PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z, z^5+z^2+z+1)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z, z^5+z^2+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z, z^5+z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z) &= z^{20}+z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^9+z^8+z^7+z^5+z^4+z+1,\\
p_1(z) &= z^{23}+z^{21}+z^{19}+z^{17}+z^{16}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7\\
&\quad +z^6+z^5+z^3+z^2+z,\\
p_2(z) &= z^{24}+z^{22}+z^{14}+z^{12}+z^{10}+z^2,\\
p_3(z) &= z^{23}+z^{21}+z^{19}+z^{17}+z^{16}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7\\
&\quad +z^6+z^5+z^3+z^2+z,\\
p_4(z) &= z^{22}+z^{18}+z^{16}+z^{15}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^5+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z) &= z^{18}+z^{16}+z^{15}+z^{14}+z^{12}+z^{11}+z^9+z^7+z^5+z^4+z,\\
p_1(z) &= z^{23}+z^{21}+z^{19}+z^{17}+z^{16}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7\\
&\quad +z^6+z^5+z^3+z^2+z,\\
p_2(z) &= z^{24}+z^{22}+z^{14}+z^{12}+z^{10}+z^2,\\
p_3(z) &= z^{23}+z^{21}+z^{19}+z^{17}+z^{16}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7\\
&\quad +z^6+z^5+z^3+z^2+z,\\
p_4(z) &= z^{22}+z^{20}+z^{18}+z^{16}+z^{15}+z^{12}+z^{11}+z^{10}+z^9+z^7+z^5+z^2+z+1\\
&\quad .
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots \times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150,150,150,150,150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z) \right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 540, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z) \right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 660, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{2u} M^e[:6u] + x^{4u} M^e[:4u] \\ & + x^{5u} M^e[:3u] + x^{3u} M^e[:6u] + x^{5u} M^e[:4u] + x^{6u} M^e[:3u] \\ & + x^{4u} M^e[:6u] + x^{6u} M^e[:4u] + x^{7u} M^e[:3u] + x^{7u} M^e[:5u] \\ & + x^{8u} M^e[:4u] + x^{10u} M^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u}) R^e, \end{aligned}$$

$$\begin{aligned}W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{2u}W^e[:6u] + x^{4u}W^e[:4u]\\ &+ x^{5u}W^e[:3u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u]\\ &+ x^{4u}W^e[:6u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{7u}W^e[:5u]\\ &+ x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^e.\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{2u}M^o[:6u] + x^{4u}M^o[:4u]\\ &+ x^{5u}M^o[:3u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u]\\ &+ x^{4u}M^o[:6u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{7u}M^o[:5u]\\ &+ x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^o,\\ W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{2u}W^o[:6u] + x^{4u}W^o[:4u]\\ &+ x^{5u}W^o[:3u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u]\\ &+ x^{4u}W^o[:6u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{7u}W^o[:5u]\\ &+ x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^o.\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{2u}T[:6u] + x^{4u}T[:4u] + x^{5u}T[:3u]\\ &+ x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{4u}T[:6u] + x^{6u}T[:4u]\\ &+ x^{7u}T[:3u] + x^{7u}T[:5u] + x^{8u}T[:4u] + x^{10u}T[:2u].\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}0 &= x^{3u}T[3u\!:\!4u] + x^{2u}T[4u\!:\!5u] + T[6u\!:\!7u],\\ 0 &= x^{3u}T[4u\!:\!5u] + x^{2u}T[5u\!:\!6u] + T[7u\!:\!8u],\\ 0 &= x^{8u}T[:u] + x^{5u}T[3u\!:\!4u] + x^{4u}T[4u\!:\!5u] + x^{3u}T[5u\!:\!6u] + T[8u\!:\!9u],\\ 0 &= x^{8u}T[u\!:\!2u] + x^{6u}T[3u\!:\!4u] + x^{4u}T[5u\!:\!6u] + T[9u\!:\!10u],\\ 0 &= x^{10u}T[:u] + x^{8u}T[2u\!:\!3u] + x^{7u}T[3u\!:\!4u] + T[10u\!:\!11u],\\ 0 &= x^{10u}T[u\!:\!2u] + x^{8u}T[3u\!:\!4u] + x^{7u}T[4u\!:\!5u] + T[11u\!:\!12u],\end{aligned}$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 696 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 110)), \tag{2.19}$$
$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.20}$$
$$\begin{aligned} 0 = {}& \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1000)), \end{aligned} \tag{2.21}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.23}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 110)), \tag{2.25}$$
$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.26}$$
$$\begin{aligned} 0 = {}& \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1000)), \end{aligned} \tag{2.27}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1010)), \tag{2.29}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{20}), E_{20}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 10$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{6u}R^e,$$
$$W_{2k} = W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{6u}R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{6u}R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{6u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = &\big(W^e[:6v] + x^{2v}W^e[:4v] + x^{3v}W^e[:3v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ &+ x^{2v}M^e[:4v] + x^{3v}M^e[:3v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{6v}W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24}+x^{23}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{13}+x^{12}+x^{10}+x^9+x^8\\
&\quad +x^6+x^4,\\
q_1(x) &= x^{23}+x^{22}+x^{21}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}\\
&\quad +x^9+x^8+x^7+x^5+x^3+x,\\
q_2(x) &= x^{22}+x^{14}+x^{12}+x^{10}+x^2+1,\\
q_3(x) &= x^{23}+x^{22}+x^{21}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}\\
&\quad +x^9+x^8+x^7+x^5+x^3+x,\\
q_4(x) &= x^{23}+x^{22}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^9+x^8+x^6\\
&\quad +x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{138}+x^{136}+x^{134}+x^{132}+x^{130}+x^{124}+x^{122}+x^{120}+x^{114}+x^{110}\\
&\quad +x^{108}+x^{86}+x^{82}+x^{80}+x^{76}+x^{74}+x^{68}+x^{66}+x^{64}+x^{62}+x^{56}\\
&\quad +x^{54}+x^{48}+x^{46}+x^{44}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}\\
&\quad +x^{24}+x^{20}+x^{12},\\
p_3(x) &= x^{134}+x^{132}+x^{130}+x^{128}+x^{126}+x^{124}+x^{122}+x^{120}+x^{118}+x^{110}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{108} + x^{106} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{66} + x^{58}\\
&+ x^{52} + x^{48} + x^{46} + x^{42} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{8},\\
p_6(x) &= x^{128} + x^{126} + x^{124} + x^{120} + x^{114} + x^{106} + x^{102} + x^{92} + x^{86} + x^{84}\\
&+ x^{78} + x^{76} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{40} + x^{38}\\
