PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^4+z^3+z^2, z^2+z)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^4+z^3+z^2, z^2+z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z), t_1(z), t_2(z), \ldots) = (a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^4+z^3+z^2, z^2+z)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z) &= z^{14}+z^{11}+z^{10}+z^8+z^7+z^2+z+1,\\
p_1(z) &= z^{18}+z^{17}+z^{13}+z^{12}+z^{10}+z^7+z^4+z^2,\\
p_2(z) &= z^{20}+z^{16}+z^{14}+z^4,\\
p_3(z) &= z^{18}+z^{17}+z^{13}+z^{12}+z^{10}+z^7+z^4+z^2,\\
p_4(z) &= z^{16}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^4+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z) &= z^{14}+z^{12}+z^{11}+z^{10}+z^7+z^2+z,\\
p_1(z) &= z^{18}+z^{17}+z^{13}+z^{12}+z^{10}+z^7+z^4+z^2,\\
p_2(z) &= z^{20}+z^{16}+z^{14}+z^4,\\
p_3(z) &= z^{18}+z^{17}+z^{13}+z^{12}+z^{10}+z^7+z^4+z^2,\\
p_4(z) &= z^{16}+z^{11}+z^{10}+z^7+z^4+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{3u}M^e[:6u] + x^{6u}M^e[:3u] \\ & + x^{7u}M^e[:2u] + x^{4u}M^e[:6u] + x^{7u}M^e[:3u] + x^{8u}M^e[:2u] \\ & + x^{7u}M^e[:5u] + x^{11u}M^e[:u] + (x^{6u}+x^{9u}+x^{10u})R^e, \\ W^e[:12u] = {} & W^e[:6u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{3u}W^e[:6u] + x^{6u}W^e[:3u] \\ & + x^{7u}W^e[:2u] + x^{4u}W^e[:6u] + x^{7u}W^e[:3u] + x^{8u}W^e[:2u] \\ & + x^{7u}W^e[:5u] + x^{11u}W^e[:u] + (x^{6u}+x^{9u}+x^{10u})R^e. \end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u] \\
&\quad + x^{7u}M^o[:2u] + x^{4u}M^o[:6u] + x^{7u}M^o[:3u] + x^{8u}M^o[:2u] \\
&\quad + x^{7u}M^o[:5u] + x^{11u}M^o[:u] + (x^{6u} + x^{9u} + x^{10u})R^o, \\
W^o[:12u] &= W^o[:6u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u] \\
&\quad + x^{7u}W^o[:2u] + x^{4u}W^o[:6u] + x^{7u}W^o[:3u] + x^{8u}W^o[:2u] \\
&\quad + x^{7u}W^o[:5u] + x^{11u}W^o[:u] + (x^{6u} + x^{9u} + x^{10u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] &= T[:6u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{7u}T[:2u] \\
&\quad + x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{7u}T[:5u] + x^{11u}T[:u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
0 &= x^{6u}T[:u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] + T[6u:7u], \\
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{4u}T[4u:5u] \\
&\quad + x^{3u}T[5u:6u] + T[8u:9u], \\
0 &= x^{8u}T[u:2u] + x^{4u}T[5u:6u] + T[9u:10u], \\
0 &= x^{7u}T[3u:4u] + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{7u}T[4u:5u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1000w]_2],
\end{aligned} \tag{2.9}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[11w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1000w]_2],
\end{aligned} \tag{2.15}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[11w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1196 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)), \tag{2.19}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.20}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.22}$$

$$0 = \tau(A(s,11)) + \tau(A(s,1010)), \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)), \tag{2.25}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.27}$$

$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.28}$$

$$0 = \tau(A(s,11)) + \tau(A(s,1010)), \tag{2.29}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{30}), E_{30}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 15$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{6u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u} M^o[:3u] + x^{4u} M^o[:2u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u} W^o[:3u] + x^{4u} W^o[:2u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = {} & \big(W^e[:6v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ & + x^{3v}M^e[:3v] + x^{4v}M^e[:2v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v] + x^{8v}W^e[:4v]M^e[:4v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v : (n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v : (n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j,k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{20} + x^{19} + x^{18} + x^{13} + x^{12} + x^{10} + x^9 + x^6,\\
q_1(x) &= x^{18} + x^{16} + x^{13} + x^{10} + x^8 + x^7 + x^3 + x^2,\\
q_2(x) &= x^{16} + x^6 + x^4 + 1,\\
q_3(x) &= x^{18} + x^{16} + x^{13} + x^{10} + x^8 + x^7 + x^3 + x^2,\\
q_4(x) &= x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) = x^{102} &+ x^{98} + x^{92} + x^{88} + x^{84} + x^{80} + x^{74} + x^{68} + x^{66} + x^{58} + x^{56}\\
&+ x^{44} + x^{42} + x^{34} + x^{28} + x^{26} + x^{24},\\
p_3(x) = x^{94} &+ x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64}\\
&+ x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{36} + x^{32} + x^{30}\\
&+ x^{24} + x^{18},\\
p_6(x) = x^{94} &+ x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{72} + x^{70} + x^{64} + x^{62}\\
&+ x^{60} + x^{58} + x^{52} + x^{48} + x^{46} + x^{42} + x^{40} + x^{30} + x^{28} + x^{24} + x^{18}\\
&+ x^{14} + x^{10} + x^2 + 1,\\
p_9(x) = x^{92} &+ x^{90} + x^{86} + x^{82} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{60} + x^{56}\\
&+ x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{28} + x^{22} + x^{18} + x^{16} + x^{12}\\
&+ x^8 + x^4,\\
p_{12}(x) = x^{92} &+ x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34}
\end{aligned}$$

$$+ x^{32} + x^{12} + x^{10} + x^8 + x^4 + x^2 + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} \\
&\quad + x^{86} + x^{81} + x^{80} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{65} + x^{62} \\
&\quad + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{39} + x^{35} + x^{30}, \\
p_3(x) &= x^{101} + x^{99} + x^{98} + x^{91} + x^{88} + x^{87} + x^{86} + x^{75} + x^{72} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{53} + x^{51} + x^{50} + x^{45} + x^{42} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} \\
&\quad + x^{18}, \\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{81} + x^{79} + x^{78} \\
&\quad + x^{76} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} + x^{52} + x^{49} + x^{46} \\
&\quad + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{25} + x^{22} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_9(x) &= x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} \\
&\quad + x^{54} + x^{53} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{27} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} \\
&\quad + x^{14} + x^{13} + x^{10} + x^8 + x^7 + x^6, \\
p_{12}(x) &= x^{95} + x^{92} + x^{83} + x^{80} + x^{47} + x^{44} + x^{35} + x^{32} + x^{15} + x^{12} + x^3 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} \\
&\quad + x^{86} + x^{81} + x^{80} + x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{65} + x^{62} \\
&\quad + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{39} + x^{35} + x^{30}, \\
p_3(x) &= x^{101} + x^{99} + x^{98} + x^{91} + x^{88} + x^{87} + x^{86} + x^{75} + x^{72} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{53} + x^{51} + x^{50} + x^{45} + x^{42} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} \\
&\quad + x^{18}, \\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{81} + x^{79} + x^{78} \\
&\quad + x^{76} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} + x^{52} + x^{49} + x^{46} \\
&\quad + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{25} + x^{22} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12},
\end{aligned}$$

$$
\begin{aligned}
p_9(x) &= x^{97}+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}+x^{85}+x^{82}+x^{80}+x^{79}+x^{78}\\
&\quad +x^{77}+x^{75}+x^{74}+x^{73}+x^{71}+x^{70}+x^{63}+x^{61}+x^{60}+x^{57}+x^{55}\\
&\quad +x^{54}+x^{53}+x^{50}+x^{48}+x^{47}+x^{46}+x^{43}+x^{40}+x^{38}+x^{37}+x^{36}\\
&\quad +x^{35}+x^{32}+x^{31}+x^{30}+x^{27}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{15}\\
&\quad +x^{14}+x^{13}+x^{10}+x^{8}+x^{7}+x^{6},\\
p_{12}(x) &= x^{95}+x^{92}+x^{83}+x^{80}+x^{47}+x^{44}+x^{35}+x^{32}+x^{15}+x^{12}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{114}+x^{110}+x^{108}+x^{104}+x^{102}+x^{100}+x^{96}+x^{92}+x^{82}+x^{78}\\
&\quad +x^{72}+x^{68}+x^{66}+x^{64}+x^{52}+x^{44}+x^{42}+x^{38}+x^{36},\\
p_3(x) &= x^{102}+x^{92}+x^{90}+x^{86}+x^{84}+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}+x^{72}\\
&\quad +x^{68}+x^{66}+x^{60}+x^{56}+x^{52}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{36}\\
&\quad +x^{32}+x^{28}+x^{18}+x^{12},\\
p_6(x) &= x^{102}+x^{98}+x^{78}+x^{76}+x^{68}+x^{58}+x^{52}+x^{42}+x^{40}+x^{38}+x^{36}\\
&\quad +x^{34}+x^{28}+x^{26}+x^{22}+x^{20}+x^{18}+x^{16}+x^{10}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{98}+x^{96}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{80}+x^{76}+x^{66}+x^{60}\\
&\quad +x^{56}+x^{54}+x^{50}+x^{38}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{18}+x^{16}\\
&\quad +x^{10}+x^{4},\\
p_{12}(x) &= x^{98}+x^{96}+x^{90}+x^{88}+x^{82}+x^{80}+x^{50}+x^{48}+x^{42}+x^{40}+x^{34}\\
&\quad +x^{32}+x^{18}+x^{16}+x^{10}+x^{8}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{114}+x^{110}+x^{108}+x^{106}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{76}+x^{72}\\
&\quad +x^{66}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{32}+x^{26}\\
&\quad +x^{24},\\
p_3(x) &= x^{102}+x^{94}+x^{88}+x^{86}+x^{80}+x^{72}+x^{70}+x^{68}+x^{62}+x^{60}+x^{58}\\
&\quad +x^{56}+x^{46}+x^{44}+x^{36}+x^{34}+x^{32}+x^{28}+x^{24}+x^{22}+x^{20}+x^{18},\\
p_6(x) &= x^{102}+x^{98}+x^{94}+x^{92}+x^{84}+x^{82}+x^{74}+x^{64}+x^{60}+x^{58}+x^{56}\\
&\quad +x^{54}+x^{52}+x^{48}+x^{46}+x^{44}+x^{34}+x^{32}+x^{28}+x^{22}+x^{20}+x^{14}\\
&\quad +x^{8}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{98}+x^{96}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{80}+x^{76}+x^{66}+x^{60}\\
&\quad +x^{56}+x^{54}+x^{50}+x^{38}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{18}+x^{16}\\
&\quad +x^{10}+x^{4},\\
p_{12}(x) &= x^{98}+x^{96}+x^{90}+x^{88}+x^{82}+x^{80}+x^{50}+x^{48}+x^{42}+x^{40}+x^{34}\\
&\quad +x^{32}+x^{18}+x^{16}+x^{10}+x^{8}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{96} + x^{94} + x^{93} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{85} + x^{83} + x^{82} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{62} + x^{61} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} \\
&\quad + x^{43} + x^{37} + x^{32} + x^{30}, \\
p_3(x) &= x^{101} + x^{99} + x^{98} + x^{91} + x^{88} + x^{87} + x^{86} + x^{75} + x^{72} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{53} + x^{51} + x^{50} + x^{45} + x^{42} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} \\
&\quad + x^{18}, \\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{81} + x^{79} + x^{78} \\
&\quad + x^{76} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} + x^{52} + x^{49} + x^{46} \\
&\quad + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{25} + x^{22} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_9(x) &= x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{63} + x^{61} + x^{60} + x^{57} + x^{55} \\
&\quad + x^{54} + x^{53} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{27} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} \\
&\quad + x^{14} + x^{13} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{95} + x^{92} + x^{83} + x^{80} + x^{47} + x^{44} + x^{35} + x^{32} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{96} + x^{94} + x^{93} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{85} + x^{83} + x^{82} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{62} + x^{61} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} \\
&\quad + x^{43} + x^{37} + x^{32} + x^{30}, \\
p_3(x) &= x^{101} + x^{99} + x^{98} + x^{91} + x^{88} + x^{87} + x^{86} + x^{75} + x^{72} + x^{71} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{53} + x^{51} + x^{50} + x^{45} + x^{42} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} \\
&\quad + x^{18}, \\
p_6(x) &= x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{81} + x^{79} + x^{78} \\
&\quad + x^{76} + x^{71} + x^{68} + x^{67} + x^{64} + x^{61} + x^{58} + x^{55} + x^{52} + x^{49} + x^{46} \\
&\quad + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{25} + x^{22} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_9(x) &= x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{77}+x^{75}+x^{74}+x^{73}+x^{71}+x^{70}+x^{63}+x^{61}+x^{60}+x^{57}+x^{55}\\
&+x^{54}+x^{53}+x^{50}+x^{48}+x^{47}+x^{46}+x^{43}+x^{40}+x^{38}+x^{37}+x^{36}\\
&+x^{35}+x^{32}+x^{31}+x^{30}+x^{27}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{15}\\
&+x^{14}+x^{13}+x^{10}+x^{8}+x^{7}+x^{6},\\
p_{12}(x)&=x^{95}+x^{92}+x^{83}+x^{80}+x^{47}+x^{44}+x^{35}+x^{32}+x^{15}+x^{12}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0].
For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{102}+x^{98}+x^{96}+x^{94}+x^{92}+x^{84}+x^{80}+x^{76}+x^{74}+x^{72}+x^{70}\\
&\quad+x^{68}+x^{66}+x^{64}+x^{62}+x^{58}+x^{54}+x^{52}+x^{44}+x^{38}+x^{36},\\
p_3(x)&=x^{94}+x^{92}+x^{88}+x^{84}+x^{78}+x^{72}+x^{68}+x^{66}+x^{64}+x^{60}+x^{56}\\
&\quad+x^{46}+x^{44}+x^{42}+x^{38}+x^{30}+x^{28}+x^{26}+x^{16}+x^{12},\\
p_6(x)&=x^{94}+x^{92}+x^{90}+x^{88}+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}\\
&\quad+x^{68}+x^{64}+x^{62}+x^{58}+x^{56}+x^{52}+x^{48}+x^{42}+x^{36}+x^{34}+x^{32}\\
&\quad+x^{28}+x^{26}+x^{24}+x^{22}+x^{18}+x^{16}+x^{12}+x^{8}+x^{2}+1,\\
p_9(x)&=x^{92}+x^{90}+x^{86}+x^{82}+x^{74}+x^{72}+x^{68}+x^{66}+x^{64}+x^{60}+x^{56}\\
&\quad+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{28}+x^{22}+x^{18}+x^{16}+x^{12}\\
&\quad+x^{8}+x^{4},\\
p_{12}(x)&=x^{92}+x^{90}+x^{88}+x^{84}+x^{82}+x^{80}+x^{44}+x^{42}+x^{40}+x^{36}+x^{34}\\
&\quad+x^{32}+x^{12}+x^{10}+x^{8}+x^{4}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].
For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{108}+x^{105}+x^{102}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}\\
&\quad+x^{92}+x^{86}+x^{82}+x^{81}+x^{77}+x^{76}+x^{72}+x^{70}+x^{68}+x^{67}+x^{65}\\
&\quad+x^{64}+x^{61}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}\\
&\quad+x^{46}+x^{45}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{33}+x^{31}+x^{28}+x^{27}\\
&\quad+x^{24},\\
p_3(x)&=x^{103}+x^{102}+x^{100}+x^{99}+x^{96}+x^{95}+x^{93}+x^{91}+x^{89}+x^{88}+x^{87}\\
&\quad+x^{86}+x^{85}+x^{84}+x^{82}+x^{76}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}\\
&\quad+x^{61}+x^{58}+x^{54}+x^{53}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{43}+x^{42}\\
&\quad+x^{41}+x^{40}+x^{39}+x^{38}+x^{35}+x^{34}+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}\\
&\quad+x^{25}+x^{24}+x^{19}+x^{16},\\
p_6(x)&=x^{103}+x^{102}+x^{100}+x^{99}+x^{96}+x^{95}+x^{94}+x^{93}+x^{89}+x^{87}+x^{83}\\
&\quad+x^{82}+x^{81}+x^{80}+x^{79}+x^{76}+x^{74}+x^{73}+x^{71}+x^{70}+x^{68}+x^{64}\\
&\quad+x^{58}+x^{55}+x^{53}+x^{50}+x^{47}+x^{45}+x^{44}+x^{42}+x^{40}+x^{36}+x^{35}\\
&\quad+x^{34}+x^{32}+x^{29}+x^{28}+x^{27}+x^{25}+x^{22}+x^{20}+x^{19}+x^{17}+x^{16}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43}\\
&\quad + x^{40} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{103} + x^{100} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{43} + x^{40} + x^{35}\\
&\quad + x^{32} + x^{23} + x^{20} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{116} + x^{114} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{98}\\
&\quad + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{75}\\
&\quad + x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48}\\
&\quad + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{30},\\
p_3(x) &= x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86}\\
&\quad + x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{67} + x^{66}\\
&\quad + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39}\\
&\quad + x^{38} + x^{37} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{19} + x^{18},\\
p_6(x) &= x^{104} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40}\\
&\quad + x^{32} + x^{28} + x^{24} + x^{22} + x^{14} + x^{12},\\
p_9(x) &= x^{106} + x^{104} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{88} + x^{87}\\
&\quad + x^{86} + x^{58} + x^{56} + x^{53} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40}\\
&\quad + x^{39} + x^{38} + x^{26} + x^{24} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9}\\
&\quad + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} + x^{60} + x^{56} + x^{52} + x^{48}\\
&\quad + x^{44} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89}\\
&\quad + x^{86} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{68} + x^{66} + x^{61}\\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{47} + x^{44} + x^{43} + x^{41} + x^{40}\\
&\quad + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30},\\
p_3(x) &= x^{98} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{80} + x^{78}\\
&\quad + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{67} + x^{66} + x^{50} + x^{48} + x^{46} + x^{45}\\
&\quad + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32}\\
&\quad + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{23} + x^{19} + x^{18},\\
p_6(x) &= x^{100} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{64} + x^{62}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{58} + x^{56} + x^{52} + x^{50} + x^{46} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28}\\
&+ x^{26} + x^{20} + x^{14} + x^{12},\\
p_9(x) &= x^{102} + x^{100} + x^{97} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{54} + x^{52} + x^{49}\\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{22} + x^{20} + x^{17} + x^{11} + x^{9} + x^{8} + x^{7}\\
&+ x^{6},\\
p_{12}(x) &= x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} + x^{56} + x^{52} + x^{44} + x^{40} + x^{36}\\
&+ x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1].
For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{83}\\
&+ x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{69} + x^{67} + x^{59} + x^{58} + x^{57}\\
&+ x^{56} + x^{55} + x^{51} + x^{46} + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{103} + x^{102} + x^{100} + x^{95} + x^{93} + x^{91} + x^{88} + x^{85} + x^{82} + x^{76} + x^{75}\\
&+ x^{74} + x^{73} + x^{72} + x^{67} + x^{64} + x^{61} + x^{58} + x^{54} + x^{53} + x^{50} + x^{49}\\
&+ x^{47} + x^{42} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{23} + x^{22}\\
&+ x^{20},\\
p_6(x) &= x^{103} + x^{102} + x^{100} + x^{95} + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} + x^{84} + x^{82}\\
&+ x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{64} + x^{61} + x^{53} + x^{52} + x^{50}\\
&+ x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33}\\
&+ x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{23} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14}\\
&+ x^{3} + 1,\\
p_9(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43}\\
&+ x^{40} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{103} + x^{100} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{43} + x^{40} + x^{35}\\
&+ x^{32} + x^{23} + x^{20} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0].
For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93}\\
&+ x^{92} + x^{86} + x^{82} + x^{81} + x^{77} + x^{76} + x^{72} + x^{70} + x^{68} + x^{67} + x^{65}\\
&+ x^{64} + x^{61} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{33} + x^{31} + x^{28} + x^{27}\\
&+ x^{24},\\
p_3(x) &= x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{82} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68}\\
&+ x^{61} + x^{58} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{42}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} \\
&+ x^{25} + x^{24} + x^{19} + x^{16}, \\
p_6(x) = {} & x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} + x^{87} + x^{83} \\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{68} + x^{64} \\
&+ x^{58} + x^{55} + x^{53} + x^{50} + x^{47} + x^{45} + x^{44} + x^{42} + x^{40} + x^{36} + x^{35} \\
&+ x^{34} + x^{32} + x^{29} + x^{28} + x^{27} + x^{25} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} \\
&+ x^{11} + x^{8} + x^{3} + 1, \\
p_9(x) = {} & x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43} \\
&+ x^{40} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8}, \\
p_{12}(x) = {} & x^{103} + x^{100} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{43} + x^{40} + x^{35} \\
&+ x^{32} + x^{23} + x^{20} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} \\
&+ x^{86} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{72} + x^{68} + x^{66} + x^{61} \\
&+ x^{59} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{47} + x^{44} + x^{43} + x^{41} + x^{40} \\
&+ x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30}, \\
p_3(x) = {} & x^{98} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{83} + x^{80} + x^{78} \\
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{67} + x^{66} + x^{50} + x^{48} + x^{46} + x^{45} \\
&+ x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} \\
&+ x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{23} + x^{19} + x^{18}, \\
p_6(x) = {} & x^{100} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{64} + x^{62} \\
&+ x^{58} + x^{56} + x^{52} + x^{50} + x^{46} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} \\
&+ x^{26} + x^{20} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{102} + x^{100} + x^{97} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{54} + x^{52} + x^{49} \\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{22} + x^{20} + x^{17} + x^{11} + x^{9} + x^{8} + x^{7} \\
&+ x^{6}, \\
p_{12}(x) = {} & x^{104} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} + x^{56} + x^{52} + x^{44} + x^{40} + x^{36} \\
&+ x^{32} + x^{24} + x^{20} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{116} + x^{114} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{98} \\
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{84} + x^{83} + x^{81} + x^{78} + x^{76} + x^{75} \\
&+ x^{63} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} \\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33} + x^{30},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} \\
&\quad + x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{67} + x^{66} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{19} + x^{18}, \\
p_6(x) &= x^{104} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{32} + x^{28} + x^{24} + x^{22} + x^{14} + x^{12}, \\
p_9(x) &= x^{106} + x^{104} + x^{101} + x^{98} + x^{96} + x^{95} + x^{94} + x^{91} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{58} + x^{56} + x^{53} + x^{50} + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} \\
&\quad + x^{39} + x^{38} + x^{26} + x^{24} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^{9} \\
&\quad + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} + x^{60} + x^{56} + x^{52} + x^{48} \\
&\quad + x^{44} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{83} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{69} + x^{67} + x^{59} + x^{58} + x^{57} \\
&\quad + x^{56} + x^{55} + x^{51} + x^{46} + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{103} + x^{102} + x^{100} + x^{95} + x^{93} + x^{91} + x^{88} + x^{85} + x^{82} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{67} + x^{64} + x^{61} + x^{58} + x^{54} + x^{53} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{42} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{23} + x^{22} \\
&\quad + x^{20}, \\
p_6(x) &= x^{103} + x^{102} + x^{100} + x^{95} + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} + x^{84} + x^{82} \\
&\quad + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{64} + x^{61} + x^{53} + x^{52} + x^{50} \\
&\quad + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{37} + x^{36} + x^{35} + x^{33} \\
&\quad + x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{23} + x^{21} + x^{19} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{3} + 1, \\
p_9(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43} \\
&\quad + x^{40} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8}, \\
p_{12}(x) &= x^{103} + x^{100} + x^{91} + x^{88} + x^{83} + x^{80} + x^{55} + x^{52} + x^{43} + x^{40} + x^{35} \\
&\quad + x^{32} + x^{23} + x^{20} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	300	[465, 466]	600	[792, 172]	900	[1071, 772]
1	[3, 4]	301	[167, 60]	601	[506, 488]	901	[790, 741]
2	[5, 6]	302	[467, 12]	602	[430, 442]	902	[1072, 60]
3	[7, 8]	303	[129, 466]	603	[793, 194]	903	[821, 412]
4	[9, 10]	304	[398, 434]	604	[47, 470]	904	[1073, 531]
5	[11, 12]	305	[197, 56]	605	[794, 795]	905	[1074, 150]
6	[13, 14]	306	[468, 469]	606	[196, 44]	906	[434, 532]
7	[15, 16]	307	[436, 146]	607	[534, 162]	907	[469, 171]
8	[17, 18]	308	[219, 404]	608	[137, 200]	908	[1075, 146]
9	[19, 20]	309	[470, 471]	609	[796, 475]	909	[411, 822]
10	[21, 22]	310	[203, 58]	610	[515, 142]	910	[1076, 152]
11	[23, 24]	311	[472, 197]	611	[797, 35]	911	[482, 526]
12	[25, 26]	312	[462, 148]	612	[798, 799]	912	[1077, 167]
13	[27, 28]	313	[473, 474]	613	[192, 421]	913	[1078, 205]
14	[29, 30]	314	[224, 209]	614	[800, 173]	914	[871, 35]
15	[31, 32]	315	[388, 475]	615	[801, 802]	915	[1079, 478]
16	[33, 34]	316	[476, 197]	616	[156, 803]	916	[1074, 35]
17	[35, 36]	317	[477, 478]	617	[130, 45]	917	[1080, 787]
18	[37, 38]	318	[479, 480]	618	[804, 805]	918	[1081, 802]
19	[39, 40]	319	[481, 436]	619	[52, 59]	919	[1082, 416]
20	[41, 42]	320	[482, 483]	620	[806, 807]	920	[877, 775]
21	[43, 44]	321	[55, 198]	621	[808, 132]	921	[831, 831]
22	[45, 46]	322	[484, 410]	622	[809, 810]	922	[1083, 13]
23	[47, 48]	323	[485, 125]	623	[811, 416]	923	[1084, 437]
24	[49, 50]	324	[37, 486]	624	[812, 748]	924	[1085, 449]
25	[51, 52]	325	[464, 185]	625	[813, 202]	925	[762, 848]
26	[53, 54]	326	[171, 13]	626	[142, 428]	926	[792, 13]
27	[55, 56]	327	[424, 62]	627	[493, 213]	927	[144, 501]
28	[57, 58]	328	[487, 488]	628	[814, 815]	928	[769, 761]
29	[59, 60]	329	[168, 456]	629	[742, 10]	929	[518, 193]
30	[61, 62]	330	[489, 490]	630	[154, 751]	930	[814, 209]
31	[63, 64]	331	[210, 491]	631	[523, 799]	931	[786, 1086]
32	[65, 66]	332	[492, 493]	632	[146, 159]	932	[1087, 1065]
33	[67, 68]	333	[494, 410]	633	[223, 208]	933	[1088, 818]
34	[69, 70]	334	[495, 193]	634	[34, 215]	934	[816, 44]
35	[71, 72]	335	[496, 49]	635	[186, 470]	935	[422, 851]
36	[73, 74]	336	[189, 434]	636	[816, 461]	936	[224, 815]
37	[75, 76]	337	[497, 498]	637	[519, 805]	937	[1089, 215]
38	[77, 78]	338	[212, 499]	638	[817, 818]	938	[463, 424]
39	[79, 80]	339	[500, 501]	639	[420, 803]	939	[1090, 209]
40	[81, 82]	340	[152, 457]	640	[819, 489]	940	[182, 818]
41	[83, 84]	341	[502, 210]	641	[444, 446]	941	[1091, 461]
42	[85, 86]	342	[48, 503]	642	[820, 223]	942	[46, 155]

43	[87, 75]	343	[504, 173]	643	[821, 822]	943	[1092, 426]
44	[88, 89]	344	[136, 199]	644	[823, 59]	944	[747, 395]
45	[90, 91]	345	[126, 176]	645	[214, 222]	945	[808, 46]
46	[92, 93]	346	[505, 407]	646	[449, 139]	946	[172, 14]
47	[94, 95]	347	[506, 507]	647	[173, 217]	947	[831, 830]
48	[96, 97]	348	[437, 147]	648	[824, 171]	948	[881, 1087]
49	[98, 99]	349	[409, 508]	649	[33, 481]	949	[864, 398]
50	[100, 101]	350	[494, 508]	650	[132, 751]	950	[1093, 848]
51	[102, 103]	351	[141, 491]	651	[825, 47]	951	[1094, 838]
52	[104, 105]	352	[509, 212]	652	[397, 152]	952	[1095, 2]
53	[16, 106]	353	[510, 511]	653	[826, 442]	953	[62, 43]
54	[107, 108]	354	[164, 393]	654	[784, 395]	954	[1096, 398]
55	[109, 110]	355	[512, 513]	655	[766, 403]	955	[793, 401]
56	[111, 112]	356	[514, 37]	656	[827, 828]	956	[1097, 446]
57	[113, 114]	357	[515, 441]	657	[829, 455]	957	[502, 141]
58	[115, 116]	358	[493, 499]	658	[830, 831]	958	[1098, 186]
59	[117, 118]	359	[516, 390]	659	[832, 833]	959	[470, 503]
60	[119, 120]	360	[517, 498]	660	[834, 459]	960	[481, 215]
61	[121, 122]	361	[518, 513]	661	[54, 165]	961	[1084, 146]
62	[123, 124]	362	[137, 432]	662	[134, 440]	962	[1099, 455]
63	[125, 126]	363	[48, 471]	663	[216, 174]	963	[1100, 777]
64	[127, 128]	364	[32, 43]	664	[835, 526]	964	[131, 46]
65	[129, 130]	365	[222, 223]	665	[836, 513]	965	[495, 513]
66	[131, 132]	366	[519, 520]	666	[837, 838]	966	[1101, 763]
67	[133, 134]	367	[61, 32]	667	[839, 839]	967	[485, 416]
68	[51, 135]	368	[521, 522]	668	[840, 224]	968	[1102, 839]
69	[136, 137]	369	[523, 524]	669	[841, 138]	969	[1103, 771]
70	[138, 139]	370	[466, 45]	670	[842, 508]	970	[421, 440]
71	[140, 141]	371	[525, 526]	671	[492, 212]	971	[1104, 483]
72	[142, 143]	372	[527, 421]	672	[843, 398]	972	[1100, 1105]
73	[144, 145]	373	[528, 481]	673	[844, 845]	973	[1087, 773]
74	[146, 147]	374	[529, 424]	674	[469, 163]	974	[1106, 499]
75	[148, 149]	375	[45, 132]	675	[846, 828]	975	[443, 445]
76	[150, 36]	376	[513, 184]	676	[847, 848]	976	[148, 758]
77	[151, 152]	377	[530, 216]	677	[194, 402]	977	[1107, 54]
78	[150, 153]	378	[513, 464]	678	[849, 401]	978	[1107, 164]
79	[154, 155]	379	[444, 531]	679	[521, 761]	979	[192, 134]
80	[156, 157]	380	[190, 532]	680	[850, 851]	980	[1108, 407]
81	[130, 131]	381	[193, 464]	681	[135, 59]	981	[744, 802]
82	[132, 155]	382	[533, 131]	682	[852, 161]	982	[1109, 213]
83	[158, 159]	383	[534, 469]	683	[853, 43]	983	[1072, 168]
84	[160, 161]	384	[401, 195]	684	[854, 403]	984	[794, 451]
85	[162, 163]	385	[421, 182]	685	[756, 197]	985	[1069, 522]
86	[164, 165]	386	[449, 535]	686	[855, 49]	986	[460, 883]

87	[166, 38]	387	[536, 537]	687	[856, 857]	987	[1110, 1080]
88	[167, 168]	388	[538, 539]	688	[160, 858]	988	[1090, 815]
89	[52, 167]	389	[540, 5]	689	[34, 436]	989	[773, 1071]
90	[169, 170]	390	[541, 237]	690	[418, 859]	990	[1080, 1086]
91	[171, 172]	391	[542, 543]	691	[166, 486]	991	[1111, 1080]
92	[173, 174]	392	[544, 545]	692	[860, 490]	992	[1112, 751]
93	[175, 176]	393	[546, 547]	693	[861, 47]	993	[208, 815]
94	[177, 142]	394	[379, 548]	694	[862, 194]	994	[1113, 393]
95	[178, 179]	395	[549, 270]	695	[445, 531]	995	[1114, 857]
96	[180, 181]	396	[550, 551]	696	[768, 125]	996	[797, 150]
97	[182, 183]	397	[552, 553]	697	[863, 188]	997	[1115, 132]
98	[184, 185]	398	[554, 555]	698	[864, 189]	998	[438, 511]
99	[186, 48]	399	[556, 557]	699	[865, 211]	999	[1112, 155]
100	[187, 188]	400	[101, 295]	700	[866, 136]	1000	[43, 461]
101	[189, 190]	401	[265, 558]	701	[175, 394]	1001	[1071, 1087]
102	[191, 32]	402	[559, 560]	702	[138, 535]	1002	[1116, 395]
103	[169, 192]	403	[561, 562]	703	[867, 219]	1003	[530, 173]
104	[193, 184]	404	[563, 564]	704	[868, 143]	1004	[850, 423]
105	[194, 195]	405	[565, 241]	705	[836, 193]	1005	[11, 223]
106	[32, 196]	406	[557, 297]	706	[869, 182]	1006	[853, 196]
107	[197, 198]	407	[566, 304]	707	[749, 207]	1007	[1117, 782]
108	[199, 200]	408	[567, 568]	708	[870, 445]	1008	[1070, 780]
109	[57, 201]	409	[4, 569]	709	[498, 390]	1009	[406, 164]
110	[202, 203]	410	[570, 119]	710	[871, 150]	1010	[1092, 739]
111	[204, 37]	411	[571, 572]	711	[872, 216]	1011	[765, 136]
112	[205, 57]	412	[573, 249]	712	[806, 857]	1012	[1114, 807]
113	[206, 207]	413	[22, 574]	713	[798, 524]	1013	[1098, 47]
114	[208, 209]	414	[575, 576]	714	[873, 166]	1014	[825, 186]
115	[210, 211]	415	[577, 343]	715	[846, 478]	1015	[187, 219]
116	[212, 213]	416	[578, 579]	716	[874, 168]	1016	[1118, 1105]
117	[214, 11]	417	[580, 581]	717	[875, 876]	1017	[837, 771]
118	[215, 146]	418	[582, 583]	718	[877, 782]	1018	[215, 437]
119	[216, 217]	419	[584, 585]	719	[168, 878]	1019	[1119, 785]
120	[36, 218]	420	[586, 587]	720	[879, 426]	1020	[1120, 780]
121	[205, 203]	421	[588, 589]	721	[880, 189]	1021	[755, 52]
122	[219, 220]	422	[590, 591]	722	[41, 833]	1022	[1121, 224]
123	[221, 222]	423	[551, 88]	723	[773, 881]	1023	[510, 439]
124	[223, 224]	424	[564, 87]	724	[882, 883]	1024	[447, 47]
125	[225, 226]	425	[592, 593]	725	[884, 859]	1025	[1122, 419]
126	[227, 228]	426	[594, 595]	726	[885, 2]	1026	[843, 189]
127	[229, 230]	427	[84, 547]	727	[881, 772]	1027	[1086, 1111]
128	[231, 232]	428	[317, 107]	728	[886, 45]	1028	[1095, 876]
129	[233, 234]	429	[596, 597]	729	[887, 53]	1029	[787, 1111]
130	[235, 92]	430	[598, 599]	730	[778, 400]	1030	[804, 520]

131	[234, 236]	431	[600, 601]	731	[888, 462]	1031	[1123, 182]
132	[237, 238]	432	[351, 602]	732	[489, 475]	1032	[1078, 202]
133	[239, 240]	433	[603, 604]	733	[49, 433]	1033	[1124, 35]
134	[241, 242]	434	[605, 606]	734	[889, 436]	1034	[841, 449]
135	[243, 244]	435	[338, 575]	735	[863, 219]	1035	[874, 60]
136	[245, 246]	436	[607, 287]	736	[35, 153]	1036	[479, 845]
137	[247, 248]	437	[70, 608]	737	[890, 669]	1037	[1125, 54]
138	[249, 239]	438	[609, 610]	738	[891, 625]	1038	[206, 750]
139	[250, 251]	439	[611, 342]	739	[660, 892]	1039	[888, 395]
140	[252, 253]	440	[340, 276]	740	[893, 894]	1040	[468, 162]
141	[254, 255]	441	[612, 613]	741	[895, 326]	1041	[1126, 55]
142	[256, 257]	442	[597, 64]	742	[896, 726]	1042	[425, 739]
143	[258, 259]	443	[614, 615]	743	[6, 897]	1043	[1083, 172]
144	[260, 261]	444	[616, 617]	744	[898, 693]	1044	[1124, 150]
145	[262, 263]	445	[618, 619]	745	[899, 900]	1045	[415, 194]
146	[264, 265]	446	[620, 386]	746	[901, 89]	1046	[1116, 462]
147	[266, 267]	447	[621, 622]	747	[902, 903]	1047	[496, 407]
148	[268, 269]	448	[623, 250]	748	[687, 733]	1048	[528, 34]
149	[270, 271]	449	[93, 121]	749	[904, 708]	1049	[1127, 14]
150	[272, 273]	450	[624, 245]	750	[905, 906]	1050	[852, 858]
151	[274, 275]	451	[625, 109]	751	[662, 907]	1051	[437, 159]
152	[276, 104]	452	[626, 627]	752	[908, 909]	1052	[1127, 413]
153	[277, 278]	453	[628, 629]	753	[910, 911]	1053	[844, 480]
154	[240, 279]	454	[630, 631]	754	[604, 308]	1054	[1104, 526]
155	[236, 280]	455	[257, 632]	755	[912, 300]	1055	[869, 440]
156	[281, 282]	456	[633, 634]	756	[913, 111]	1056	[809, 474]
157	[283, 284]	457	[275, 635]	757	[344, 107]	1057	[787, 1110]
158	[285, 286]	458	[636, 637]	758	[315, 350]	1058	[1128, 413]
159	[287, 288]	459	[619, 291]	759	[914, 735]	1059	[60, 878]
160	[289, 290]	460	[638, 639]	760	[915, 916]	1060	[879, 739]
161	[291, 292]	461	[66, 81]	761	[591, 917]	1061	[1129, 165]
162	[293, 285]	462	[640, 641]	762	[918, 919]	1062	[9, 743]
163	[294, 29]	463	[642, 123]	763	[920, 921]	1063	[1123, 440]
164	[295, 296]	464	[643, 21]	764	[637, 280]	1064	[1130, 705]
165	[297, 298]	465	[644, 645]	765	[922, 247]	1065	[1131, 1132]
166	[299, 235]	466	[635, 646]	766	[923, 347]	1066	[1133, 254]
167	[300, 301]	467	[312, 647]	767	[924, 925]	1067	[1134, 616]
168	[105, 98]	468	[648, 649]	768	[926, 927]	1068	[1135, 673]
169	[302, 303]	469	[560, 27]	769	[928, 929]	1069	[1136, 573]
170	[304, 305]	470	[622, 650]	770	[930, 901]	1070	[1137, 540]
171	[306, 307]	471	[95, 626]	771	[539, 719]	1071	[931, 1131]
172	[308, 309]	472	[651, 652]	772	[900, 931]	1072	[68, 964]
173	[310, 293]	473	[653, 337]	773	[667, 932]	1073	[1138, 30]
174	[311, 312]	474	[654, 229]	774	[933, 256]	1074	[1139, 1140]

175	[313, 314]	475	[537, 324]	775	[907, 895]	1075	[103, 292]
176	[226, 315]	476	[655, 346]	776	[934, 935]	1076	[1141, 1142]
177	[316, 317]	477	[656, 361]	777	[370, 936]	1077	[76, 119]
178	[318, 319]	478	[657, 658]	778	[937, 938]	1078	[1143, 113]
179	[320, 321]	479	[659, 660]	779	[939, 688]	1079	[1144, 1145]
180	[322, 94]	480	[661, 662]	780	[255, 940]	1080	[917, 1146]
181	[323, 299]	481	[378, 663]	781	[941, 612]	1081	[1147, 262]
182	[324, 325]	482	[664, 665]	782	[319, 942]	1082	[1148, 1149]
183	[326, 327]	483	[666, 667]	783	[943, 944]	1083	[1150, 22]
184	[328, 329]	484	[631, 634]	784	[945, 550]	1084	[1151, 266]
185	[330, 331]	485	[668, 313]	785	[298, 946]	1085	[1152, 1153]
186	[332, 333]	486	[669, 63]	786	[947, 948]	1086	[1154, 1155]
187	[334, 335]	487	[670, 671]	787	[892, 947]	1087	[1132, 1156]
188	[309, 336]	488	[672, 566]	788	[949, 231]	1088	[646, 229]
189	[337, 338]	489	[673, 674]	789	[950, 544]	1089	[736, 677]
190	[339, 283]	490	[675, 240]	790	[951, 952]	1090	[259, 362]
191	[99, 321]	491	[676, 677]	791	[354, 953]	1091	[244, 299]
192	[78, 340]	492	[678, 679]	792	[730, 897]	1092	[1157, 1158]
193	[341, 330]	493	[680, 681]	793	[954, 955]	1093	[1159, 654]
194	[342, 17]	494	[682, 85]	794	[956, 957]	1094	[1160, 1036]
195	[343, 344]	495	[683, 643]	795	[958, 959]	1095	[1161, 366]
196	[64, 345]	496	[684, 685]	796	[80, 960]	1096	[1162, 1163]
197	[108, 69]	497	[686, 687]	797	[961, 73]	1097	[1053, 574]
198	[346, 243]	498	[688, 689]	798	[962, 963]	1098	[1164, 1165]
199	[347, 348]	499	[690, 691]	799	[82, 964]	1099	[1166, 1004]
200	[246, 349]	500	[692, 352]	800	[965, 377]	1100	[1167, 538]
201	[350, 351]	501	[693, 380]	801	[966, 967]	1101	[1168, 1169]
202	[352, 115]	502	[694, 676]	802	[587, 968]	1102	[1102, 1102]
203	[353, 354]	503	[627, 695]	803	[632, 364]	1103	[1170, 369]
204	[355, 356]	504	[696, 697]	804	[969, 584]	1104	[1171, 672]
205	[357, 358]	505	[698, 100]	805	[348, 970]	1105	[979, 661]
206	[359, 360]	506	[699, 578]	806	[971, 611]	1106	[1062, 327]
207	[361, 362]	507	[700, 701]	807	[972, 973]	1107	[303, 546]
208	[24, 363]	508	[568, 702]	808	[383, 662]	1108	[1172, 1173]
209	[364, 365]	509	[703, 371]	809	[974, 975]	1109	[1050, 979]
210	[366, 367]	510	[704, 640]	810	[919, 976]	1110	[948, 1174]
211	[368, 264]	511	[705, 286]	811	[977, 227]	1111	[1029, 1154]
212	[369, 370]	512	[706, 707]	812	[935, 267]	1112	[305, 249]
213	[371, 308]	513	[708, 709]	813	[978, 353]	1113	[647, 109]
214	[372, 373]	514	[710, 77]	814	[18, 906]	1114	[1175, 1176]
215	[374, 375]	515	[711, 712]	815	[385, 960]	1115	[1177, 1009]
216	[362, 376]	516	[713, 288]	816	[702, 545]	1116	[1178, 1179]
217	[377, 378]	517	[714, 715]	817	[733, 976]	1117	[1180, 1138]
218	[273, 379]	518	[716, 328]	818	[242, 979]	1118	[1181, 713]

219	[380, 23]	519	[717, 718]	819	[980, 981]	1119	[1182, 598]
220	[335, 381]	520	[719, 546]	820	[110, 386]	1120	[1183, 1030]
221	[382, 383]	521	[720, 721]	821	[982, 71]	1121	[1184, 284]
222	[251, 384]	522	[722, 723]	822	[721, 279]	1122	[1185, 1186]
223	[373, 385]	523	[724, 657]	823	[232, 702]	1123	[938, 326]
224	[386, 310]	524	[314, 633]	824	[558, 691]	1124	[1187, 1188]
225	[387, 388]	525	[725, 700]	825	[983, 984]	1125	[1189, 970]
226	[389, 390]	526	[726, 727]	826	[639, 907]	1126	[1190, 1191]
227	[391, 392]	527	[381, 324]	827	[985, 905]	1127	[663, 263]
228	[54, 393]	528	[728, 374]	828	[986, 987]	1128	[677, 380]
229	[126, 394]	529	[729, 730]	829	[988, 318]	1129	[384, 959]
230	[395, 396]	530	[731, 311]	830	[989, 990]	1130	[1064, 807]
231	[397, 127]	531	[615, 647]	831	[991, 989]	1131	[772, 1065]
232	[398, 190]	532	[602, 732]	832	[992, 618]	1132	[1086, 1110]
233	[31, 62]	533	[112, 733]	833	[634, 993]	1133	[1091, 44]
234	[399, 400]	534	[734, 306]	834	[994, 682]	1134	[1192, 456]
235	[125, 175]	535	[735, 736]	835	[995, 996]	1135	[1113, 165]
236	[62, 196]	536	[737, 37]	836	[997, 998]	1136	[1093, 763]
237	[401, 402]	537	[738, 739]	837	[999, 680]	1137	[1193, 147]
238	[396, 149]	538	[740, 741]	838	[365, 1000]	1138	[740, 791]
239	[403, 137]	539	[400, 53]	839	[973, 1001]	1139	[1121, 208]
240	[188, 404]	540	[742, 743]	840	[645, 364]	1140	[391, 181]
241	[405, 192]	541	[744, 745]	841	[1002, 1003]	1141	[854, 136]
242	[406, 54]	542	[746, 491]	842	[1004, 1005]	1142	[204, 166]
243	[403, 199]	543	[387, 489]	843	[1006, 605]	1143	[840, 208]
244	[407, 50]	544	[747, 462]	844	[1007, 1008]	1144	[1101, 848]
245	[408, 409]	545	[153, 414]	845	[97, 1009]	1145	[880, 398]
246	[409, 410]	546	[53, 164]	846	[1010, 1011]	1146	[839, 1102]
247	[411, 412]	547	[466, 131]	847	[1012, 1013]	1147	[827, 478]
248	[13, 413]	548	[389, 748]	848	[894, 991]	1148	[1075, 437]
249	[36, 414]	549	[429, 178]	849	[1014, 1015]	1149	[450, 795]
250	[415, 401]	550	[749, 750]	850	[1016, 1017]	1150	[405, 170]
251	[416, 126]	551	[46, 751]	851	[993, 1018]	1151	[399, 8]
252	[417, 188]	552	[752, 202]	852	[1019, 1020]	1152	[512, 193]
253	[418, 419]	553	[505, 49]	853	[1021, 88]	1153	[472, 55]
254	[420, 157]	554	[753, 494]	854	[1022, 1023]	1154	[1110, 786]
255	[170, 421]	555	[141, 211]	855	[1024, 603]	1155	[1111, 786]
256	[422, 423]	556	[754, 163]	856	[1025, 1026]	1156	[1065, 1071]
257	[424, 32]	557	[755, 135]	857	[1020, 1027]	1157	[1122, 859]
258	[425, 426]	558	[416, 175]	858	[278, 266]	1158	[1082, 125]
259	[152, 128]	559	[756, 55]	859	[1028, 1029]	1159	[884, 419]
260	[427, 428]	560	[202, 57]	860	[1030, 365]	1160	[158, 147]
261	[429, 430]	561	[517, 389]	861	[1031, 96]	1161	[1088, 183]
262	[431, 205]	562	[430, 179]	862	[1032, 559]	1162	[1115, 46]

263	[199, 432]	563	[514, 166]	863	[1033, 1034]	1163	[180, 392]
264	[407, 433]	564	[395, 148]	864	[1035, 339]	1164	[1194, 43]
265	[434, 435]	565	[757, 406]	865	[1036, 81]	1165	[500, 145]
266	[436, 437]	566	[396, 758]	866	[1037, 1038]	1166	[1195, 456]
267	[8, 406]	567	[759, 138]	867	[1039, 563]	1167	[1128, 14]
268	[438, 439]	568	[760, 761]	868	[629, 298]	1168	[835, 483]
269	[440, 183]	569	[8, 53]	869	[1040, 953]	1169	[813, 205]
270	[441, 143]	570	[762, 763]	870	[1041, 1042]	1170	[1192, 878]
271	[178, 442]	571	[764, 179]	871	[1043, 277]	1171	[1081, 745]
272	[443, 444]	572	[753, 409]	872	[1044, 1045]	1172	[752, 205]
273	[445, 446]	573	[765, 403]	873	[1046, 1047]	1173	[504, 216]
274	[447, 186]	574	[170, 134]	874	[1048, 633]	1174	[830, 830]
275	[448, 449]	575	[498, 748]	875	[1049, 1050]	1175	[783, 802]
276	[153, 218]	576	[441, 428]	876	[267, 1051]	1176	[866, 403]
277	[450, 451]	577	[766, 136]	877	[1052, 1053]	1177	[529, 61]
278	[172, 413]	578	[767, 430]	878	[118, 1005]	1178	[1194, 196]
279	[184, 452]	579	[388, 490]	879	[1054, 958]	1179	[487, 507]
280	[400, 406]	580	[768, 416]	880	[1055, 1056]	1180	[1129, 393]
281	[453, 42]	581	[151, 127]	881	[658, 1057]	1181	[817, 183]
282	[454, 455]	582	[769, 522]	882	[1058, 4]	1182	[1193, 159]
283	[60, 456]	583	[431, 202]	883	[30, 1059]	1183	[1195, 878]
284	[163, 13]	584	[770, 771]	884	[1060, 323]	1184	[465, 130]
285	[127, 457]	585	[772, 773]	885	[1061, 1062]	1185	[1079, 828]
286	[203, 201]	586	[774, 775]	886	[230, 237]	1186	[861, 186]
287	[12, 224]	587	[776, 777]	887	[376, 259]	1187	[1125, 164]
288	[162, 171]	588	[778, 8]	888	[1063, 268]	1188	[473, 810]
289	[458, 459]	589	[134, 182]	889	[336, 264]	1189	[221, 11]
290	[460, 459]	590	[779, 780]	890	[811, 125]	1190	[1096, 189]
291	[196, 461]	591	[781, 782]	891	[1064, 857]	1191	[448, 138]
292	[59, 168]	592	[783, 745]	892	[1065, 881]	1192	[122, 566]
293	[462, 396]	593	[784, 462]	893	[1066, 2]	1193	[691, 342]
294	[463, 61]	594	[454, 785]	894	[1067, 42]	1194	[649, 1018]
295	[188, 220]	595	[786, 787]	895	[440, 818]	1195	[321, 701]
296	[190, 435]	596	[788, 127]	896	[1068, 777]		
297	[163, 172]	597	[789, 419]	897	[12, 208]		
298	[135, 167]	598	[790, 791]	898	[1069, 761]		
299	[464, 452]	599	[222, 12]	899	[1070, 40]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	200	1	400	0	600	1	800	0	1000	0
1	0	201	0	401	1	601	0	801	1	1001	0
2	0	202	0	402	0	602	1	802	1	1002	1
3	0	203	0	403	1	603	1	803	0	1003	0

4	1	204	1	404	0	604	0	804	0	1004	1
5	0	205	1	405	0	605	1	805	1	1005	0
6	1	206	1	406	0	606	1	806	0	1006	0
7	0	207	0	407	0	607	0	807	0	1007	1
8	1	208	1	408	0	608	1	808	0	1008	0
9	1	209	0	409	1	609	1	809	1	1009	0
10	0	210	1	410	0	610	1	810	1	1010	1
11	0	211	0	411	1	611	0	811	0	1011	0
12	1	212	0	412	0	612	1	812	0	1012	1
13	1	213	0	413	1	613	0	813	0	1013	1
14	0	214	0	414	1	614	0	814	1	1014	1
15	0	215	1	415	0	615	1	815	1	1015	0
16	1	216	1	416	1	616	1	816	1	1016	1
17	1	217	0	417	1	617	0	817	1	1017	0
18	1	218	0	418	1	618	0	818	0	1018	1
19	1	219	0	419	1	619	0	819	1	1019	0
20	1	220	1	420	0	620	0	820	0	1020	1
21	0	221	1	421	1	621	0	821	0	1021	0
22	1	222	1	422	0	622	1	822	1	1022	1
23	0	223	0	423	0	623	0	823	1	1023	0
24	1	224	0	424	0	624	0	824	1	1024	0
25	1	225	0	425	1	625	0	825	1	1025	1
26	1	226	0	426	1	626	0	826	0	1026	0
27	1	227	1	427	1	627	1	827	0	1027	1
28	1	228	0	428	0	628	1	828	1	1028	0
29	0	229	1	429	0	629	0	829	0	1029	0
30	1	230	0	430	1	630	1	830	1	1030	0
31	0	231	1	431	1	631	0	831	0	1031	0
32	0	232	1	432	0	632	0	832	0	1032	0
33	1	233	0	433	1	633	0	833	0	1033	1
34	0	234	0	434	1	634	0	834	0	1034	1
35	1	235	0	435	0	635	1	835	1	1035	0
36	1	236	1	436	0	636	1	836	1	1036	0
37	1	237	1	437	1	637	1	837	0	1037	1
38	0	238	0	438	1	638	1	838	1	1038	1
39	1	239	1	439	0	639	0	839	1	1039	0
40	0	240	1	440	0	640	1	840	0	1040	1
41	1	241	0	441	1	641	1	841	1	1041	1
42	1	242	0	442	0	642	0	842	1	1042	1
43	0	243	1	443	0	643	0	843	0	1043	0
44	1	244	0	444	1	644	1	844	1	1044	1
45	1	245	0	445	0	645	0	845	1	1045	0
46	0	246	1	446	0	646	0	846	1	1046	1
47	0	247	1	447	0	647	0	847	1	1047	1

48	0	248	1	448	0	648	1	848	0	1048	0
49	1	249	1	449	0	649	1	849	1	1049	1
50	0	250	0	450	0	650	1	850	1	1050	0
51	1	251	1	451	0	651	1	851	1	1051	1
52	0	252	1	452	0	652	1	852	0	1052	1
53	1	253	1	453	1	653	0	853	0	1053	1
54	0	254	0	454	1	654	0	854	1	1054	0
55	1	255	1	455	0	655	0	855	0	1055	1
56	1	256	0	456	0	656	0	856	1	1056	1
57	1	257	0	457	0	657	0	857	1	1057	0
58	1	258	1	458	1	658	1	858	0	1058	0
59	0	259	0	459	0	659	0	859	0	1059	1
60	1	260	1	460	1	660	0	860	0	1060	0
61	1	261	0	461	0	661	0	861	0	1061	1
62	1	262	1	462	1	662	0	862	0	1062	1
63	0	263	0	463	0	663	1	863	1	1063	0
64	1	264	0	464	0	664	1	864	0	1064	0
65	0	265	1	465	1	665	1	865	0	1065	0
66	0	266	0	466	1	666	0	866	1	1066	0
67	1	267	1	467	1	667	1	867	0	1067	0
68	1	268	1	468	1	668	0	868	0	1068	0
69	0	269	0	469	0	669	1	869	1	1069	0
70	1	270	1	470	1	670	1	870	1	1070	0
71	1	271	0	471	0	671	1	871	0	1071	0
72	0	272	0	472	1	672	0	872	1	1072	1
73	1	273	0	473	0	673	1	873	1	1073	0
74	0	274	0	474	0	674	0	874	0	1074	1
75	1	275	0	475	0	675	1	875	1	1075	1
76	0	276	0	476	0	676	1	876	1	1076	1
77	0	277	0	477	0	677	0	877	1	1077	0
78	0	278	0	478	0	678	1	878	1	1078	0
79	1	279	1	479	0	679	0	879	0	1079	1
80	1	280	0	480	0	680	1	880	1	1080	1
81	0	281	1	481	1	681	1	881	1	1081	0
82	1	282	1	482	1	682	0	882	0	1082	1
83	1	283	1	483	0	683	0	883	1	1083	0
84	1	284	0	484	0	684	1	884	0	1084	0
85	1	285	1	485	0	685	0	885	1	1085	1
86	1	286	0	486	1	686	0	886	0	1086	1
87	0	287	1	487	1	687	1	887	1	1087	1
88	1	288	1	488	0	688	1	888	0	1088	0
89	0	289	1	489	1	689	0	889	0	1089	1
90	1	290	1	490	1	690	1	890	0	1090	0
91	1	291	1	491	1	691	0	891	0	1091	0

92	0	292	0	492	1	692	0	892	0	1092	1
93	0	293	1	493	1	693	0	893	0	1093	0
94	0	294	0	494	0	694	0	894	0	1094	1
95	0	295	1	495	0	695	0	895	0	1095	0
96	0	296	0	496	1	696	1	896	0	1096	1
97	1	297	0	497	0	697	1	897	1	1097	1
98	1	298	1	498	1	698	0	898	0	1098	1
99	1	299	0	499	1	699	0	899	0	1099	1
100	0	300	1	500	0	700	1	900	0	1100	0
101	0	301	1	501	0	701	0	901	1	1101	1
102	1	302	1	502	0	702	1	902	1	1102	0
103	1	303	0	503	1	703	0	903	0	1103	0
104	0	304	1	504	1	704	0	904	0	1104	0
105	0	305	0	505	0	705	1	905	1	1105	0
106	0	306	1	506	0	706	1	906	1	1106	1
107	0	307	0	507	1	707	0	907	0	1107	0
108	0	308	0	508	1	708	1	908	1	1108	1
109	1	309	1	509	0	709	1	909	1	1109	0
110	0	310	0	510	0	710	0	910	1	1110	1
111	1	311	1	511	1	711	1	911	1	1111	0
112	1	312	1	512	1	712	0	912	0	1112	0
113	1	313	0	513	1	713	1	913	0	1113	0
114	1	314	0	514	0	714	1	914	0	1114	1
115	1	315	0	515	1	715	1	915	1	1115	1
116	0	316	0	516	1	716	0	916	1	1116	1
117	0	317	0	517	1	717	1	917	1	1117	1
118	1	318	0	518	0	718	1	918	0	1118	1
119	1	319	1	519	1	719	0	919	1	1119	0
120	1	320	1	520	0	720	0	920	1	1120	1
121	1	321	1	521	0	721	1	921	0	1121	1
122	0	322	0	522	1	722	1	922	0	1122	1
123	1	323	0	523	0	723	1	923	0	1123	0
124	0	324	1	524	0	724	0	924	1	1124	1
125	0	325	0	525	0	725	0	925	0	1125	1
126	1	326	1	526	1	726	1	926	1	1126	1
127	1	327	0	527	0	727	1	927	1	1127	1
128	1	328	1	528	0	728	0	928	1	1128	0
129	0	329	0	529	1	729	1	929	0	1129	1
130	0	330	1	530	0	730	1	930	1	1130	0
131	0	331	1	531	1	731	0	931	0	1131	0
132	1	332	1	532	1	732	1	932	1	1132	1
133	1	333	0	533	1	733	1	933	0	1133	0
134	0	334	0	534	0	734	0	934	1	1134	0
135	1	335	1	535	1	735	1	935	0	1135	0

136	0	336	0	536	0	736	1	936	0	1136	0
137	1	337	0	537	0	737	0	937	1	1137	0
138	1	338	0	538	0	738	0	938	0	1138	0
139	0	339	0	539	0	739	0	939	0	1139	1
140	1	340	0	540	0	740	0	940	1	1140	1
141	0	341	0	541	0	741	0	941	0	1141	1
142	0	342	0	542	1	742	0	942	0	1142	1
143	1	343	1	543	0	743	1	943	1	1143	0
144	1	344	0	544	1	744	0	944	1	1144	1
145	1	345	1	545	0	745	0	945	0	1145	1
146	0	346	0	546	1	746	1	946	0	1146	1
147	0	347	0	547	1	747	1	947	0	1147	0
148	1	348	1	548	0	748	1	948	1	1148	1
149	1	349	1	549	0	749	0	949	0	1149	0
150	0	350	0	550	0	750	1	950	0	1150	0
151	0	351	0	551	0	751	0	951	1	1151	0
152	0	352	0	552	1	752	1	952	0	1152	1
153	0	353	0	553	0	753	1	953	1	1153	1
154	1	354	1	554	1	754	0	954	1	1154	1
155	1	355	1	555	0	755	0	955	1	1155	0
156	1	356	0	556	0	756	0	956	1	1156	0
157	1	357	1	557	0	757	0	957	0	1157	1
158	1	358	1	558	1	758	0	958	1	1158	1
159	1	359	1	559	0	759	0	959	1	1159	0
160	1	360	1	560	0	760	1	960	1	1160	1
161	1	361	0	561	1	761	0	961	0	1161	0
162	1	362	1	562	1	762	0	962	1	1162	1
163	0	363	0	563	0	763	1	963	0	1163	0
164	1	364	0	564	0	764	1	964	0	1164	1
165	0	365	1	565	0	765	0	965	0	1165	0
166	0	366	1	566	0	766	0	966	1	1166	1
167	1	367	1	567	0	767	1	967	0	1167	0
168	0	368	0	568	1	768	1	968	0	1168	1
169	1	369	0	569	1	769	1	969	0	1169	0
170	1	370	1	570	0	770	1	970	1	1170	0
171	1	371	0	571	1	771	0	971	0	1171	0
172	0	372	0	572	1	772	0	972	0	1172	1
173	0	373	0	573	0	773	1	973	1	1173	1
174	1	374	1	574	1	774	0	974	1	1174	1
175	0	375	1	575	1	775	0	975	0	1175	1
176	0	376	1	576	1	776	1	976	1	1176	1
177	0	377	0	577	0	777	1	977	0	1177	1
178	0	378	1	578	1	778	1	978	0	1178	1
179	1	379	1	579	0	779	0	979	0	1179	1

180	0	380	0	580	1	780	1	980	1	1180	1
181	0	381	0	581	0	781	0	981	0	1181	1
182	1	382	1	582	1	782	1	982	0	1182	0
183	1	383	0	583	1	783	1	983	1	1183	1
184	1	384	1	584	1	784	0	984	1	1184	1
185	1	385	1	585	0	785	1	985	0	1185	1
186	1	386	0	586	0	786	0	986	1	1186	0
187	0	387	0	587	1	787	0	987	1	1187	1
188	1	388	0	588	1	788	0	988	0	1188	0
189	0	389	0	589	0	789	0	989	1	1189	1
190	0	390	0	590	0	790	1	990	1	1190	1
191	1	391	1	591	0	791	1	991	0	1191	0
192	0	392	1	592	1	792	1	992	0	1192	0
193	0	393	1	593	0	793	1	993	1	1193	0
194	0	394	1	594	1	794	1	994	0	1194	1
195	1	395	0	595	0	795	1	995	1	1195	1
196	1	396	0	596	0	796	1	996	0		
197	0	397	1	597	0	797	0	997	1		
198	0	398	1	598	1	798	1	998	1		
199	0	399	0	599	1	799	1	999	0		

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