PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z,z^5+z^3+z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z,z^5+z^3+z^2+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z,z^5+z^3+z^2+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$p_0(z)=z^{20}+z^{18}+z^{15}+z^{12}+z^{10}+z^9+z^8+z^7+z^5+z^4+z^3+z^2+z+1,$$
$$p_1(z)=z^{23}+z^{21}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^2+z$$
$$,$$
$$p_2(z)=z^{24}+z^{22}+z^{16}+z^{12}+z^8+z^2,$$
$$p_3(z)=z^{23}+z^{21}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^2+z$$
$$,$$
$$p_4(z)=z^{22}+z^{18}+z^{15}+z^{10}+z^9+z^8+z^7+z^5+z^4+z^3+z,$$

and for $\mathrm{CF}(b,a)$*,*

$$p_0(z)=z^{18}+z^{15}+z^{10}+z^9+z^7+z^5+z^3+z^2+z,$$
$$p_1(z)=z^{23}+z^{21}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^2+z$$
$$,$$
$$p_2(z)=z^{24}+z^{22}+z^{16}+z^{12}+z^8+z^2,$$
$$p_3(z)=z^{23}+z^{21}+z^{19}+z^{16}+z^{15}+z^{13}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^2+z$$
$$,$$
$$p_4(z)=z^{22}+z^{20}+z^{18}+z^{15}+z^{12}+z^{10}+z^9+z^7+z^5+z^3+z+1.$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\ &\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\ W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\ &\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] &= M^e[:6u] + x^{3u} M^e[:3u] + x^{3u} M^e[:6u] + x^{6u} M^e[:3u] + x^{6u} M^e[:6u] \\ &\quad + x^{8u} M^e[:4u] + x^{10u} M^e[:2u] + (x^{6u} + x^{9u}) R^e, \\ W^e[:12u] &= W^e[:6u] + x^{3u} W^e[:3u] + x^{3u} W^e[:6u] + x^{6u} W^e[:3u] + x^{6u} W^e[:6u] \\ &\quad + x^{8u} W^e[:4u] + x^{10u} W^e[:2u] + (x^{6u} + x^{9u}) R^e. \end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^{3u}M^o[:3u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u] + x^{6u}M^o[:6u] \\ &\quad + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + (x^{6u} + x^{9u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^{3u}W^o[:3u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u] + x^{6u}W^o[:6u] \\ &\quad + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + (x^{6u} + x^{9u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] &= T[:6u] + x^{3u}T[:3u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{6u}T[:6u] + x^{8u}T[:4u] \\ &\quad + x^{10u}T[:2u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{3u}T[3u:4u] + T[6u:7u], \\ 0 &= x^{3u}T[4u:5u] + T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + T[9u:10u], \\ 0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u], \\ 0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[100w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[11w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[100w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1474 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,11)) + \tau(A(s,110)), \tag{2.19}$$

$$0 = \tau(A(s,100)) + \tau(A(s,111)), \tag{2.20}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.21}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1001)), \tag{2.22}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.23}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,11)) + \tau(A(s,110)), \tag{2.25}$$

$$0 = \tau(A(s,100)) + \tau(A(s,111)), \tag{2.26}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.27}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1001)), \tag{2.28}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.29}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{42}), E_{42}) = (A(i,0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 21$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u}M^e[:3u] + x^{6u}R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u}W^e[:3u] + x^{6u}R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u}M^o[:3u] + x^{6u}R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u}W^o[:3u] + x^{6u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \left(W^e[:6v] + x^{3v}W^e[:3v] + x^{6u}R^e\right) \times \left(M^e[:6v] + x^{3v}M^e[:3v]\right.$$

$$+ x^{6u}R^e); \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^9 \\ &\quad + x^6 + x^4, \\ q_1(x) &= x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^9 + x^8 + x^5 \\ &\quad + x^3 + x, \end{aligned}$$

$$
\begin{aligned}
q_2(x) &= x^{22} + x^{16} + x^{12} + x^8 + x^2 + 1,\\
q_3(x) &= x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{11} + x^9 + x^8 + x^5\\
&\quad + x^3 + x,\\
q_4(x) &= x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^9 + x^6 + x^2.
\end{aligned}
$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{134} + x^{132} + x^{126} + x^{118} + x^{110} + x^{108} + x^{104} + x^{102} + x^{98}\\
&\quad + x^{96} + x^{92} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50}\\
&\quad + x^{46} + x^{38} + x^{28} + x^{26} + x^{20} + x^{12},\\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96}\\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{70} + x^{68} + x^{60} + x^{58} + x^{56} + x^{52}\\
&\quad + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24}\\
&\quad + x^{20} + x^{16} + x^{14} + x^8,\\
p_6(x) &= x^{134} + x^{128} + x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{112} + x^{106} + x^{102}\\
&\quad + x^{100} + x^{96} + x^{94} + x^{88} + x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{64} + x^{62}\\
&\quad + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{34} + x^{30}\\
&\quad + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^4 + x^2 + 1,\\
p_9(x) &= x^{136} + x^{134} + x^{128} + x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{106} + x^{102}\\
&\quad + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72}\\
&\quad + x^{70} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} + x^{40}\\
&\quad + x^{38} + x^{36} + x^{34} + x^{28} + x^{26} + x^{22} + x^{14} + x^{12} + x^8 + x^6 + x^4 + x^2,\\
p_{12}(x) &= x^{136} + x^{120} + x^{112} + x^{104} + x^{96} + x^{88} + x^{80} + x^{72} + x^{64} + x^{48} + x^{24}\\
&\quad + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$p_0(x) = x^{126} + x^{125} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}$$

$$
\begin{aligned}
&+x^{112}+x^{110}+x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}\\
&+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{89}+x^{88}\\
&+x^{86}+x^{84}+x^{83}+x^{79}+x^{78}+x^{76}+x^{75}+x^{73}+x^{71}+x^{68}+x^{65}\\
&+x^{63}+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}+x^{47}+x^{46}+x^{45}\\
&+x^{43}+x^{40}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{29}+x^{28}+x^{26}+x^{25}\\
&+x^{23}+x^{22}+x^{21}+x^{18},\\
p_3(x)=&\,x^{124}+x^{123}+x^{122}+x^{121}+x^{119}+x^{118}+x^{117}+x^{113}+x^{112}+x^{109}\\
&+x^{106}+x^{105}+x^{102}+x^{99}+x^{96}+x^{95}+x^{92}+x^{89}+x^{76}+x^{75}+x^{74}\\
&+x^{73}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}+x^{62}+x^{60}\\
&+x^{59}+x^{57}+x^{56}+x^{55}+x^{52}+x^{49}+x^{44}+x^{43}+x^{42}+x^{41}+x^{39}\\
&+x^{38}+x^{37}+x^{33}+x^{32}+x^{29}+x^{26}+x^{25}+x^{22}+x^{19}+x^{16}+x^{15}\\
&+x^{12}+x^{9},\\
p_6(x)=&\,x^{123}+x^{120}+x^{117}+x^{114}+x^{113}+x^{111}+x^{110}+x^{108}+x^{107}+x^{104}\\
&+x^{103}+x^{100}+x^{93}+x^{90}+x^{87}+x^{84}+x^{79}+x^{76}+x^{71}+x^{68}+x^{67}\\
&+x^{64}+x^{63}+x^{61}+x^{60}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}+x^{49}+x^{47}\\
&+x^{46}+x^{44}+x^{43}+x^{40}+x^{35}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}\\
&+x^{27}+x^{26}+x^{24}+x^{21}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}\\
&+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{6},\\
p_9(x)=&\,x^{122}+x^{121}+x^{118}+x^{115}+x^{108}+x^{107}+x^{103}+x^{102}+x^{99}+x^{98}\\
&+x^{97}+x^{96}+x^{95}+x^{94}+x^{92}+x^{91}+x^{90}+x^{88}+x^{87}+x^{85}+x^{82}\\
&+x^{81}+x^{78}+x^{76}+x^{74}+x^{73}+x^{72}+x^{67}+x^{65}+x^{64}+x^{62}+x^{60}\\
&+x^{59}+x^{58}+x^{54}+x^{52}+x^{47}+x^{45}+x^{43}+x^{41}+x^{39}+x^{38}+x^{36}\\
&+x^{31}+x^{29}+x^{28}+x^{26}+x^{24}+x^{23}+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}\\
&+x^{16}+x^{11}+x^{9}+x^{7}+x^{6}+x^{3},\\
p_{12}(x)=&\,x^{121}+x^{119}+x^{118}+x^{116}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{104}+x^{101}+x^{99}+x^{98}+x^{96}+x^{89}+x^{87}+x^{86}+x^{84}+x^{81}\\
&+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{72}+x^{69}+x^{67}+x^{66}+x^{64}\\
&+x^{57}+x^{55}+x^{54}+x^{52}+x^{49}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}\\
&+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}\\
&+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}\\
&+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^{8}\\
&+x^{5}+x^{3}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{126}+x^{125}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}$$

$$
\begin{aligned}
&+ x^{112} + x^{110} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88}\\
&+ x^{86} + x^{84} + x^{83} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} + x^{71} + x^{68} + x^{65}\\
&+ x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{47} + x^{46} + x^{45}\\
&+ x^{43} + x^{40} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{26} + x^{25}\\
&+ x^{23} + x^{22} + x^{21} + x^{18},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{113} + x^{112} + x^{109}\\
&+ x^{106} + x^{105} + x^{102} + x^{99} + x^{96} + x^{95} + x^{92} + x^{89} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{38} + x^{37} + x^{33} + x^{32} + x^{29} + x^{26} + x^{25} + x^{22} + x^{19} + x^{16} + x^{15}\\
&+ x^{12} + x^{9},\\
p_6(x) &= x^{123} + x^{120} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{104}\\
&+ x^{103} + x^{100} + x^{93} + x^{90} + x^{87} + x^{84} + x^{79} + x^{76} + x^{71} + x^{68} + x^{67}\\
&+ x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{47}\\
&+ x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28}\\
&+ x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{122} + x^{121} + x^{118} + x^{115} + x^{108} + x^{107} + x^{103} + x^{102} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{82}\\
&+ x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{54} + x^{52} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36}\\
&+ x^{31} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17}\\
&+ x^{16} + x^{11} + x^{9} + x^{7} + x^{6} + x^{3},\\
p_{12}(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{101} + x^{99} + x^{98} + x^{96} + x^{89} + x^{87} + x^{86} + x^{84} + x^{81}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{64}\\
&+ x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8}\\
&+ x^{5} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$p_0(x) = x^{114} + x^{112} + x^{110} + x^{102} + x^{98} + x^{94} + x^{92} + x^{88} + x^{84} + x^{78} + x^{76}$$