&+ x^{28} + x^{20} + x^{16} + x^{14} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{136} + x^{134} + x^{130} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{114}\\
&+ x^{112} + x^{110} + x^{108} + x^{106} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84}\\
&+ x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58}\\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{28} + x^{20} + x^{12}\\
&+ x^{8} + x^{6} + x^{4} + x^{2},\\
p_{12}(x) &= x^{136} + x^{128} + x^{112} + x^{104} + x^{96} + x^{88} + x^{72} + x^{56} + x^{32} + x^{24} + x^{16}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111}\\
&+ x^{110} + x^{109} + x^{107} + x^{103} + x^{102} + x^{101} + x^{100} + x^{96} + x^{94} + x^{93}\\
&+ x^{91} + x^{90} + x^{89} + x^{87} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{67} + x^{63}\\
&+ x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{51} + x^{47} + x^{44} + x^{43} + x^{41} + x^{38}\\
&+ x^{37} + x^{35} + x^{34} + x^{33} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{20} + x^{18},\\
p_3(x) &= x^{124} + x^{122} + x^{119} + x^{114} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102}\\
&+ x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{83} + x^{82}\\
&+ x^{81} + x^{79} + x^{76} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63}\\
&+ x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17}\\
&+ x^{16} + x^{12} + x^{11} + x^{9},\\
p_6(x) &= x^{123} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{112} + x^{107} + x^{106} + x^{104}\\
&+ x^{103} + x^{102} + x^{101} + x^{98} + x^{95} + x^{94} + x^{93} + x^{90} + x^{83} + x^{82} + x^{80}\\
&+ x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{63} + x^{62}\\
&+ x^{61} + x^{58} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{37} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{18}\\
&+ x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{6},\\
p_9(x) &= x^{122} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{89}\\
&+ x^{88} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{72} + x^{67} + x^{66} + x^{65} + x^{63}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{62}+x^{60}+x^{57}+x^{56}+x^{54}+x^{52}+x^{51}+x^{50}+x^{47}+x^{40}+x^{35}\\
&+x^{34}+x^{33}+x^{31}+x^{30}+x^{28}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{17}\\
&+x^{16}+x^{15}+x^{14}+x^{13}+x^{9}+x^{7}+x^{6}+x^{5}+x^{3},\\
p_{12}(x)=\ &x^{121}+x^{120}+x^{119}+x^{117}+x^{115}+x^{113}+x^{111}+x^{108}+x^{105}+x^{104}\\
&+x^{103}+x^{100}+x^{97}+x^{96}+x^{95}+x^{92}+x^{85}+x^{84}+x^{83}+x^{80}+x^{73}\\
&+x^{72}+x^{71}+x^{69}+x^{67}+x^{65}+x^{63}+x^{60}+x^{53}+x^{52}+x^{51}+x^{48}\\
&+x^{41}+x^{40}+x^{39}+x^{37}+x^{35}+x^{33}+x^{31}+x^{28}+x^{25}+x^{24}+x^{23}\\
&+x^{20}+x^{17}+x^{16}+x^{15}+x^{12}+x^{5}+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x)=\ &x^{126}+x^{125}+x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{115}+x^{114}+x^{111}\\
&+x^{110}+x^{109}+x^{107}+x^{103}+x^{102}+x^{101}+x^{100}+x^{96}+x^{94}+x^{93}\\
&+x^{91}+x^{90}+x^{89}+x^{87}+x^{79}+x^{78}+x^{77}+x^{75}+x^{73}+x^{67}+x^{63}\\
&+x^{61}+x^{60}+x^{58}+x^{56}+x^{55}+x^{51}+x^{47}+x^{44}+x^{43}+x^{41}+x^{38}\\
&+x^{37}+x^{35}+x^{34}+x^{33}+x^{29}+x^{28}+x^{26}+x^{25}+x^{24}+x^{23}+x^{21}\\
&+x^{20}+x^{18},\\
p_3(x)=\ &x^{124}+x^{122}+x^{119}+x^{114}+x^{110}+x^{109}+x^{107}+x^{106}+x^{104}+x^{102}\\
&+x^{101}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{89}+x^{88}+x^{83}+x^{82}\\
&+x^{81}+x^{79}+x^{76}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}\\
&+x^{59}+x^{57}+x^{55}+x^{54}+x^{53}+x^{51}+x^{48}+x^{43}+x^{42}+x^{41}+x^{39}\\
&+x^{34}+x^{30}+x^{29}+x^{28}+x^{27}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{17}\\
&+x^{16}+x^{12}+x^{11}+x^{9},\\
p_6(x)=\ &x^{123}+x^{122}+x^{120}+x^{117}+x^{116}+x^{115}+x^{112}+x^{107}+x^{106}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{98}+x^{95}+x^{94}+x^{93}+x^{90}+x^{83}+x^{82}+x^{80}\\
&+x^{79}+x^{78}+x^{77}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{66}+x^{63}+x^{62}\\
&+x^{61}+x^{58}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}+x^{45}+x^{42}+x^{41}+x^{40}\\
&+x^{39}+x^{37}+x^{34}+x^{31}+x^{30}+x^{29}+x^{27}+x^{24}+x^{21}+x^{20}+x^{18}\\
&+x^{15}+x^{14}+x^{13}+x^{11}+x^{9}+x^{6},\\
p_9(x)=\ &x^{122}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}+x^{110}+x^{109}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{101}+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}+x^{89}\\
&+x^{88}+x^{86}+x^{84}+x^{83}+x^{82}+x^{79}+x^{72}+x^{67}+x^{66}+x^{65}+x^{63}\\
&+x^{62}+x^{60}+x^{57}+x^{56}+x^{54}+x^{52}+x^{51}+x^{50}+x^{47}+x^{40}+x^{35}\\
&+x^{34}+x^{33}+x^{31}+x^{30}+x^{28}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{17}\\
&+x^{16}+x^{15}+x^{14}+x^{13}+x^{9}+x^{7}+x^{6}+x^{5}+x^{3},\\
p_{12}(x)=\ &x^{121}+x^{120}+x^{119}+x^{117}+x^{115}+x^{113}+x^{111}+x^{108}+x^{105}+x^{104}
\end{aligned}
$$

$$+ x^{103} + x^{100} + x^{97} + x^{96} + x^{95} + x^{92} + x^{85} + x^{84} + x^{83} + x^{80} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{60} + x^{53} + x^{52} + x^{51} + x^{48} + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{28} + x^{25} + x^{24} + x^{23} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12} + x^{5} + x^{4} + x^{3} + 1,$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{108} + x^{102} + x^{98} + x^{88} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{42} + x^{38} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{24}, \\
p_3(x) &= x^{108} + x^{104} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{80} + x^{78} + x^{74} + x^{72} \\
&\quad + x^{70} + x^{66} + x^{64} + x^{62} + x^{60} + x^{46} + x^{40} + x^{38} + x^{30} + x^{28} + x^{24} \\
&\quad + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6}, \\
p_6(x) &= x^{108} + x^{106} + x^{104} + x^{98} + x^{94} + x^{90} + x^{88} + x^{78} + x^{76} + x^{74} + x^{70} \\
&\quad + x^{68} + x^{52} + x^{50} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{22} \\
&\quad + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{2} + 1, \\
p_9(x) &= x^{106} + x^{98} + x^{92} + x^{86} + x^{78} + x^{74} + x^{70} + x^{68} + x^{60} + x^{54} + x^{46} \\
&\quad + x^{42} + x^{38} + x^{36} + x^{28} + x^{22} + x^{14} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{106} + x^{104} + x^{102} + x^{96} + x^{90} + x^{88} + x^{86} + x^{80} + x^{58} + x^{56} + x^{54} \\
&\quad + x^{48} + x^{26} + x^{24} + x^{22} + x^{16} + x^{10} + x^{8} + x^{6} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{84} + x^{82} + x^{80} + x^{78} + x^{74} \\
