PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^4 + z^3 + z^2, z^2 + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^4 + z^3 + z^2, z^2 + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 15:46:58 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 14.6 minutes CPU time used.

Theorem 1.1. *When $(a,b)=(z^4+z^3+z^2, z^2+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{16}+z^{11}+z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2+1,\\
p_1(z)&=z^{22}+z^{19}+z^{18}+z^{17}+z^{14}+z^{13}+z^{10}+z^8+z^3+z^2,\\
p_2(z)&=z^{24}+z^{16}+z^{14}+z^8+z^6+z^4,\\
p_3(z)&=z^{22}+z^{19}+z^{18}+z^{17}+z^{14}+z^{13}+z^{10}+z^8+z^3+z^2,\\
p_4(z)&=z^{20}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^9+z^8+z^7+z^5+z^3+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2,\\
p_1(z)&=z^{22}+z^{19}+z^{18}+z^{17}+z^{14}+z^{13}+z^{10}+z^8+z^3+z^2,\\
p_2(z)&=z^{24}+z^{16}+z^{14}+z^8+z^6+z^4,\\
p_3(z)&=z^{22}+z^{19}+z^{18}+z^{17}+z^{14}+z^{13}+z^{10}+z^8+z^3+z^2,\\
p_4(z)&=z^{20}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^5+z^3+z^2+1\\
&.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n}\tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n}\tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:12u] = {} & M^e[:6u] + x^uM^e[:5u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^uM^e[:6u]\\ & + x^{2u}M^e[:5u] + x^{4u}M^e[:3u] + x^{6u}M^e[:u] + x^{2u}M^e[:6u]\\ & + x^{3u}M^e[:5u] + x^{5u}M^e[:3u] + x^{7u}M^e[:u] + x^{4u}M^e[:6u]\\ & + x^{5u}M^e[:5u] + x^{7u}M^e[:3u] + x^{9u}M^e[:u] + x^{5u}M^e[:6u]\\ & + x^{6u}M^e[:5u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] + x^{7u}M^e[:5u]\\ & + x^{10u}M^e[:2u] + x^{9u}M^e[:3u] + x^{11u}M^e[:u]\end{aligned}$$

$$
\begin{aligned}
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^e,\\
W^e[:12u] &= W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^u W^e[:6u]\\
&+ x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{6u} W^e[:u] + x^{2u} W^e[:6u]\\
&+ x^{3u} W^e[:5u] + x^{5u} W^e[:3u] + x^{7u} W^e[:u] + x^{4u} W^e[:6u]\\
&+ x^{5u} W^e[:5u] + x^{7u} W^e[:3u] + x^{9u} W^e[:u] + x^{5u} W^e[:6u]\\
&+ x^{6u} W^e[:5u] + x^{8u} W^e[:3u] + x^{10u} W^e[:u] + x^{7u} W^e[:5u]\\
&+ x^{10u} W^e[:2u] + x^{9u} W^e[:3u] + x^{11u} W^e[:u]\\
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{5u} M^o[:u] + x^u M^o[:6u]\\
&+ x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{6u} M^o[:u] + x^{2u} M^o[:6u]\\
&+ x^{3u} M^o[:5u] + x^{5u} M^o[:3u] + x^{7u} M^o[:u] + x^{4u} M^o[:6u]\\
&+ x^{5u} M^o[:5u] + x^{7u} M^o[:3u] + x^{9u} M^o[:u] + x^{5u} M^o[:6u]\\
&+ x^{6u} M^o[:5u] + x^{8u} M^o[:3u] + x^{10u} M^o[:u] + x^{7u} M^o[:5u]\\
&+ x^{10u} M^o[:2u] + x^{9u} M^o[:3u] + x^{11u} M^o[:u]\\
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{5u} W^o[:u] + x^u W^o[:6u]\\
&+ x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{6u} W^o[:u] + x^{2u} W^o[:6u]\\
&+ x^{3u} W^o[:5u] + x^{5u} W^o[:3u] + x^{7u} W^o[:u] + x^{4u} W^o[:6u]\\
&+ x^{5u} W^o[:5u] + x^{7u} W^o[:3u] + x^{9u} W^o[:u] + x^{5u} W^o[:6u]\\
&+ x^{6u} W^o[:5u] + x^{8u} W^o[:3u] + x^{10u} W^o[:u] + x^{7u} W^o[:5u]\\
&+ x^{10u} W^o[:2u] + x^{9u} W^o[:3u] + x^{11u} W^o[:u]\\
&+ (x^{6u} + x^{7u} + x^{8u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] &= T[:6u] + x^u T[:5u] + x^{3u} T[:3u] + x^{5u} T[:u] + x^u T[:6u] + x^{2u} T[:5u]\\
&+ x^{4u} T[:3u] + x^{6u} T[:u] + x^{2u} T[:6u] + x^{3u} T[:5u] + x^{5u} T[:3u]\\
&+ x^{7u} T[:u] + x^{4u} T[:6u] + x^{5u} T[:5u] + x^{7u} T[:3u] + x^{9u} T[:u]\\
&+ x^{5u} T[:6u] + x^{6u} T[:5u] + x^{8u} T[:3u] + x^{10u} T[:u] + x^{7u} T[:5u]\\
&+ x^{10u} T[:2u] + x^{9u} T[:3u] + x^{11u} T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
0 &= x^{5u} T[u:2u] + x^{3u} T[3u:4u] + x^u T[5u:6u] + T[6u:7u],\\
0 &= x^{7u} T[:u] + x^{6u} T[u:2u] + x^{5u} T[2u:3u] + x^{4u} T[3u:4u]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + x^{3u}T[4u\!:\!5u] + x^{2u}T[5u\!:\!6u] + T[7u\!:\!8u],\\
0 &= x^{8u}T[:\!u] + x^{6u}T[2u\!:\!3u] + x^{4u}T[4u\!:\!5u] + T[8u\!:\!9u],\\
0 &= x^{8u}T[u\!:\!2u] + x^{6u}T[3u\!:\!4u] + x^{4u}T[5u\!:\!6u] + T[9u\!:\!10u],\\
0 &= x^{9u}T[u\!:\!2u] + x^{8u}T[2u\!:\!3u] + x^{7u}T[3u\!:\!4u] + x^{6u}T[4u\!:\!5u]\\
&\quad + x^{5u}T[5u\!:\!6u] + T[10u\!:\!11u],\\
0 &= x^{11u}T[:\!u] + x^{10u}T[u\!:\!2u] + x^{9u}T[2u\!:\!3u] + x^{7u}T[4u\!:\!5u]\\
&\quad + T[11u\!:\!12u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[101w]_2] + T[[111w]_2],\end{aligned} \tag{2.8}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$\begin{aligned}0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1010w]_2],\end{aligned} \tag{2.11}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\ &\quad + T[[101w]_2] + T[[111w]_2],\end{aligned} \tag{2.14}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$\begin{aligned}0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1010w]_2],\end{aligned} \tag{2.17}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 626 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)), \tag{2.19}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,111)),\end{aligned} \tag{2.20}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1000)), \tag{2.21}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)), \tag{2.25}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1000)), \tag{2.27}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{14}), E_{14}) = (A(i, 0^{12}), E_{12})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 7$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{6u} R^e, \\ W_{2k} &= W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^{6u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{5u} M^o[:u] + x^{6u} R^o, \\ W_{2k+1} &= W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{5u} W^o[:u] + x^{6u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(&W^e[:6v] + x^v W^e[:5v] + x^{3v} W^e[:3v] + x^{5v} W^e[:v] + x^{6u} R^e\big) \times \big(\\ &M^e[:6v] + x^v M^e[:5v] + x^{3v} M^e[:3v] + x^{5v} M^e[:v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is