$$
\begin{aligned}
&+ x^{70} + x^{58} + x^{54} + x^{48} + x^{46} + x^{42} + x^{38} + x^{34} + x^{30} + x^{24},\\
p_3(x) &= x^{108} + x^{106} + x^{100} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76}\\
&+ x^{66} + x^{64} + x^{62} + x^{56} + x^{52} + x^{50} + x^{44} + x^{38} + x^{30} + x^{28} + x^{26}\\
&+ x^{20} + x^{18} + x^{16} + x^{14} + x^{8} + x^{6},\\
p_6(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{90} + x^{86} + x^{84} + x^{80} + x^{76} + x^{72}\\
&+ x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42}\\
&+ x^{40} + x^{38} + x^{22} + x^{10} + x^{6} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{106} + x^{104} + x^{102} + x^{94} + x^{84} + x^{82} + x^{80} + x^{78} + x^{70} + x^{68} + x^{64}\\
&+ x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{28} + x^{26}\\
&+ x^{14} + x^{8} + x^{4} + x^{2},\\
p_{12}(x) &= x^{106} + x^{102} + x^{100} + x^{96} + x^{74} + x^{70} + x^{68} + x^{64} + x^{42} + x^{38} + x^{36}\\
&+ x^{32} + x^{26} + x^{22} + x^{20} + x^{16} + x^{10} + x^{6} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{94} + x^{92} + x^{90} + x^{80} + x^{78} + x^{68} + x^{66} + x^{64} + x^{56}\\
&+ x^{50} + x^{48} + x^{44} + x^{36} + x^{34} + x^{32} + x^{24} + x^{20} + x^{18} + x^{12},\\
p_3(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{88} + x^{86} + x^{78} + x^{62}\\
&+ x^{58} + x^{54} + x^{48} + x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{24} + x^{14} + x^{12}\\
&+ x^{10} + x^{8},\\
p_6(x) &= x^{108} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} + x^{78} + x^{74}\\
&+ x^{68} + x^{64} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{34}\\
&+ x^{28} + x^{14} + x^{10} + x^{8} + x^{2} + 1,\\
p_9(x) &= x^{106} + x^{104} + x^{102} + x^{94} + x^{84} + x^{82} + x^{80} + x^{78} + x^{70} + x^{68} + x^{64}\\
&+ x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{28} + x^{26}\\
&+ x^{14} + x^{8} + x^{4} + x^{2},\\
p_{12}(x) &= x^{106} + x^{102} + x^{100} + x^{96} + x^{74} + x^{70} + x^{68} + x^{64} + x^{42} + x^{38} + x^{36}\\
&+ x^{32} + x^{26} + x^{22} + x^{20} + x^{16} + x^{10} + x^{6} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{119} + x^{113} + x^{112} + x^{110} + x^{107} + x^{106} + x^{105}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{82} + x^{80} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{69} + x^{66} + x^{65} + x^{63} + x^{60} + x^{55} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42}\\
&+ x^{40} + x^{36} + x^{35} + x^{32} + x^{31} + x^{30} + x^{28} + x^{24},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{113} + x^{112} + x^{109}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{106} + x^{105} + x^{102} + x^{99} + x^{96} + x^{95} + x^{92} + x^{89} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{38} + x^{37} + x^{33} + x^{32} + x^{29} + x^{26} + x^{25} + x^{22} + x^{19} + x^{16} + x^{15}\\
&+ x^{12} + x^{9},\\
p_6(x) = {}& x^{123} + x^{120} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{104}\\
&+ x^{103} + x^{100} + x^{93} + x^{90} + x^{87} + x^{84} + x^{79} + x^{76} + x^{71} + x^{68} + x^{67}\\
&+ x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{47}\\
&+ x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28}\\
&+ x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) = {}& x^{122} + x^{121} + x^{118} + x^{115} + x^{108} + x^{107} + x^{103} + x^{102} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{82}\\
&+ x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60}\\
&+ x^{59} + x^{58} + x^{54} + x^{52} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36}\\
&+ x^{31} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17}\\
&+ x^{16} + x^{11} + x^{9} + x^{7} + x^{6} + x^{3},\\
p_{12}(x) = {}& x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{104} + x^{101} + x^{99} + x^{98} + x^{96} + x^{89} + x^{87} + x^{86} + x^{84} + x^{81}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{64}\\
&+ x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8}\\
&+ x^{5} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{124} + x^{123} + x^{119} + x^{113} + x^{112} + x^{110} + x^{107} + x^{106} + x^{105}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{82} + x^{80} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{69} + x^{66} + x^{65} + x^{63} + x^{60} + x^{55} + x^{51} + x^{47} + x^{46} + x^{45} + x^{42}\\
&+ x^{40} + x^{36} + x^{35} + x^{32} + x^{31} + x^{30} + x^{28} + x^{24},\\
p_3(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{113} + x^{112} + x^{109}\\
&+ x^{106} + x^{105} + x^{102} + x^{99} + x^{96} + x^{95} + x^{92} + x^{89} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60}
\end{aligned}
$$

$$+ x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{33} + x^{32} + x^{29} + x^{26} + x^{25} + x^{22} + x^{19} + x^{16} + x^{15} + x^{12} + x^{9},$$

$$p_6(x) = x^{123} + x^{120} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{104} + x^{103} + x^{100} + x^{93} + x^{90} + x^{87} + x^{84} + x^{79} + x^{76} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},$$

$$p_9(x) = x^{122} + x^{121} + x^{118} + x^{115} + x^{108} + x^{107} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{54} + x^{52} + x^{47} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36} + x^{31} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{11} + x^{9} + x^{7} + x^{6} + x^{3},$$

$$p_{12}(x) = x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{101} + x^{99} + x^{98} + x^{96} + x^{89} + x^{87} + x^{86} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{64} + x^{57} + x^{55} + x^{54} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$p_0(x) = x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} + x^{118} + x^{116} + x^{114} + x^{110} + x^{104} + x^{98} + x^{94} + x^{84} + x^{82} + x^{72} + x^{64} + x^{62} + x^{58} + x^{52} + x^{48} + x^{42} + x^{36},$$

$$p_3(x) = x^{134} + x^{132} + x^{130} + x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{86} + x^{80} + x^{74} + x^{72} + x^{62} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{42} + x^{34} + x^{28} + x^{18} + x^{16} + x^{14} + x^{8} + x^{6},$$

$$p_6(x) = x^{130} + x^{124} + x^{122} + x^{118} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{90} + x^{88} + x^{82} + x^{76} + x^{72} + x^{70} + x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{38} + x^{28} + x^{26} + x^{22} + x^{18} + x^{16} + x^{12} + x^{6} + x^{2} + 1,$$

$$p_9(x) = x^{136} + x^{134} + x^{128} + x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{106} + x^{102}$$