&\quad + x^{72} + x^{64} + x^{60} + x^{48} + x^{44} + x^{36} + x^{34} + x^{24} + x^{20} + x^{18} + x^{16} \\
&\quad + x^{12}, \\
p_3(x) &= x^{108} + x^{96} + x^{86} + x^{80} + x^{72} + x^{64} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} \\
&\quad + x^{48} + x^{40} + x^{36} + x^{34} + x^{32} + x^{22} + x^{20} + x^{18} + x^{16} + x^{10} + x^{8}, \\
p_6(x) &= x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{80} + x^{76} \\
&\quad + x^{68} + x^{66} + x^{62} + x^{58} + x^{54} + x^{50} + x^{42} + x^{38} + x^{34} + x^{32} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{14} + x^{8} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{106} + x^{98} + x^{92} + x^{86} + x^{78} + x^{74} + x^{70} + x^{68} + x^{60} + x^{54} + x^{46} \\
&\quad + x^{42} + x^{38} + x^{36} + x^{28} + x^{22} + x^{14} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{106} + x^{104} + x^{102} + x^{96} + x^{90} + x^{88} + x^{86} + x^{80} + x^{58} + x^{56} + x^{54} \\
&\quad + x^{48} + x^{26} + x^{24} + x^{22} + x^{16} + x^{10} + x^{8} + x^{6} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$p_0(x) = x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{108}$$

$$
\begin{aligned}
&+ x^{107} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{73} + x^{71}\\
&+ x^{69} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{47} + x^{45} + x^{43} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{24},\\
p_3(x) = &\; x^{124} + x^{122} + x^{119} + x^{114} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102}\\
&+ x^{101} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{83} + x^{82}\\
&+ x^{81} + x^{79} + x^{76} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63}\\
&+ x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17}\\
&+ x^{16} + x^{12} + x^{11} + x^{9},\\
p_6(x) = &\; x^{123} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{112} + x^{107} + x^{106} + x^{104}\\
&+ x^{103} + x^{102} + x^{101} + x^{98} + x^{95} + x^{94} + x^{93} + x^{90} + x^{83} + x^{82} + x^{80}\\
&+ x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{63} + x^{62}\\
&+ x^{61} + x^{58} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{37} + x^{34} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{18}\\
&+ x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{6},\\
p_9(x) = &\; x^{122} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} + x^{89}\\
&+ x^{88} + x^{86} + x^{84} + x^{83} + x^{82} + x^{79} + x^{72} + x^{67} + x^{66} + x^{65} + x^{63}\\
&+ x^{62} + x^{60} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{40} + x^{35}\\
&+ x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{17}\\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{9} + x^{7} + x^{6} + x^{5} + x^{3},\\
p_{12}(x) = &\; x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{108} + x^{105} + x^{104}\\
&+ x^{103} + x^{100} + x^{97} + x^{96} + x^{95} + x^{92} + x^{85} + x^{84} + x^{83} + x^{80} + x^{73}\\
&+ x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{60} + x^{53} + x^{52} + x^{51} + x^{48}\\
&+ x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{28} + x^{25} + x^{24} + x^{23}\\
&+ x^{20} + x^{17} + x^{16} + x^{15} + x^{12} + x^{5} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = &\; x^{126} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{108}\\
&+ x^{107} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{73} + x^{71}\\
&+ x^{69} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{47} + x^{45} + x^{43} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{24},\\
p_3(x) = &\; x^{124} + x^{122} + x^{119} + x^{114} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{101}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{89}+x^{88}+x^{83}+x^{82}\\
&+x^{81}+x^{79}+x^{76}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}\\
&+x^{59}+x^{57}+x^{55}+x^{54}+x^{53}+x^{51}+x^{48}+x^{43}+x^{42}+x^{41}+x^{39}\\
&+x^{34}+x^{30}+x^{29}+x^{28}+x^{27}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{17}\\
&+x^{16}+x^{12}+x^{11}+x^{9},\\
p_6(x)=\ &x^{123}+x^{122}+x^{120}+x^{117}+x^{116}+x^{115}+x^{112}+x^{107}+x^{106}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{98}+x^{95}+x^{94}+x^{93}+x^{90}+x^{83}+x^{82}+x^{80}\\
&+x^{79}+x^{78}+x^{77}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{66}+x^{63}+x^{62}\\
&+x^{61}+x^{58}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}+x^{45}+x^{42}+x^{41}+x^{40}\\
&+x^{39}+x^{37}+x^{34}+x^{31}+x^{30}+x^{29}+x^{27}+x^{24}+x^{21}+x^{20}+x^{18}\\
&+x^{15}+x^{14}+x^{13}+x^{11}+x^{9}+x^{6},\\
p_9(x)=\ &x^{122}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}+x^{110}+x^{109}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{101}+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}+x^{89}\\
&+x^{88}+x^{86}+x^{84}+x^{83}+x^{82}+x^{79}+x^{72}+x^{67}+x^{66}+x^{65}+x^{63}\\
&+x^{62}+x^{60}+x^{57}+x^{56}+x^{54}+x^{52}+x^{51}+x^{50}+x^{47}+x^{40}+x^{35}\\
&+x^{34}+x^{33}+x^{31}+x^{30}+x^{28}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{17}\\
&+x^{16}+x^{15}+x^{14}+x^{13}+x^{9}+x^{7}+x^{6}+x^{5}+x^{3},\\
p_{12}(x)=\ &x^{121}+x^{120}+x^{119}+x^{117}+x^{115}+x^{113}+x^{111}+x^{108}+x^{105}+x^{104}\\
&+x^{103}+x^{100}+x^{97}+x^{96}+x^{95}+x^{92}+x^{85}+x^{84}+x^{83}+x^{80}+x^{73}\\
&+x^{72}+x^{71}+x^{69}+x^{67}+x^{65}+x^{63}+x^{60}+x^{53}+x^{52}+x^{51}+x^{48}\\
&+x^{41}+x^{40}+x^{39}+x^{37}+x^{35}+x^{33}+x^{31}+x^{28}+x^{25}+x^{24}+x^{23}\\
&+x^{20}+x^{17}+x^{16}+x^{15}+x^{12}+x^{5}+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)=\ &x^{138}+x^{132}+x^{130}+x^{120}+x^{114}+x^{112}+x^{110}+x^{100}+x^{88}+x^{86}\\
&+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}+x^{70}+x^{62}+x^{60}+x^{58}+x^{56}+x^{52}\\
&+x^{48}+x^{42}+x^{40}+x^{36},\\
p_3(x)=\ &x^{130}+x^{126}+x^{114}+x^{112}+x^{110}+x^{106}+x^{100}+x^{96}+x^{94}+x^{92}\\
&+x^{88}+x^{86}+x^{82}+x^{80}+x^{68}+x^{64}+x^{60}+x^{58}+x^{54}+x^{46}+x^{38}\\
&+x^{36}+x^{30}+x^{28}+x^{24}+x^{18}+x^{16}+x^{14}+x^{8}+x^{6},\\
p_6(x)=\ &x^{134}+x^{128}+x^{126}+x^{122}+x^{120}+x^{118}+x^{116}+x^{112}+x^{106}+x^{102}\\
&+x^{92}+x^{90}+x^{86}+x^{84}+x^{80}+x^{76}+x^{72}+x^{68}+x^{64}+x^{62}+x^{58}\\
&+x^{56}+x^{52}+x^{50}+x^{42}+x^{40}+x^{32}+x^{30}+x^{26}+x^{24}+x^{22}+x^{20}\\
&+x^{16}+x^{14}+x^{12}+x^{6}+x^{2}+1,\\
p_9(x)=\ &x^{136}+x^{134}+x^{130}+x^{128}+x^{126}+x^{124}+x^{122}+x^{120}+x^{118}+x^{114}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{112} + x^{110} + x^{108} + x^{106} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} \\
