^{38}+x^{37}+x^{33}+x^{32}+x^{30}+x^{29}+x^{25}+x^{24}+x^{22}+x^{20}\\
&+x^{19}+x^{11}+x^{10}+x^{9},\\
p_6(x)=\,&x^{99}+x^{96}+x^{94}+x^{92}+x^{90}+x^{89}+x^{88}+x^{85}+x^{82}+x^{81}+x^{80}\\
&+x^{75}+x^{73}+x^{72}+x^{65}+x^{62}+x^{60}+x^{59}+x^{58}+x^{55}+x^{53}+x^{52}\\
&+x^{51}+x^{48}+x^{46}+x^{45}+x^{44}+x^{43}+x^{41}+x^{40}+x^{37}+x^{34}+x^{33}\\
&+x^{32}+x^{25}+x^{23}+x^{22}+x^{19}+x^{16}+x^{15}+x^{14}+x^{9}+x^{7}+x^{6},\\
p_9(x)=\,&x^{97}+x^{96}+x^{94}+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}+x^{83}+x^{82}+x^{80}\\
&+x^{78}+x^{77}+x^{73}+x^{72}+x^{70}+x^{68}+x^{67}+x^{59}+x^{58}+x^{57}+x^{53}\\
&+x^{52}+x^{51}+x^{47}+x^{46}+x^{45}+x^{39}+x^{38}+x^{37}+x^{29}+x^{28}+x^{27}\\
&+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{16}+x^{15}+x^{13}+x^{12}+x^{11}\\
&+x^{9}+x^{8}+x^{7}+x^{5}+x^{4}+x^{3},\\
p_{12}(x)=\,&x^{95}+x^{93}+x^{92}+x^{87}+x^{85}+x^{84}+x^{83}+x^{81}+x^{80}+x^{79}+x^{77}\\
&+x^{76}+x^{71}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{55}\\
&+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}+x^{39}+x^{37}+x^{36}\\
&+x^{35}+x^{33}+x^{32}+x^{15}+x^{13}+x^{12}+x^{7}+x^{5}+x^{4}+x^{3}+x+1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0].

For $\phi(W^e,1,1)$:

$$\begin{aligned}
p_0(x)=\,&x^{96}+x^{94}+x^{90}+x^{84}+x^{76}+x^{74}+x^{70}+x^{66}+x^{62}+x^{58}+x^{50}\\
&+x^{42}+x^{34}+x^{30}+x^{28}+x^{26}+x^{24},
\end{aligned}$$

$$
\begin{aligned}
p_3(x) &= x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{68} + x^{62} + x^{60} + x^{50} \\
&\quad + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{34} + x^{32} + x^{30} + x^{24} + x^{16} + x^{6}, \\
p_6(x) &= x^{88} + x^{86} + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{50} \\
&\quad + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{16} + x^{8} \\
&\quad + x^{4} + 1, \\
p_9(x) &= x^{86} + x^{72} + x^{64} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} \\
&\quad + x^{36} + x^{32} + x^{28} + x^{26} + x^{24} + x^{16} + x^{12} + x^{8} + x^{2}, \\
p_{12}(x) &= x^{86} + x^{82} + x^{80} + x^{70} + x^{66} + x^{64} + x^{54} + x^{50} + x^{48} + x^{38} + x^{34} \\
&\quad + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{77} + x^{76} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{47} + x^{42} + x^{40} + x^{37} + x^{30} + x^{26} + x^{25} + x^{23} + x^{22} \\
&\quad + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{103} + x^{102} + x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{85} \\
&\quad + x^{84} + x^{79} + x^{78} + x^{77} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} \\
&\quad + x^{60} + x^{59} + x^{51} + x^{50} + x^{49} + x^{48} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{34} + x^{33} + x^{30} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{15} + x^{13} + x^{10} \\
&\quad + x^{9} + x^{8}, \\
p_6(x) &= x^{103} + x^{102} + x^{101} + x^{97} + x^{96} + x^{95} + x^{88} + x^{86} + x^{85} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{77} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68} + x^{63} + x^{62} + x^{61} + x^{56} \\
&\quad + x^{55} + x^{52} + x^{50} + x^{49} + x^{44} + x^{41} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24} \\
&\quad + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{10} + x^{8} + x^{7} + x^{6} \\
&\quad + x^{3} + x + 1, \\
p_9(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} \\
&\quad + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{23} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{23} + x^{21} \\
&\quad + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3} + x
\end{aligned}
$$