$$
\begin{aligned}
&+x^{100}+x^{98}+x^{96}+x^{94}+x^{90}+x^{84}+x^{82}+x^{80}+x^{76}+x^{74}+x^{72}\\
&+x^{70}+x^{62}+x^{60}+x^{58}+x^{56}+x^{54}+x^{52}+x^{48}+x^{44}+x^{42}+x^{40}\\
&+x^{38}+x^{36}+x^{34}+x^{28}+x^{26}+x^{22}+x^{14}+x^{12}+x^{8}+x^{6}+x^{4}+x^{2},\\
p_{12}(x)=&\;x^{136}+x^{120}+x^{112}+x^{104}+x^{96}+x^{88}+x^{80}+x^{72}+x^{64}+x^{48}+x^{24}\\
&+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)=&\;x^{126}+x^{125}+x^{124}+x^{122}+x^{120}+x^{119}+x^{112}+x^{111}+x^{109}+x^{103}\\
&+x^{99}+x^{96}+x^{95}+x^{93}+x^{91}+x^{90}+x^{84}+x^{81}+x^{80}+x^{76}+x^{75}\\
&+x^{74}+x^{72}+x^{67}+x^{66}+x^{65}+x^{60}+x^{59}+x^{57}+x^{56}+x^{55}+x^{51}\\
&+x^{49}+x^{46}+x^{44}+x^{43}+x^{41}+x^{36}+x^{35}+x^{31}+x^{30}+x^{29}+x^{28}\\
&+x^{27}+x^{23}+x^{22}+x^{18}+x^{15}+x^{12},\\
p_3(x)=&\;x^{125}+x^{124}+x^{122}+x^{118}+x^{116}+x^{114}+x^{113}+x^{111}+x^{110}+x^{108}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{100}+x^{99}+x^{97}+x^{95}+x^{93}\\
&+x^{92}+x^{90}+x^{87}+x^{85}+x^{84}+x^{83}+x^{82}+x^{80}+x^{79}+x^{73}+x^{71}\\
&+x^{69}+x^{68}+x^{67}+x^{64}+x^{62}+x^{61}+x^{58}+x^{56}+x^{54}+x^{53}+x^{51}\\
&+x^{50}+x^{49}+x^{48}+x^{46}+x^{44}+x^{43}+x^{40}+x^{39}+x^{38}+x^{37}+x^{31}\\
&+x^{29}+x^{28}+x^{27}+x^{25}+x^{23}+x^{22}+x^{20}+x^{19}+x^{17}+x^{11}+x^{8},\\
p_6(x)=&\;x^{125}+x^{124}+x^{122}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}\\
&+x^{110}+x^{108}+x^{106}+x^{98}+x^{95}+x^{94}+x^{93}+x^{91}+x^{90}+x^{89}+x^{88}\\
&+x^{87}+x^{84}+x^{83}+x^{82}+x^{81}+x^{79}+x^{78}+x^{73}+x^{71}+x^{66}+x^{65}\\
&+x^{60}+x^{59}+x^{57}+x^{52}+x^{50}+x^{49}+x^{48}+x^{45}+x^{42}+x^{41}+x^{39}\\
&+x^{34}+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{20}+x^{19}+x^{14}\\
&+x^{12}+x^{10}+x^{7}+x^{5}+x^{4}+x^{3}+x^{2}+1,\\
p_9(x)=&\;x^{125}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}\\
&+x^{114}+x^{113}+x^{112}+x^{111}+x^{110}+x^{108}+x^{105}+x^{103}+x^{102}\\
&+x^{100}+x^{93}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{76}+x^{73}+x^{71}+x^{70}+x^{68}+x^{61}\\
&+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}\\
&+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}+x^{37}+x^{35}+x^{34}\\
&+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{24}+x^{21}+x^{19}\\
&+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}+x^{9}+x^{7}+x^{6}+x^{4},\\
p_{12}(x)=&\;x^{125}+x^{123}+x^{122}+x^{120}+x^{117}+x^{115}+x^{114}+x^{112}+x^{109}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{96}\\
&+x^{93}+x^{91}+x^{90}+x^{88}+x^{85}+x^{83}+x^{82}+x^{80}+x^{77}+x^{75}+x^{74}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{59} \\
&+ x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{41} + x^{39} + x^{38} + x^{36} + x^{25} \\
&+ x^{23} + x^{22} + x^{20} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} \\
&+ x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1].
For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{124} + x^{121} + x^{120} + x^{116} + x^{113} + x^{110} + x^{108} + x^{103} + x^{101} + x^{100} \\
&+ x^{98} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{82} + x^{81} + x^{79} \\
&+ x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{68} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} \\
&+ x^{60} + x^{59} + x^{58} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47} + x^{43} + x^{38} + x^{37} \\
&+ x^{35} + x^{29} + x^{24} + x^{23} + x^{18}, \\
p_3(x) = {}& x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{105} + x^{101} + x^{100} \\
&+ x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{89} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} \\
&+ x^{63} + x^{60} + x^{58} + x^{56} + x^{55} + x^{53} + x^{49} + x^{37} + x^{36} + x^{34} + x^{33} \\
&+ x^{32} + x^{31} + x^{29} + x^{25} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} \\
&+ x^{9}, \\
p_6(x) = {}& x^{118} + x^{116} + x^{114} + x^{108} + x^{102} + x^{100} + x^{88} + x^{86} + x^{84} + x^{74} \\
&+ x^{72} + x^{70} + x^{68} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} \\
&+ x^{40} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{10} + x^{6}, \\
p_9(x) = {}& x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{110} + x^{108} + x^{107} + x^{106} + x^{105} \\
&+ x^{103} + x^{99} + x^{87} + x^{86} + x^{85} + x^{83} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} \\
&+ x^{73} + x^{71} + x^{67} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} \\
&+ x^{42} + x^{41} + x^{38} + x^{37} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} \\
&+ x^{21} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{3}, \\
p_{12}(x) = {}& x^{120} + x^{112} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56} \\
&+ x^{48} + x^{36} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1].
For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{138} + x^{137} + x^{136} + x^{135} + x^{131} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} \\
&+ x^{121} + x^{120} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} \\
&+ x^{102} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{85} \\
&+ x^{84} + x^{83} + x^{82} + x^{74} + x^{72} + x^{70} + x^{69} + x^{64} + x^{62} + x^{61} + x^{59} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{44} + x^{43} + x^{42} + x^{37} \\
&+ x^{36} + x^{30} + x^{28} + x^{27} + x^{24}, \\
p_3(x) = {}& x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{131} + x^{128} + x^{126} + x^{125} + x^{123}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{122}+x^{121}+x^{119}+x^{118}+x^{115}+x^{114}+x^{111}+x^{110}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{101}+x^{100}+x^{98}+x^{96}+x^{95}+x^{93}\\
&+x^{88}+x^{86}+x^{85}+x^{84}+x^{83}+x^{81}+x^{76}+x^{74}+x^{71}+x^{70}+x^{68}\\
&+x^{67}+x^{60}+x^{58}+x^{55}+x^{54}+x^{52}+x^{51}+x^{49}+x^{48}+x^{46}+x^{45}\\
&+x^{43}+x^{42}+x^{41}+x^{39}+x^{38}+x^{35}+x^{34}+x^{31}+x^{30}+x^{27}+x^{26}\\
&+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{16}+x^{15}+x^{13}+x^{9},\\
p_6(x)=&\,x^{138}+x^{136}+x^{134}+x^{130}+x^{124}+x^{118}+x^{112}+x^{110}+x^{108}+x^{106}\\
&+x^{104}+x^{102}+x^{98}+x^{90}+x^{82}+x^{80}+x^{78}+x^{74}+x^{70}+x^{64}+x^{60}\\
&+x^{54}+x^{52}+x^{50}+x^{42}+x^{38}+x^{34}+x^{32}+x^{28}+x^{22}+x^{20}+x^{18}\\
&+x^{10}+x^{6},\\
p_9(x)=&\,x^{139}+x^{138}+x^{137}+x^{135}+x^{133}+x^{131}+x^{129}+x^{128}+x^{127}+x^{125}\\
&+x^{122}+x^{121}+x^{120}+x^{119}+x^{118}+x^{116}+x^{115}+x^{114}+x^{110}\\
&+x^{108}+x^{107}+x^{101}+x^{97}+x^{96}+x^{95}+x^{93}+x^{90}+x^{89}+x^{88}+x^{87}\\
&+x^{86}+x^{84}+x^{83}+x^{82}+x^{78}+x^{76}+x^{75}+x^{69}+x^{65}+x^{64}+x^{63}\\
&+x^{61}+x^{59}+x^{56}+x^{54}+x^{53}+x^{52}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}\\
&+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{35}\\
&+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{24}+x^{22}+x^{20}+x^{18}\\
&+x^{14}+x^{12}+x^{10}+x^{9}+x^{7}+x^{3},\\
p_{12}(x)=&\,x^{140}+x^{128}+x^{120}+x^{112}+x^{108}+x^{100}+x^{88}+x^{80}+x^{76}+x^{68}\\
&+x^{60}+x^{56}+x^{44}+x^{40}+x^{36}+x^{32}+x^{24}+x^{20}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{121}+x^{119}+x^{116}+x^{113}+x^{105}+x^{103}+x^{102}+x^{100}+x^{99}\\
&+x^{96}+x^{94}+x^{93}+x^{92}+x^{91}+x^{88}+x^{87}+x^{86}+x^{82}+x^{80}+x^{79}\\
&+x^{78}+x^{76}+x^{75}+x^{74}+x^{72}+x^{69}+x^{68}+x^{66}+x^{64}+x^{60}+x^{58}\\
&+x^{57}+x^{56}+x^{53}+x^{51}+x^{48}+x^{44}+x^{43}+x^{40}+x^{37}+x^{35}+x^{30},\\
p_3(x)=&\,x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{117}+x^{116}+x^{113}+x^{110}\\
&+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{93}+x^{91}\\
&+x^{90}+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}+x^{82}+x^{80}+x^{79}+x^{75}+x^{72}\\
&+x^{71}+x^{70}+x^{68}+x^{66}+x^{65}+x^{63}+x^{60}+x^{59}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{50}+x^{49}+x^{46}+x^{44}+x^{41}+x^{39}+x^{34}+x^{29}+x^{27}+x^{23}\\
&+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{15}+x^{12},\\
p_6(x)=&\,x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{117}+x^{115}+x^{114}+x^{113}+x^{112}\\
&+x^{110}+x^{108}+x^{107}+x^{106}+x^{104}+x^{103}+x^{100}+x^{98}+x^{97}+x^{92}\\
&+x^{91}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{80}+x^{79}+x^{77}+x^{75}+x^{74}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{55} + x^{54} \\
&+ x^{50} + x^{49} + x^{48} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{32} \\
&+ x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{17} + x^{15} + x^{10} \\
&+ x^{5} + x^{3} + x^{2} + 1, \\
p_9(x) = {}& x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} \\
&+ x^{100} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{68} + x^{61} \\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} \\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} \\
&+ x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6} + x^{4}, \\
p_{12}(x) = {}& x^{125} + x^{123} + x^{122} + x^{120} + x^{117} + x^{115} + x^{114} + x^{112} + x^{109} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} \\
&+ x^{93} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74} \\
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{59} \\
&+ x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{41} + x^{39} + x^{38} + x^{36} + x^{25} \\
&+ x^{23} + x^{22} + x^{20} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} \\
&+ x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{125} + x^{124} + x^{122} + x^{120} + x^{119} + x^{112} + x^{111} + x^{109} + x^{103} \\
&+ x^{99} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{84} + x^{81} + x^{80} + x^{76} + x^{75} \\
&+ x^{74} + x^{72} + x^{67} + x^{66} + x^{65} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{51} \\
&+ x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{36} + x^{35} + x^{31} + x^{30} + x^{29} + x^{28} \\
&+ x^{27} + x^{23} + x^{22} + x^{18} + x^{15} + x^{12}, \\
p_3(x) = {}& x^{125} + x^{124} + x^{122} + x^{118} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} \\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} \\
&+ x^{92} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{73} + x^{71} \\
&+ x^{69} + x^{68} + x^{67} + x^{64} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} \\
&+ x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{31} \\
&+ x^{29} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{11} + x^{8}, \\
p_6(x) = {}& x^{125} + x^{124} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} \\
&+ x^{110} + x^{108} + x^{106} + x^{98} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} \\
&+ x^{87} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{73} + x^{71} + x^{66} + x^{65}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{60} + x^{59} + x^{57} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} + x^{39}\\
&+ x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{20} + x^{19} + x^{14}\\
&+ x^{12} + x^{10} + x^{7} + x^{5} + x^{4} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102}\\
&+ x^{100} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{68} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19}\\
&+ x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6} + x^{4},\\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{117} + x^{115} + x^{114} + x^{112} + x^{109} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96}\\
&+ x^{93} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74}\\
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{59}\\
&+ x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{41} + x^{39} + x^{38} + x^{36} + x^{25}\\
&+ x^{23} + x^{22} + x^{20} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4}\\
&+ x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{137} + x^{136} + x^{135} + x^{131} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124}\\
&+ x^{121} + x^{120} + x^{114} + x^{113} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104}\\
&+ x^{102} + x^{101} + x^{100} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{74} + x^{72} + x^{70} + x^{69} + x^{64} + x^{62} + x^{61} + x^{59}\\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{44} + x^{43} + x^{42} + x^{37}\\
&+ x^{36} + x^{30} + x^{28} + x^{27} + x^{24},\\
p_3(x) &= x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{131} + x^{128} + x^{126} + x^{125} + x^{123}\\
&+ x^{122} + x^{121} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{93}\\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{76} + x^{74} + x^{71} + x^{70} + x^{68}\\
&+ x^{67} + x^{60} + x^{58} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45}\\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{9},\\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{130} + x^{124} + x^{118} + x^{112} + x^{110} + x^{108} + x^{106}\\
&+ x^{104} + x^{102} + x^{98} + x^{90} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{64} + x^{60}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{54} + x^{52} + x^{50} + x^{42} + x^{38} + x^{34} + x^{32} + x^{28} + x^{22} + x^{20} + x^{18}\\
&+ x^{10} + x^{6},\\
p_9(x) = {}& x^{139} + x^{138} + x^{137} + x^{135} + x^{133} + x^{131} + x^{129} + x^{128} + x^{127} + x^{125}\\
&+ x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{110}\\
&+ x^{108} + x^{107} + x^{101} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{84} + x^{83} + x^{82} + x^{78} + x^{76} + x^{75} + x^{69} + x^{65} + x^{64} + x^{63}\\
&+ x^{61} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46}\\
&+ x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35}\\
&+ x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18}\\
&+ x^{14} + x^{12} + x^{10} + x^{9} + x^{7} + x^{3},\\
p_{12}(x) = {}& x^{140} + x^{128} + x^{120} + x^{112} + x^{108} + x^{100} + x^{88} + x^{80} + x^{76} + x^{68}\\
&+ x^{60} + x^{56} + x^{44} + x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{124} + x^{121} + x^{120} + x^{116} + x^{113} + x^{110} + x^{108} + x^{103} + x^{101} + x^{100}\\
&+ x^{98} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85} + x^{82} + x^{81} + x^{79}\\
&+ x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{68} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{55} + x^{52} + x^{51} + x^{48} + x^{47} + x^{43} + x^{38} + x^{37}\\
&+ x^{35} + x^{29} + x^{24} + x^{23} + x^{18},\\
p_3(x) = {}& x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{105} + x^{101} + x^{100}\\
&+ x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{89} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{60} + x^{58} + x^{56} + x^{55} + x^{53} + x^{49} + x^{37} + x^{36} + x^{34} + x^{33}\\
&+ x^{32} + x^{31} + x^{29} + x^{25} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13}\\
&+ x^{9},\\
p_6(x) = {}& x^{118} + x^{116} + x^{114} + x^{108} + x^{102} + x^{100} + x^{88} + x^{86} + x^{84} + x^{74}\\
&+ x^{72} + x^{70} + x^{68} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42}\\
&+ x^{40} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{10} + x^{6},\\
p_9(x) = {}& x^{119} + x^{118} + x^{117} + x^{115} + x^{113} + x^{110} + x^{108} + x^{107} + x^{106} + x^{105}\\
&+ x^{103} + x^{99} + x^{87} + x^{86} + x^{85} + x^{83} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{71} + x^{67} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43}\\
&+ x^{42} + x^{41} + x^{38} + x^{37} + x^{33} + x^{30} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22}\\
&+ x^{21} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9} + x^{7} + x^{3},\\
p_{12}(x) = {}& x^{120} + x^{112} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56}\\
&+ x^{48} + x^{36} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{119} + x^{116} + x^{113} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} \\
&\quad + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{82} + x^{80} + x^{79} \\
&\quad + x^{78} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} \\
&\quad + x^{57} + x^{56} + x^{53} + x^{51} + x^{48} + x^{44} + x^{43} + x^{40} + x^{37} + x^{35} + x^{30}, \\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{113} + x^{110} \\
&\quad + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} + x^{91} \\
&\quad + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{72} \\
&\quad + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63} + x^{60} + x^{59} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{50} + x^{49} + x^{46} + x^{44} + x^{41} + x^{39} + x^{34} + x^{29} + x^{27} + x^{23} \\
&\quad + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{12}, \\
p_6(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{92} \\
&\quad + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} \\
&\quad + x^{72} + x^{71} + x^{70} + x^{69} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{55} + x^{54} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{32} \\
&\quad + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{17} + x^{15} + x^{10} \\
&\quad + x^{5} + x^{3} + x^{2} + 1, \\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} \\
&\quad + x^{100} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{71} + x^{70} + x^{68} + x^{61} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} \\
&\quad + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{19} \\
&\quad + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{117} + x^{115} + x^{114} + x^{112} + x^{109} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} \\
&\quad + x^{93} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{75} + x^{74} \\
&\quad + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{61} + x^{59} \\
&\quad + x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{41} + x^{39} + x^{38} + x^{36} + x^{25} \\
&\quad + x^{23} + x^{22} + x^{20} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} \\
&\quad + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	369	[533, 534]	738	[894, 9]	1107	[694, 354]
1	[3, 4]	370	[526, 535]	739	[895, 863]	1108	[1228, 587]
2	[5, 6]	371	[536, 537]	740	[872, 200]	1109	[1229, 1230]
3	[7, 8]	372	[340, 117]	741	[896, 897]	1110	[1231, 1075]
4	[9, 10]	373	[84, 538]	742	[456, 391]	1111	[1232, 705]
5	[11, 12]	374	[539, 540]	743	[38, 145]	1112	[706, 4]
6	[13, 14]	375	[541, 542]	744	[797, 898]	1113	[1233, 210]
7	[15, 16]	376	[543, 544]	745	[899, 478]	1114	[1234, 923]
8	[17, 18]	377	[545, 222]	746	[173, 512]	1115	[1235, 734]
9	[19, 20]	378	[546, 547]	747	[900, 12]	1116	[1236, 583]
10	[21, 22]	379	[548, 549]	748	[901, 203]	1117	[574, 66]
11	[23, 24]	380	[550, 551]	749	[369, 808]	1118	[1237, 650]
12	[25, 26]	381	[552, 553]	750	[795, 503]	1119	[1238, 341]
13	[27, 28]	382	[554, 555]	751	[767, 435]	1120	[1239, 1240]
14	[29, 30]	383	[338, 27]	752	[902, 903]	1121	[1241, 699]
15	[31, 32]	384	[238, 556]	753	[904, 905]	1122	[1242, 1243]
16	[33, 34]	385	[557, 558]	754	[32, 820]	1123	[1244, 294]
17	[35, 36]	386	[559, 560]	755	[303, 358]	1124	[1245, 1246]
18	[37, 38]	387	[561, 302]	756	[28, 627]	1125	[1247, 1248]
19	[39, 40]	388	[562, 563]	757	[906, 359]	1126	[1249, 1250]
20	[41, 42]	389	[564, 565]	758	[907, 908]	1127	[1251, 1252]
21	[43, 44]	390	[566, 567]	759	[702, 79]	1128	[1253, 574]
22	[45, 46]	391	[568, 569]	760	[909, 910]	1129	[1254, 1255]
23	[47, 48]	392	[328, 570]	761	[911, 912]	1130	[688, 951]
24	[49, 50]	393	[571, 572]	762	[317, 698]	1131	[1256, 1257]
25	[51, 52]	394	[573, 574]	763	[913, 914]	1132	[1258, 560]
26	[53, 54]	395	[575, 576]	764	[915, 916]	1133	[1259, 1260]
27	[55, 56]	396	[577, 578]	765	[917, 565]	1134	[231, 943]
28	[57, 46]	397	[579, 580]	766	[918, 919]	1135	[1261, 1084]
29	[58, 59]	398	[211, 526]	767	[920, 271]	1136	[742, 540]
30	[60, 61]	399	[581, 582]	768	[921, 922]	1137	[1262, 236]
31	[62, 63]	400	[583, 584]	769	[923, 924]	1138	[1263, 701]
32	[64, 65]	401	[585, 320]	770	[925, 622]	1139	[939, 726]
33	[66, 67]	402	[586, 322]	771	[926, 927]	1140	[1264, 1265]
34	[68, 69]	403	[587, 588]	772	[928, 579]	1141	[1003, 339]
35	[70, 71]	404	[589, 590]	773	[929, 249]	1142	[1266, 1267]
36	[72, 73]	405	[591, 592]	774	[930, 562]	1143	[1268, 1027]
37	[74, 75]	406	[593, 594]	775	[931, 932]	1144	[1269, 247]
38	[76, 77]	407	[595, 596]	776	[638, 933]	1145	[1270, 1271]
39	[78, 79]	408	[597, 598]	777	[934, 935]	1146	[1272, 69]