&+ x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} \\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{28} + x^{20} + x^{12} \\
&+ x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) = &\, x^{136} + x^{128} + x^{112} + x^{104} + x^{96} + x^{88} + x^{72} + x^{56} + x^{32} + x^{24} + x^{16} \\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = &\, x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} \\
&+ x^{110} + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} \\
&+ x^{96} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{86} + x^{81} + x^{80} + x^{73} + x^{69} \\
&+ x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{52} + x^{51} \\
&+ x^{50} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{34} \\
&+ x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{15} + x^{14} \\
&+ x^{12}, \\
p_3(x) = &\, x^{125} + x^{122} + x^{121} + x^{120} + x^{119} + x^{116} + x^{115} + x^{111} + x^{110} + x^{109} \\
&+ x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} \\
&+ x^{91} + x^{89} + x^{88} + x^{85} + x^{83} + x^{81} + x^{78} + x^{77} + x^{72} + x^{68} + x^{67} \\
&+ x^{66} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} \\
&+ x^{44} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} \\
&+ x^{26} + x^{24} + x^{22} + x^{17} + x^{11} + x^{10} + x^{8}, \\
p_6(x) = &\, x^{125} + x^{122} + x^{121} + x^{118} + x^{117} + x^{115} + x^{114} + x^{110} + x^{107} + x^{106} \\
&+ x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{93} + x^{90} + x^{89} \\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} \\
&+ x^{71} + x^{68} + x^{64} + x^{63} + x^{61} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
&+ x^{51} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{36} + x^{32} + x^{31} \\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} \\
&+ x^{8} + x^{7} + x^{6} + x^{5} + x^{3} + 1, \\
p_9(x) = &\, x^{125} + x^{124} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{112} + x^{109} + x^{108} \\
&+ x^{107} + x^{105} + x^{103} + x^{101} + x^{99} + x^{96} + x^{89} + x^{88} + x^{87} + x^{84} \\
&+ x^{77} + x^{76} + x^{75} + x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{57} + x^{56} + x^{55} \\
&+ x^{52} + x^{45} + x^{44} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} \\
&+ x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{16} + x^{9} + x^{8} + x^{7} + x^{4}, \\
p_{12}(x) = &\, x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{116} + x^{113} + x^{112} + x^{111} + x^{109} \\
&+ x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{92} + x^{89} + x^{88}
\end{aligned}
$$

$$+ x^{87} + x^{85} + x^{83} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{48} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{15} + x^{12} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3} + 1,$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{124} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{110} + x^{107} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{95} + x^{92} + x^{89} + x^{87} + x^{86} + x^{85} + x^{82} + x^{80} + x^{79} \\
&\quad + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} + x^{70} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} \\
&\quad + x^{59} + x^{56} + x^{53} + x^{52} + x^{50} + x^{47} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} \\
&\quad + x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{18}, \\
p_3(x) &= x^{117} + x^{116} + x^{115} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} \\
&\quad + x^{100} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{85} + x^{81} + x^{77} \\
&\quad + x^{76} + x^{75} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{60} + x^{59} + x^{57} + x^{54} \\
&\quad + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{41} + x^{37} + x^{36} + x^{35} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{18} + x^{16} + x^{13} + x^{9}, \\
p_6(x) &= x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{90} + x^{78} \\
&\quad + x^{76} + x^{72} + x^{70} + x^{66} + x^{58} + x^{46} + x^{44} + x^{40} + x^{38} + x^{34} + x^{26} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{10} + x^{6}, \\
p_9(x) &= x^{119} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106} \\
&\quad + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{87} \\
&\quad + x^{83} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} \\
&\quad + x^{55} + x^{51} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{26} + x^{22} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} \\
&\quad + x^{7} + x^{3}, \\
p_{12}(x) &= x^{120} + x^{116} + x^{108} + x^{104} + x^{96} + x^{92} + x^{84} + x^{80} + x^{72} + x^{68} + x^{60} \\
&\quad + x^{52} + x^{48} + x^{40} + x^{36} + x^{28} + x^{24} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{138} + x^{137} + x^{136} + x^{134} + x^{131} + x^{127} + x^{126} + x^{123} + x^{122} + x^{119} \\
&\quad + x^{118} + x^{117} + x^{116} + x^{115} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{103} + x^{101} + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{88} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{81} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} + x^{68} + x^{66} + x^{63} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{26} + x^{24},\\
p_3(x) &= x^{137} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{127} + x^{122} + x^{120}\\
&+ x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{108} + x^{107} + x^{106} + x^{105}\\
&+ x^{103} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{84} + x^{83} + x^{78}\\
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63} + x^{62}\\
&+ x^{61} + x^{59} + x^{56} + x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{22}\\
&+ x^{19} + x^{18} + x^{16} + x^{13} + x^{9},\\
p_6(x) &= x^{138} + x^{136} + x^{132} + x^{130} + x^{126} + x^{118} + x^{110} + x^{108} + x^{104} + x^{102}\\
&+ x^{96} + x^{94} + x^{90} + x^{84} + x^{78} + x^{76} + x^{74} + x^{72} + x^{64} + x^{62} + x^{58}\\
&+ x^{52} + x^{46} + x^{44} + x^{36} + x^{34} + x^{32} + x^{26} + x^{22} + x^{20} + x^{10} + x^{6},\\
p_9(x) &= x^{139} + x^{138} + x^{136} + x^{135} + x^{133} + x^{128} + x^{127} + x^{123} + x^{121} + x^{117}\\
&+ x^{116} + x^{114} + x^{113} + x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{100}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88} + x^{85} + x^{83} + x^{80}\\
&+ x^{79} + x^{74} + x^{73} + x^{72} + x^{71} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60}\\
&+ x^{59} + x^{56} + x^{53} + x^{51} + x^{48} + x^{47} + x^{43} + x^{41} + x^{37} + x^{36} + x^{34}\\