$$+ 1,$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{108} + x^{106} + x^{105} + x^{104} \\
&+ x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} \\
&+ x^{88} + x^{83} + x^{81} + x^{80} + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{63} + x^{57} \\
&+ x^{54} + x^{52} + x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{38} + x^{35} \\
&+ x^{33} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24}, \\
p_3(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{97} + x^{96} + x^{95} + x^{92} + x^{87} \\
&+ x^{86} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} \\
&+ x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} \\
&+ x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{22} + x^{21} + x^{20} \\
&+ x^{17} + x^{15} + x^{14} + x^{13} + x^{9}, \\
p_6(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} \\
&+ x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{52} + x^{50} + x^{44} + x^{40} + x^{38} + x^{36} \\
&+ x^{32} + x^{28} + x^{22} + x^{20} + x^{16} + x^{14} + x^{10} + x^{6}, \\
p_9(x) &= x^{110} + x^{109} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{96} + x^{91} \\
&+ x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{75} + x^{74} + x^{73} + x^{70} \\
&+ x^{68} + x^{67} + x^{65} + x^{64} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{49} \\
&+ x^{48} + x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{39} + x^{35} + x^{30} + x^{29} + x^{26} \\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{7} \\
&+ x^{3}, \\
p_{12}(x) &= x^{112} + x^{108} + x^{100} + x^{96} + x^{88} + x^{72} + x^{56} + x^{48} + x^{44} + x^{40} + x^{36} \\
&+ x^{28} + x^{20} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} \\
&+ x^{84} + x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{66} + x^{62} \\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{35} \\
&+ x^{30} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_3(x) &= x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{81} + x^{70} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} \\
&+ x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} \\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{9},\\
p_6(x) &= x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{72} + x^{62} + x^{60} + x^{58}\\
&\quad + x^{52} + x^{48} + x^{46} + x^{44} + x^{40} + x^{34} + x^{32} + x^{22} + x^{16} + x^{14} + x^{6},\\
p_9(x) &= x^{98} + x^{97} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{83} + x^{82} + x^{81} + x^{78}\\
&\quad + x^{76} + x^{75} + x^{73} + x^{72} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57}\\
&\quad + x^{56} + x^{51} + x^{50} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{35} + x^{18}\\
&\quad + x^{17} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3},\\
p_{12}(x) &= x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{76} + x^{72} + x^{68} + x^{60} + x^{56} + x^{52}\\
&\quad + x^{44} + x^{40} + x^{32} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92}\\
&\quad + x^{91} + x^{89} + x^{88} + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{72}\\
&\quad + x^{70} + x^{69} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{53} + x^{52} + x^{50} + x^{49}\\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34}\\
&\quad + x^{33} + x^{32} + x^{30},\\
p_3(x) &= x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{93} + x^{92} + x^{89} + x^{86} + x^{85} + x^{82}\\
&\quad + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{68} + x^{65} + x^{64} + x^{61} + x^{59}\\
&\quad + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{24}\\
&\quad + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12},\\
p_6(x) &= x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{83}\\
&\quad + x^{82} + x^{80} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64}\\
&\quad + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48}\\
&\quad + x^{46} + x^{45} + x^{43} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29}\\
&\quad + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5}\\
&\quad + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81}\\
&\quad + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{51}\\
&\quad + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{23} + x^{21} + x^{20}\\
&\quad + x^{19} + x^{17} + x^{16} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89}\\
&\quad + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71}\\
&\quad + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52}\\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{23} + x^{21}
\end{aligned}
$$

$$+ x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^3 + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{77} + x^{76} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{47} + x^{42} + x^{40} + x^{37} + x^{30} + x^{26} + x^{25} + x^{23} + x^{22} \\
&\quad + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{103} + x^{102} + x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{85} \\
&\quad + x^{84} + x^{79} + x^{78} + x^{77} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} \\
&\quad + x^{60} + x^{59} + x^{51} + x^{50} + x^{49} + x^{48} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{34} + x^{33} + x^{30} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{15} + x^{13} + x^{10} \\
&\quad + x^9 + x^8, \\
p_6(x) &= x^{103} + x^{102} + x^{101} + x^{97} + x^{96} + x^{95} + x^{88} + x^{86} + x^{85} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{77} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68} + x^{63} + x^{62} + x^{61} + x^{56} \\
&\quad + x^{55} + x^{52} + x^{50} + x^{49} + x^{44} + x^{41} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24} \\
&\quad + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^6 \\
&\quad + x^3 + x + 1, \\
p_9(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} \\
&\quad + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{23} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{11} + x^9 + x^8 + x^7 + x^5 + x^4, \\
p_{12}(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{23} + x^{21} \\
&\quad + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^3 + x \\
&\quad + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{87} \\
&\quad + x^{84} + x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{66} + x^{62} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{35} \\
&\quad + x^{30} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_3(x) &= x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{81} + x^{70} + x^{69} + x^{68}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{67} + x^{66} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} \\
&+ x^{51} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} \\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} \\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{9}, \\
p_6(x) &= x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{72} + x^{62} + x^{60} + x^{58} \\
&+ x^{52} + x^{48} + x^{46} + x^{44} + x^{40} + x^{34} + x^{32} + x^{22} + x^{16} + x^{14} + x^{6}, \\
p_9(x) &= x^{98} + x^{97} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{83} + x^{82} + x^{81} + x^{78} \\
&+ x^{76} + x^{75} + x^{73} + x^{72} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{59} + x^{57} \\
&+ x^{56} + x^{51} + x^{50} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{35} + x^{18} \\
&+ x^{17} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{76} + x^{72} + x^{68} + x^{60} + x^{56} + x^{52} \\
&+ x^{44} + x^{40} + x^{32} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{108} + x^{106} + x^{105} + x^{104} \\
&+ x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} \\
&+ x^{88} + x^{83} + x^{81} + x^{80} + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{63} + x^{57} \\
&+ x^{54} + x^{52} + x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{38} + x^{35} \\
&+ x^{33} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24}, \\
p_3(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{97} + x^{96} + x^{95} + x^{92} + x^{87} \\
&+ x^{86} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} \\
&+ x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} \\
&+ x^{35} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{22} + x^{21} + x^{20} \\
&+ x^{17} + x^{15} + x^{14} + x^{13} + x^{9}, \\
p_6(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} \\
&+ x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{52} + x^{50} + x^{44} + x^{40} + x^{38} + x^{36} \\
&+ x^{32} + x^{28} + x^{22} + x^{20} + x^{16} + x^{14} + x^{10} + x^{6}, \\
p_9(x) &= x^{110} + x^{109} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{96} + x^{91} \\
&+ x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{75} + x^{74} + x^{73} + x^{70} \\
&+ x^{68} + x^{67} + x^{65} + x^{64} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{49} \\
&+ x^{48} + x^{46} + x^{45} + x^{43} + x^{41} + x^{40} + x^{39} + x^{35} + x^{30} + x^{29} + x^{26} \\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{9} + x^{8} + x^{7} \\
&+ x^{3}, \\
p_{12}(x) &= x^{112} + x^{108} + x^{100} + x^{96} + x^{88} + x^{72} + x^{56} + x^{48} + x^{44} + x^{40} + x^{36}
\end{aligned}
$$