$O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{2v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:6v]M^e[:6v] + x^{10v}W^e[:2v]M^e[:2v].$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^8 \\ &\quad + x^7 + x^6, \\ q_1(x) &= x^{22} + x^{21} + x^{16} + x^{14} + x^{11} + x^{10} + x^7 + x^6 + x^5 + x^2, \\ q_2(x) &= x^{20} + x^{18} + x^{16} + x^{10} + x^8 + 1, \\ q_3(x) &= x^{22} + x^{21} + x^{16} + x^{14} + x^{11} + x^{10} + x^7 + x^6 + x^5 + x^2, \end{aligned}$$

$$q_4(x) = x^{22} + x^{21} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^8 + x^7 + x^6 + x^4.$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{116} + x^{108} + x^{106} + x^{100} + x^{98} + x^{94} + x^{88} + x^{86} + x^{82} + x^{76} \\
&\quad + x^{74} + x^{68} + x^{66} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{42} + x^{40} + x^{36} \\
&\quad + x^{32} + x^{26} + x^{24}, \\
p_3(x) &= x^{112} + x^{110} + x^{108} + x^{102} + x^{100} + x^{92} + x^{90} + x^{86} + x^{84} + x^{78} + x^{74} \\
&\quad + x^{72} + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{48} + x^{44} + x^{42} + x^{38} + x^{34} \\
&\quad + x^{30} + x^{26} + x^{20} + x^{18}, \\
p_6(x) &= x^{110} + x^{108} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{84} + x^{78} + x^{72} \\
&\quad + x^{70} + x^{68} + x^{64} + x^{60} + x^{56} + x^{46} + x^{44} + x^{40} + x^{38} + x^{34} + x^{30} \\
&\quad + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} + x^6 + x^2 + 1, \\
p_9(x) &= x^{116} + x^{114} + x^{108} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} \\
&\quad + x^{78} + x^{74} + x^{70} + x^{62} + x^{60} + x^{54} + x^{52} + x^{48} + x^{44} + x^{38} + x^{32} \\
&\quad + x^{30} + x^{24} + x^{20} + x^{18} + x^6 + x^4, \\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96} + x^{84} + x^{82} + x^{80} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{52} + x^{50} + x^{48} + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} \\
&\quad + x^{109} + x^{106} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{68} \\
&\quad + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{51} \\
&\quad + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34}
\end{aligned}$$

$$
\begin{aligned}
&+x^{33}+x^{30},\\
p_3(x) &= x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{110}+x^{109}\\
&+x^{108}+x^{107}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{95}+x^{94}\\
&+x^{88}+x^{85}+x^{84}+x^{78}+x^{75}+x^{74}+x^{66}+x^{64}+x^{63}+x^{61}+x^{60}\\
&+x^{59}+x^{58}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{46}+x^{43}+x^{40}\\
&+x^{39}+x^{37}+x^{36}+x^{35}+x^{34}+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}\\
&+x^{22}+x^{19}+x^{18},\\
p_6(x) &= x^{120}+x^{119}+x^{118}+x^{116}+x^{115}+x^{114}+x^{106}+x^{105}+x^{104}+x^{102}\\
&+x^{101}+x^{100}+x^{98}+x^{97}+x^{95}+x^{94}+x^{88}+x^{87}+x^{85}+x^{83}+x^{81}\\
&+x^{79}+x^{77}+x^{75}+x^{73}+x^{72}+x^{70}+x^{69}+x^{67}+x^{65}+x^{64}+x^{62}\\
&+x^{61}+x^{59}+x^{58}+x^{54}+x^{53}+x^{52}+x^{50}+x^{49}+x^{48}+x^{44}+x^{43}\\
&+x^{42}+x^{40}+x^{39}+x^{37}+x^{35}+x^{34}+x^{32}+x^{31}+x^{30}+x^{22}+x^{21}\\
&+x^{20}+x^{18}+x^{17}+x^{15}+x^{13}+x^{12},\\
p_9(x) &= x^{118}+x^{115}+x^{114}+x^{112}+x^{109}+x^{106}+x^{105}+x^{104}+x^{103}+x^{100}\\
&+x^{99}+x^{98}+x^{97}+x^{96}+x^{94}+x^{91}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{80}+x^{77}+x^{76}+x^{74}+x^{72}+x^{71}+x^{69}+x^{67}+x^{65}+x^{64}\\
&+x^{63}+x^{60}+x^{59}+x^{58}+x^{57}+x^{54}+x^{53}+x^{51}+x^{50}+x^{49}+x^{48}\\
&+x^{46}+x^{43}+x^{40}+x^{39}+x^{38}+x^{37}+x^{33}+x^{29}+x^{28}+x^{26}+x^{23}\\
&+x^{22}+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{10}+x^{7}+x^{6},\\
p_{12}(x) &= x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}+x^{104}+x^{103}+x^{101}+x^{99}\\
&+x^{97}+x^{96}+x^{84}+x^{83}+x^{81}+x^{79}+x^{77}+x^{76}+x^{72}+x^{71}+x^{69}\\
&+x^{67}+x^{65}+x^{64}+x^{52}+x^{51}+x^{49}+x^{47}+x^{45}+x^{44}+x^{40}+x^{39}\\
&+x^{37}+x^{36}+x^{32}+x^{31}+x^{29}+x^{28}+x^{24}+x^{23}+x^{21}+x^{20}+x^{16}\\
&+x^{15}+x^{13}+x^{12}+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1].