&+ x^{33} + x^{31} + x^{29} + x^{28} + x^{25} + x^{24} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16}\\
&+ x^{13} + x^{12} + x^{10} + x^{7} + x^{3},\\
p_{12}(x) &= x^{140} + x^{120} + x^{116} + x^{108} + x^{104} + x^{96} + x^{92} + x^{84} + x^{80} + x^{76} + x^{72}\\
&+ x^{68} + x^{60} + x^{52} + x^{48} + x^{40} + x^{36} + x^{28} + x^{24} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{114} + x^{111}\\
&+ x^{110} + x^{107} + x^{106} + x^{105} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{93}\\
&+ x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75}\\
&+ x^{74} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{60} + x^{59} + x^{58}\\
&+ x^{57} + x^{53} + x^{50} + x^{48} + x^{45} + x^{44} + x^{37} + x^{35} + x^{30},\\
p_3(x) &= x^{125} + x^{123} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111}\\
&+ x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{94} + x^{89} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{77} + x^{75} + x^{74} + x^{69} + x^{66} + x^{65}\\
&+ x^{63} + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48}\\
&+ x^{47} + x^{46} + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31}\\
&+ x^{26} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_6(x) &= x^{125} + x^{123} + x^{119} + x^{118} + x^{114} + x^{112} + x^{110} + x^{105} + x^{103} + x^{100}\\
&+ x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{82}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{81}+x^{80}+x^{79}+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}\\
&+x^{66}+x^{64}+x^{62}+x^{61}+x^{60}+x^{59}+x^{57}+x^{55}+x^{53}+x^{52}+x^{50}\\
&+x^{49}+x^{48}+x^{47}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}\\
&+x^{34}+x^{32}+x^{30}+x^{28}+x^{25}+x^{22}+x^{21}+x^{20}+x^{17}+x^{15}+x^{14}\\
&+x^{13}+x^{12}+x^{11}+x^{10}+x^{8}+x^{5}+x^{4}+x^{3}+1,\\
p_9(x) = {}&x^{125}+x^{124}+x^{123}+x^{120}+x^{117}+x^{116}+x^{115}+x^{112}+x^{109}+x^{108}\\
&+x^{107}+x^{105}+x^{103}+x^{101}+x^{99}+x^{96}+x^{89}+x^{88}+x^{87}+x^{84}\\
&+x^{77}+x^{76}+x^{75}+x^{72}+x^{69}+x^{68}+x^{67}+x^{64}+x^{57}+x^{56}+x^{55}\\
&+x^{52}+x^{45}+x^{44}+x^{43}+x^{40}+x^{37}+x^{36}+x^{35}+x^{32}+x^{29}+x^{28}\\
&+x^{27}+x^{25}+x^{23}+x^{21}+x^{19}+x^{16}+x^{9}+x^{8}+x^{7}+x^{4},\\
p_{12}(x) = {}&x^{125}+x^{124}+x^{123}+x^{121}+x^{119}+x^{116}+x^{113}+x^{112}+x^{111}+x^{109}\\
&+x^{107}+x^{104}+x^{101}+x^{100}+x^{99}+x^{97}+x^{95}+x^{92}+x^{89}+x^{88}\\
&+x^{87}+x^{85}+x^{83}+x^{80}+x^{77}+x^{76}+x^{75}+x^{73}+x^{71}+x^{68}+x^{65}\\
&+x^{64}+x^{63}+x^{60}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{48}+x^{45}+x^{44}\\
&+x^{43}+x^{41}+x^{39}+x^{36}+x^{33}+x^{32}+x^{31}+x^{29}+x^{27}+x^{24}+x^{21}\\
&+x^{20}+x^{19}+x^{17}+x^{15}+x^{12}+x^{9}+x^{8}+x^{7}+x^{5}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}&x^{126}+x^{125}+x^{124}+x^{123}+x^{122}+x^{118}+x^{115}+x^{114}+x^{113}+x^{111}\\
&+x^{110}+x^{108}+x^{107}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}\\
&+x^{96}+x^{94}+x^{93}+x^{92}+x^{89}+x^{88}+x^{86}+x^{81}+x^{80}+x^{73}+x^{69}\\
&+x^{68}+x^{67}+x^{66}+x^{63}+x^{62}+x^{61}+x^{60}+x^{58}+x^{56}+x^{52}+x^{51}\\
&+x^{50}+x^{47}+x^{46}+x^{45}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{35}+x^{34}\\
&+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}+x^{15}+x^{14}\\
&+x^{12},\\
p_3(x) = {}&x^{125}+x^{122}+x^{121}+x^{120}+x^{119}+x^{116}+x^{115}+x^{111}+x^{110}+x^{109}\\
&+x^{107}+x^{106}+x^{104}+x^{102}+x^{101}+x^{99}+x^{98}+x^{96}+x^{95}+x^{93}\\
&+x^{91}+x^{89}+x^{88}+x^{85}+x^{83}+x^{81}+x^{78}+x^{77}+x^{72}+x^{68}+x^{67}\\
&+x^{66}+x^{62}+x^{60}+x^{59}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{49}+x^{48}\\
&+x^{44}+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}\\
&+x^{26}+x^{24}+x^{22}+x^{17}+x^{11}+x^{10}+x^{8},\\
p_6(x) = {}&x^{125}+x^{122}+x^{121}+x^{118}+x^{117}+x^{115}+x^{114}+x^{110}+x^{107}+x^{106}\\
&+x^{105}+x^{104}+x^{102}+x^{101}+x^{100}+x^{99}+x^{95}+x^{93}+x^{90}+x^{89}\\
&+x^{88}+x^{87}+x^{85}+x^{84}+x^{83}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{73}\\
&+x^{71}+x^{68}+x^{64}+x^{63}+x^{61}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{51}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{41}+x^{39}+x^{36}+x^{32}+x^{31}\\
&+x^{24}+x^{23}+x^{22}+x^{20}+x^{18}+x^{16}+x^{15}+x^{13}+x^{12}+x^{11}+x^{10}\\
&+x^{8}+x^{7}+x^{6}+x^{5}+x^{3}+1,\\
p_9(x)=\,&x^{125}+x^{124}+x^{123}+x^{120}+x^{117}+x^{116}+x^{115}+x^{112}+x^{109}+x^{108}\\
&+x^{107}+x^{105}+x^{103}+x^{101}+x^{99}+x^{96}+x^{89}+x^{88}+x^{87}+x^{84}\\
&+x^{77}+x^{76}+x^{75}+x^{72}+x^{69}+x^{68}+x^{67}+x^{64}+x^{57}+x^{56}+x^{55}\\
&+x^{52}+x^{45}+x^{44}+x^{43}+x^{40}+x^{37}+x^{36}+x^{35}+x^{32}+x^{29}+x^{28}\\
&+x^{27}+x^{25}+x^{23}+x^{21}+x^{19}+x^{16}+x^{9}+x^{8}+x^{7}+x^{4},\\
p_{12}(x)=\,&x^{125}+x^{124}+x^{123}+x^{121}+x^{119}+x^{116}+x^{113}+x^{112}+x^{111}+x^{109}\\
&+x^{107}+x^{104}+x^{101}+x^{100}+x^{99}+x^{97}+x^{95}+x^{92}+x^{89}+x^{88}\\
&+x^{87}+x^{85}+x^{83}+x^{80}+x^{77}+x^{76}+x^{75}+x^{73}+x^{71}+x^{68}+x^{65}\\
&+x^{64}+x^{63}+x^{60}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{48}+x^{45}+x^{44}\\
&+x^{43}+x^{41}+x^{39}+x^{36}+x^{33}+x^{32}+x^{31}+x^{29}+x^{27}+x^{24}+x^{21}\\
&+x^{20}+x^{19}+x^{17}+x^{15}+x^{12}+x^{9}+x^{8}+x^{7}+x^{5}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x)=\,&x^{138}+x^{137}+x^{136}+x^{134}+x^{131}+x^{127}+x^{126}+x^{123}+x^{122}+x^{119}\\
&+x^{118}+x^{117}+x^{116}+x^{115}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{103}+x^{101}+x^{99}+x^{98}+x^{96}+x^{93}+x^{92}+x^{88}+x^{86}+x^{85}\\
&+x^{84}+x^{81}+x^{80}+x^{78}+x^{77}+x^{74}+x^{73}+x^{68}+x^{66}+x^{63}+x^{60}\\
&+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{47}+x^{46}\\
&+x^{42}+x^{41}+x^{40}+x^{37}+x^{36}+x^{35}+x^{34}+x^{32}+x^{30}+x^{28}+x^{27}\\
&+x^{26}+x^{24},\\
p_3(x)=\,&x^{137}+x^{136}+x^{135}+x^{133}+x^{132}+x^{131}+x^{130}+x^{127}+x^{122}+x^{120}\\
&+x^{118}+x^{115}+x^{114}+x^{113}+x^{111}+x^{108}+x^{107}+x^{106}+x^{105}\\
&+x^{103}+x^{98}+x^{97}+x^{95}+x^{94}+x^{93}+x^{92}+x^{89}+x^{84}+x^{83}+x^{78}\\
&+x^{77}+x^{76}+x^{74}+x^{73}+x^{72}+x^{70}+x^{68}+x^{66}+x^{65}+x^{63}+x^{62}\\
&+x^{61}+x^{59}+x^{56}+x^{55}+x^{54}+x^{52}+x^{49}+x^{48}+x^{47}+x^{45}+x^{42}\\
&+x^{39}+x^{38}+x^{37}+x^{36}+x^{32}+x^{31}+x^{29}+x^{26}+x^{25}+x^{24}+x^{22}\\
&+x^{19}+x^{18}+x^{16}+x^{13}+x^{9},\\
p_6(x)=\,&x^{138}+x^{136}+x^{132}+x^{130}+x^{126}+x^{118}+x^{110}+x^{108}+x^{104}+x^{102}\\
&+x^{96}+x^{94}+x^{90}+x^{84}+x^{78}+x^{76}+x^{74}+x^{72}+x^{64}+x^{62}+x^{58}\\