$$+ x^{28} + x^{20} + x^{12} + x^{8} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{88} + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{72} \\
&\quad + x^{70} + x^{69} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{53} + x^{52} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{93} + x^{92} + x^{89} + x^{86} + x^{85} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{68} + x^{65} + x^{64} + x^{61} + x^{59} \\
&\quad + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{24} \\
&\quad + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12}, \\
p_6(x) &= x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{83} \\
&\quad + x^{82} + x^{80} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} \\
&\quad + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{46} + x^{45} + x^{43} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5} \\
&\quad + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} \\
&\quad + x^{80} + x^{75} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{59} + x^{57} + x^{56} + x^{51} \\
&\quad + x^{49} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{23} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{11} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{35} + x^{33} + x^{32} + x^{23} + x^{21} \\
&\quad + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3} + x \\
&\quad + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	262	[122, 43]	524	[358, 717]	786	[901, 115]
1	[3, 4]	263	[10, 428]	525	[718, 719]	787	[679, 902]

2	[5, 6]	264	[118, 413]	526	[32, 460]	788	[297, 284]
3	[7, 8]	265	[429, 430]	527	[39, 401]	789	[903, 640]
4	[9, 10]	266	[354, 150]	528	[434, 720]	790	[824, 904]
5	[11, 12]	267	[121, 431]	529	[720, 721]	791	[905, 861]
6	[13, 14]	268	[127, 163]	530	[406, 433]	792	[906, 907]
7	[15, 16]	269	[129, 196]	531	[720, 481]	793	[908, 909]
8	[17, 18]	270	[432, 433]	532	[391, 153]	794	[910, 550]
9	[19, 20]	271	[434, 435]	533	[123, 722]	795	[911, 912]
10	[21, 22]	272	[410, 177]	534	[723, 363]	796	[913, 338]
11	[23, 24]	273	[185, 155]	535	[724, 725]	797	[914, 276]
12	[25, 26]	274	[436, 372]	536	[475, 186]	798	[662, 867]
13	[27, 28]	275	[437, 355]	537	[726, 452]	799	[915, 870]
14	[29, 30]	276	[174, 438]	538	[468, 727]	800	[916, 578]
15	[31, 32]	277	[202, 435]	539	[728, 14]	801	[917, 508]
16	[9, 33]	278	[152, 392]	540	[46, 355]	802	[918, 236]
17	[34, 35]	279	[157, 411]	541	[729, 134]	803	[527, 919]
18	[36, 37]	280	[439, 440]	542	[478, 704]	804	[920, 84]
19	[38, 39]	281	[441, 125]	543	[147, 188]	805	[921, 922]
20	[40, 41]	282	[442, 443]	544	[721, 203]	806	[923, 607]
21	[14, 42]	283	[444, 445]	545	[730, 127]	807	[924, 591]
22	[36, 43]	284	[179, 446]	546	[731, 389]	808	[925, 272]
23	[44, 45]	285	[447, 448]	547	[732, 35]	809	[608, 531]
24	[46, 47]	286	[32, 194]	548	[168, 733]	810	[866, 833]
25	[48, 49]	287	[378, 449]	549	[734, 129]	811	[116, 354]
26	[50, 51]	288	[429, 400]	550	[735, 143]	812	[926, 429]
27	[52, 53]	289	[135, 450]	551	[736, 737]	813	[769, 927]
28	[54, 55]	290	[451, 452]	552	[492, 417]	814	[380, 474]
29	[56, 57]	291	[54, 376]	553	[164, 738]	815	[928, 792]
30	[58, 59]	292	[453, 144]	554	[53, 739]	816	[789, 482]
31	[60, 61]	293	[441, 131]	555	[740, 710]	817	[176, 124]
32	[62, 63]	294	[454, 390]	556	[137, 362]	818	[117, 150]
33	[64, 65]	295	[455, 456]	557	[741, 354]	819	[927, 461]
34	[66, 67]	296	[457, 363]	558	[742, 39]	820	[929, 179]
35	[68, 69]	297	[458, 415]	559	[38, 148]	821	[796, 445]
36	[70, 71]	298	[50, 410]	560	[142, 743]	822	[930, 411]
37	[72, 73]	299	[459, 185]	561	[719, 384]	823	[483, 747]
38	[74, 75]	300	[35, 460]	562	[193, 164]	824	[787, 176]
39	[76, 77]	301	[357, 461]	563	[744, 379]	825	[931, 381]
40	[78, 79]	302	[190, 201]	564	[482, 449]	826	[932, 129]
41	[80, 81]	303	[173, 202]	565	[484, 183]	827	[933, 934]
42	[82, 83]	304	[173, 434]	566	[474, 483]	828	[935, 763]
43	[84, 85]	305	[462, 425]	567	[745, 727]	829	[490, 151]
44	[86, 87]	306	[463, 119]	568	[746, 14]	830	[439, 936]
45	[88, 89]	307	[464, 465]	569	[454, 170]	831	[709, 937]