For $\phi(M^e,1,0)$:

$$
\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{118}+x^{117}+x^{115}+x^{114}\\
&+x^{109}+x^{106}+x^{102}+x^{100}+x^{96}+x^{94}+x^{92}+x^{91}+x^{88}+x^{87}\\
&+x^{86}+x^{82}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{68}\\
&+x^{66}+x^{65}+x^{63}+x^{62}+x^{61}+x^{59}+x^{58}+x^{56}+x^{55}+x^{52}+x^{51}\\
&+x^{50}+x^{49}+x^{46}+x^{45}+x^{44}+x^{40}+x^{39}+x^{38}+x^{36}+x^{35}+x^{34}\\
&+x^{33}+x^{30},\\
p_3(x) &= x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{110}+x^{109}\\
&+x^{108}+x^{107}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{95}+x^{94}\\
&+x^{88}+x^{85}+x^{84}+x^{78}+x^{75}+x^{74}+x^{66}+x^{64}+x^{63}+x^{61}+x^{60}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{40}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&+ x^{22} + x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}\\
&+ x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43}\\
&+ x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{118} + x^{115} + x^{114} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{33} + x^{29} + x^{28} + x^{26} + x^{23}\\
&+ x^{22} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6},\\
p_{12}(x) &= x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99}\\
&+ x^{97} + x^{96} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69}\\
&+ x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{112} + x^{108} + x^{100} + x^{96}\\
&+ x^{94} + x^{92} + x^{90} + x^{88} + x^{70} + x^{66} + x^{60} + x^{58} + x^{52} + x^{48} + x^{40}\\
&+ x^{38} + x^{36},\\
p_3(x) &= x^{120} + x^{118} + x^{110} + x^{104} + x^{98} + x^{96} + x^{92} + x^{84} + x^{82} + x^{80} + x^{76}\\
&+ x^{70} + x^{68} + x^{66} + x^{64} + x^{58} + x^{50} + x^{44} + x^{38} + x^{36} + x^{30} + x^{28}\\
&+ x^{24} + x^{20} + x^{16} + x^{12},\\
p_6(x) &= x^{120} + x^{118} + x^{116} + x^{108} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{84} + x^{80}\\
&+ x^{78} + x^{72} + x^{64} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{28}\\
&+ x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + 1,\\
p_9(x) &= x^{116} + x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{72} + x^{68} + x^{60} + x^{56} + x^{52} + x^{50} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28}\\
&+ x^{22} + x^{20} + x^{18} + x^{16} + x^{10} + x^{4},\\
p_{12}(x) &= x^{116} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{84} + x^{80} + x^{76} + x^{72} + x^{68}
\end{aligned}
$$