&+x^{52}+x^{46}+x^{44}+x^{36}+x^{34}+x^{32}+x^{26}+x^{22}+x^{20}+x^{10}+x^{6},\\
p_9(x)=\,&x^{139}+x^{138}+x^{136}+x^{135}+x^{133}+x^{128}+x^{127}+x^{123}+x^{121}+x^{117}\\
&+x^{116}+x^{114}+x^{113}+x^{111}+x^{109}+x^{108}+x^{105}+x^{104}+x^{100}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{99}+x^{98}+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{88}+x^{85}+x^{83}+x^{80}\\
&+x^{79}+x^{74}+x^{73}+x^{72}+x^{71}+x^{68}+x^{66}+x^{65}+x^{64}+x^{61}+x^{60}\\
&+x^{59}+x^{56}+x^{53}+x^{51}+x^{48}+x^{47}+x^{43}+x^{41}+x^{37}+x^{36}+x^{34}\\
&+x^{33}+x^{31}+x^{29}+x^{28}+x^{25}+x^{24}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}\\
&+x^{13}+x^{12}+x^{10}+x^{7}+x^{3},\\
p_{12}(x)=&\,x^{140}+x^{120}+x^{116}+x^{108}+x^{104}+x^{96}+x^{92}+x^{84}+x^{80}+x^{76}+x^{72}\\
&+x^{68}+x^{60}+x^{52}+x^{48}+x^{40}+x^{36}+x^{28}+x^{24}+x^{16}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{124}+x^{121}+x^{119}+x^{116}+x^{115}+x^{114}+x^{110}+x^{107}+x^{104}+x^{103}\\
&+x^{102}+x^{101}+x^{95}+x^{92}+x^{89}+x^{87}+x^{86}+x^{85}+x^{82}+x^{80}+x^{79}\\
&+x^{78}+x^{77}+x^{75}+x^{74}+x^{72}+x^{70}+x^{67}+x^{64}+x^{62}+x^{61}+x^{60}\\
&+x^{59}+x^{56}+x^{53}+x^{52}+x^{50}+x^{47}+x^{44}+x^{43}+x^{42}+x^{41}+x^{39}\\
&+x^{37}+x^{36}+x^{32}+x^{31}+x^{29}+x^{28}+x^{27}+x^{26}+x^{23}+x^{22}+x^{18},\\
p_3(x)=&\,x^{117}+x^{116}+x^{115}+x^{110}+x^{109}+x^{108}+x^{106}+x^{105}+x^{104}+x^{102}\\
&+x^{100}+x^{98}+x^{96}+x^{94}+x^{93}+x^{91}+x^{90}+x^{88}+x^{85}+x^{81}+x^{77}\\
&+x^{76}+x^{75}+x^{70}+x^{67}+x^{66}+x^{65}+x^{64}+x^{60}+x^{59}+x^{57}+x^{54}\\
&+x^{51}+x^{50}+x^{49}+x^{48}+x^{45}+x^{41}+x^{37}+x^{36}+x^{35}+x^{30}+x^{29}\\
&+x^{28}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{19}+x^{18}+x^{16}+x^{13}+x^{9},\\
p_6(x)=&\,x^{118}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}+x^{104}+x^{98}+x^{90}+x^{78}\\
&+x^{76}+x^{72}+x^{70}+x^{66}+x^{58}+x^{46}+x^{44}+x^{40}+x^{38}+x^{34}+x^{26}\\
&+x^{22}+x^{20}+x^{18}+x^{10}+x^{6},\\
p_9(x)=&\,x^{119}+x^{118}+x^{116}+x^{114}+x^{113}+x^{112}+x^{111}+x^{109}+x^{108}+x^{106}\\
&+x^{102}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{93}+x^{92}+x^{90}+x^{87}\\
&+x^{83}+x^{71}+x^{70}+x^{68}+x^{66}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{58}\\
&+x^{55}+x^{51}+x^{39}+x^{38}+x^{36}+x^{34}+x^{33}+x^{32}+x^{31}+x^{29}+x^{28}\\
&+x^{26}+x^{22}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{13}+x^{12}+x^{10}\\
&+x^{7}+x^{3},\\
p_{12}(x)=&\,x^{120}+x^{116}+x^{108}+x^{104}+x^{96}+x^{92}+x^{84}+x^{80}+x^{72}+x^{68}+x^{60}\\
&+x^{52}+x^{48}+x^{40}+x^{36}+x^{28}+x^{24}+x^{16}+x^{12}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{124}+x^{122}+x^{121}+x^{120}+x^{119}+x^{118}+x^{116}+x^{114}+x^{111}\\
&+x^{110}+x^{107}+x^{106}+x^{105}+x^{101}+x^{100}+x^{99}+x^{96}+x^{94}+x^{93}\\
&+x^{90}+x^{87}+x^{86}+x^{85}+x^{84}+x^{82}+x^{80}+x^{79}+x^{77}+x^{76}+x^{75}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{74} + x^{73} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{60} + x^{59} + x^{58} \\
&+ x^{57} + x^{53} + x^{50} + x^{48} + x^{45} + x^{44} + x^{37} + x^{35} + x^{30}, \\
p_3(x) &= x^{125} + x^{123} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} \\
&+ x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{94} + x^{89} + x^{87} \\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{77} + x^{75} + x^{74} + x^{69} + x^{66} + x^{65} \\
&+ x^{63} + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
&+ x^{47} + x^{46} + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} \\
&+ x^{26} + x^{23} + x^{21} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12}, \\
p_6(x) &= x^{125} + x^{123} + x^{119} + x^{118} + x^{114} + x^{112} + x^{110} + x^{105} + x^{103} + x^{100} \\
&+ x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{84} + x^{82} \\
&+ x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} \\
&+ x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{53} + x^{52} + x^{50} \\
&+ x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} \\
&+ x^{34} + x^{32} + x^{30} + x^{28} + x^{25} + x^{22} + x^{21} + x^{20} + x^{17} + x^{15} + x^{14} \\
&+ x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{5} + x^{4} + x^{3} + 1, \\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{112} + x^{109} + x^{108} \\
&+ x^{107} + x^{105} + x^{103} + x^{101} + x^{99} + x^{96} + x^{89} + x^{88} + x^{87} + x^{84} \\
&+ x^{77} + x^{76} + x^{75} + x^{72} + x^{69} + x^{68} + x^{67} + x^{64} + x^{57} + x^{56} + x^{55} \\
&+ x^{52} + x^{45} + x^{44} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} \\
&+ x^{27} + x^{25} + x^{23} + x^{21} + x^{19} + x^{16} + x^{9} + x^{8} + x^{7} + x^{4}, \\
p_{12}(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{116} + x^{113} + x^{112} + x^{111} + x^{109} \\
&+ x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{92} + x^{89} + x^{88} \\
&+ x^{87} + x^{85} + x^{83} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{68} + x^{65} \\
&+ x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{48} + x^{45} + x^{44} \\
&+ x^{43} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{24} + x^{21} \\
&+ x^{20} + x^{19} + x^{17} + x^{15} + x^{12} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	140	[235, 236]	280	[400, 401]	420	[534, 535]	560	[633, 102]
1	[3, 4]	141	[237, 77]	281	[402, 386]	421	[536, 95]	561	[233, 634]
2	[5, 6]	142	[238, 19]	282	[403, 352]	422	[524, 104]	562	[28, 635]
3	[7, 8]	143	[239, 240]	283	[404, 55]	423	[479, 236]	563	[636, 485]
4	[9, 10]	144	[241, 77]	284	[366, 405]	424	[296, 85]	564	[476, 479]
5	[11, 12]	145	[242, 105]	285	[192, 406]	425	[537, 491]	565	[637, 238]

6	[13, 14]	146	[243, 244]	286	[407, 197]	426	[538, 116]	566	[230, 476]
7	[15, 16]	147	[21, 245]	287	[408, 179]	427	[321, 539]	567	[74, 638]
8	[17, 18]	148	[246, 82]	288	[160, 183]	428	[131, 312]	568	[231, 239]
9	[19, 20]	149	[247, 248]	289	[392, 409]	429	[540, 125]	569	[636, 639]
10	[21, 22]	150	[111, 249]	290	[410, 402]	430	[311, 305]	570	[640, 490]
11	[23, 24]	151	[250, 246]	291	[411, 152]	431	[306, 320]	571	[486, 641]
12	[25, 26]	152	[251, 252]	292	[412, 54]	432	[130, 541]	572	[283, 235]
13	[27, 28]	153	[253, 85]	293	[413, 414]	433	[314, 329]	573	[523, 481]
14	[29, 30]	154	[22, 254]	294	[415, 168]	434	[34, 331]	574	[283, 479]
15	[31, 31]	155	[255, 256]	295	[399, 157]	435	[123, 307]	575	[642, 510]
16	[32, 33]	156	[257, 258]	296	[416, 151]	436	[542, 310]	576	[643, 505]
17	[34, 35]	157	[259, 260]	297	[417, 154]	437	[543, 34]	577	[464, 244]
18	[36, 37]	158	[261, 99]	298	[418, 419]	438	[115, 544]	578	[644, 645]