46	[90, 91]	308	[184, 154]	570	[747, 370]	832	[432, 407]
47	[92, 93]	309	[179, 466]	571	[59, 59]	833	[435, 721]
48	[94, 95]	310	[467, 468]	572	[748, 749]	834	[938, 939]
49	[96, 97]	311	[458, 469]	573	[181, 703]	835	[940, 493]
50	[98, 99]	312	[139, 427]	574	[701, 447]	836	[941, 445]
51	[100, 101]	313	[470, 471]	575	[732, 32]	837	[183, 703]
52	[102, 103]	314	[353, 117]	576	[362, 702]	838	[735, 743]
53	[104, 105]	315	[378, 472]	577	[8, 163]	839	[927, 491]
54	[106, 107]	316	[178, 473]	578	[750, 751]	840	[942, 471]
55	[108, 109]	317	[474, 475]	579	[752, 753]	841	[713, 725]
56	[110, 111]	318	[476, 362]	580	[754, 719]	842	[943, 474]
57	[112, 113]	319	[452, 425]	581	[755, 36]	843	[944, 39]
58	[58, 58]	320	[477, 398]	582	[756, 139]	844	[774, 745]
59	[114, 115]	321	[52, 200]	583	[389, 757]	845	[945, 484]
60	[116, 117]	322	[478, 479]	584	[758, 411]	846	[426, 927]
61	[118, 119]	323	[420, 480]	585	[388, 752]	847	[476, 138]
62	[120, 121]	324	[481, 203]	586	[759, 760]	848	[53, 768]
63	[122, 37]	325	[482, 472]	587	[761, 144]	849	[802, 179]
64	[123, 124]	326	[381, 483]	588	[202, 720]	850	[946, 773]
65	[125, 126]	327	[484, 181]	589	[408, 189]	851	[775, 417]
66	[7, 127]	328	[413, 485]	590	[701, 431]	852	[947, 140]
67	[128, 129]	329	[11, 486]	591	[419, 738]	853	[948, 468]
68	[130, 117]	330	[451, 372]	592	[141, 33]	854	[783, 805]
69	[131, 132]	331	[189, 371]	593	[40, 397]	855	[33, 418]
70	[133, 134]	332	[59, 58]	594	[176, 722]	856	[200, 739]
71	[135, 136]	333	[409, 371]	595	[442, 2]	857	[949, 471]
72	[137, 138]	334	[487, 488]	596	[762, 763]	858	[943, 381]
73	[139, 140]	335	[448, 136]	597	[483, 186]	859	[799, 752]
74	[141, 10]	336	[393, 191]	598	[764, 361]	860	[431, 448]
75	[142, 143]	337	[14, 361]	599	[765, 766]	861	[455, 937]
76	[144, 145]	338	[489, 179]	600	[767, 768]	862	[772, 360]
77	[131, 126]	339	[399, 430]	601	[769, 357]	863	[365, 50]
78	[146, 2]	340	[117, 490]	602	[770, 771]	864	[457, 702]
79	[147, 57]	341	[394, 192]	603	[772, 14]	865	[940, 417]
80	[148, 149]	342	[357, 491]	604	[762, 773]	866	[950, 763]
81	[150, 151]	343	[492, 493]	605	[774, 468]	867	[473, 446]
82	[152, 153]	344	[494, 45]	606	[724, 714]	868	[398, 410]
83	[154, 155]	345	[495, 13]	607	[775, 493]	869	[138, 363]
84	[156, 131]	346	[447, 135]	608	[721, 408]	870	[750, 803]
85	[157, 158]	347	[496, 438]	609	[452, 373]	871	[119, 779]
86	[159, 160]	348	[435, 481]	610	[776, 148]	872	[436, 452]
87	[161, 162]	349	[371, 201]	611	[777, 778]	873	[951, 385]
88	[163, 164]	350	[201, 434]	612	[448, 450]	874	[952, 953]
89	[165, 53]	351	[409, 190]	613	[413, 779]	875	[755, 122]

90	[166, 157]	352	[481, 408]	614	[780, 474]	876	[954, 744]
91	[167, 168]	353	[497, 264]	615	[47, 160]	877	[930, 158]
92	[169, 170]	354	[498, 499]	616	[489, 473]	878	[759, 199]
93	[51, 171]	355	[500, 501]	617	[354, 490]	879	[444, 737]
94	[172, 173]	356	[502, 503]	618	[365, 398]	880	[955, 956]
95	[174, 175]	357	[337, 106]	619	[781, 768]	881	[957, 52]
96	[49, 176]	358	[504, 505]	620	[782, 12]	882	[958, 443]
97	[51, 177]	359	[506, 507]	621	[49, 123]	883	[467, 745]
98	[178, 179]	360	[508, 509]	622	[744, 157]	884	[959, 956]
99	[180, 181]	361	[510, 511]	623	[402, 452]	885	[437, 47]
100	[47, 182]	362	[512, 513]	624	[783, 766]	886	[960, 784]
101	[180, 183]	363	[243, 514]	625	[784, 157]	887	[400, 455]
102	[184, 185]	364	[515, 516]	626	[723, 702]	888	[459, 154]
103	[186, 187]	365	[517, 518]	627	[161, 707]	889	[782, 486]
104	[56, 188]	366	[519, 520]	628	[193, 419]	890	[961, 10]
105	[189, 190]	367	[521, 522]	629	[761, 719]	891	[777, 934]
106	[191, 192]	368	[523, 524]	630	[785, 379]	892	[758, 158]
107	[8, 193]	369	[81, 525]	631	[786, 384]	893	[948, 745]
108	[35, 194]	370	[526, 527]	632	[742, 148]	894	[929, 473]
109	[195, 196]	371	[528, 529]	633	[198, 760]	895	[736, 445]
110	[197, 155]	372	[530, 531]	634	[756, 395]	896	[962, 2]
111	[128, 195]	373	[532, 533]	635	[197, 715]	897	[963, 484]
112	[198, 199]	374	[534, 535]	636	[464, 771]	898	[964, 953]
113	[165, 200]	375	[536, 537]	637	[787, 123]	899	[965, 440]
114	[201, 202]	376	[101, 538]	638	[195, 788]	900	[804, 122]
115	[203, 189]	377	[539, 540]	639	[381, 475]	901	[950, 773]
116	[204, 111]	378	[541, 542]	640	[789, 378]	902	[703, 479]
117	[205, 206]	379	[543, 544]	641	[719, 145]	903	[453, 719]
118	[207, 208]	380	[545, 546]	642	[119, 485]	904	[747, 187]
119	[209, 210]	381	[547, 548]	643	[790, 139]	905	[966, 795]
120	[211, 212]	382	[549, 550]	644	[151, 55]	906	[156, 125]
121	[213, 214]	383	[551, 552]	645	[708, 480]	907	[154, 715]
122	[215, 216]	384	[553, 554]	646	[718, 144]	908	[946, 763]
123	[79, 217]	385	[555, 278]	647	[791, 792]	909	[958, 2]
124	[71, 218]	386	[556, 209]	648	[793, 717]	910	[396, 126]
125	[219, 220]	387	[557, 558]	649	[477, 50]	911	[150, 54]
126	[221, 222]	388	[559, 560]	650	[752, 757]	912	[784, 379]
127	[223, 224]	389	[561, 239]	651	[794, 795]	913	[967, 384]
128	[225, 226]	390	[562, 563]	652	[796, 737]	914	[935, 773]
129	[227, 228]	391	[564, 565]	653	[797, 751]	915	[412, 119]
130	[229, 230]	392	[566, 567]	654	[745, 798]	916	[968, 472]
131	[231, 232]	393	[568, 336]	655	[799, 389]	917	[969, 203]
132	[233, 234]	394	[569, 570]	656	[414, 469]	918	[734, 195]
133	[235, 236]	395	[505, 571]	657	[800, 440]	919	[460, 152]

134	[237, 238]	396	[572, 553]	658	[801, 772]	920	[785, 157]
135	[239, 240]	397	[309, 573]	659	[475, 747]	921	[970, 787]
136	[107, 241]	398	[516, 574]	660	[468, 798]	922	[183, 478]
137	[242, 243]	399	[575, 576]	661	[802, 473]	923	[971, 173]
138	[244, 245]	400	[4, 577]	662	[797, 803]	924	[972, 939]
139	[246, 247]	401	[578, 579]	663	[168, 471]	925	[973, 440]
140	[248, 249]	402	[537, 245]	664	[33, 428]	926	[974, 893]
141	[250, 251]	403	[580, 581]	665	[430, 709]	927	[533, 975]
142	[252, 29]	404	[254, 582]	666	[487, 375]	928	[976, 291]
143	[253, 254]	405	[565, 583]	667	[403, 394]	929	[977, 841]
144	[255, 256]	406	[584, 585]	668	[200, 768]	930	[978, 333]
145	[232, 227]	407	[586, 587]	669	[780, 381]	931	[979, 670]
146	[257, 258]	408	[588, 528]	670	[366, 195]	932	[980, 648]
147	[259, 260]	409	[271, 589]	671	[121, 447]	933	[981, 255]
148	[261, 262]	410	[518, 590]	672	[166, 379]	934	[855, 982]
149	[75, 263]	411	[591, 312]	673	[804, 36]	935	[983, 98]
150	[264, 265]	412	[592, 593]	674	[130, 354]	936	[26, 96]
151	[206, 266]	413	[594, 100]	675	[385, 792]	937	[577, 253]
152	[63, 267]	414	[595, 596]	676	[127, 193]	938	[984, 827]
153	[268, 269]	415	[567, 597]	677	[473, 466]	939	[868, 290]
154	[270, 271]	416	[598, 4]	678	[765, 805]	940	[985, 986]
155	[272, 273]	417	[599, 321]	679	[790, 395]	941	[987, 843]
156	[274, 275]	418	[20, 506]	680	[740, 369]	942	[988, 28]
157	[276, 277]	419	[18, 300]	681	[463, 413]	943	[989, 601]
158	[278, 279]	420	[600, 601]	682	[700, 121]	944	[990, 991]
159	[280, 103]	421	[602, 603]	683	[470, 733]	945	[992, 993]
160	[281, 64]	422	[604, 304]	684	[389, 753]	946	[994, 995]
161	[282, 283]	423	[605, 606]	685	[806, 165]	947	[851, 349]
162	[284, 285]	424	[607, 608]	686	[194, 391]	948	[996, 838]
163	[16, 286]	425	[245, 609]	687	[807, 792]	949	[997, 82]
164	[287, 288]	426	[610, 611]	688	[490, 54]	950	[998, 558]
165	[113, 289]	427	[612, 261]	689	[776, 39]	951	[999, 556]
166	[290, 291]	428	[251, 613]	690	[793, 359]	952	[1000, 611]
167	[292, 293]	429	[614, 615]	691	[129, 788]	953	[597, 102]
168	[294, 295]	430	[616, 617]	692	[781, 739]	954	[1001, 543]
169	[296, 297]	431	[212, 618]	693	[770, 465]	955	[1002, 838]
170	[298, 299]	432	[619, 61]	694	[726, 372]	956	[659, 862]
171	[300, 301]	433	[620, 621]	695	[808, 192]	957	[1003, 104]
172	[302, 277]	434	[333, 302]	696	[809, 721]	958	[1004, 993]
173	[303, 304]	435	[571, 608]	697	[810, 155]	959	[1005, 1006]
174	[305, 306]	436	[622, 513]	698	[731, 752]	960	[1007, 343]
175	[307, 308]	437	[623, 318]	699	[811, 616]	961	[1008, 524]
176	[95, 303]	438	[624, 625]	700	[812, 813]	962	[1009, 316]
177	[99, 309]	439	[626, 627]	701	[814, 815]	963	[1010, 316]