40	[80, 74]	409	[599, 600]	778	[936, 937]	1147	[1090, 600]
41	[81, 82]	410	[601, 602]	779	[938, 939]	1148	[1273, 85]
42	[83, 84]	411	[603, 604]	780	[604, 940]	1149	[1274, 1275]
43	[85, 86]	412	[584, 605]	781	[941, 596]	1150	[1276, 226]
44	[87, 88]	413	[86, 89]	782	[942, 943]	1151	[553, 1277]
45	[89, 90]	414	[250, 255]	783	[944, 945]	1152	[1278, 643]
46	[91, 92]	415	[606, 258]	784	[312, 658]	1153	[232, 6]
47	[93, 94]	416	[607, 608]	785	[946, 531]	1154	[1279, 231]
48	[95, 96]	417	[256, 76]	786	[947, 682]	1155	[1280, 743]
49	[97, 29]	418	[609, 251]	787	[648, 598]	1156	[1281, 1282]
50	[98, 99]	419	[610, 239]	788	[948, 607]	1157	[1283, 350]
51	[100, 101]	420	[346, 611]	789	[949, 303]	1158	[1284, 669]
52	[102, 103]	421	[612, 613]	790	[950, 951]	1159	[1285, 654]
53	[104, 105]	422	[614, 615]	791	[952, 953]	1160	[594, 543]
54	[106, 107]	423	[616, 617]	792	[954, 955]	1161	[1286, 666]
55	[108, 68]	424	[618, 583]	793	[956, 538]	1162	[1287, 1288]
56	[109, 110]	425	[270, 619]	794	[957, 958]	1163	[1289, 1057]
57	[111, 112]	426	[620, 621]	795	[959, 609]	1164	[1290, 1291]
58	[79, 113]	427	[295, 601]	796	[960, 595]	1165	[1292, 1293]
59	[18, 114]	428	[622, 623]	797	[961, 750]	1166	[73, 712]
60	[115, 116]	429	[624, 625]	798	[560, 707]	1167	[1271, 218]
61	[117, 118]	430	[626, 253]	799	[962, 308]	1168	[1294, 360]
62	[119, 120]	431	[627, 628]	800	[99, 963]	1169	[677, 26]
63	[121, 122]	432	[629, 630]	801	[964, 965]	1170	[1295, 716]
64	[123, 124]	433	[631, 629]	802	[966, 329]	1171	[1296, 947]
65	[125, 126]	434	[632, 633]	803	[967, 968]	1172	[1297, 1298]
66	[57, 127]	435	[634, 635]	804	[969, 930]	1173	[1299, 347]
67	[128, 129]	436	[636, 62]	805	[970, 971]	1174	[1300, 1301]
68	[130, 131]	437	[637, 638]	806	[972, 973]	1175	[1302, 1303]
69	[132, 133]	438	[639, 640]	807	[714, 235]	1176	[746, 1304]
70	[134, 135]	439	[641, 642]	808	[582, 974]	1177	[1305, 1306]
71	[55, 136]	440	[359, 273]	809	[975, 606]	1178	[1307, 1308]
72	[137, 138]	441	[643, 644]	810	[635, 930]	1179	[1309, 967]
73	[139, 140]	442	[645, 646]	811	[976, 977]	1180	[1310, 1311]
74	[141, 142]	443	[647, 648]	812	[978, 979]	1181	[556, 98]
75	[143, 144]	444	[649, 650]	813	[980, 286]	1182	[360, 573]
76	[145, 144]	445	[651, 652]	814	[981, 557]	1183	[1312, 109]
77	[146, 147]	446	[333, 548]	815	[744, 553]	1184	[605, 1313]
78	[148, 149]	447	[544, 653]	816	[982, 983]	1185	[570, 21]
79	[150, 122]	448	[22, 654]	817	[984, 318]	1186	[1314, 234]
80	[151, 152]	449	[655, 591]	818	[985, 700]	1187	[733, 261]
81	[153, 154]	450	[600, 656]	819	[986, 987]	1188	[1315, 1065]
82	[155, 156]	451	[558, 657]	820	[63, 577]	1189	[1316, 1317]
83	[157, 146]	452	[658, 298]	821	[691, 988]	1190	[1318, 1319]

84	[36, 158]	453	[659, 660]	822	[989, 990]	1191	[1001, 1306]
85	[159, 160]	454	[661, 559]	823	[991, 992]	1192	[646, 1320]
86	[161, 59]	455	[662, 663]	824	[993, 994]	1193	[1321, 920]
87	[162, 163]	456	[664, 665]	825	[995, 996]	1194	[1322, 295]
88	[164, 45]	457	[666, 659]	826	[997, 327]	1195	[1323, 1324]
89	[160, 165]	458	[667, 668]	827	[998, 999]	1196	[1255, 641]
90	[166, 167]	459	[669, 670]	828	[1000, 1001]	1197	[653, 108]
91	[168, 169]	460	[528, 671]	829	[1002, 1003]	1198	[1325, 940]
92	[170, 171]	461	[672, 673]	830	[1004, 1005]	1199	[633, 1326]
93	[172, 171]	462	[657, 672]	831	[1006, 1004]	1200	[1327, 1313]
94	[173, 174]	463	[463, 463]	832	[1007, 1008]	1201	[1328, 1284]
95	[175, 176]	464	[674, 675]	833	[958, 638]	1202	[92, 1260]
96	[177, 120]	465	[676, 677]	834	[1009, 592]	1203	[88, 345]
97	[178, 60]	466	[678, 679]	835	[1010, 577]	1204	[1329, 1330]
98	[179, 180]	467	[680, 288]	836	[1011, 1012]	1205	[1331, 1332]
99	[181, 182]	468	[681, 634]	837	[1013, 549]	1206	[1333, 907]
100	[183, 184]	469	[682, 564]	838	[1014, 1015]	1207	[1334, 107]
101	[185, 186]	470	[683, 93]	839	[1016, 1017]	1208	[1335, 660]
102	[187, 188]	471	[684, 81]	840	[1018, 972]	1209	[535, 1336]
103	[189, 181]	472	[685, 686]	841	[1019, 279]	1210	[1337, 1338]
104	[190, 191]	473	[687, 688]	842	[1020, 1021]	1211	[1339, 84]
105	[176, 45]	474	[689, 349]	843	[1022, 595]	1212	[502, 384]
106	[192, 193]	475	[690, 691]	844	[1023, 230]	1213	[1340, 506]
107	[194, 195]	476	[692, 294]	845	[1024, 1005]	1214	[41, 808]
108	[196, 132]	477	[693, 694]	846	[1025, 1026]	1215	[1341, 1342]
109	[197, 198]	478	[695, 696]	847	[352, 674]	1216	[1343, 192]
110	[199, 200]	479	[335, 296]	848	[740, 690]	1217	[1194, 489]
111	[201, 133]	480	[697, 698]	849	[323, 1027]	1218	[1344, 897]
112	[202, 203]	481	[699, 677]	850	[1028, 1029]	1219	[901, 516]
113	[149, 156]	482	[700, 221]	851	[1030, 1031]	1220	[1345, 1346]
114	[36, 204]	483	[701, 702]	852	[1032, 309]	1221	[1347, 1348]
115	[205, 206]	484	[703, 704]	853	[1033, 1034]	1222	[1114, 158]
116	[157, 147]	485	[246, 625]	854	[686, 1035]	1223	[1102, 138]
117	[180, 207]	486	[705, 706]	855	[572, 1036]	1224	[1349, 422]
118	[149, 126]	487	[707, 308]	856	[1037, 345]	1225	[783, 820]
119	[208, 209]	488	[708, 709]	857	[1038, 1039]	1226	[1350, 842]
120	[210, 211]	489	[710, 711]	858	[1040, 618]	1227	[761, 762]
121	[209, 212]	490	[712, 649]	859	[1041, 659]	1228	[1351, 10]
122	[213, 214]	491	[713, 627]	860	[1042, 228]	1229	[1352, 790]
123	[215, 216]	492	[569, 714]	861	[1043, 1044]	1230	[1179, 167]
124	[217, 218]	493	[715, 688]	862	[1045, 98]	1231	[1353, 152]
125	[219, 220]	494	[555, 716]	863	[1046, 1007]	1232	[1354, 390]
126	[221, 222]	495	[717, 718]	864	[1047, 316]	1233	[1165, 897]
127	[223, 224]	496	[719, 289]	865	[979, 1048]	1234	[1355, 1123]

128	[225, 91]	497	[670, 720]	866	[534, 1049]	1235	[1356, 903]
129	[226, 227]	498	[721, 656]	867	[1050, 737]	1236	[1357, 770]
130	[228, 229]	499	[722, 723]	868	[988, 1051]	1237	[1358, 519]
131	[230, 231]	500	[628, 724]	869	[1052, 1053]	1238	[1359, 504]
132	[20, 232]	501	[725, 630]	870	[268, 911]	1239	[1360, 853]
133	[233, 234]	502	[726, 364]	871	[1054, 1055]	1240	[388, 759]
134	[235, 236]	503	[654, 727]	872	[1056, 715]	1241	[1361, 52]
135	[237, 238]	504	[728, 83]	873	[1057, 579]	1242	[1362, 2]
136	[239, 240]	505	[729, 730]	874	[1058, 265]	1243	[1210, 806]
137	[241, 242]	506	[731, 732]	875	[656, 213]	1244	[1355, 523]
138	[243, 244]	507	[679, 733]	876	[1059, 993]	1245	[206, 407]
139	[245, 246]	508	[732, 734]	877	[749, 1060]	1246	[1129, 379]
140	[247, 248]	509	[101, 667]	878	[1061, 224]	1247	[1359, 1110]
141	[249, 250]	510	[642, 735]	879	[1049, 618]	1248	[1190, 475]
142	[251, 252]	511	[736, 559]	880	[1062, 87]	1249	[1363, 1364]
143	[253, 254]	512	[737, 663]	881	[118, 1063]	1250	[874, 439]
144	[255, 256]	513	[738, 569]	882	[1064, 21]	1251	[1138, 805]
145	[75, 257]	514	[739, 740]	883	[266, 1065]	1252	[779, 396]
146	[258, 259]	515	[741, 742]	884	[1066, 1067]	1253	[1365, 886]
147	[259, 260]	516	[244, 106]	885	[1068, 708]	1254	[1366, 905]
148	[261, 262]	517	[743, 744]	886	[1069, 1070]	1255	[1155, 524]
149	[263, 264]	518	[745, 746]	887	[754, 1071]	1256	[1120, 366]
150	[265, 266]	519	[747, 748]	888	[1072, 1051]	1257	[1174, 502]
151	[267, 268]	520	[236, 749]	889	[1073, 278]	1258	[1367, 850]
152	[269, 270]	521	[750, 751]	890	[578, 632]	1259	[1148, 489]
153	[271, 272]	522	[752, 744]	891	[1074, 548]	1260	[1172, 171]
154	[273, 274]	523	[753, 643]	892	[1075, 1076]	1261	[1368, 119]
155	[275, 276]	524	[16, 754]	893	[1077, 1043]	1262	[1132, 897]
156	[262, 71]	525	[755, 515]	894	[711, 1078]	1263	[1369, 426]
157	[116, 277]	526	[368, 756]	895	[1079, 1080]	1264	[1370, 1113]
158	[278, 279]	527	[757, 478]	896	[1081, 1039]	1265	[1371, 780]
159	[280, 281]	528	[488, 451]	897	[1082, 1083]	1266	[1372, 1373]
160	[282, 283]	529	[758, 759]	898	[1084, 62]	1267	[49, 475]
161	[284, 285]	530	[760, 511]	899	[1085, 106]	1268	[831, 478]
162	[286, 287]	531	[507, 485]	900	[1086, 1087]	1269	[1374, 32]
163	[288, 289]	532	[40, 524]	901	[1088, 1089]	1270	[1375, 1348]
164	[290, 223]	533	[761, 207]	902	[530, 1090]	1271	[1371, 396]
165	[281, 291]	534	[393, 446]	903	[1091, 1092]	1272	[1376, 1100]
166	[292, 293]	535	[515, 203]	904	[1093, 1094]	1273	[1377, 869]
167	[294, 295]	536	[384, 762]	905	[1095, 1059]	1274	[1378, 466]
168	[296, 297]	537	[763, 383]	906	[1096, 1097]	1275	[885, 45]
169	[298, 299]	538	[764, 407]	907	[61, 485]	1276	[1379, 390]
170	[300, 301]	539	[765, 40]	908	[504, 503]	1277	[152, 780]
171	[302, 303]	540	[766, 767]	909	[1098, 1099]	1278	[1097, 767]

172	[304, 305]	541	[768, 472]	910	[1100, 391]	1279	[1380, 1381]
173	[306, 307]	542	[486, 432]	911	[876, 481]	1280	[1135, 797]
174	[308, 309]	543	[195, 193]	912	[812, 200]	1281	[1382, 1364]
175	[310, 89]	544	[140, 508]	913	[1101, 819]	1282	[375, 55]
176	[311, 312]	545	[769, 770]	914	[842, 424]	1283	[1383, 882]
177	[313, 314]	546	[771, 772]	915	[1102, 1103]	1284	[1121, 370]
178	[315, 117]	547	[773, 431]	916	[8, 58]	1285	[1384, 828]
179	[316, 317]	548	[774, 44]	917	[1104, 1105]	1286	[1122, 523]
180	[318, 319]	549	[775, 55]	918	[1106, 8]	1287	[827, 204]
181	[320, 321]	550	[776, 777]	919	[431, 787]	1288	[827, 158]
182	[322, 323]	551	[778, 774]	920	[1107, 136]	1289	[1385, 443]
183	[324, 325]	552	[411, 152]	921	[1108, 1109]	1290	[1386, 816]
184	[326, 282]	553	[779, 780]	922	[510, 372]	1291	[778, 490]
185	[327, 328]	554	[781, 782]	923	[1110, 503]	1292	[1387, 1205]
186	[329, 330]	555	[783, 383]	924	[10, 186]	1293	[1343, 195]
187	[331, 323]	556	[402, 181]	925	[1111, 402]	1294	[1388, 1346]
188	[332, 333]	557	[452, 462]	926	[1112, 1113]	1295	[1389, 366]
189	[334, 335]	558	[784, 169]	927	[1114, 204]	1296	[496, 1390]
190	[336, 92]	559	[469, 758]	928	[1115, 138]	1297	[501, 1391]
191	[337, 338]	560	[785, 475]	929	[1116, 206]	1298	[176, 129]
192	[339, 340]	561	[786, 468]	930	[1117, 758]	1299	[1392, 449]
193	[341, 342]	562	[130, 787]	931	[1118, 176]	1300	[1389, 1121]
194	[343, 344]	563	[788, 58]	932	[403, 127]	1301	[197, 493]
195	[345, 346]	564	[200, 198]	933	[152, 396]	1302	[1393, 438]
196	[222, 233]	565	[44, 444]	934	[1106, 40]	1303	[33, 154]
197	[347, 348]	566	[789, 790]	935	[1119, 192]	1304	[478, 54]
198	[349, 350]	567	[791, 511]	936	[1120, 1121]	1305	[1394, 474]
199	[351, 352]	568	[14, 756]	937	[129, 127]	1306	[880, 490]
200	[274, 353]	569	[475, 14]	938	[1122, 1123]	1307	[1395, 1342]
201	[354, 355]	570	[128, 45]	939	[8, 524]	1308	[1206, 767]
202	[356, 357]	571	[792, 793]	940	[811, 774]	1309	[1208, 1099]
203	[358, 359]	572	[125, 156]	941	[1124, 1125]	1310	[1396, 1193]
204	[71, 360]	573	[136, 34]	942	[826, 36]	1311	[1397, 830]
205	[361, 280]	574	[520, 384]	943	[764, 898]	1312	[1394, 199]
206	[362, 363]	575	[794, 409]	944	[1126, 1127]	1313	[1107, 56]
207	[364, 320]	576	[795, 369]	945	[1105, 394]	1314	[1398, 382]
208	[365, 366]	577	[120, 121]	946	[7, 40]	1315	[1126, 422]
209	[367, 182]	578	[775, 376]	947	[474, 200]	1316	[1399, 191]
210	[13, 368]	579	[458, 372]	948	[1128, 1129]	1317	[1121, 162]
211	[369, 42]	580	[202, 516]	949	[1130, 1131]	1318	[1385, 773]
212	[366, 370]	581	[796, 797]	950	[387, 514]	1319	[503, 808]
213	[371, 131]	582	[417, 144]	951	[53, 830]	1320	[426, 520]
214	[160, 372]	583	[774, 428]	952	[1132, 1133]	1321	[436, 782]
215	[373, 374]	584	[397, 432]	953	[853, 167]	1322	[1400, 1097]