19	[38, 39]	159	[262, 22]	299	[420, 421]	439	[545, 34]	579	[287, 247]
20	[40, 9]	160	[263, 101]	300	[422, 54]	440	[546, 134]	580	[646, 647]
21	[41, 42]	161	[264, 26]	301	[377, 423]	441	[321, 547]	581	[463, 243]
22	[43, 44]	162	[265, 266]	302	[374, 424]	442	[320, 321]	582	[638, 483]
23	[45, 46]	163	[84, 30]	303	[425, 391]	443	[548, 547]	583	[648, 238]
24	[47, 48]	164	[80, 111]	304	[224, 202]	444	[549, 306]	584	[20, 641]
25	[49, 50]	165	[267, 268]	305	[426, 60]	445	[550, 332]	585	[638, 528]
26	[51, 52]	166	[269, 270]	306	[217, 427]	446	[551, 322]	586	[649, 465]
27	[13, 53]	167	[271, 272]	307	[210, 206]	447	[117, 552]	587	[513, 634]
28	[54, 55]	168	[273, 274]	308	[200, 208]	448	[118, 553]	588	[642, 459]
29	[56, 57]	169	[275, 260]	309	[428, 199]	449	[545, 330]	589	[235, 480]
30	[58, 59]	170	[99, 276]	310	[429, 64]	450	[554, 552]	590	[243, 75]
31	[15, 60]	171	[277, 278]	311	[430, 224]	451	[555, 332]	591	[488, 470]
32	[61, 62]	172	[85, 279]	312	[430, 201]	452	[556, 455]	592	[273, 471]
33	[63, 64]	173	[280, 281]	313	[431, 432]	453	[320, 548]	593	[650, 517]
34	[65, 66]	174	[282, 283]	314	[433, 210]	454	[557, 129]	594	[651, 93]
35	[67, 68]	175	[26, 247]	315	[204, 434]	455	[455, 455]	595	[652, 107]
36	[69, 70]	176	[97, 80]	316	[66, 435]	456	[118, 558]	596	[536, 492]
37	[71, 72]	177	[30, 252]	317	[63, 436]	457	[335, 31]	597	[653, 654]
38	[73, 74]	178	[254, 284]	318	[64, 226]	458	[559, 341]	598	[655, 461]
39	[75, 76]	179	[93, 246]	319	[437, 438]	459	[560, 422]	599	[301, 470]
40	[77, 78]	180	[269, 88]	320	[439, 440]	460	[194, 561]	600	[108, 249]
41	[79, 80]	181	[285, 286]	321	[441, 66]	461	[14, 562]	601	[656, 292]
42	[81, 82]	182	[287, 92]	322	[436, 213]	462	[563, 352]	602	[91, 657]
43	[83, 84]	183	[266, 288]	323	[442, 217]	463	[564, 561]	603	[658, 530]
44	[85, 86]	184	[289, 290]	324	[443, 444]	464	[565, 141]	604	[271, 512]
45	[87, 88]	185	[291, 292]	325	[445, 220]	465	[566, 362]	605	[537, 286]
46	[89, 84]	186	[293, 228]	326	[216, 446]	466	[39, 567]	606	[659, 266]
47	[90, 91]	187	[294, 20]	327	[447, 448]	467	[568, 569]	607	[23, 514]
48	[26, 92]	188	[230, 283]	328	[215, 449]	468	[570, 423]	608	[660, 657]
49	[93, 81]	189	[295, 296]	329	[450, 451]	469	[571, 572]	609	[661, 291]

50	[94, 95]	190	[258, 261]	330	[452, 453]	470	[573, 357]	610	[662, 268]
51	[96, 97]	191	[297, 284]	331	[446, 427]	471	[574, 377]	611	[663, 645]
52	[98, 99]	192	[245, 254]	332	[454, 455]	472	[575, 38]	612	[484, 639]
53	[100, 101]	193	[277, 298]	333	[456, 438]	473	[576, 577]	613	[268, 647]
54	[102, 103]	194	[299, 300]	334	[221, 435]	474	[578, 136]	614	[664, 488]
55	[104, 105]	195	[301, 231]	335	[60, 223]	475	[579, 580]	615	[286, 277]
56	[106, 107]	196	[99, 302]	336	[457, 62]	476	[581, 582]	616	[470, 239]
57	[80, 108]	197	[303, 108]	337	[208, 207]	477	[583, 572]	617	[665, 230]
58	[109, 110]	198	[304, 305]	338	[458, 459]	478	[141, 364]	618	[666, 271]
59	[111, 112]	199	[306, 307]	339	[460, 461]	479	[584, 141]	619	[667, 288]
60	[113, 114]	200	[308, 33]	340	[74, 462]	480	[585, 577]	620	[472, 632]
61	[115, 116]	201	[309, 310]	341	[463, 464]	481	[586, 347]	621	[668, 102]
62	[117, 118]	202	[311, 131]	342	[465, 241]	482	[144, 587]	622	[669, 638]
63	[119, 120]	203	[305, 312]	343	[20, 466]	483	[9, 342]	623	[533, 261]
64	[121, 119]	204	[313, 308]	344	[467, 74]	484	[585, 588]	624	[670, 635]
65	[122, 123]	205	[314, 315]	345	[468, 274]	485	[420, 159]	625	[552, 553]
66	[124, 125]	206	[35, 316]	346	[469, 78]	486	[589, 138]	626	[331, 323]
67	[126, 127]	207	[317, 37]	347	[470, 232]	487	[590, 409]	627	[671, 127]
68	[128, 129]	208	[135, 308]	348	[468, 471]	488	[591, 346]	628	[552, 558]
69	[130, 8]	209	[318, 114]	349	[472, 473]	489	[592, 412]	629	[330, 35]
70	[131, 132]	210	[319, 135]	350	[474, 475]	490	[593, 594]	630	[549, 123]
71	[133, 134]	211	[123, 320]	351	[107, 93]	491	[595, 192]	631	[603, 672]
72	[114, 135]	212	[307, 321]	352	[232, 240]	492	[578, 342]	632	[618, 354]
73	[9, 136]	213	[125, 322]	353	[476, 235]	493	[596, 383]	633	[673, 184]
74	[137, 138]	214	[35, 323]	354	[477, 468]	494	[406, 178]	634	[400, 419]
75	[139, 140]	215	[324, 325]	355	[478, 243]	495	[597, 395]	635	[53, 562]
76	[141, 142]	216	[326, 8]	356	[479, 480]	496	[153, 43]	636	[55, 603]
77	[143, 144]	217	[327, 121]	357	[244, 76]	497	[53, 189]	637	[674, 38]
78	[145, 146]	218	[328, 329]	358	[481, 104]	498	[598, 157]	638	[136, 343]
79	[147, 148]	219	[330, 331]	359	[482, 477]	499	[599, 140]	639	[598, 41]
80	[46, 149]	220	[332, 325]	360	[462, 483]	500	[350, 406]	640	[39, 675]
81	[150, 151]	221	[333, 120]	361	[484, 485]	501	[379, 143]	641	[592, 422]
82	[152, 153]	222	[126, 334]	362	[486, 466]	502	[158, 421]	642	[412, 404]
83	[154, 155]	223	[31, 335]	363	[487, 488]	503	[600, 52]	643	[576, 588]
84	[156, 154]	224	[336, 119]	364	[489, 490]	504	[559, 601]	644	[593, 58]
85	[157, 42]	225	[305, 132]	365	[285, 491]	505	[602, 44]	645	[674, 676]
86	[158, 159]	226	[337, 121]	366	[92, 248]	506	[603, 604]	646	[677, 357]
87	[160, 161]	227	[145, 338]	367	[94, 492]	507	[375, 605]	647	[375, 613]
88	[162, 163]	228	[339, 10]	368	[493, 279]	508	[606, 607]	648	[354, 672]
89	[164, 58]	229	[340, 341]	369	[286, 494]	509	[45, 608]	649	[358, 678]
90	[165, 166]	230	[342, 343]	370	[295, 253]	510	[609, 567]	650	[378, 186]
91	[167, 168]	231	[344, 345]	371	[255, 495]	511	[608, 149]	651	[594, 59]
92	[169, 170]	232	[346, 347]	372	[258, 98]	512	[414, 386]	652	[679, 12]
93	[12, 171]	233	[348, 349]	373	[496, 262]	513	[357, 184]	653	[339, 50]

94	[172, 53]	234	[350, 351]	374	[494, 298]	514	[610, 424]	654	[680, 681]
95	[173, 174]	235	[352, 353]	375	[497, 6]	515	[177, 385]	655	[682, 364]
96	[175, 46]	236	[55, 354]	376	[498, 300]	516	[611, 612]	656	[619, 171]
97	[176, 170]	237	[355, 145]	377	[499, 19]	517	[424, 613]	657	[2, 602]
98	[161, 177]	238	[142, 40]	378	[500, 281]	518	[614, 144]	658	[184, 678]
99	[178, 179]	239	[356, 187]	379	[501, 241]	519	[574, 570]	659	[389, 396]
100	[180, 181]	240	[357, 358]	380	[502, 253]	520	[152, 2]	660	[676, 39]
101	[165, 182]	241	[359, 360]	381	[494, 278]	521	[615, 388]	661	[609, 675]
102	[183, 177]	242	[361, 362]	382	[262, 245]	522	[579, 181]	662	[610, 375]
103	[184, 185]	243	[363, 353]	383	[503, 276]	523	[616, 186]	663	[683, 155]
104	[186, 143]	244	[364, 40]	384	[291, 504]	524	[617, 187]	664	[402, 344]
105	[187, 188]	245	[365, 366]	385	[87, 270]	525	[618, 603]	665	[422, 404]
106	[14, 189]	246	[191, 158]	386	[505, 506]	526	[11, 619]	666	[410, 414]
107	[190, 191]	247	[367, 148]	387	[507, 24]	527	[414, 344]	667	[681, 360]