$$+ x^{64} + x^{52} + x^{48} + x^{44} + x^{40} + x^{28} + x^{24} + x^{12} + x^{8} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{126} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{102} + x^{100}\\
&\quad + x^{86} + x^{84} + x^{78} + x^{76} + x^{66} + x^{64} + x^{54} + x^{52} + x^{42} + x^{38} + x^{34}\\
&\quad + x^{30} + x^{24},\\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{94} + x^{92} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60}\\
&\quad + x^{58} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{34} + x^{32} + x^{30} + x^{28}\\
&\quad + x^{20} + x^{18},\\
p_6(x) &= x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{106} + x^{100} + x^{92} + x^{86}\\
&\quad + x^{84} + x^{80} + x^{74} + x^{62} + x^{56} + x^{54} + x^{48} + x^{44} + x^{42} + x^{40} + x^{32}\\
&\quad + x^{28} + x^{26} + x^{22} + x^{20} + x^{14} + x^{12} + x^{10} + x^{8} + 1,\\
p_9(x) &= x^{116} + x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78}\\
&\quad + x^{72} + x^{68} + x^{60} + x^{56} + x^{52} + x^{50} + x^{46} + x^{38} + x^{34} + x^{32} + x^{28}\\
&\quad + x^{22} + x^{20} + x^{18} + x^{16} + x^{10} + x^{4},\\
p_{12}(x) &= x^{116} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{84} + x^{80} + x^{76} + x^{72} + x^{68}\\
&\quad + x^{64} + x^{52} + x^{48} + x^{44} + x^{40} + x^{28} + x^{24} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}\\
&\quad + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{93} + x^{92}\\
&\quad + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{75} + x^{70}\\
&\quad + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{53} + x^{50} + x^{49}\\
&\quad + x^{44} + x^{43} + x^{42} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30},\\
p_3(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109}\\
&\quad + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94}\\
&\quad + x^{88} + x^{85} + x^{84} + x^{78} + x^{75} + x^{74} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60}\\
&\quad + x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{40}\\
&\quad + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&\quad + x^{22} + x^{19} + x^{18},\\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102}\\
&\quad + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81}\\
&\quad + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}\\
&\quad + x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12},\\
p_9(x) = {}& x^{118} + x^{115} + x^{114} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48}\\
&+ x^{46} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{33} + x^{29} + x^{28} + x^{26} + x^{23}\\
&+ x^{22} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6},\\
p_{12}(x) = {}& x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99}\\
&+ x^{97} + x^{96} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69}\\
&+ x^{67} + x^{65} + x^{64} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} + x^{93} + x^{92}\\
&+ x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{75} + x^{70}\\
&+ x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{53} + x^{50} + x^{49}\\
&+ x^{44} + x^{43} + x^{42} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30},\\
p_3(x) = {}& x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109}\\
&+ x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94}\\
&+ x^{88} + x^{85} + x^{84} + x^{78} + x^{75} + x^{74} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{40}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&+ x^{22} + x^{19} + x^{18},\\
p_6(x) = {}& x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81}\\
&+ x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62}\\
&+ x^{61} + x^{59} + x^{58} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{44} + x^{43}\\
&+ x^{42} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{15} + x^{13} + x^{12},\\
p_9(x) = {}& x^{118} + x^{115} + x^{114} + x^{112} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{63}+x^{60}+x^{59}+x^{58}+x^{57}+x^{54}+x^{53}+x^{51}+x^{50}+x^{49}+x^{48}\\
&+x^{46}+x^{43}+x^{40}+x^{39}+x^{38}+x^{37}+x^{33}+x^{29}+x^{28}+x^{26}+x^{23}\\
&+x^{22}+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{10}+x^{7}+x^{6},\\
p_{12}(x)=&\,x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{108}+x^{104}+x^{103}+x^{101}+x^{99}\\
&+x^{97}+x^{96}+x^{84}+x^{83}+x^{81}+x^{79}+x^{77}+x^{76}+x^{72}+x^{71}+x^{69}\\
&+x^{67}+x^{65}+x^{64}+x^{52}+x^{51}+x^{49}+x^{47}+x^{45}+x^{44}+x^{40}+x^{39}\\
&+x^{37}+x^{36}+x^{32}+x^{31}+x^{29}+x^{28}+x^{24}+x^{23}+x^{21}+x^{20}+x^{16}\\
&+x^{15}+x^{13}+x^{12}+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{120}+x^{114}+x^{98}+x^{96}+x^{92}+x^{86}+x^{84}+x^{80}+x^{76}+x^{74}+x^{72}\\
&+x^{70}+x^{68}+x^{62}+x^{56}+x^{50}+x^{46}+x^{44}+x^{42}+x^{40}+x^{36},\\
p_3(x)=&\,x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{96}+x^{92}+x^{86}+x^{82}\\
&+x^{80}+x^{76}+x^{74}+x^{68}+x^{64}+x^{60}+x^{58}+x^{56}+x^{48}+x^{42}+x^{40}\\
&+x^{36}+x^{32}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{16}+x^{14}+x^{12},\\
p_6(x)=&\,x^{112}+x^{110}+x^{104}+x^{98}+x^{90}+x^{86}+x^{74}+x^{72}+x^{70}+x^{62}+x^{56}\\
&+x^{46}+x^{40}+x^{38}+x^{24}+x^{22}+x^{18}+x^{10}+x^{8}+x^{6}+x^{2}+1,\\
p_9(x)=&\,x^{116}+x^{114}+x^{108}+x^{102}+x^{98}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{82}\\
&+x^{78}+x^{74}+x^{70}+x^{62}+x^{60}+x^{54}+x^{52}+x^{48}+x^{44}+x^{38}+x^{32}\\
&+x^{30}+x^{24}+x^{20}+x^{18}+x^{6}+x^{4},\\
p_{12}(x)=&\,x^{116}+x^{114}+x^{112}+x^{100}+x^{98}+x^{96}+x^{84}+x^{82}+x^{80}+x^{68}+x^{66}\\
&+x^{64}+x^{52}+x^{50}+x^{48}+x^{4}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{124}+x^{123}+x^{121}+x^{117}+x^{116}+x^{115}+x^{113}+x^{111}+x^{110}\\
&+x^{109}+x^{108}+x^{106}+x^{105}+x^{101}+x^{100}+x^{99}+x^{96}+x^{92}+x^{90}\\
&+x^{86}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}\\
&+x^{67}+x^{66}+x^{64}+x^{62}+x^{58}+x^{54}+x^{51}+x^{47}+x^{46}+x^{45}+x^{44}\\
&+x^{43}+x^{39}+x^{36}+x^{35}+x^{34}+x^{29}+x^{27}+x^{25}+x^{24},\\
p_3(x)=&\,x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{114}+x^{113}+x^{110}+x^{106}\\
&+x^{104}+x^{98}+x^{92}+x^{91}+x^{89}+x^{84}+x^{83}+x^{82}+x^{80}+x^{79}+x^{78}\\
&+x^{77}+x^{76}+x^{70}+x^{69}+x^{68}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{59}\\
&+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{45}+x^{44}+x^{42}+x^{40}+x^{38}\\
&+x^{32}+x^{30}+x^{26}+x^{25}+x^{24}+x^{21}+x^{19}+x^{17}+x^{16},\\
p_6(x)=&\,x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{113}+x^{111}+x^{107}+x^{105}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{103} + x^{100} + x^{93} + x^{92} + x^{88} + x^{87} + x^{82} + x^{77} + x^{75} + x^{73} + x^{72} \\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{58} + x^{56} + x^{55} \\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} \\
&+ x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} \\
&+ x^{19} + x^{18} + x^{16} + x^{15} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104} \\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60} \\
&+ x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28} \\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107} \\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84} \\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} \\
&+ x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37} \\
&+ x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{136} + x^{130} + x^{129} + x^{126} + x^{124} + x^{122} + x^{119} + x^{115} + x^{114} + x^{110} \\
&+ x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89} \\
&+ x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{72} + x^{71} + x^{64} + x^{63} + x^{62} \\
&+ x^{61} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{44} + x^{43} + x^{42} \\
&+ x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30}, \\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} \\
&+ x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} \\
&+ x^{74} + x^{66} + x^{65} + x^{64} + x^{63} + x^{59} + x^{58} + x^{57} + x^{53} + x^{52} + x^{51} \\
&+ x^{47} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{35} + x^{34} + x^{33} + x^{29} + x^{28} \\
&+ x^{27} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_6(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} \\
&+ x^{100} + x^{98} + x^{94} + x^{92} + x^{84} + x^{80} + x^{74} + x^{72} + x^{68} + x^{60} + x^{58} \\
&+ x^{48} + x^{42} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{18} + x^{12}, \\
p_9(x) &= x^{126} + x^{125} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} \\
&+ x^{107} + x^{106} + x^{105} + x^{102} + x^{94} + x^{93} + x^{87} + x^{85} + x^{84} + x^{82} \\
&+ x^{81} + x^{80} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{62} + x^{61} + x^{55} \\
&+ x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} \\
&+ x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{10}\\
&+ x^{9} + x^{6},\\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{104} + x^{92} + x^{84} + x^{72} + x^{60} + x^{52} + x^{48} + x^{44}\\
&+ x^{40} + x^{36} + x^{28} + x^{24} + x^{20} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0].
For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{124} + x^{121} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{109} + x^{108} + x^{106}\\
&+ x^{105} + x^{103} + x^{102} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{76} + x^{75} + x^{68} + x^{63} + x^{62}\\
&+ x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{45} + x^{43}\\
&+ x^{42} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30},\\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{110} + x^{108}\\
&+ x^{106} + x^{104} + x^{102} + x^{100} + x^{97} + x^{95} + x^{93} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{74} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{52} + x^{49} + x^{47} + x^{45} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{32} + x^{30} + x^{28} + x^{25} + x^{23} + x^{21}\\
&+ x^{18},\\
p_6(x) &= x^{124} + x^{122} + x^{116} + x^{114} + x^{110} + x^{108} + x^{100} + x^{94} + x^{92} + x^{88}\\
&+ x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{50} + x^{46} + x^{42}\\
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12},\\
p_9(x) &= x^{126} + x^{125} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{111}\\
&+ x^{110} + x^{109} + x^{108} + x^{107} + x^{105} + x^{102} + x^{94} + x^{93} + x^{90} + x^{89}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73}\\
&+ x^{70} + x^{62} + x^{61} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47}\\
&+ x^{44} + x^{43} + x^{42} + x^{39} + x^{36} + x^{35} + x^{34} + x^{31} + x^{28} + x^{27} + x^{26}\\
&+ x^{23} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9} + x^{6},\\
p_{12}(x) &= x^{128} + x^{108} + x^{100} + x^{76} + x^{68} + x^{48} + x^{44} + x^{36} + x^{28} + x^{20} + x^{16}\\
&+ x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].
For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{105} + x^{104} + x^{102} + x^{98} + x^{93} + x^{92} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83}\\
&+ x^{82} + x^{76} + x^{73} + x^{72} + x^{69} + x^{66} + x^{63} + x^{62} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{112} + x^{109} + x^{108} + x^{105} + x^{104}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{86} + x^{85}\\
&+ x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68}\\
&+ x^{65} + x^{61} + x^{59} + x^{58} + x^{55} + x^{50} + x^{49} + x^{48} + x^{45} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{34} + x^{33} + x^{29} + x^{28} + x^{21} + x^{20},\\
p_6(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{114} + x^{109} + x^{106} + x^{105} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{89} + x^{87} + x^{84}\\
&+ x^{83} + x^{82} + x^{81} + x^{79} + x^{71} + x^{67} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60}\\
&+ x^{59} + x^{56} + x^{55} + x^{52} + x^{49} + x^{42} + x^{38} + x^{37} + x^{36} + x^{35} + x^{29}\\
&+ x^{28} + x^{26} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) = {}& x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28}\\
&+ x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) = {}& x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107}\\
&+ x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84}\\
&+ x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{124} + x^{123} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{110}\\
&+ x^{109} + x^{108} + x^{106} + x^{105} + x^{101} + x^{100} + x^{99} + x^{96} + x^{92} + x^{90}\\
&+ x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72}\\
&+ x^{67} + x^{66} + x^{64} + x^{62} + x^{58} + x^{54} + x^{51} + x^{47} + x^{46} + x^{45} + x^{44}\\
&+ x^{43} + x^{39} + x^{36} + x^{35} + x^{34} + x^{29} + x^{27} + x^{25} + x^{24},\\
p_3(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{114} + x^{113} + x^{110} + x^{106}\\
&+ x^{104} + x^{98} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78}\\
&+ x^{77} + x^{76} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59}\\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{45} + x^{44} + x^{42} + x^{40} + x^{38}\\
&+ x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{21} + x^{19} + x^{17} + x^{16},\\
p_6(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{113} + x^{111} + x^{107} + x^{105}\\
&+ x^{103} + x^{100} + x^{93} + x^{92} + x^{88} + x^{87} + x^{82} + x^{77} + x^{75} + x^{73} + x^{72}\\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{58} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38}\\
&+ x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{19}+x^{18}+x^{16}+x^{15}+x^{12}+x^{11}+x^{9}+x^{7}+x^{5}+x^{3}+x+1,\\
p_9(x)&=x^{124}+x^{123}+x^{121}+x^{120}+x^{112}+x^{111}+x^{109}+x^{107}+x^{105}+x^{104}\\
&+x^{92}+x^{91}+x^{89}+x^{88}+x^{80}+x^{79}+x^{77}+x^{75}+x^{73}+x^{72}+x^{60}\\
&+x^{59}+x^{57}+x^{56}+x^{48}+x^{47}+x^{45}+x^{44}+x^{32}+x^{31}+x^{29}+x^{28}\\
&+x^{16}+x^{15}+x^{13}+x^{11}+x^{9}+x^{8},\\
p_{12}(x)&=x^{124}+x^{123}+x^{121}+x^{120}+x^{116}+x^{115}+x^{113}+x^{112}+x^{108}+x^{107}\\
&+x^{105}+x^{103}+x^{101}+x^{99}+x^{97}+x^{96}+x^{92}+x^{91}+x^{89}+x^{88}+x^{84}\\
&+x^{83}+x^{81}+x^{80}+x^{76}+x^{75}+x^{73}+x^{71}+x^{69}+x^{67}+x^{65}+x^{64}\\
&+x^{60}+x^{59}+x^{57}+x^{56}+x^{52}+x^{51}+x^{49}+x^{48}+x^{40}+x^{39}+x^{37}\\
&+x^{36}+x^{24}+x^{23}+x^{21}+x^{20}+x^{12}+x^{11}+x^{9}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{124}+x^{121}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{109}+x^{108}+x^{106}\\
&+x^{105}+x^{103}+x^{102}+x^{98}+x^{96}+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}\\
&+x^{87}+x^{86}+x^{84}+x^{83}+x^{82}+x^{81}+x^{76}+x^{75}+x^{68}+x^{63}+x^{62}\\
&+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}+x^{52}+x^{51}+x^{49}+x^{45}+x^{43}\\
&+x^{42}+x^{39}+x^{36}+x^{35}+x^{34}+x^{33}+x^{31}+x^{30},\\
p_3(x)&=x^{122}+x^{121}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{112}+x^{110}+x^{108}\\
&+x^{106}+x^{104}+x^{102}+x^{100}+x^{97}+x^{95}+x^{93}+x^{89}+x^{88}+x^{87}\\
&+x^{86}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{77}+x^{74}+x^{66}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{56}+x^{54}+x^{52}+x^{49}+x^{47}+x^{45}+x^{41}\\
&+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{32}+x^{30}+x^{28}+x^{25}+x^{23}+x^{21}\\
&+x^{18},\\
p_6(x)&=x^{124}+x^{122}+x^{116}+x^{114}+x^{110}+x^{108}+x^{100}+x^{94}+x^{92}+x^{88}\\
&+x^{76}+x^{74}+x^{72}+x^{70}+x^{68}+x^{66}+x^{64}+x^{56}+x^{50}+x^{46}+x^{42}\\
&+x^{40}+x^{38}+x^{36}+x^{32}+x^{30}+x^{26}+x^{24}+x^{20}+x^{18}+x^{16}+x^{12},\\
p_9(x)&=x^{126}+x^{125}+x^{122}+x^{121}+x^{119}+x^{118}+x^{116}+x^{115}+x^{114}+x^{111}\\
&+x^{110}+x^{109}+x^{108}+x^{107}+x^{105}+x^{102}+x^{94}+x^{93}+x^{90}+x^{89}\\
&+x^{87}+x^{86}+x^{84}+x^{83}+x^{82}+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}\\
&+x^{70}+x^{62}+x^{61}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{47}\\
&+x^{44}+x^{43}+x^{42}+x^{39}+x^{36}+x^{35}+x^{34}+x^{31}+x^{28}+x^{27}+x^{26}\\
&+x^{23}+x^{20}+x^{19}+x^{18}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{9}+x^{6},\\
p_{12}(x)&=x^{128}+x^{108}+x^{100}+x^{76}+x^{68}+x^{48}+x^{44}+x^{36}+x^{28}+x^{20}+x^{16}\\
&+x^{12}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{130} + x^{129} + x^{126} + x^{124} + x^{122} + x^{119} + x^{115} + x^{114} + x^{110} \\
&\quad + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{97} + x^{95} + x^{92} + x^{91} + x^{89} \\
&\quad + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{72} + x^{71} + x^{64} + x^{63} + x^{62} \\
&\quad + x^{61} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{49} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30}, \\
p_3(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} \\
&\quad + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{74} + x^{66} + x^{65} + x^{64} + x^{63} + x^{59} + x^{58} + x^{57} + x^{53} + x^{52} + x^{51} \\
&\quad + x^{47} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{35} + x^{34} + x^{33} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_6(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} \\
&\quad + x^{100} + x^{98} + x^{94} + x^{92} + x^{84} + x^{80} + x^{74} + x^{72} + x^{68} + x^{60} + x^{58} \\
&\quad + x^{48} + x^{42} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{18} + x^{12}, \\
p_9(x) &= x^{126} + x^{125} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{102} + x^{94} + x^{93} + x^{87} + x^{85} + x^{84} + x^{82} \\
&\quad + x^{81} + x^{80} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{62} + x^{61} + x^{55} \\
&\quad + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{10} \\
&\quad + x^{9} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{104} + x^{92} + x^{84} + x^{72} + x^{60} + x^{52} + x^{48} + x^{44} \\
&\quad + x^{40} + x^{36} + x^{28} + x^{24} + x^{20} + x^{16} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{98} + x^{93} + x^{92} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} \\
&\quad + x^{82} + x^{76} + x^{73} + x^{72} + x^{69} + x^{66} + x^{63} + x^{62} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{112} + x^{109} + x^{108} + x^{105} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} \\
&\quad + x^{65} + x^{61} + x^{59} + x^{58} + x^{55} + x^{50} + x^{49} + x^{48} + x^{45} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{34} + x^{33} + x^{29} + x^{28} + x^{21} + x^{20},
\end{aligned}$$