216	[375, 376]	585	[798, 166]	954	[1134, 777]	1323	[1401, 1109]
217	[377, 36]	586	[799, 489]	955	[817, 502]	1324	[447, 133]
218	[378, 379]	587	[14, 800]	956	[1135, 206]	1325	[1402, 1125]
219	[380, 381]	588	[760, 450]	957	[1136, 1137]	1326	[419, 376]
220	[382, 383]	589	[193, 372]	958	[406, 898]	1327	[1198, 422]
221	[384, 207]	590	[801, 391]	959	[1138, 1129]	1328	[1403, 1391]
222	[120, 150]	591	[791, 450]	960	[1139, 135]	1329	[514, 453]
223	[385, 168]	592	[802, 10]	961	[1140, 905]	1330	[499, 386]
224	[52, 386]	593	[408, 772]	962	[1141, 194]	1331	[1150, 176]
225	[387, 170]	594	[506, 140]	963	[180, 762]	1332	[1183, 376]
226	[388, 389]	595	[520, 180]	964	[1142, 1143]	1333	[1159, 504]
227	[390, 391]	596	[135, 502]	965	[1100, 457]	1334	[1404, 1131]
228	[392, 122]	597	[803, 804]	966	[1144, 60]	1335	[1405, 1170]
229	[393, 394]	598	[805, 806]	967	[1109, 516]	1336	[9, 459]
230	[147, 146]	599	[771, 409]	968	[1119, 195]	1337	[1115, 1103]
231	[395, 396]	600	[807, 756]	969	[1142, 846]	1338	[1110, 369]
232	[40, 58]	601	[434, 808]	970	[1145, 523]	1339	[1406, 1381]
233	[397, 182]	602	[809, 140]	971	[139, 61]	1340	[1407, 291]
234	[150, 398]	603	[810, 811]	972	[440, 136]	1341	[1055, 1026]
235	[399, 369]	604	[492, 180]	973	[811, 490]	1342	[1408, 1036]
236	[400, 193]	605	[812, 197]	974	[797, 407]	1343	[1409, 1410]
237	[401, 402]	606	[417, 142]	975	[377, 827]	1344	[1411, 1412]
238	[403, 46]	607	[147, 147]	976	[1146, 474]	1345	[1222, 615]
239	[404, 163]	608	[416, 414]	977	[367, 432]	1346	[1413, 692]
240	[366, 162]	609	[144, 414]	978	[51, 499]	1347	[348, 1278]
241	[405, 2]	610	[813, 366]	979	[168, 451]	1348	[1414, 1415]
242	[406, 407]	611	[814, 370]	980	[1147, 449]	1349	[1416, 578]
243	[408, 409]	612	[815, 816]	981	[1148, 1149]	1350	[1036, 81]
244	[410, 389]	613	[817, 520]	982	[178, 506]	1351	[1417, 1418]
245	[411, 395]	614	[796, 206]	983	[766, 469]	1352	[1419, 1228]
246	[144, 38]	615	[378, 806]	984	[1150, 164]	1353	[1420, 27]
247	[412, 413]	616	[818, 819]	985	[1151, 374]	1354	[1336, 666]
248	[32, 383]	617	[382, 820]	986	[1128, 805]	1355	[1421, 114]
249	[37, 414]	618	[821, 207]	987	[836, 158]	1356	[665, 1057]
250	[415, 146]	619	[50, 14]	988	[814, 162]	1357	[563, 1415]
251	[416, 38]	620	[822, 474]	989	[1152, 853]	1358	[663, 529]
252	[414, 417]	621	[404, 481]	990	[877, 800]	1359	[1422, 533]
253	[38, 417]	622	[823, 198]	991	[186, 133]	1360	[1293, 991]
254	[418, 417]	623	[402, 486]	992	[801, 457]	1361	[1423, 1038]
255	[414, 145]	624	[824, 825]	993	[839, 167]	1362	[90, 1424]
256	[146, 146]	625	[395, 780]	994	[184, 195]	1363	[538, 1425]
257	[142, 414]	626	[826, 827]	995	[1098, 790]	1364	[1426, 968]
258	[418, 145]	627	[56, 34]	996	[1117, 435]	1365	[547, 1427]
259	[142, 38]	628	[828, 424]	997	[1153, 419]	1366	[1428, 974]

260	[145, 142]	629	[452, 168]	998	[1154, 409]	1367	[1027, 1429]
261	[419, 55]	630	[829, 830]	999	[1155, 58]	1368	[342, 210]
262	[420, 413]	631	[831, 829]	1000	[1156, 782]	1369	[30, 996]
263	[421, 422]	632	[121, 398]	1001	[423, 843]	1370	[1219, 983]
264	[423, 424]	633	[763, 820]	1002	[1157, 858]	1371	[1430, 1431]
265	[129, 46]	634	[832, 59]	1003	[467, 370]	1372	[1432, 1083]
266	[199, 197]	635	[443, 130]	1004	[185, 132]	1373	[1433, 1044]
267	[425, 426]	636	[833, 834]	1005	[851, 459]	1374	[1044, 338]
268	[427, 428]	637	[835, 152]	1006	[1158, 851]	1375	[1034, 692]
269	[429, 138]	638	[836, 204]	1007	[821, 762]	1376	[107, 589]
270	[430, 431]	639	[837, 793]	1008	[893, 843]	1377	[1298, 1434]
271	[181, 432]	640	[838, 394]	1009	[1159, 1110]	1378	[1435, 1436]
272	[433, 181]	641	[767, 758]	1010	[1160, 775]	1379	[1267, 568]
273	[434, 42]	642	[524, 517]	1011	[904, 1161]	1380	[112, 1437]
274	[435, 389]	643	[807, 800]	1012	[773, 130]	1381	[1438, 108]
275	[436, 437]	644	[839, 487]	1013	[1162, 1163]	1382	[734, 1252]
276	[438, 126]	645	[420, 444]	1014	[1164, 437]	1383	[640, 96]
277	[415, 147]	646	[155, 126]	1015	[1105, 446]	1384	[613, 1022]
278	[427, 44]	647	[840, 805]	1016	[1165, 1133]	1385	[1439, 1440]
279	[119, 439]	648	[141, 144]	1017	[519, 511]	1386	[615, 1245]
280	[440, 56]	649	[140, 485]	1018	[1166, 811]	1387	[1436, 746]
281	[441, 198]	650	[368, 800]	1019	[1167, 1163]	1388	[1277, 1287]
282	[442, 131]	651	[841, 381]	1020	[1168, 124]	1389	[1441, 214]
283	[443, 431]	652	[842, 843]	1021	[838, 446]	1390	[1442, 262]
284	[428, 444]	653	[844, 130]	1022	[428, 413]	1391	[1443, 645]
285	[445, 446]	654	[44, 413]	1023	[835, 395]	1392	[1087, 736]
286	[447, 448]	655	[845, 846]	1024	[1169, 1170]	1393	[973, 703]
287	[449, 450]	656	[192, 160]	1025	[498, 52]	1394	[1444, 274]
288	[451, 452]	657	[462, 169]	1026	[1171, 857]	1395	[1429, 1445]
289	[170, 453]	658	[847, 463]	1027	[1172, 453]	1396	[945, 1332]
290	[454, 52]	659	[459, 132]	1028	[1173, 474]	1397	[1446, 324]
291	[56, 154]	660	[785, 50]	1029	[1174, 520]	1398	[1447, 1007]
292	[455, 12]	661	[848, 785]	1030	[1175, 1110]	1399	[1448, 1432]
293	[456, 457]	662	[849, 850]	1031	[832, 517]	1400	[240, 718]
294	[458, 165]	663	[851, 10]	1032	[1176, 1177]	1401	[1437, 111]
295	[459, 186]	664	[852, 518]	1033	[1178, 846]	1402	[1094, 987]
296	[460, 168]	665	[853, 487]	1034	[1179, 487]	1403	[1449, 1450]
297	[461, 462]	666	[201, 448]	1035	[854, 461]	1404	[1083, 1329]
298	[461, 168]	667	[484, 508]	1036	[177, 439]	1405	[951, 1451]
299	[463, 464]	668	[809, 61]	1037	[818, 793]	1406	[910, 935]
300	[465, 466]	669	[376, 136]	1038	[870, 368]	1407	[1452, 1105]
301	[467, 162]	670	[370, 481]	1039	[468, 767]	1408	[480, 402]
302	[159, 193]	671	[847, 464]	1040	[1180, 466]	1409	[765, 8]
303	[468, 469]	672	[169, 452]	1041	[1181, 785]	1410	[58, 517]