178	[310, 311]	440	[628, 629]	702	[816, 817]	964	[1011, 653]
179	[312, 313]	441	[630, 221]	703	[315, 818]	965	[1012, 991]
180	[314, 315]	442	[631, 632]	704	[676, 819]	966	[1013, 503]
181	[316, 317]	443	[633, 634]	705	[820, 535]	967	[552, 58]
182	[318, 319]	444	[635, 16]	706	[821, 530]	968	[1014, 612]
183	[258, 320]	445	[636, 637]	707	[241, 628]	969	[352, 271]
184	[321, 322]	446	[236, 638]	708	[822, 670]	970	[1015, 901]
185	[323, 324]	447	[306, 639]	709	[615, 566]	971	[509, 544]
186	[325, 326]	448	[214, 526]	710	[823, 824]	972	[1016, 870]
187	[327, 328]	449	[640, 641]	711	[825, 306]	973	[1017, 75]
188	[329, 330]	450	[639, 642]	712	[826, 827]	974	[1018, 701]
189	[331, 332]	451	[643, 644]	713	[828, 270]	975	[928, 386]
190	[105, 333]	452	[645, 351]	714	[263, 829]	976	[729, 404]
191	[334, 335]	453	[646, 232]	715	[335, 21]	977	[705, 413]
192	[336, 337]	454	[647, 648]	716	[830, 831]	978	[962, 443]
193	[338, 339]	455	[341, 649]	717	[677, 536]	979	[711, 121]
194	[61, 340]	456	[326, 650]	718	[89, 553]	980	[961, 33]
195	[234, 341]	457	[651, 532]	719	[832, 114]	981	[146, 443]
196	[226, 342]	458	[652, 347]	720	[304, 588]	982	[194, 152]
197	[343, 331]	459	[299, 272]	721	[350, 833]	983	[1019, 722]
198	[344, 233]	460	[67, 268]	722	[97, 594]	984	[716, 126]
199	[345, 346]	461	[653, 654]	723	[834, 644]	985	[1020, 722]
200	[347, 348]	462	[596, 588]	724	[835, 836]	986	[944, 148]
201	[332, 349]	463	[655, 656]	725	[538, 837]	987	[424, 373]
202	[277, 350]	464	[657, 521]	726	[621, 532]	988	[1021, 749]
203	[351, 352]	465	[658, 659]	727	[838, 839]	989	[741, 117]
204	[31, 35]	466	[311, 660]	728	[587, 295]	990	[356, 927]
205	[353, 354]	467	[661, 662]	729	[840, 841]	991	[933, 778]
206	[355, 160]	468	[511, 663]	730	[842, 843]	992	[1022, 32]
207	[356, 357]	469	[664, 665]	731	[844, 87]	993	[1023, 745]
208	[358, 359]	470	[666, 510]	732	[845, 846]	994	[1024, 361]
209	[360, 361]	471	[293, 76]	733	[847, 848]	995	[1023, 468]
210	[362, 363]	472	[667, 668]	734	[849, 297]	996	[382, 927]
211	[364, 365]	473	[641, 287]	735	[850, 851]	997	[973, 936]
212	[366, 129]	474	[669, 325]	736	[852, 853]	998	[767, 739]
213	[34, 32]	475	[670, 671]	737	[854, 646]	999	[746, 360]
214	[191, 367]	476	[672, 673]	738	[286, 855]	1000	[1025, 192]
215	[368, 369]	477	[674, 675]	739	[289, 856]	1001	[1026, 721]
216	[186, 370]	478	[22, 676]	740	[857, 251]	1002	[1027, 43]
217	[371, 173]	479	[317, 677]	741	[858, 260]	1003	[1028, 189]
218	[372, 373]	480	[678, 679]	742	[859, 578]	1004	[810, 715]
219	[374, 375]	481	[30, 331]	743	[660, 860]	1005	[1029, 472]
220	[151, 376]	482	[680, 248]	744	[637, 861]	1006	[706, 162]
221	[377, 378]	483	[681, 327]	745	[817, 564]	1007	[1030, 203]

222	[379, 158]	484	[682, 683]	746	[862, 510]	1008	[932, 195]
223	[380, 381]	485	[656, 684]	747	[262, 863]	1009	[462, 373]
224	[125, 132]	486	[685, 686]	748	[864, 662]	1010	[931, 474]
225	[382, 357]	487	[687, 593]	749	[829, 643]	1011	[1031, 160]
226	[383, 41]	488	[688, 27]	750	[865, 866]	1012	[1032, 43]
227	[144, 384]	489	[689, 690]	751	[867, 868]	1013	[942, 733]
228	[385, 386]	490	[111, 108]	752	[73, 869]	1014	[972, 45]
229	[387, 365]	491	[550, 691]	753	[870, 871]	1015	[1033, 189]
230	[388, 389]	492	[692, 616]	754	[238, 220]	1016	[949, 733]
231	[169, 390]	493	[693, 694]	755	[872, 873]	1017	[807, 386]
232	[391, 392]	494	[695, 560]	756	[634, 612]	1018	[1034, 698]
233	[393, 394]	495	[696, 604]	757	[560, 80]	1019	[896, 332]
234	[395, 140]	496	[697, 698]	758	[836, 529]	1020	[1035, 589]
235	[396, 132]	497	[699, 429]	759	[874, 875]	1021	[1036, 891]
236	[383, 397]	498	[700, 701]	760	[876, 562]	1022	[1037, 585]
237	[398, 51]	499	[138, 702]	761	[308, 831]	1023	[1038, 1006]
238	[13, 360]	500	[374, 488]	762	[877, 516]	1024	[909, 302]
239	[355, 182]	501	[703, 704]	763	[878, 622]	1025	[1039, 322]
240	[399, 400]	502	[705, 119]	764	[879, 350]	1026	[531, 351]
241	[148, 401]	503	[706, 707]	765	[880, 667]	1027	[1040, 97]
242	[402, 372]	504	[416, 493]	766	[881, 688]	1028	[247, 105]
243	[403, 191]	505	[708, 421]	767	[882, 352]	1029	[1041, 904]
244	[133, 404]	506	[39, 149]	768	[93, 561]	1030	[348, 324]
245	[164, 405]	507	[400, 709]	769	[883, 653]	1031	[1042, 289]
246	[406, 407]	508	[496, 175]	770	[884, 885]	1032	[1043, 902]
247	[408, 409]	509	[203, 409]	771	[886, 887]	1033	[349, 348]
248	[410, 171]	510	[419, 405]	772	[888, 82]	1034	[1022, 35]
249	[379, 411]	511	[360, 42]	773	[889, 888]	1035	[941, 737]
250	[412, 413]	512	[368, 710]	774	[890, 891]	1036	[791, 386]
251	[414, 415]	513	[709, 456]	775	[892, 893]	1037	[1018, 121]
252	[416, 417]	514	[372, 425]	776	[894, 627]	1038	[712, 752]
253	[10, 418]	515	[711, 701]	777	[895, 896]	1039	[1021, 956]
254	[163, 419]	516	[423, 33]	778	[654, 298]	1040	[794, 953]
255	[420, 421]	517	[120, 701]	779	[593, 664]	1041	[966, 953]
256	[190, 173]	518	[394, 367]	780	[897, 681]	1042	[748, 956]
257	[422, 140]	519	[712, 389]	781	[283, 528]	1043	[938, 45]
258	[423, 10]	520	[713, 714]	782	[898, 816]		
259	[424, 425]	521	[377, 482]	783	[899, 900]		
260	[426, 357]	522	[185, 715]	784	[582, 591]		
261	[395, 427]	523	[716, 132]	785	[625, 278]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	175	0	350	1	525	0	700	1	875	1

1	0	176	0	351	1	526	0	701	1	876	0
2	1	177	0	352	0	527	1	702	0	877	0
3	0	178	0	353	0	528	1	703	1	878	1
4	0	179	1	354	1	529	0	704	1	879	0
5	1	180	0	355	1	530	1	705	1	880	0
6	1	181	0	356	1	531	0	706	0	881	0
7	0	182	1	357	0	532	0	707	0	882	1
8	0	183	1	358	0	533	1	708	0	883	0
9	0	184	1	359	1	534	1	709	0	884	1
10	0	185	1	360	1	535	1	710	0	885	0
11	1	186	0	361	0	536	1	711	0	886	1
12	0	187	1	362	1	537	0	712	1	887	0
13	1	188	1	363	1	538	1	713	0	888	0
14	0	189	0	364	0	539	1	714	0	889	0
15	0	190	0	365	0	540	1	715	0	890	1
16	0	191	0	366	1	541	1	716	1	891	1
17	0	192	0	367	1	542	0	717	0	892	1
18	0	193	1	368	1	543	1	718	0	893	0
19	0	194	1	369	1	544	1	719	0	894	1
20	0	195	0	370	0	545	1	720	0	895	1
21	0	196	1	371	1	546	1	721	1	896	0
22	0	197	0	372	1	547	1	722	0	897	0
23	1	198	0	373	0	548	1	723	1	898	0
24	1	199	1	374	1	549	0	724	1	899	1
25	0	200	1	375	1	550	1	725	1	900	0
26	0	201	1	376	0	551	1	726	0	901	1
27	1	202	0	377	1	552	0	727	1	902	1
28	0	203	1	378	1	553	1	728	1	903	0
29	0	204	0	379	1	554	0	729	1	904	1
30	0	205	0	380	1	555	0	730	1	905	1
31	0	206	1	381	1	556	0	731	1	906	1
32	0	207	1	382	0	557	1	732	1	907	0
33	1	208	0	383	1	558	0	733	1	908	1
34	0	209	1	384	1	559	0	734	0	909	1
35	1	210	1	385	0	560	0	735	1	910	0
36	0	211	0	386	0	561	0	736	1	911	1
37	0	212	1	387	1	562	1	737	1	912	0
38	0	213	0	388	0	563	1	738	0	913	0
39	1	214	0	389	0	564	0	739	1	914	0
40	0	215	1	390	1	565	1	740	0	915	0
41	0	216	0	391	0	566	0	741	1	916	0
42	1	217	1	392	0	567	0	742	0	917	0
43	1	218	1	393	0	568	0	743	1	918	0
44	1	219	1	394	1	569	1	744	1	919	0

45	0	220	1	395	0	570	1	745	0	920	0
46	1	221	1	396	0	571	1	746	0	921	1
47	0	222	1	397	1	572	0	747	1	922	1
48	0	223	1	398	1	573	0	748	0	923	1
49	0	224	1	399	1	574	1	749	0	924	0
50	0	225	0	400	0	575	1	750	1	925	0
51	0	226	1	401	1	576	1	751	0	926	1
52	1	227	1	402	0	577	0	752	1	927	1
53	0	228	0	403	1	578	1	753	1	928	1
54	0	229	1	404	0	579	1	754	1	929	1
55	1	230	0	405	1	580	1	755	1	930	0
56	0	231	0	406	1	581	1	756	0	931	0
57	0	232	0	407	1	582	0	757	0	932	1
58	0	233	0	408	0	583	0	758	1	933	0
59	1	234	0	409	1	584	1	759	1	934	1
60	0	235	0	410	1	585	0	760	0	935	0
61	1	236	1	411	0	586	1	761	1	936	0
62	0	237	1	412	0	587	1	762	0	937	0
63	1	238	1	413	0	588	0	763	1	938	1
64	1	239	1	414	1	589	0	764	0	939	1
65	1	240	1	415	0	590	1	765	0	940	1
66	0	241	0	416	0	591	0	766	0	941	1
67	0	242	0	417	0	592	0	767	1	942	1
68	1	243	1	418	0	593	0	768	0	943	1
69	0	244	0	419	0	594	0	769	0	944	1
70	0	245	1	420	1	595	1	770	1	945	1
71	1	246	1	421	1	596	0	771	1	946	1
72	0	247	0	422	0	597	0	772	0	947	1
73	1	248	1	423	1	598	0	773	0	948	0
74	0	249	1	424	1	599	0	774	1	949	0
75	0	250	0	425	1	600	1	775	1	950	1
76	1	251	1	426	1	601	0	776	1	951	0
77	0	252	0	427	0	602	1	777	1	952	1
78	0	253	0	428	1	603	0	778	0	953	0
79	1	254	0	429	0	604	0	779	0	954	0
80	0	255	1	430	1	605	1	780	0	955	0
81	1	256	0	431	1	606	1	781	0	956	1
82	1	257	0	432	0	607	1	782	0	957	0
83	0	258	1	433	0	608	1	783	1	958	1
84	1	259	1	434	1	609	0	784	0	959	1
85	0	260	1	435	1	610	1	785	0	960	1
86	1	261	0	436	1	611	1	786	1	961	1
87	1	262	1	437	0	612	0	787	1	962	0
88	0	263	0	438	1	613	0	788	0	963	0