$$
\begin{aligned}
p_6(x) = {} & x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{114} + x^{109} + x^{106} + x^{105} + x^{104} \\
& + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{89} + x^{87} + x^{84} \\
& + x^{83} + x^{82} + x^{81} + x^{79} + x^{71} + x^{67} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{59} + x^{56} + x^{55} + x^{52} + x^{49} + x^{42} + x^{38} + x^{37} + x^{36} + x^{35} + x^{29} \\
& + x^{28} + x^{26} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) = {} & x^{124} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{107} + x^{105} + x^{104} \\
& + x^{92} + x^{91} + x^{89} + x^{88} + x^{80} + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{60} \\
& + x^{59} + x^{57} + x^{56} + x^{48} + x^{47} + x^{45} + x^{44} + x^{32} + x^{31} + x^{29} + x^{28} \\
& + x^{16} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) = {} & x^{124} + x^{123} + x^{121} + x^{120} + x^{116} + x^{115} + x^{113} + x^{112} + x^{108} + x^{107} \\
& + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} + x^{92} + x^{91} + x^{89} + x^{88} + x^{84} \\
& + x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} \\
& + x^{60} + x^{59} + x^{57} + x^{56} + x^{52} + x^{51} + x^{49} + x^{48} + x^{40} + x^{39} + x^{37} \\
& + x^{36} + x^{24} + x^{23} + x^{21} + x^{20} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	126	[210, 211]	252	[388, 363]	378	[482, 483]	504	[374, 150]
1	[3, 4]	127	[212, 213]	253	[374, 157]	379	[484, 485]	505	[14, 153]
2	[5, 6]	128	[214, 215]	254	[156, 375]	380	[486, 208]	506	[391, 51]
3	[7, 8]	129	[216, 217]	255	[389, 146]	381	[487, 488]	507	[412, 409]
4	[9, 10]	130	[218, 81]	256	[126, 390]	382	[489, 490]	508	[344, 344]
5	[11, 12]	131	[219, 220]	257	[391, 196]	383	[491, 492]	509	[48, 572]
6	[13, 14]	132	[221, 222]	258	[380, 392]	384	[493, 494]	510	[160, 576]
7	[15, 16]	133	[223, 224]	259	[393, 137]	385	[217, 262]	511	[127, 395]
8	[17, 18]	134	[102, 225]	260	[385, 347]	386	[495, 496]	512	[388, 40]
9	[19, 20]	135	[226, 227]	261	[394, 56]	387	[497, 498]	513	[577, 57]
10	[21, 22]	136	[228, 229]	262	[345, 395]	388	[499, 284]	514	[167, 37]
11	[23, 24]	137	[230, 231]	263	[396, 146]	389	[500, 501]	515	[345, 578]
12	[25, 26]	138	[232, 233]	264	[397, 51]	390	[502, 503]	516	[176, 393]
13	[27, 28]	139	[234, 235]	265	[389, 398]	391	[309, 504]	517	[396, 398]
14	[29, 30]	140	[236, 237]	266	[399, 400]	392	[492, 505]	518	[372, 571]
15	[31, 32]	141	[238, 239]	267	[349, 155]	393	[506, 478]	519	[579, 42]
16	[33, 34]	142	[240, 241]	268	[401, 402]	394	[507, 508]	520	[175, 410]
17	[35, 36]	143	[242, 243]	269	[45, 153]	395	[450, 509]	521	[187, 578]
18	[37, 38]	144	[24, 244]	270	[403, 59]	396	[510, 511]	522	[580, 162]
19	[39, 40]	145	[245, 246]	271	[404, 191]	397	[512, 212]	523	[197, 351]
20	[41, 42]	146	[247, 248]	272	[405, 135]	398	[513, 514]	524	[183, 422]

21	[43, 44]	147	[109, 249]	273	[384, 163]	399	[515, 516]	525	[200, 425]
22	[45, 46]	148	[250, 251]	274	[136, 406]	400	[74, 517]	526	[581, 161]
23	[47, 48]	149	[252, 253]	275	[345, 128]	401	[518, 4]	527	[426, 362]
24	[47, 49]	150	[254, 255]	276	[407, 159]	402	[519, 520]	528	[582, 366]
25	[50, 51]	151	[256, 257]	277	[9, 44]	403	[521, 522]	529	[583, 184]
26	[52, 53]	152	[258, 259]	278	[361, 167]	404	[253, 523]	530	[167, 56]
27	[54, 55]	153	[260, 261]	279	[177, 406]	405	[524, 77]	531	[132, 189]
28	[56, 57]	154	[262, 263]	280	[408, 409]	406	[314, 525]	532	[377, 140]
29	[58, 59]	155	[264, 265]	281	[181, 32]	407	[526, 338]	533	[142, 168]
30	[60, 61]	156	[266, 267]	282	[36, 169]	408	[527, 299]	534	[584, 165]
31	[62, 63]	157	[268, 269]	283	[195, 410]	409	[333, 528]	535	[162, 191]
32	[64, 65]	158	[270, 271]	284	[176, 136]	410	[529, 530]	536	[141, 169]
33	[66, 67]	159	[272, 273]	285	[411, 49]	411	[288, 256]	537	[386, 575]
34	[68, 69]	160	[274, 216]	286	[32, 180]	412	[243, 440]	538	[579, 402]
35	[70, 71]	161	[275, 276]	287	[172, 402]	413	[531, 532]	539	[415, 398]
36	[4, 72]	162	[277, 278]	288	[361, 360]	414	[316, 533]	540	[12, 413]
37	[73, 74]	163	[106, 279]	289	[130, 189]	415	[534, 535]	541	[563, 162]
38	[75, 76]	164	[280, 281]	290	[133, 354]	416	[536, 537]	542	[585, 153]
39	[77, 78]	165	[282, 283]	291	[412, 368]	417	[538, 539]	543	[560, 122]
40	[79, 80]	166	[284, 73]	292	[138, 181]	418	[540, 541]	544	[570, 370]
41	[81, 82]	167	[285, 286]	293	[143, 413]	419	[225, 542]	545	[411, 570]
42	[83, 84]	168	[287, 288]	294	[182, 393]	420	[543, 544]	546	[344, 121]
43	[85, 86]	169	[289, 290]	295	[414, 400]	421	[545, 15]	547	[49, 572]
44	[87, 88]	170	[291, 292]	296	[41, 173]	422	[546, 547]	548	[580, 12]
45	[89, 90]	171	[293, 294]	297	[174, 44]	423	[548, 549]	549	[134, 586]
46	[28, 91]	172	[295, 296]	298	[415, 146]	424	[209, 550]	550	[345, 188]
47	[92, 93]	173	[297, 298]	299	[414, 416]	425	[483, 551]	551	[561, 373]
48	[90, 94]	174	[299, 300]	300	[50, 417]	426	[552, 553]	552	[580, 143]
49	[95, 96]	175	[301, 302]	301	[418, 419]	427	[554, 548]	553	[178, 38]
50	[97, 98]	176	[303, 230]	302	[358, 406]	428	[479, 555]	554	[177, 574]
51	[99, 100]	177	[304, 305]	303	[47, 369]	429	[556, 447]	555	[391, 417]
52	[101, 102]	178	[306, 307]	304	[149, 157]	430	[124, 122]	556	[197, 561]
53	[103, 104]	179	[308, 309]	305	[357, 55]	431	[124, 180]	557	[587, 588]
54	[105, 106]	180	[310, 311]	306	[37, 420]	432	[342, 343]	558	[511, 589]
55	[16, 107]	181	[312, 310]	307	[12, 144]	433	[343, 180]	559	[590, 591]
56	[108, 109]	182	[313, 314]	308	[357, 421]	434	[342, 32]	560	[592, 312]
57	[110, 111]	183	[315, 316]	309	[399, 416]	435	[41, 402]	561	[593, 594]
58	[112, 113]	184	[317, 318]	310	[342, 124]	436	[346, 557]	562	[595, 596]
59	[114, 115]	185	[63, 64]	311	[123, 32]	437	[34, 190]	563	[597, 598]
60	[116, 117]	186	[319, 274]	312	[185, 123]	438	[367, 409]	564	[494, 599]
61	[118, 119]	187	[257, 218]	313	[378, 200]	439	[558, 559]	565	[454, 236]
62	[120, 121]	188	[320, 321]	314	[130, 34]	440	[372, 390]	566	[600, 601]
63	[122, 123]	189	[322, 323]	315	[186, 422]	441	[355, 187]	567	[594, 602]
64	[121, 120]	190	[67, 324]	316	[12, 191]	442	[37, 57]	568	[603, 590]

65	[123, 124]	191	[325, 326]	317	[180, 123]	443	[408, 368]	569	[473, 604]
66	[125, 53]	192	[327, 320]	318	[34, 354]	444	[560, 180]	570	[259, 605]
67	[126, 59]	193	[328, 329]	319	[163, 347]	445	[8, 421]	571	[606, 607]
68	[127, 128]	194	[330, 331]	320	[378, 368]	446	[423, 53]	572	[337, 608]
69	[129, 55]	195	[332, 333]	321	[379, 201]	447	[399, 161]	573	[474, 609]
70	[130, 131]	196	[334, 335]	322	[404, 144]	448	[561, 186]	574	[610, 611]
71	[132, 133]	197	[211, 319]	323	[423, 366]	449	[199, 48]	575	[612, 613]
72	[8, 55]	198	[336, 317]	324	[53, 424]	450	[132, 34]	576	[307, 614]
73	[134, 135]	199	[117, 337]	325	[352, 139]	451	[356, 562]	577	[615, 486]
74	[136, 137]	200	[338, 339]	326	[379, 425]	452	[563, 12]	578	[616, 617]
75	[138, 123]	201	[340, 341]	327	[367, 379]	453	[163, 564]	579	[222, 116]
76	[139, 140]	202	[185, 342]	328	[149, 375]	454	[350, 198]	580	[71, 327]
77	[141, 61]	203	[124, 185]	329	[426, 167]	455	[192, 187]	581	[618, 452]
78	[142, 141]	204	[121, 121]	330	[378, 409]	456	[565, 139]	582	[619, 620]
79	[143, 144]	205	[343, 185]	331	[131, 190]	457	[346, 564]	583	[621, 606]
80	[145, 146]	206	[120, 344]	332	[427, 392]	458	[33, 189]	584	[550, 622]
81	[147, 148]	207	[121, 344]	333	[193, 395]	459	[187, 128]	585	[623, 624]
82	[149, 150]	208	[192, 345]	334	[428, 375]	460	[566, 387]	586	[466, 625]
83	[151, 51]	209	[346, 347]	335	[152, 46]	461	[563, 143]	587	[181, 343]
84	[152, 153]	210	[36, 348]	336	[429, 34]	462	[183, 567]	588	[49, 370]
85	[147, 154]	211	[182, 136]	337	[147, 381]	463	[123, 343]	589	[359, 167]
86	[9, 155]	212	[349, 10]	338	[168, 382]	464	[122, 181]	590	[414, 161]
87	[156, 157]	213	[350, 351]	339	[166, 360]	465	[568, 139]	591	[164, 134]
88	[158, 159]	214	[352, 353]	340	[171, 381]	466	[560, 185]	592	[122, 342]
89	[160, 161]	215	[131, 354]	341	[53, 134]	467	[162, 194]	593	[568, 353]
90	[162, 144]	216	[355, 345]	342	[430, 431]	468	[193, 128]	594	[181, 124]
91	[55, 163]	217	[356, 135]	343	[432, 433]	469	[569, 351]	595	[138, 342]
92	[125, 164]	218	[357, 170]	344	[203, 434]	470	[164, 424]	596	[570, 572]
93	[39, 165]	219	[358, 137]	345	[435, 66]	471	[199, 570]	597	[352, 558]
94	[166, 167]	220	[359, 360]	346	[436, 112]	472	[133, 190]	598	[130, 133]
95	[168, 169]	221	[361, 362]	347	[107, 437]	473	[35, 141]	599	[353, 140]
96	[54, 170]	222	[39, 363]	348	[438, 439]	474	[187, 395]	600	[418, 165]
97	[171, 154]	223	[60, 169]	349	[440, 304]	475	[126, 571]	601	[168, 61]
98	[172, 173]	224	[364, 198]	350	[441, 442]	476	[397, 196]	602	[353, 559]
99	[174, 10]	225	[177, 137]	351	[443, 444]	477	[158, 410]	603	[52, 164]
100	[175, 159]	226	[365, 120]	352	[445, 252]	478	[426, 360]	604	[356, 586]
101	[176, 177]	227	[189, 354]	353	[246, 446]	479	[171, 148]	605	[582, 53]
102	[178, 57]	228	[349, 44]	354	[447, 448]	480	[404, 194]	606	[36, 382]
103	[179, 49]	229	[125, 366]	355	[449, 450]	481	[364, 351]	607	[145, 398]
104	[180, 181]	230	[367, 368]	356	[451, 79]	482	[350, 561]	608	[198, 186]
105	[182, 177]	231	[369, 370]	357	[452, 453]	483	[388, 165]	609	[585, 46]
106	[183, 184]	232	[343, 122]	358	[20, 454]	484	[393, 371]	610	[376, 353]
107	[32, 185]	233	[185, 181]	359	[455, 108]	485	[569, 198]	611	[558, 140]
108	[186, 184]	234	[358, 371]	360	[456, 75]	486	[36, 61]	612	[424, 135]

109	[187, 188]	235	[129, 170]	361	[78, 457]	487	[136, 371]	613	[394, 37]
110	[31, 124]	236	[372, 59]	362	[458, 110]	488	[407, 410]	614	[14, 46]
111	[189, 190]	237	[198, 373]	363	[459, 460]	489	[376, 139]	615	[163, 557]
112	[143, 191]	238	[374, 375]	364	[461, 462]	490	[369, 572]	616	[47, 570]
113	[192, 193]	239	[52, 366]	365	[463, 464]	491	[178, 420]	617	[377, 559]
114	[143, 194]	240	[376, 377]	366	[465, 466]	492	[141, 382]	618	[141, 348]
115	[195, 159]	241	[378, 379]	367	[26, 340]	493	[565, 353]	619	[142, 36]
116	[50, 196]	242	[172, 42]	368	[467, 27]	494	[344, 120]	620	[186, 567]
117	[197, 198]	243	[8, 170]	369	[468, 469]	495	[403, 573]	621	[134, 562]
118	[199, 49]	244	[48, 370]	370	[93, 470]	496	[193, 188]	622	[355, 193]
119	[200, 201]	245	[380, 381]	371	[471, 472]	497	[373, 184]	623	[584, 419]
120	[120, 120]	246	[60, 382]	372	[279, 473]	498	[55, 385]	624	[127, 188]
121	[202, 203]	247	[383, 347]	373	[442, 474]	499	[136, 574]	625	[139, 559]
122	[204, 205]	248	[384, 385]	374	[475, 242]	500	[58, 573]		
123	[65, 206]	249	[35, 168]	375	[476, 477]	501	[393, 406]		
124	[205, 207]	250	[12, 194]	376	[478, 479]	502	[177, 371]		
125	[208, 209]	251	[386, 387]	377	[480, 481]	503	[566, 575]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	90	1	180	1	270	0	360	0	450	1	540	0
1	0	91	0	181	0	271	1	361	1	451	1	541	1
2	0	92	0	182	0	272	0	362	0	452	1	542	0
3	0	93	1	183	0	273	0	363	0	453	0	543	0
4	1	94	0	184	1	274	1	364	1	454	0	544	0
5	0	95	0	185	0	275	0	365	1	455	0	545	1
6	0	96	0	186	0	276	1	366	0	456	0	546	1
7	0	97	0	187	0	277	1	367	0	457	0	547	0
8	1	98	0	188	0	278	1	368	1	458	0	548	1
9	1	99	0	189	1	279	1	369	1	459	0	549	1
10	0	100	0	190	1	280	1	370	1	460	0	550	0
11	0	101	0	191	1	281	0	371	0	461	1	551	0
12	0	102	1	192	0	282	1	372	1	462	0	552	1
13	0	103	1	193	1	283	1	373	1	463	1	553	1
14	0	104	1	194	0	284	0	374	1	464	0	554	1
15	0	105	0	195	1	285	1	375	1	465	0	555	0
16	0	106	0	196	0	286	0	376	1	466	0	556	0
17	1	107	0	197	0	287	0	377	1	467	1	557	0
18	1	108	0	198	0	288	1	378	0	468	1	558	1
19	1	109	0	199	0	289	1	379	0	469	1	559	0
20	0	110	0	200	0	290	0	380	1	470	1	560	0
21	0	111	1	201	0	291	1	381	1	471	0	561	0
22	1	112	0	202	0	292	1	382	1	472	0	562	1
23	0	113	0	203	1	293	0	383	1	473	1	563	1

24	0	114	0	204	0	294	0	384	0	474	0	564	1
25	0	115	1	205	1	295	0	385	1	475	1	565	0
26	0	116	0	206	0	296	0	386	0	476	1	566	0
27	0	117	0	207	0	297	0	387	1	477	0	567	0
28	0	118	0	208	0	298	0	388	1	478	1	568	0
29	0	119	0	209	0	299	0	389	0	479	0	569	1
30	0	120	0	210	1	300	0	390	1	480	1	570	0
31	0	121	0	211	0	301	0	391	0	481	1	571	1
32	0	122	0	212	1	302	0	392	1	482	0	572	0
33	0	123	1	213	0	303	0	393	0	483	1	573	0
34	1	124	1	214	1	304	1	394	1	484	0	574	1
35	1	125	0	215	0	305	1	395	1	485	1	575	0
36	1	126	1	216	0	306	1	396	1	486	1	576	0
37	1	127	1	217	1	307	0	397	1	487	1	577	0
38	1	128	1	218	1	308	1	398	0	488	1	578	0
39	1	129	0	219	0	309	0	399	0	489	1	579	1
40	0	130	1	220	0	310	1	400	1	490	1	580	1
41	0	131	0	221	1	311	1	401	1	491	1	581	1
42	1	132	1	222	1	312	0	402	1	492	1	582	1
43	0	133	0	223	0	313	0	403	0	493	0	583	1
44	0	134	1	224	1	314	1	404	1	494	1	584	0
45	1	135	1	225	1	315	0	405	0	495	0	585	0
46	0	136	1	226	1	316	0	406	1	496	1	586	0
47	0	137	0	227	1	317	1	407	1	497	1	587	0
48	1	138	1	228	1	318	1	408	1	498	0	588	0
49	0	139	0	229	0	319	0	409	1	499	1	589	0
50	0	140	1	230	0	320	0	410	1	500	0	590	0
51	0	141	1	231	1	321	0	411	1	501	0	591	1
52	0	142	1	232	1	322	1	412	1	502	1	592	0
53	1	143	0	233	0	323	1	413	1	503	0	593	0
54	0	144	0	234	0	324	1	414	0	504	1	594	0
55	0	145	1	235	0	325	1	415	0	505	0	595	1
56	0	146	1	236	1	326	0	416	1	506	0	596	0
57	0	147	0	237	0	327	0	417	1	507	1	597	1
58	0	148	0	238	1	328	1	418	0	508	1	598	1
59	0	149	1	239	0	329	1	419	1	509	1	599	0
60	0	150	0	240	1	330	0	420	0	510	1	600	0
61	0	151	1	241	0	331	0	421	1	511	1	601	0
62	0	152	1	242	0	332	1	422	1	512	1	602	0
63	0	153	1	243	1	333	1	423	1	513	0	603	0
64	0	154	0	244	1	334	0	424	0	514	1	604	1
65	1	155	1	245	1	335	1	425	1	515	0	605	1
66	0	156	0	246	0	336	0	426	1	516	0	606	1
67	1	157	1	247	1	337	0	427	1	517	1	607	1

68	1	158	0	248	0	338	0	428	0	518	1	608	0
69	0	159	0	249	1	339	0	429	0	519	1	609	0
70	1	160	1	250	0	340	0	430	1	520	0	610	1
71	1	161	0	251	0	341	1	431	1	521	0	611	1
72	1	162	1	252	1	342	1	432	1	522	1	612	0
73	1	163	0	253	1	343	1	433	1	523	0	613	1
74	1	164	1	254	0	344	1	434	1	524	0	614	0
75	1	165	1	255	0	345	0	435	0	525	0	615	0
76	0	166	0	256	1	346	0	436	0	526	1	616	0
77	1	167	1	257	0	347	0	437	1	527	1	617	1
78	1	168	0	258	1	348	0	438	0	528	1	618	1
79	0	169	1	259	0	349	1	439	1	529	1	619	1
80	1	170	1	260	1	350	0	440	1	530	1	620	0
81	0	171	0	261	1	351	1	441	0	531	1	621	1
82	1	172	0	262	0	352	1	442	1	532	1	622	0
83	1	173	0	263	1	353	0	443	1	533	1	623	0
84	1	174	0	264	1	354	0	444	0	534	0	624	1
85	0	175	0	265	0	355	0	445	1	535	1	625	0
86	1	176	0	266	0	356	1	446	1	536	1		
87	0	177	1	267	1	357	1	447	0	537	0		
88	0	178	1	268	1	358	0	448	0	538	1		
89	1	179	1	269	1	359	0	449	0	539	0		

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