304	[470, 471]	673	[169, 461]	1042	[1182, 393]	1411	[1453, 48]
305	[472, 181]	674	[854, 452]	1043	[438, 156]	1412	[1351, 459]
306	[473, 471]	675	[462, 451]	1044	[1183, 55]	1413	[1454, 443]
307	[474, 197]	676	[855, 856]	1045	[1164, 782]	1414	[852, 12]
308	[475, 368]	677	[52, 857]	1046	[470, 858]	1415	[400, 160]
309	[194, 192]	678	[473, 858]	1047	[1184, 184]	1416	[1455, 834]
310	[476, 166]	679	[497, 398]	1048	[460, 462]	1417	[1454, 773]
311	[477, 478]	680	[859, 170]	1049	[770, 774]	1418	[61, 508]
312	[479, 461]	681	[860, 443]	1050	[1185, 851]	1419	[1456, 1373]
313	[480, 472]	682	[490, 44]	1051	[43, 428]	1420	[1457, 886]
314	[162, 481]	683	[861, 862]	1052	[1186, 1110]	1421	[1458, 825]
315	[482, 149]	684	[190, 863]	1053	[442, 787]	1422	[1459, 393]
316	[483, 435]	685	[864, 514]	1054	[1187, 1188]	1423	[1460, 468]
317	[484, 485]	686	[865, 464]	1055	[1189, 857]	1424	[1097, 469]
318	[486, 182]	687	[866, 856]	1056	[899, 829]	1425	[206, 898]
319	[164, 129]	688	[829, 54]	1057	[1190, 50]	1426	[1144, 506]
320	[132, 448]	689	[867, 489]	1058	[1173, 199]	1427	[195, 160]
321	[166, 487]	690	[439, 121]	1059	[410, 759]	1428	[1461, 804]
322	[488, 169]	691	[200, 493]	1060	[844, 431]	1429	[478, 830]
323	[489, 174]	692	[802, 459]	1061	[1191, 888]	1430	[1211, 1161]
324	[472, 486]	693	[868, 869]	1062	[175, 164]	1431	[506, 61]
325	[490, 428]	694	[870, 14]	1063	[770, 490]	1432	[1397, 54]
326	[491, 443]	695	[871, 188]	1064	[1101, 793]	1433	[768, 402]
327	[492, 384]	696	[872, 197]	1065	[179, 384]	1434	[376, 56]
328	[441, 493]	697	[873, 519]	1066	[1192, 1193]	1435	[1462, 862]
329	[431, 131]	698	[435, 759]	1067	[1189, 386]	1436	[1201, 174]
330	[60, 140]	699	[865, 463]	1068	[1194, 1149]	1437	[869, 50]
331	[494, 495]	700	[874, 120]	1069	[1195, 176]	1438	[1207, 846]
332	[496, 466]	701	[875, 150]	1070	[823, 493]	1439	[1463, 438]
333	[497, 122]	702	[876, 163]	1071	[886, 120]	1440	[412, 444]
334	[498, 499]	703	[136, 154]	1072	[841, 124]	1441	[1464, 1100]
335	[451, 461]	704	[445, 394]	1073	[1196, 119]	1442	[1403, 188]
336	[500, 495]	705	[877, 756]	1074	[1197, 775]	1443	[1399, 863]
337	[501, 188]	706	[390, 457]	1075	[892, 417]	1444	[1465, 515]
338	[153, 34]	707	[758, 389]	1076	[143, 142]	1445	[1149, 174]
339	[502, 180]	708	[464, 464]	1077	[1198, 1127]	1446	[1180, 1390]
340	[370, 163]	709	[784, 451]	1078	[439, 150]	1447	[426, 502]
341	[503, 42]	710	[878, 863]	1079	[1199, 1200]	1448	[1466, 1188]
342	[504, 369]	711	[189, 486]	1080	[1201, 512]	1449	[1467, 1200]
343	[505, 506]	712	[50, 368]	1081	[1202, 1177]	1450	[172, 453]
344	[507, 508]	713	[879, 828]	1082	[859, 514]	1451	[514, 171]
345	[509, 121]	714	[371, 787]	1083	[1171, 386]	1452	[724, 645]
346	[45, 127]	715	[464, 463]	1084	[875, 121]	1453	[1468, 1469]
347	[510, 165]	716	[880, 774]	1085	[1203, 194]	1454	[1470, 963]

348	[449, 511]	717	[380, 124]	1086	[1204, 1205]	1455	[1471, 1471]
349	[479, 452]	718	[509, 150]	1087	[1206, 469]	1456	[1330, 1468]
350	[489, 512]	719	[881, 882]	1088	[1207, 1143]	1457	[668, 1092]
351	[513, 514]	720	[433, 486]	1089	[1109, 203]	1458	[357, 1248]
352	[385, 462]	721	[883, 9]	1090	[184, 192]	1459	[716, 333]
353	[515, 516]	722	[884, 858]	1091	[1208, 790]	1460	[272, 920]
354	[161, 517]	723	[885, 129]	1092	[196, 186]	1461	[992, 1472]
355	[399, 503]	724	[886, 439]	1093	[1209, 1137]	1462	[285, 1447]
356	[11, 518]	725	[887, 888]	1094	[1210, 379]	1463	[1257, 679]
357	[519, 450]	726	[10, 132]	1095	[1211, 905]	1464	[540, 1059]
358	[193, 165]	727	[828, 843]	1096	[723, 1212]	1465	[935, 244]
359	[469, 435]	728	[889, 36]	1097	[1213, 1214]	1466	[220, 1257]
360	[135, 520]	729	[890, 842]	1098	[1215, 1216]	1467	[652, 1029]
361	[521, 419]	730	[392, 398]	1099	[1217, 94]	1468	[1149, 512]
362	[522, 523]	731	[891, 805]	1100	[1218, 1219]	1469	[499, 857]
363	[430, 130]	732	[892, 145]	1101	[1220, 973]	1470	[1473, 382]
364	[524, 59]	733	[893, 424]	1102	[1221, 1222]	1471	[1129, 806]
365	[525, 526]	734	[805, 379]	1103	[1223, 111]	1472	[869, 475]
366	[527, 528]	735	[788, 524]	1104	[216, 536]	1473	[1291, 284]
367	[529, 530]	736	[186, 448]	1105	[1224, 1225]		
368	[531, 532]	737	[483, 758]	1106	[1226, 1227]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	246	0	492	1	738	1	984	0	1230	0
1	0	247	0	493	1	739	1	985	1	1231	1
2	0	248	1	494	0	740	0	986	1	1232	0
3	0	249	0	495	0	741	0	987	0	1233	1
4	0	250	0	496	1	742	0	988	1	1234	1
5	0	251	1	497	1	743	1	989	1	1235	0
6	0	252	0	498	1	744	1	990	1	1236	1
7	0	253	1	499	1	745	0	991	1	1237	1
8	0	254	0	500	0	746	0	992	1	1238	0
9	0	255	0	501	1	747	1	993	1	1239	1
10	0	256	0	502	0	748	1	994	0	1240	0
11	0	257	1	503	0	749	1	995	0	1241	0
12	1	258	0	504	0	750	1	996	1	1242	1
13	0	259	1	505	1	751	1	997	1	1243	0
14	0	260	1	506	0	752	0	998	0	1244	1
15	0	261	0	507	1	753	0	999	0	1245	0
16	1	262	1	508	1	754	1	1000	0	1246	1
17	0	263	1	509	1	755	0	1001	1	1247	0
18	0	264	1	510	1	756	1	1002	1	1248	1
19	0	265	0	511	1	757	1	1003	0	1249	0

20	0	266	1	512	0	758	1	1004	1	1250	0
21	0	267	1	513	1	759	1	1005	1	1251	1
22	1	268	1	514	1	760	0	1006	0	1252	1
23	0	269	1	515	0	761	1	1007	0	1253	1
24	0	270	1	516	1	762	0	1008	0	1254	1
25	1	271	0	517	1	763	1	1009	0	1255	0
26	0	272	0	518	0	764	0	1010	0	1256	1
27	0	273	1	519	1	765	1	1011	0	1257	1
28	1	274	0	520	1	766	1	1012	1	1258	1
29	0	275	1	521	1	767	1	1013	0	1259	1
30	1	276	0	522	0	768	1	1014	1	1260	1
31	0	277	0	523	0	769	1	1015	1	1261	0
32	1	278	1	524	1	770	0	1016	1	1262	1
33	1	279	0	525	0	771	1	1017	1	1263	1
34	0	280	0	526	1	772	0	1018	0	1264	1
35	0	281	0	527	1	773	1	1019	0	1265	0
36	1	282	1	528	1	774	1	1020	1	1266	1
37	0	283	0	529	1	775	1	1021	1	1267	0
38	1	284	1	530	0	776	0	1022	1	1268	0
39	0	285	0	531	1	777	1	1023	0	1269	0
40	1	286	0	532	1	778	1	1024	0	1270	0
41	0	287	0	533	1	779	1	1025	1	1271	0
42	1	288	0	534	0	780	1	1026	1	1272	1
43	0	289	0	535	0	781	0	1027	1	1273	1
44	0	290	0	536	0	782	1	1028	0	1274	0
45	1	291	1	537	1	783	0	1029	1	1275	1
46	1	292	1	538	0	784	0	1030	1	1276	0
47	0	293	0	539	1	785	0	1031	0	1277	1
48	0	294	0	540	1	786	0	1032	0	1278	1
49	0	295	1	541	1	787	0	1033	0	1279	0
50	0	296	1	542	1	788	1	1034	0	1280	0
51	1	297	1	543	1	789	1	1035	1	1281	0
52	0	298	1	544	0	790	0	1036	1	1282	1
53	0	299	0	545	1	791	1	1037	1	1283	1
54	0	300	0	546	1	792	0	1038	1	1284	0
55	0	301	0	547	1	793	0	1039	0	1285	0
56	1	302	0	548	1	794	0	1040	1	1286	1
57	1	303	0	549	1	795	1	1041	0	1287	0
58	0	304	0	550	0	796	0	1042	0	1288	0
59	0	305	1	551	1	797	1	1043	0	1289	1
60	1	306	0	552	0	798	0	1044	0	1290	1
61	1	307	0	553	1	799	0	1045	1	1291	1
62	0	308	1	554	0	800	0	1046	0	1292	1
63	1	309	1	555	0	801	1	1047	1	1293	1

64	1	310	0	556	0	802	0	1048	1	1294	1
65	0	311	1	557	0	803	1	1049	0	1295	1
66	1	312	0	558	0	804	1	1050	0	1296	1
67	0	313	1	559	0	805	0	1051	0	1297	1
68	0	314	0	560	0	806	0	1052	1	1298	1
69	0	315	0	561	0	807	0	1053	1	1299	0
70	0	316	0	562	0	808	0	1054	0	1300	1
71	0	317	0	563	1	809	1	1055	0	1301	1
72	1	318	1	564	0	810	0	1056	0	1302	1
73	0	319	0	565	0	811	1	1057	1	1303	1
74	0	320	0	566	1	812	1	1058	0	1304	0
75	1	321	1	567	1	813	0	1059	1	1305	0
76	1	322	1	568	0	814	1	1060	0	1306	0
77	0	323	1	569	1	815	1	1061	1	1307	0
78	0	324	1	570	0	816	0	1062	0	1308	0
79	0	325	0	571	0	817	0	1063	0	1309	0
80	1	326	0	572	0	818	1	1064	1	1310	1
81	0	327	1	573	0	819	1	1065	0	1311	1
82	1	328	0	574	1	820	1	1066	1	1312	0
83	1	329	1	575	0	821	0	1067	0	1313	1
84	1	330	1	576	1	822	1	1068	1	1314	1
85	0	331	0	577	0	823	1	1069	1	1315	0
86	1	332	1	578	1	824	1	1070	1	1316	0
87	0	333	1	579	0	825	0	1071	1	1317	0
88	0	334	1	580	0	826	1	1072	0	1318	1
89	1	335	0	581	0	827	0	1073	0	1319	0
90	1	336	0	582	0	828	0	1074	0	1320	1
91	1	337	1	583	1	829	1	1075	1	1321	1
92	0	338	0	584	0	830	1	1076	1	1322	1
93	0	339	0	585	0	831	0	1077	0	1323	1
94	0	340	1	586	0	832	0	1078	1	1324	0
95	0	341	0	587	0	833	1	1079	1	1325	0
96	1	342	0	588	0	834	0	1080	1	1326	0
97	0	343	1	589	0	835	0	1081	0	1327	0
98	0	344	1	590	1	836	0	1082	0	1328	1
99	0	345	1	591	1	837	0	1083	1	1329	1
100	1	346	1	592	0	838	1	1084	0	1330	1
101	1	347	1	593	1	839	1	1085	0	1331	0
102	0	348	0	594	0	840	0	1086	1	1332	0
103	1	349	0	595	1	841	0	1087	0	1333	0
104	0	350	1	596	0	842	1	1088	1	1334	1
105	1	351	1	597	1	843	1	1089	1	1335	0
106	0	352	0	598	0	844	0	1090	0	1336	0
107	1	353	0	599	1	845	0	1091	0	1337	0

108	0	354	1	600	0	846	1	1092	0	1338	1
109	1	355	0	601	1	847	0	1093	0	1339	0
110	1	356	0	602	1	848	0	1094	0	1340	1
111	1	357	1	603	0	849	1	1095	0	1341	0
112	0	358	0	604	1	850	0	1096	1	1342	1
113	1	359	0	605	1	851	1	1097	1	1343	1
114	1	360	0	606	0	852	0	1098	0	1344	0
115	1	361	1	607	1	853	0	1099	1	1345	1
116	1	362	0	608	1	854	1	1100	0	1346	1
117	1	363	1	609	0	855	0	1101	1	1347	0
118	1	364	1	610	0	856	1	1102	0	1348	0
119	0	365	0	611	1	857	1	1103	0	1349	1
120	0	366	1	612	1	858	1	1104	1	1350	1
121	1	367	1	613	0	859	0	1105	1	1351	1
122	0	368	1	614	0	860	0	1106	1	1352	1
123	1	369	1	615	1	861	0	1107	1	1353	1
124	1	370	1	616	1	862	1	1108	1	1354	0
125	0	371	0	617	0	863	0	1109	1	1355	1
126	0	372	1	618	0	864	1	1110	1	1356	0
127	0	373	1	619	0	865	1	1111	0	1357	1
128	0	374	1	620	1	866	0	1112	1	1358	1
129	0	375	1	621	0	867	0	1113	1	1359	0
130	0	376	1	622	1	868	1	1114	1	1360	1
131	1	377	1	623	0	869	1	1115	0	1361	0
132	0	378	1	624	1	870	1	1116	1	1362	1
133	0	379	1	625	0	871	0	1117	1	1363	0
134	0	380	0	626	1	872	0	1118	1	1364	0
135	0	381	0	627	1	873	1	1119	0	1365	1
136	0	382	0	628	0	874	0	1120	1	1366	1
137	1	383	0	629	0	875	0	1121	0	1367	1
138	1	384	0	630	1	876	1	1122	1	1368	0
139	0	385	0	631	0	877	1	1123	1	1369	1
140	0	386	0	632	1	878	1	1124	0	1370	1
141	0	387	0	633	1	879	0	1125	0	1371	0
142	1	388	0	634	0	880	0	1126	0	1372	1
143	1	389	0	635	0	881	1	1127	1	1373	1
144	0	390	1	636	1	882	1	1128	1	1374	0
145	1	391	0	637	0	883	1	1129	1	1375	0
146	0	392	0	638	0	884	1	1130	1	1376	1
147	1	393	0	639	0	885	1	1131	1	1377	1
148	0	394	0	640	1	886	1	1132	1	1378	0
149	1	395	0	641	1	887	1	1133	1	1379	0
150	0	396	0	642	1	888	0	1134	0	1380	0
151	1	397	0	643	0	889	0	1135	0	1381	1

152	1	398	1	644	1	890	1	1136	0	1382	0
153	0	399	0	645	1	891	0	1137	1	1383	1
154	1	400	1	646	1	892	1	1138	1	1384	0
155	1	401	0	647	0	893	0	1139	0	1385	1
156	1	402	0	648	0	894	1	1140	1	1386	1
157	1	403	0	649	0	895	1	1141	0	1387	1
158	1	404	0	650	1	896	0	1142	1	1388	1
159	0	405	1	651	0	897	0	1143	0	1389	1
160	1	406	1	652	1	898	0	1144	0	1390	1
161	1	407	1	653	0	899	0	1145	0	1391	0
162	0	408	1	654	0	900	1	1146	1	1392	0
163	0	409	1	655	0	901	1	1147	0	1393	1
164	0	410	1	656	0	902	0	1148	1	1394	0
165	0	411	0	657	0	903	0	1149	0	1395	0
166	1	412	0	658	0	904	0	1150	0	1396	1
167	0	413	1	659	1	905	0	1151	1	1397	1
168	1	414	0	660	0	906	1	1152	1	1398	1
169	1	415	0	661	0	907	1	1153	1	1399	0
170	0	416	1	662	1	908	0	1154	0	1400	1
171	0	417	0	663	1	909	0	1155	0	1401	1
172	0	418	0	664	0	910	0	1156	0	1402	0
173	0	419	0	665	0	911	1	1157	1	1403	1
174	1	420	1	666	1	912	1	1158	0	1404	1
175	0	421	1	667	0	913	1	1159	0	1405	0
176	1	422	0	668	1	914	1	1160	0	1406	0
177	1	423	1	669	1	915	0	1161	1	1407	1
178	0	424	0	670	1	916	0	1162	0	1408	1
179	0	425	1	671	0	917	1	1163	1	1409	1
180	1	426	1	672	1	918	1	1164	1	1410	0
181	0	427	1	673	1	919	1	1165	1	1411	0
182	1	428	1	674	1	920	1	1166	0	1412	1
183	1	429	1	675	0	921	1	1167	0	1413	1
184	0	430	1	676	0	922	1	1168	1	1414	0
185	1	431	1	677	0	923	1	1169	0	1415	1
186	1	432	0	678	0	924	0	1170	1	1416	1
187	0	433	0	679	1	925	0	1171	1	1417	1
188	1	434	1	680	0	926	1	1172	1	1418	1
189	1	435	0	681	0	927	1	1173	0	1419	1
190	0	436	1	682	0	928	0	1174	1	1420	1
191	1	437	0	683	0	929	1	1175	1	1421	1
192	0	438	0	684	0	930	1	1176	0	1422	0
193	0	439	1	685	1	931	1	1177	0	1423	0
194	1	440	0	686	1	932	0	1178	0	1424	1
195	1	441	0	687	0	933	1	1179	0	1425	0

196	0	442	1	688	1	934	1	1180	1	1426	0
197	1	443	0	689	0	935	0	1181	0	1427	1
198	0	444	0	690	1	936	1	1182	0	1428	1
199	1	445	0	691	0	937	0	1183	0	1429	0
200	0	446	1	692	0	938	1	1184	1	1430	0
201	1	447	0	693	1	939	0	1185	0	1431	0
202	0	448	1	694	1	940	1	1186	1	1432	1
203	0	449	0	695	0	941	0	1187	0	1433	1
204	0	450	0	696	0	942	1	1188	0	1434	1
205	1	451	0	697	1	943	0	1189	0	1435	0
206	0	452	0	698	0	944	0	1190	1	1436	1
207	1	453	1	699	1	945	1	1191	1	1437	1
208	0	454	0	700	0	946	0	1192	1	1438	1
209	1	455	1	701	0	947	0	1193	1	1439	1
210	0	456	0	702	1	948	1	1194	1	1440	0
211	1	457	1	703	0	949	1	1195	1	1441	1
212	1	458	0	704	0	950	0	1196	0	1442	1
213	0	459	1	705	1	951	0	1197	0	1443	0
214	1	460	1	706	1	952	1	1198	0	1444	0
215	1	461	1	707	1	953	0	1199	1	1445	0
216	1	462	0	708	1	954	0	1200	0	1446	1
217	1	463	0	709	0	955	0	1201	1	1447	1
218	1	464	1	710	1	956	0	1202	0	1448	0
219	0	465	0	711	1	957	0	1203	0	1449	1
220	0	466	0	712	0	958	1	1204	1	1450	0
221	0	467	0	713	0	959	1	1205	0	1451	1
222	0	468	0	714	0	960	0	1206	0	1452	1
223	0	469	0	715	1	961	1	1207	1	1453	0
224	0	470	0	716	0	962	0	1208	0	1454	1
225	0	471	0	717	0	963	1	1209	0	1455	1
226	0	472	1	718	1	964	1	1210	0	1456	1
227	1	473	0	719	1	965	0	1211	0	1457	1
228	0	474	0	720	0	966	0	1212	0	1458	1
229	0	475	1	721	1	967	1	1213	1	1459	0
230	1	476	0	722	1	968	0	1214	0	1460	0
231	0	477	1	723	1	969	1	1215	0	1461	1
232	1	478	0	724	1	970	0	1216	1	1462	0
233	0	479	0	725	1	971	0	1217	1	1463	1
234	0	480	1	726	0	972	0	1218	0	1464	1
235	0	481	1	727	0	973	1	1219	1	1465	0
236	1	482	0	728	0	974	1	1220	1	1466	0
237	0	483	0	729	1	975	1	1221	0	1467	1
238	0	484	0	730	0	976	1	1222	1	1468	0
239	0	485	0	731	0	977	1	1223	0	1469	1

240	1	486	1	732	1	978	1	1224	1	1470	1
241	1	487	1	733	0	979	1	1225	0	1471	1
242	1	488	1	734	0	980	0	1226	1	1472	1
243	1	489	1	735	1	981	1	1227	1	1473	1
244	1	490	0	736	1	982	0	1228	1		
245	0	491	0	737	0	983	1	1229	1		

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