89	0	264	1	439	1	614	0	789	0	964	0
90	1	265	0	440	1	615	0	790	1	965	1
91	0	266	1	441	0	616	1	791	1	966	1
92	0	267	0	442	1	617	1	792	1	967	0
93	0	268	1	443	0	618	0	793	1	968	0
94	0	269	1	444	0	619	0	794	0	969	0
95	0	270	0	445	0	620	0	795	1	970	1
96	0	271	1	446	1	621	0	796	0	971	1
97	0	272	1	447	0	622	1	797	0	972	0
98	0	273	1	448	0	623	0	798	0	973	0
99	0	274	1	449	0	624	1	799	0	974	1
100	0	275	0	450	1	625	0	800	0	975	1
101	0	276	0	451	1	626	1	801	0	976	1
102	1	277	0	452	0	627	1	802	0	977	1
103	0	278	1	453	0	628	1	803	1	978	0
104	0	279	0	454	1	629	1	804	0	979	0
105	0	280	1	455	1	630	0	805	1	980	1
106	0	281	0	456	1	631	1	806	1	981	0
107	0	282	1	457	0	632	0	807	0	982	1
108	1	283	0	458	0	633	0	808	0	983	0
109	0	284	1	459	0	634	0	809	1	984	1
110	0	285	0	460	0	635	0	810	1	985	1
111	0	286	0	461	0	636	0	811	0	986	1
112	0	287	1	462	0	637	1	812	1	987	1
113	0	288	0	463	0	638	0	813	0	988	1
114	1	289	1	464	0	639	1	814	1	989	1
115	1	290	1	465	0	640	0	815	1	990	1
116	0	291	0	466	0	641	0	816	0	991	0
117	0	292	0	467	0	642	1	817	0	992	1
118	1	293	0	468	1	643	1	818	0	993	1
119	1	294	1	469	1	644	1	819	1	994	1
120	0	295	1	470	0	645	0	820	1	995	1
121	0	296	0	471	0	646	0	821	0	996	0
122	1	297	0	472	1	647	1	822	0	997	0
123	1	298	0	473	0	648	1	823	0	998	1
124	1	299	0	474	0	649	1	824	1	999	0
125	1	300	1	475	1	650	1	825	0	1000	1
126	1	301	0	476	1	651	0	826	1	1001	0
127	1	302	0	477	1	652	0	827	0	1002	0
128	0	303	0	478	0	653	0	828	0	1003	0
129	1	304	0	479	0	654	0	829	0	1004	1
130	1	305	0	480	0	655	0	830	1	1005	1
131	0	306	0	481	0	656	1	831	0	1006	0
132	0	307	0	482	0	657	0	832	0	1007	1

133	0	308	1	483	0	658	0	833	1	1008	1
134	1	309	1	484	1	659	1	834	1	1009	0
135	1	310	0	485	1	660	1	835	1	1010	0
136	0	311	0	486	1	661	0	836	1	1011	0
137	0	312	1	487	0	662	0	837	1	1012	1
138	0	313	0	488	0	663	1	838	1	1013	1
139	1	314	0	489	1	664	1	839	1	1014	0
140	1	315	1	490	0	665	1	840	1	1015	1
141	0	316	0	491	1	666	0	841	0	1016	0
142	0	317	0	492	0	667	1	842	1	1017	0
143	0	318	1	493	1	668	1	843	1	1018	1
144	1	319	0	494	0	669	0	844	1	1019	0
145	0	320	1	495	1	670	1	845	1	1020	1
146	0	321	1	496	1	671	0	846	1	1021	1
147	1	322	0	497	0	672	1	847	1	1022	1
148	0	323	1	498	1	673	0	848	0	1023	1
149	0	324	0	499	0	674	1	849	0	1024	1
150	1	325	0	500	1	675	0	850	1	1025	1
151	1	326	1	501	1	676	1	851	1	1026	0
152	1	327	1	502	1	677	0	852	1	1027	0
153	1	328	0	503	0	678	0	853	0	1028	0
154	0	329	1	504	0	679	1	854	1	1029	1
155	1	330	1	505	0	680	0	855	1	1030	1
156	1	331	0	506	1	681	0	856	1	1031	0
157	0	332	1	507	0	682	1	857	0	1032	1
158	1	333	1	508	1	683	0	858	1	1033	1
159	1	334	0	509	1	684	0	859	0	1034	1
160	0	335	0	510	0	685	1	860	1	1035	1
161	1	336	0	511	1	686	1	861	1	1036	1
162	1	337	0	512	1	687	0	862	0	1037	1
163	0	338	1	513	0	688	0	863	0	1038	1
164	1	339	1	514	1	689	1	864	0	1039	1
165	0	340	0	515	0	690	1	865	1	1040	0
166	1	341	1	516	1	691	1	866	1	1041	1
167	0	342	0	517	0	692	0	867	0	1042	0
168	1	343	0	518	1	693	1	868	1	1043	1
169	0	344	0	519	1	694	0	869	0		
170	0	345	1	520	0	695	0	870	1		
171	1	346	0	521	1	696	1	871	1		
172	0	347	1	522	1	697	1	872	1		
173	0	348	1	523	1	698	1	873	0		
174	0	349	1	524	0	699	0	874	1		

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