108	[148, 192]	248	[52, 176]	388	[502, 296]	528	[568, 174]	668	[621, 349]
109	[193, 166]	249	[368, 369]	389	[508, 28]	529	[620, 359]	669	[684, 9]
110	[194, 195]	250	[370, 152]	390	[268, 509]	530	[621, 622]	670	[685, 582]
111	[196, 197]	251	[371, 191]	391	[493, 86]	531	[187, 612]	671	[440, 205]
112	[149, 163]	252	[154, 372]	392	[458, 510]	532	[424, 605]	672	[531, 524]
113	[198, 199]	253	[373, 158]	393	[511, 512]	533	[623, 398]	673	[686, 291]
114	[72, 200]	254	[42, 57]	394	[513, 234]	534	[597, 587]	674	[464, 75]
115	[201, 202]	255	[374, 375]	395	[488, 231]	535	[615, 624]	675	[631, 473]
116	[203, 201]	256	[376, 377]	396	[507, 514]	536	[366, 607]	676	[687, 506]
117	[204, 205]	257	[163, 161]	397	[515, 495]	537	[418, 401]	677	[511, 272]
118	[18, 206]	258	[351, 178]	398	[108, 112]	538	[203, 224]	678	[478, 464]
119	[207, 200]	259	[149, 160]	399	[250, 81]	539	[440, 434]	679	[491, 277]
120	[208, 72]	260	[378, 379]	400	[516, 517]	540	[429, 436]	680	[669, 462]
121	[209, 208]	261	[380, 381]	401	[515, 256]	541	[451, 625]	681	[688, 103]
122	[17, 210]	262	[382, 383]	402	[518, 468]	542	[436, 226]	682	[689, 654]
123	[211, 68]	263	[2, 43]	403	[282, 476]	543	[66, 222]	683	[690, 475]
124	[212, 70]	264	[52, 169]	404	[519, 477]	544	[201, 225]	684	[691, 465]
125	[213, 63]	265	[162, 160]	405	[491, 494]	545	[442, 446]	685	[656, 504]
126	[214, 215]	266	[384, 145]	406	[520, 110]	546	[626, 448]	686	[560, 412]
127	[216, 217]	267	[47, 385]	407	[521, 258]	547	[214, 451]	687	[692, 366]
128	[218, 219]	268	[386, 345]	408	[522, 262]	548	[68, 453]	688	[693, 381]
129	[220, 220]	269	[387, 388]	409	[523, 524]	549	[451, 449]	689	[694, 169]
130	[221, 222]	270	[389, 369]	410	[525, 505]	550	[445, 445]	690	[173, 569]
131	[16, 223]	271	[390, 391]	411	[28, 526]	551	[627, 432]	691	[680, 359]
132	[224, 225]	272	[392, 393]	412	[527, 290]	552	[628, 440]	692	[646, 509]
133	[71, 207]	273	[394, 346]	413	[462, 528]	553	[215, 625]	693	[670, 526]
134	[64, 213]	274	[144, 395]	414	[529, 530]	554	[450, 215]	694	[695, 535]
135	[226, 63]	275	[368, 396]	415	[531, 481]	555	[629, 219]	695	[376, 570]
136	[227, 228]	276	[177, 48]	416	[532, 97]	556	[630, 444]		
137	[229, 230]	277	[397, 398]	417	[503, 302]	557	[220, 445]		

138	[231, 232]	278	[192, 351]	418	[303, 111]	558	[433, 18]		
139	[233, 234]	279	[399, 41]	419	[533, 98]	559	[631, 632]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	100	1	200	1	300	0	400	1	500	0	600	0
1	0	101	1	201	1	301	1	401	1	501	0	601	1
2	0	102	0	202	0	302	0	402	0	502	1	602	1
3	0	103	1	203	1	303	1	403	0	503	0	603	1
4	0	104	0	204	0	304	0	404	0	504	1	604	1
5	0	105	1	205	1	305	1	405	0	505	1	605	1
6	0	106	1	206	1	306	0	406	0	506	1	606	1
7	0	107	0	207	1	307	0	407	1	507	1	607	0
8	0	108	0	208	0	308	1	408	1	508	1	608	0
9	0	109	1	209	0	309	1	409	0	509	0	609	1
10	0	110	1	210	0	310	0	410	0	510	1	610	0
11	0	111	1	211	0	311	0	411	0	511	0	611	1
12	1	112	0	212	0	312	0	412	0	512	0	612	1
13	0	113	0	213	1	313	0	413	0	513	0	613	1
14	1	114	0	214	1	314	1	414	0	514	0	614	0
15	0	115	1	215	1	315	0	415	1	515	1	615	1
16	1	116	1	216	1	316	0	416	1	516	1	616	0
17	0	117	0	217	0	317	1	417	0	517	1	617	0
18	1	118	1	218	1	318	0	418	1	518	0	618	0
19	0	119	1	219	0	319	0	419	1	519	0	619	1
20	0	120	0	220	1	320	0	420	1	520	0	620	0
21	0	121	0	221	1	321	1	421	1	521	1	621	1
22	1	122	0	222	1	322	1	422	0	522	1	622	0
23	0	123	0	223	0	323	0	423	0	523	0	623	1
24	1	124	0	224	0	324	1	424	1	524	0	624	1
25	1	125	1	225	1	325	0	425	1	525	0	625	1
26	0	126	1	226	0	326	1	426	1	526	0	626	1
27	0	127	1	227	1	327	0	427	1	527	0	627	1
28	0	128	1	228	1	328	1	428	1	528	1	628	1
29	1	129	1	229	1	329	0	429	0	529	0	629	0
30	1	130	1	230	1	330	0	430	0	530	1	630	0
31	0	131	1	231	1	331	1	431	0	531	1	631	1
32	1	132	0	232	1	332	1	432	1	532	1	632	0
33	1	133	1	233	1	333	1	433	1	533	1	633	0
34	0	134	0	234	0	334	1	434	0	534	1	634	1
35	1	135	0	235	1	335	0	435	0	535	1	635	1
36	1	136	1	236	0	336	0	436	1	536	1	636	0
37	1	137	1	237	0	337	0	437	0	537	1	637	0
38	0	138	1	238	0	338	1	438	1	538	1	638	1

39	1	139	1	239	0	339	1	439	0	539	1	639	0
40	0	140	1	240	0	340	1	440	1	540	0	640	1
41	0	141	0	241	1	341	1	441	1	541	0	641	0
42	1	142	0	242	1	342	1	442	0	542	1	642	0
43	1	143	0	243	1	343	0	443	1	543	0	643	1
44	0	144	1	244	0	344	1	444	0	544	1	644	0
45	0	145	1	245	0	345	1	445	0	545	0	645	0
46	0	146	1	246	0	346	1	446	1	546	1	646	0
47	1	147	0	247	0	347	0	447	0	547	1	647	1
48	0	148	0	248	0	348	1	448	1	548	1	648	1
49	1	149	0	249	1	349	0	449	0	549	0	649	1
50	0	150	1	250	0	350	0	450	0	550	0	650	0
51	0	151	0	251	0	351	0	451	0	551	1	651	1
52	0	152	0	252	1	352	1	452	0	552	1	652	0
53	1	153	0	253	0	353	1	453	0	553	1	653	1
54	0	154	1	254	1	354	1	454	1	554	0	654	0
55	0	155	0	255	0	355	0	455	0	555	0	655	0
56	1	156	0	256	0	356	0	456	1	556	0	656	1
57	0	157	0	257	0	357	0	457	0	557	1	657	0
58	1	158	1	258	0	358	1	458	1	558	1	658	1
59	1	159	1	259	0	359	1	459	0	559	1	659	1
60	0	160	0	260	0	360	0	460	1	560	0	660	0
61	1	161	0	261	1	361	1	461	1	561	1	661	1
62	0	162	0	262	1	362	1	462	0	562	0	662	0
63	1	163	0	263	0	363	1	463	1	563	0	663	1
64	0	164	0	264	0	364	0	464	0	564	1	664	0
65	0	165	1	265	0	365	0	465	1	565	0	665	0
66	0	166	1	266	0	366	1	466	1	566	1	666	0
67	1	167	1	267	1	367	0	467	1	567	1	667	1
68	1	168	0	268	1	368	1	468	1	568	1	668	1
69	1	169	1	269	1	369	1	469	1	569	0	669	0
70	1	170	1	270	1	370	0	470	0	570	1	670	1
71	1	171	1	271	1	371	0	471	0	571	1	671	1
72	0	172	0	272	1	372	0	472	0	572	0	672	1
73	0	173	1	273	0	373	0	473	1	573	0	673	0
74	1	174	0	274	1	374	0	474	0	574	0	674	0
75	1	175	0	275	1	375	1	475	1	575	0	675	1
76	0	176	1	276	1	376	0	476	1	576	1	676	0
77	0	177	1	277	1	377	1	477	1	577	0	677	0
78	1	178	1	278	0	378	0	478	0	578	0	678	0
79	0	179	1	279	0	379	0	479	0	579	1	679	0
80	0	180	1	280	1	380	1	480	1	580	0	680	0
81	1	181	0	281	0	381	0	481	1	581	1	681	1
82	0	182	1	282	0	382	1	482	1	582	1	682	0

83	1	183	0	283	0	383	0	483	0	583	1	683	1
84	0	184	1	284	1	384	0	484	1	584	0	684	0
85	0	185	0	285	0	385	0	485	1	585	1	685	1
86	1	186	0	286	1	386	1	486	1	586	1	686	0
87	0	187	1	287	1	387	1	487	1	587	0	687	0
88	0	188	1	288	0	388	1	488	0	588	0	688	1
89	0	189	0	289	1	389	1	489	0	589	1	689	0
90	1	190	0	290	0	390	1	490	0	590	1	690	1
91	1	191	0	291	0	391	1	491	0	591	0	691	0
92	1	192	0	292	0	392	1	492	0	592	0	692	0
93	1	193	1	293	0	393	0	493	1	593	0	693	1
94	0	194	1	294	1	394	0	494	0	594	1	694	0
95	1	195	1	295	0	395	0	495	1	595	0	695	0
96	0	196	1	296	1	396	1	496	0	596	1		
97	1	197	1	297	0	397	1	497	1	597	1		
98	0	198	0	298	1	398	0	498	0	598	0		
99	1	199	0	299	1	399	0	499	1	599	1		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn