PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^4+z,z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^4+z,z^2+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^4+z, z^2+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^6+z^3+z^2+z+1,\\
p_1(z)&=z^{18}+z^{14}+z^{13}+z^{12}+z^{10}+z^8+z^7+z^5+z^2+z,\\
p_2(z)&=z^{20}+z^{12}+z^6+z^2,\\
p_3(z)&=z^{18}+z^{14}+z^{13}+z^{12}+z^{10}+z^8+z^7+z^5+z^2+z,\\
p_4(z)&=z^{16}+z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^7+z^4+z^3+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{11}+z^{10}+z^9+z^7+z^6+z^3+z^2+z,\\
p_1(z)&=z^{18}+z^{14}+z^{13}+z^{12}+z^{10}+z^8+z^7+z^5+z^2+z,\\
p_2(z)&=z^{20}+z^{12}+z^6+z^2,\\
p_3(z)&=z^{18}+z^{14}+z^{13}+z^{12}+z^{10}+z^8+z^7+z^5+z^2+z,\\
p_4(z)&=z^{16}+z^{14}+z^{11}+z^{10}+z^9+z^7+z^4+z^3+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$
$$M^o = W^e \cdot M^e, \tag{2.4}$$
$$W^e = M^o \cdot W^o, \tag{2.5}$$
$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] = {} & M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u]\\
& + x^{2u}M^e[:6u] + x^{4u}M^e[:4u] + x^{5u}M^e[:3u] + x^{6u}M^e[:2u]\\
& + x^{7u}M^e[:u] + x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{6u}M^e[:3u]\\
& + x^{7u}M^e[:2u] + x^{8u}M^e[:u] + x^{5u}M^e[:6u] + x^{7u}M^e[:4u]\\
& + x^{8u}M^e[:3u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{6u}M^e[:6u]\\
& + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^e,
\end{aligned}$$

$$\begin{aligned} W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u] \\ &\quad + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u] \\ &\quad + x^{7u}W^e[:u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u] \\ &\quad + x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{5u}W^e[:6u] + x^{7u}W^e[:4u] \\ &\quad + x^{8u}W^e[:3u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{6u}W^e[:6u] \\ &\quad + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^e. \end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u] \\ &\quad + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u] \\ &\quad + x^{7u}M^o[:u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u] \\ &\quad + x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{5u}M^o[:6u] + x^{7u}M^o[:4u] \\ &\quad + x^{8u}M^o[:3u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{6u}M^o[:6u] \\ &\quad + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u] \\ &\quad + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u] \\ &\quad + x^{7u}W^o[:u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u] \\ &\quad + x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{5u}W^o[:6u] + x^{7u}W^o[:4u] \\ &\quad + x^{8u}W^o[:3u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{6u}W^o[:6u] \\ &\quad + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{5u}T[:u] + x^{2u}T[:6u] \\ &\quad + x^{4u}T[:4u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{3u}T[:6u] \\ &\quad + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{5u}T[:6u] \\ &\quad + x^{7u}T[:4u] + x^{8u}T[:3u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{6u}T[:6u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{6u}T[:u] + x^{5u}T[u:2u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] \\ &\quad + x^{2u}T[4u:5u] + T[6u:7u], \\ 0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] \\ &\quad + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u], \\ 0 &= x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] \\ &\quad + x^{5u}T[4u:5u] + T[9u:10u], \end{aligned}$$

$$\begin{aligned}0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] \\ &\quad + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] + T[10u:11u], \\ 0 &= x^{6u}T[5u:6u] + T[11u:12u],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[110w]_2], \quad (2.7)\\ 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[101w]_2] + T[[111w]_2], \quad (2.8)\\ 0 &= T[[101w]_2] + T[[1000w]_2], \quad (2.9)\\ 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[1001w]_2], \quad (2.10)\\ 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[101w]_2] + T[[1010w]_2], \quad (2.11)\\ 0 &= T[[101w]_2] + T[[1011w]_2], \quad (2.12)\end{aligned}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[110w]_2], \quad (2.13)\\ 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[101w]_2] + T[[111w]_2], \quad (2.14)\\ 0 &= T[[101w]_2] + T[[1000w]_2], \quad (2.15)\\ 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[1001w]_2], \quad (2.16)\\ 0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[101w]_2] + T[[1010w]_2], \quad (2.17)\\ 0 &= T[[101w]_2] + T[[1011w]_2], \quad (2.18)\end{aligned}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1166 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)), \quad (2.19)\\ 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,111)), \quad (2.20)\\ 0 &= \tau(A(s,101)) + \tau(A(s,1000)), \quad (2.21)\end{aligned}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1001)), \end{aligned} \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)), \end{aligned} \tag{2.25}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.27}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1001)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$0 = \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{46}), E_{46}) = (A(i,0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 23$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u] \\ &\quad + x^{6u}R^e, \\ W_{2k} &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u] \\ &\quad + x^{6u}R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u] \\ &\quad + x^{6u}R^o, \\ W_{2k+1} &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u] \\ &\quad + x^{6u}R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^{2v}W^e[:4v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{5v}W^e[:v] \\ &+ x^{6u}R^e\big) \times \big(M^e[:6v] + x^{2v}M^e[:4v] + x^{3v}M^e[:3v] + x^{4v}M^e[:2v] \\ &+ x^{5v}M^e[:v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{6v}W^e[:6v]M^e[:6v] \\ + x^{8v}W^e[:4v]M^e[:4v] + x^{10v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \leq 12$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v : (n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v : (n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j,k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{20} + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^6,\\
q_1(x) &= x^{19} + x^{18} + x^{15} + x^{13} + x^{12} + x^{10} + x^8 + x^7 + x^6 + x^2,\\
q_2(x) &= x^{18} + x^{14} + x^8 + 1,\\
q_3(x) &= x^{19} + x^{18} + x^{15} + x^{13} + x^{12} + x^{10} + x^8 + x^7 + x^6 + x^2,\\
q_4(x) &= x^{19} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^6 + x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{92} + x^{88} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{66} + x^{54}\\
&\quad + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{30} + x^{26} + x^{24},\\
p_3(x) &= x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{62}\\
&\quad + x^{60} + x^{54} + x^{44} + x^{42} + x^{38} + x^{36} + x^{22},\\
p_6(x) &= x^{98} + x^{96} + x^{90} + x^{78} + x^{76} + x^{72} + x^{56} + x^{52} + x^{48} + x^{46} + x^{44}\\
&\quad + x^{40} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{16} + x^{14} + x^{12} + x^{10} + x^6\\
&\quad + x^2 + 1,\\
p_9(x) &= x^{100} + x^{94} + x^{88} + x^{84} + x^{80} + x^{78} + x^{66} + x^{64} + x^{62} + x^{58} + x^{52}\\
&\quad + x^{50} + x^{46} + x^{44} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14}\\
&\quad + x^{12} + x^{10} + x^6 + x^4,\\
p_{12}(x) &= x^{100} + x^{98} + x^{96} + x^{68} + x^{66} + x^{64} + x^{52} + x^{50} + x^{48} + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} + x^{95} + x^{92} + x^{91}\\
&\quad + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{79} + x^{75} + x^{70} + x^{68} + x^{67}\\
&\quad + x^{66} + x^{65} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{46} + x^{45} + x^{43}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{42} + x^{40} + x^{35} + x^{34} + x^{30},\\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{64} + x^{63} + x^{61} + x^{58}\\
&+ x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{29} + x^{26} + x^{24}\\
&+ x^{23} + x^{21} + x^{18},\\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93}\\
&+ x^{90} + x^{89} + x^{86} + x^{85} + x^{82} + x^{79} + x^{76} + x^{75} + x^{72} + x^{69} + x^{67}\\
&+ x^{66} + x^{64} + x^{59} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{45} + x^{42} + x^{35}\\
&+ x^{32} + x^{25} + x^{23} + x^{22} + x^{20} + x^{15} + x^{12},\\
p_9(x) &= x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} + x^{86} + x^{85} + x^{83}\\
&+ x^{80} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57}\\
&+ x^{55} + x^{52} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42} + x^{39} + x^{37} + x^{33} + x^{32}\\
&+ x^{30} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9} + x^{6},\\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79}\\
&+ x^{76} + x^{75} + x^{72} + x^{67} + x^{64} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43} + x^{40}\\
&+ x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19}\\
&+ x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} + x^{95} + x^{92} + x^{91}\\
&+ x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{79} + x^{75} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{46} + x^{45} + x^{43}\\
&+ x^{42} + x^{40} + x^{35} + x^{34} + x^{30},\\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{64} + x^{63} + x^{61} + x^{58}\\
&+ x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{29} + x^{26} + x^{24}\\
&+ x^{23} + x^{21} + x^{18},\\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93}\\
&+ x^{90} + x^{89} + x^{86} + x^{85} + x^{82} + x^{79} + x^{76} + x^{75} + x^{72} + x^{69} + x^{67}\\
&+ x^{66} + x^{64} + x^{59} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{45} + x^{42} + x^{35}\\
&+ x^{32} + x^{25} + x^{23} + x^{22} + x^{20} + x^{15} + x^{12},\\
p_9(x) &= x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} + x^{86} + x^{85} + x^{83}\\
&+ x^{80} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{55}+x^{52}+x^{50}+x^{49}+x^{47}+x^{43}+x^{42}+x^{39}+x^{37}+x^{33}+x^{32}\\
&+x^{30}+x^{18}+x^{17}+x^{16}+x^{14}+x^{12}+x^{11}+x^{9}+x^{6},\\
p_{12}(x)&=x^{103}+x^{100}+x^{95}+x^{92}+x^{91}+x^{88}+x^{87}+x^{84}+x^{83}+x^{80}+x^{79}\\
&+x^{76}+x^{75}+x^{72}+x^{67}+x^{64}+x^{55}+x^{52}+x^{47}+x^{44}+x^{43}+x^{40}\\
&+x^{39}+x^{36}+x^{35}+x^{32}+x^{31}+x^{28}+x^{27}+x^{24}+x^{23}+x^{20}+x^{19}\\
&+x^{16}+x^{15}+x^{12}+x^{11}+x^{8}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{114}+x^{110}+x^{106}+x^{98}+x^{90}+x^{88}+x^{80}+x^{70}+x^{66}+x^{64}+x^{58}\\
&+x^{56}+x^{50}+x^{48}+x^{40}+x^{38}+x^{36},\\
p_3(x)&=x^{108}+x^{94}+x^{92}+x^{90}+x^{86}+x^{82}+x^{80}+x^{76}+x^{74}+x^{68}+x^{64}\\
&+x^{62}+x^{58}+x^{56}+x^{54}+x^{52}+x^{44}+x^{42}+x^{34}+x^{32}+x^{30}+x^{28}\\
&+x^{14}+x^{12},\\
p_6(x)&=x^{108}+x^{106}+x^{100}+x^{98}+x^{92}+x^{90}+x^{76}+x^{74}+x^{70}+x^{68}+x^{66}\\
&+x^{64}+x^{58}+x^{52}+x^{50}+x^{46}+x^{42}+x^{40}+x^{38}+x^{34}+x^{32}+x^{30}\\
&+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{12}+x^{6}+x^{4}+x^{2}+1,\\
p_9(x)&=x^{106}+x^{102}+x^{94}+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{78}+x^{72}\\
&+x^{70}+x^{64}+x^{60}+x^{54}+x^{48}+x^{44}+x^{42}+x^{40}+x^{34}+x^{28}+x^{20}\\
&+x^{16}+x^{14}+x^{10}+x^{8}+x^{6}+x^{4},\\
p_{12}(x)&=x^{106}+x^{104}+x^{90}+x^{88}+x^{82}+x^{80}+x^{66}+x^{64}+x^{58}+x^{56}+x^{42}\\
&+x^{40}+x^{34}+x^{32}+x^{26}+x^{24}+x^{18}+x^{16}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{114}+x^{108}+x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{88}+x^{86}+x^{84}+x^{80}\\
&+x^{76}+x^{70}+x^{62}+x^{60}+x^{58}+x^{56}+x^{54}+x^{48}+x^{46}+x^{42}+x^{40}\\
&+x^{34}+x^{30}+x^{28}+x^{26}+x^{24},\\
p_3(x)&=x^{108}+x^{104}+x^{96}+x^{92}+x^{86}+x^{80}+x^{74}+x^{70}+x^{62}+x^{58}+x^{52}\\
&+x^{50}+x^{48}+x^{44}+x^{40}+x^{38}+x^{36}+x^{32}+x^{26}+x^{20},\\
p_6(x)&=x^{108}+x^{106}+x^{104}+x^{102}+x^{94}+x^{92}+x^{86}+x^{84}+x^{82}+x^{74}+x^{64}\\
&+x^{62}+x^{60}+x^{54}+x^{52}+x^{48}+x^{46}+x^{42}+x^{40}+x^{36}+x^{32}+x^{30}\\
&+x^{28}+x^{26}+x^{22}+x^{18}+x^{16}+x^{14}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2}+1,\\
p_9(x)&=x^{106}+x^{102}+x^{94}+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{78}+x^{72}\\
&+x^{70}+x^{64}+x^{60}+x^{54}+x^{48}+x^{44}+x^{42}+x^{40}+x^{34}+x^{28}+x^{20}\\
&+x^{16}+x^{14}+x^{10}+x^{8}+x^{6}+x^{4},\\
p_{12}(x)&=x^{106}+x^{104}+x^{90}+x^{88}+x^{82}+x^{80}+x^{66}+x^{64}+x^{58}+x^{56}+x^{42}
\end{aligned}
$$

$$+ x^{40} + x^{34} + x^{32} + x^{26} + x^{24} + x^{18} + x^{16} + x^2 + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{51} + x^{48} + x^{44} + x^{41} + x^{40} + x^{36} + x^{32} + x^{30}, \\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94} \\
&\quad + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{64} + x^{63} + x^{61} + x^{58} \\
&\quad + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{29} + x^{26} + x^{24} \\
&\quad + x^{23} + x^{21} + x^{18}, \\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} \\
&\quad + x^{90} + x^{89} + x^{86} + x^{85} + x^{82} + x^{79} + x^{76} + x^{75} + x^{72} + x^{69} + x^{67} \\
&\quad + x^{66} + x^{64} + x^{59} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{45} + x^{42} + x^{35} \\
&\quad + x^{32} + x^{25} + x^{23} + x^{22} + x^{20} + x^{15} + x^{12}, \\
p_9(x) &= x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} + x^{86} + x^{85} + x^{83} \\
&\quad + x^{80} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} \\
&\quad + x^{55} + x^{52} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42} + x^{39} + x^{37} + x^{33} + x^{32} \\
&\quad + x^{30} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} \\
&\quad + x^{76} + x^{75} + x^{72} + x^{67} + x^{64} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43} + x^{40} \\
&\quad + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} \\
&\quad + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{51} + x^{48} + x^{44} + x^{41} + x^{40} + x^{36} + x^{32} + x^{30}, \\
p_3(x) &= x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94} \\
&\quad + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{64} + x^{63} + x^{61} + x^{58} \\
&\quad + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{31} + x^{29} + x^{26} + x^{24} \\
&\quad + x^{23} + x^{21} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} \\
&\quad + x^{90} + x^{89} + x^{86} + x^{85} + x^{82} + x^{79} + x^{76} + x^{75} + x^{72} + x^{69} + x^{67} \\
&\quad + x^{66} + x^{64} + x^{59} + x^{56} + x^{53} + x^{51} + x^{50} + x^{48} + x^{45} + x^{42} + x^{35} \\
&\quad + x^{32} + x^{25} + x^{23} + x^{22} + x^{20} + x^{15} + x^{12}, \\
p_9(x) &= x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{92} + x^{86} + x^{85} + x^{83} \\
&\quad + x^{80} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} \\
&\quad + x^{55} + x^{52} + x^{50} + x^{49} + x^{47} + x^{43} + x^{42} + x^{39} + x^{37} + x^{33} + x^{32} \\
&\quad + x^{30} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79} \\
&\quad + x^{76} + x^{75} + x^{72} + x^{67} + x^{64} + x^{55} + x^{52} + x^{47} + x^{44} + x^{43} + x^{40} \\
&\quad + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} \\
&\quad + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{92} + x^{88} + x^{80} + x^{74} + x^{72} + x^{68} + x^{64} + x^{58} + x^{54} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{98} + x^{90} + x^{86} + x^{84} + x^{76} + x^{72} + x^{64} + x^{60} + x^{58} + x^{56} + x^{54} \\
&\quad + x^{50} + x^{42} + x^{40} + x^{34} + x^{28} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_6(x) &= x^{92} + x^{88} + x^{64} + x^{54} + x^{50} + x^{44} + x^{38} + x^{36} + x^{34} + x^{30} + x^{28} \\
&\quad + x^{26} + x^{20} + x^{12} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{100} + x^{94} + x^{88} + x^{84} + x^{80} + x^{78} + x^{66} + x^{64} + x^{62} + x^{58} + x^{52} \\
&\quad + x^{50} + x^{46} + x^{44} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{12} + x^{10} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{100} + x^{98} + x^{96} + x^{68} + x^{66} + x^{64} + x^{52} + x^{50} + x^{48} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} \\
&\quad + x^{94} + x^{92} + x^{89} + x^{88} + x^{84} + x^{83} + x^{82} + x^{77} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{69} + x^{65} + x^{62} + x^{61} + x^{59} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{40} + x^{38} + x^{37} + x^{34} + x^{32} + x^{30} + x^{27} + x^{24}, \\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{95} + x^{93} + x^{88} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{65} + x^{61} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} \\
&\quad + x^{48} + x^{45} + x^{43} + x^{41} + x^{36} + x^{35} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{22} + x^{19} + x^{16},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{94} + x^{93} + x^{92}\\
&\quad + x^{90} + x^{89} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{70}\\
&\quad + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{52}\\
&\quad + x^{46} + x^{43} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{26} + x^{21}\\
&\quad + x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{3} + 1,\\
p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75}\\
&\quad + x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32}\\
&\quad + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8},\\
p_{12}(x) &= x^{107} + x^{104} + x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{88} + x^{79} + x^{76} + x^{67}\\
&\quad + x^{64} + x^{59} + x^{56} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28}\\
&\quad + x^{27} + x^{24} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102}\\
&\quad + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89}\\
&\quad + x^{88} + x^{85} + x^{82} + x^{81} + x^{75} + x^{72} + x^{67} + x^{63} + x^{62} + x^{61} + x^{58}\\
&\quad + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{47} + x^{42} + x^{41} + x^{38}\\
&\quad + x^{37} + x^{33} + x^{32} + x^{30},\\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{97} + x^{95} + x^{94} + x^{92}\\
&\quad + x^{90} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{60}\\
&\quad + x^{58} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31}\\
&\quad + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{108} + x^{100} + x^{98} + x^{94} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74}\\
&\quad + x^{68} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{38}\\
&\quad + x^{36} + x^{30} + x^{26} + x^{22} + x^{14} + x^{12},\\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{92}\\
&\quad + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{75}\\
&\quad + x^{74} + x^{72} + x^{70} + x^{61} + x^{60} + x^{57} + x^{55} + x^{54} + x^{52} + x^{50} + x^{49}\\
&\quad + x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{33} + x^{32}\\
&\quad + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13}\\
&\quad + x^{11} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{110} + x^{108} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66}\\
&\quad + x^{64} + x^{62} + x^{60} + x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{30} + x^{28}\\
&\quad + x^{26} + x^{24} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{96} + x^{93} + x^{92} \\
&\quad + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} + x^{76} + x^{72} \\
&\quad + x^{68} + x^{67} + x^{65} + x^{60} + x^{59} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{46} \\
&\quad + x^{41} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{105} + x^{104} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{90} + x^{73} + x^{72} + x^{69} \\
&\quad + x^{67} + x^{66} + x^{64} + x^{63} + x^{58} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} \\
&\quad + x^{31} + x^{29} + x^{27} + x^{24} + x^{23} + x^{18}, \\
p_6(x) &= x^{106} + x^{104} + x^{96} + x^{82} + x^{80} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{32} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{16} + x^{12}, \\
p_9(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{79} + x^{78} + x^{76} + x^{75} + x^{70} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{104} + x^{100} + x^{88} + x^{80} + x^{76} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} \\
&\quad + x^{40} + x^{32} + x^{24} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{103} + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} \\
&\quad + x^{88} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{67} + x^{66} \\
&\quad + x^{65} + x^{62} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{52} + x^{51} + x^{48} + x^{45} + x^{43} + x^{38} + x^{36} + x^{35} + x^{33} + x^{29} + x^{26} \\
&\quad + x^{24} + x^{23} + x^{22} + x^{20}, \\
p_6(x) &= x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{86} + x^{85} + x^{84} + x^{80} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{60} + x^{58} + x^{56} + x^{53} + x^{50} + x^{46} + x^{45} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{33} + x^{32} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{21} + x^{19} + x^{18} + x^{14} \\
&\quad + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}$$

$$
\begin{aligned}
p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} \\
&\quad + x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} \\
&\quad + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8}, \\
p_{12}(x) &= x^{107} + x^{104} + x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{88} + x^{79} + x^{76} + x^{67} \\
&\quad + x^{64} + x^{59} + x^{56} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} \\
&\quad + x^{27} + x^{24} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} \\
&\quad + x^{94} + x^{92} + x^{89} + x^{88} + x^{84} + x^{83} + x^{82} + x^{77} + x^{74} + x^{73} + x^{72} \\
&\quad + x^{69} + x^{65} + x^{62} + x^{61} + x^{59} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{40} + x^{38} + x^{37} + x^{34} + x^{32} + x^{30} + x^{27} + x^{24}, \\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{95} + x^{93} + x^{88} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{65} + x^{61} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} \\
&\quad + x^{48} + x^{45} + x^{43} + x^{41} + x^{36} + x^{35} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{22} + x^{19} + x^{16}, \\
p_6(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{94} + x^{93} + x^{92} \\
&\quad + x^{90} + x^{89} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{52} \\
&\quad + x^{46} + x^{43} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{26} + x^{21} \\
&\quad + x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{3} + 1, \\
p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} \\
&\quad + x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} \\
&\quad + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8}, \\
p_{12}(x) &= x^{107} + x^{104} + x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{88} + x^{79} + x^{76} + x^{67} \\
&\quad + x^{64} + x^{59} + x^{56} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} \\
&\quad + x^{27} + x^{24} + x^{15} + x^{12} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{96} + x^{93} + x^{92} \\
&\quad + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} + x^{76} + x^{72} \\
&\quad + x^{68} + x^{67} + x^{65} + x^{60} + x^{59} + x^{58} + x^{56} + x^{53} + x^{52} + x^{51} + x^{46} \\
&\quad + x^{41} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{105} + x^{104} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{90} + x^{73} + x^{72} + x^{69}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{67} + x^{66} + x^{64} + x^{63} + x^{58} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{31} + x^{29} + x^{27} + x^{24} + x^{23} + x^{18},\\
p_6(x) &= x^{106} + x^{104} + x^{96} + x^{82} + x^{80} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60}\\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{32} + x^{26} + x^{24}\\
&+ x^{22} + x^{20} + x^{16} + x^{12},\\
p_9(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95}\\
&+ x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{80}\\
&+ x^{79} + x^{78} + x^{76} + x^{75} + x^{70} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{52}\\
&+ x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38}\\
&+ x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{6},\\
p_{12}(x) &= x^{108} + x^{104} + x^{100} + x^{88} + x^{80} + x^{76} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52}\\
&+ x^{40} + x^{32} + x^{24} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{104} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89}\\
&+ x^{88} + x^{85} + x^{82} + x^{81} + x^{75} + x^{72} + x^{67} + x^{63} + x^{62} + x^{61} + x^{58}\\
&+ x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{47} + x^{42} + x^{41} + x^{38}\\
&+ x^{37} + x^{33} + x^{32} + x^{30},\\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{97} + x^{95} + x^{94} + x^{92}\\
&+ x^{90} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{65} + x^{63} + x^{62} + x^{60}\\
&+ x^{58} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31}\\
&+ x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{108} + x^{100} + x^{98} + x^{94} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74}\\
&+ x^{68} + x^{64} + x^{62} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42} + x^{38}\\
&+ x^{36} + x^{30} + x^{26} + x^{22} + x^{14} + x^{12},\\
p_9(x) &= x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{92}\\
&+ x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{77} + x^{75}\\
&+ x^{74} + x^{72} + x^{70} + x^{61} + x^{60} + x^{57} + x^{55} + x^{54} + x^{52} + x^{50} + x^{49}\\
&+ x^{48} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{33} + x^{32}\\
&+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13}\\
&+ x^{11} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{110} + x^{108} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66}\\
&+ x^{64} + x^{62} + x^{60} + x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{30} + x^{28}
\end{aligned}
$$

$$+ x^{26} + x^{24} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned} p_0(x) &= x^{108} + x^{106} + x^{104} + x^{103} + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} \\ &\quad + x^{88} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{67} + x^{66} \\ &\quad + x^{65} + x^{62} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\ &\quad + x^{47} + x^{45} + x^{40} + x^{39} + x^{38} + x^{36}, \\ p_3(x) &= x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} + x^{84} \\ &\quad + x^{83} + x^{81} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} \\ &\quad + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\ &\quad + x^{52} + x^{51} + x^{48} + x^{45} + x^{43} + x^{38} + x^{36} + x^{35} + x^{33} + x^{29} + x^{26} \\ &\quad + x^{24} + x^{23} + x^{22} + x^{20}, \\ p_6(x) &= x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{86} + x^{85} + x^{84} + x^{80} \\ &\quad + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} \\ &\quad + x^{61} + x^{60} + x^{58} + x^{56} + x^{53} + x^{50} + x^{46} + x^{45} + x^{42} + x^{40} + x^{38} \\ &\quad + x^{33} + x^{32} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{21} + x^{19} + x^{18} + x^{14} \\ &\quad + x^{11} + x^{8} + x^{3} + 1, \\ p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{87} + x^{84} + x^{83} + x^{80} + x^{75} \\ &\quad + x^{72} + x^{59} + x^{56} + x^{55} + x^{52} + x^{51} + x^{48} + x^{39} + x^{36} + x^{35} + x^{32} \\ &\quad + x^{23} + x^{20} + x^{19} + x^{16} + x^{11} + x^{8}, \\ p_{12}(x) &= x^{107} + x^{104} + x^{99} + x^{96} + x^{95} + x^{92} + x^{91} + x^{88} + x^{79} + x^{76} + x^{67} \\ &\quad + x^{64} + x^{59} + x^{56} + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{40} + x^{31} + x^{28} \\ &\quad + x^{27} + x^{24} + x^{15} + x^{12} + x^{3} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	292	[447, 448]	584	[757, 169]	876	[1008, 1009]
1	[3, 4]	293	[449, 450]	585	[412, 158]	877	[768, 687]
2	[5, 6]	294	[157, 190]	586	[758, 59]	878	[1010, 990]
3	[7, 8]	295	[375, 372]	587	[437, 132]	879	[1011, 1012]
4	[9, 10]	296	[413, 53]	588	[401, 37]	880	[1013, 461]
5	[11, 12]	297	[147, 56]	589	[759, 760]	881	[758, 118]
6	[13, 14]	298	[434, 51]	590	[695, 697]	882	[440, 396]
7	[15, 16]	299	[451, 435]	591	[761, 762]	883	[767, 749]
8	[17, 18]	300	[452, 453]	592	[763, 764]	884	[693, 769]
9	[19, 20]	301	[12, 424]	593	[765, 377]	885	[767, 698]

10	[21, 22]	302	[172, 198]	594	[148, 142]	886	[1014, 351]
11	[23, 24]	303	[454, 455]	595	[766, 767]	887	[1015, 444]
12	[25, 26]	304	[50, 446]	596	[768, 767]	888	[1016, 379]
13	[27, 28]	305	[389, 138]	597	[686, 767]	889	[1017, 782]
14	[29, 30]	306	[43, 9]	598	[769, 694]	890	[131, 43]
15	[31, 32]	307	[456, 179]	599	[770, 129]	891	[996, 142]
16	[32, 33]	308	[457, 163]	600	[53, 680]	892	[1018, 1009]
17	[34, 35]	309	[182, 130]	601	[55, 348]	893	[1019, 1020]
18	[36, 37]	310	[458, 155]	602	[771, 772]	894	[1021, 817]
19	[38, 39]	311	[179, 180]	603	[773, 774]	895	[1022, 117]
20	[40, 41]	312	[459, 149]	604	[775, 682]	896	[804, 441]
21	[42, 37]	313	[460, 122]	605	[776, 200]	897	[992, 698]
22	[43, 42]	314	[461, 344]	606	[465, 367]	898	[1023, 38]
23	[44, 45]	315	[462, 193]	607	[53, 705]	899	[397, 164]
24	[46, 47]	316	[356, 372]	608	[777, 403]	900	[1024, 1025]
25	[48, 49]	317	[463, 176]	609	[184, 185]	901	[1026, 146]
26	[50, 51]	318	[464, 453]	610	[386, 179]	902	[141, 149]
27	[52, 53]	319	[465, 169]	611	[36, 10]	903	[805, 53]
28	[54, 55]	320	[466, 424]	612	[778, 444]	904	[1027, 731]
29	[56, 57]	321	[467, 468]	613	[9, 37]	905	[34, 139]
30	[58, 59]	322	[177, 348]	614	[139, 399]	906	[387, 398]
31	[60, 61]	323	[374, 469]	615	[14, 779]	907	[754, 153]
32	[62, 63]	324	[470, 471]	616	[765, 146]	908	[819, 120]
33	[64, 65]	325	[472, 199]	617	[780, 135]	909	[1028, 691]
34	[66, 67]	326	[167, 473]	618	[781, 782]	910	[1029, 466]
35	[68, 69]	327	[143, 374]	619	[708, 177]	911	[1001, 760]
36	[70, 71]	328	[474, 160]	620	[783, 784]	912	[1030, 1004]
37	[72, 73]	329	[458, 41]	621	[785, 786]	913	[752, 764]
38	[74, 75]	330	[137, 390]	622	[787, 139]	914	[984, 789]
39	[76, 77]	331	[8, 180]	623	[788, 789]	915	[167, 779]
40	[78, 79]	332	[475, 433]	624	[790, 173]	916	[1031, 125]
41	[80, 81]	333	[476, 477]	625	[791, 682]	917	[675, 58]
42	[82, 83]	334	[478, 479]	626	[792, 191]	918	[1032, 171]
43	[84, 72]	335	[480, 152]	627	[151, 481]	919	[706, 677]
44	[85, 86]	336	[160, 446]	628	[793, 794]	920	[1033, 365]
45	[87, 88]	337	[442, 122]	629	[795, 366]	921	[1034, 685]
46	[89, 90]	338	[159, 434]	630	[776, 454]	922	[1003, 1035]
47	[91, 92]	339	[121, 432]	631	[428, 120]	923	[1011, 691]
48	[93, 94]	340	[188, 348]	632	[377, 147]	924	[1036, 797]
49	[95, 96]	341	[152, 118]	633	[796, 797]	925	[790, 198]
50	[97, 98]	342	[193, 481]	634	[466, 199]	926	[409, 468]
51	[99, 100]	343	[482, 483]	635	[798, 680]	927	[1037, 384]
52	[101, 102]	344	[268, 484]	636	[799, 800]	928	[1002, 694]
53	[103, 67]	345	[485, 486]	637	[801, 802]	929	[749, 686]

54	[104, 105]	346	[487, 488]	638	[356, 427]	930	[1038, 461]
55	[106, 107]	347	[489, 490]	639	[346, 187]	931	[1039, 987]
56	[108, 109]	348	[105, 491]	640	[803, 458]	932	[1029, 12]
57	[110, 111]	349	[492, 298]	641	[804, 396]	933	[1040, 351]
58	[112, 113]	350	[493, 494]	642	[805, 47]	934	[1016, 393]
59	[114, 115]	351	[495, 496]	643	[806, 354]	935	[444, 45]
60	[116, 117]	352	[497, 498]	644	[474, 434]	936	[1041, 454]
61	[58, 118]	353	[499, 500]	645	[400, 391]	937	[1000, 160]
62	[119, 120]	354	[501, 502]	646	[438, 181]	938	[1042, 2]
63	[121, 122]	355	[63, 503]	647	[704, 463]	939	[1043, 147]
64	[123, 124]	356	[504, 505]	648	[174, 158]	940	[144, 469]
65	[125, 126]	357	[506, 341]	649	[456, 8]	941	[1044, 146]
66	[127, 128]	358	[507, 289]	650	[807, 789]	942	[1045, 748]
67	[129, 130]	359	[508, 509]	651	[349, 677]	943	[1046, 1047]
68	[131, 132]	360	[510, 511]	652	[808, 683]	944	[1048, 794]
69	[133, 134]	361	[512, 329]	653	[177, 189]	945	[1015, 153]
70	[135, 130]	362	[277, 249]	654	[809, 49]	946	[378, 393]
71	[136, 130]	363	[513, 514]	655	[411, 58]	947	[1049, 135]
72	[137, 138]	364	[515, 516]	656	[347, 189]	948	[1050, 178]
73	[139, 140]	365	[517, 97]	657	[463, 124]	949	[1051, 1052]
74	[141, 142]	366	[518, 519]	658	[151, 194]	950	[1053, 1054]
75	[143, 144]	367	[520, 521]	659	[35, 140]	951	[791, 463]
76	[49, 145]	368	[26, 522]	660	[810, 705]	952	[424, 368]
77	[146, 147]	369	[258, 523]	661	[811, 812]	953	[1036, 196]
78	[148, 149]	370	[524, 525]	662	[813, 814]	954	[998, 350]
79	[150, 151]	371	[526, 527]	663	[815, 186]	955	[195, 797]
80	[152, 59]	372	[528, 529]	664	[728, 122]	956	[351, 369]
81	[153, 45]	373	[530, 531]	665	[816, 817]	957	[1055, 1056]
82	[154, 155]	374	[231, 532]	666	[140, 390]	958	[1057, 1058]
83	[128, 33]	375	[86, 76]	667	[11, 466]	959	[1033, 790]
84	[156, 139]	376	[77, 533]	668	[818, 467]	960	[1041, 200]
85	[157, 49]	377	[534, 108]	669	[819, 382]	961	[743, 396]
86	[56, 158]	378	[535, 536]	670	[451, 187]	962	[175, 124]
87	[46, 53]	379	[537, 538]	671	[146, 430]	963	[117, 344]
88	[159, 160]	380	[539, 540]	672	[820, 14]	964	[1026, 377]
89	[161, 162]	381	[541, 494]	673	[365, 173]	965	[1059, 1060]
90	[133, 163]	382	[542, 543]	674	[391, 737]	966	[1005, 375]
91	[32, 164]	383	[544, 304]	675	[821, 114]	967	[1061, 35]
92	[165, 166]	384	[235, 545]	676	[822, 638]	968	[140, 138]
93	[13, 167]	385	[546, 80]	677	[823, 208]	969	[1062, 1025]
94	[168, 169]	386	[547, 548]	678	[627, 108]	970	[1063, 194]
95	[170, 171]	387	[549, 103]	679	[824, 825]	971	[1040, 368]
96	[172, 173]	388	[550, 551]	680	[102, 826]	972	[423, 199]
97	[174, 57]	389	[552, 553]	681	[827, 828]	973	[729, 33]

98	[175, 176]	390	[69, 554]	682	[829, 830]	974	[991, 1064]
99	[54, 177]	391	[555, 232]	683	[540, 831]	975	[1065, 168]
100	[122, 178]	392	[556, 28]	684	[832, 330]	976	[727, 379]
101	[7, 179]	393	[557, 558]	685	[833, 834]	977	[123, 176]
102	[180, 181]	394	[559, 560]	686	[835, 836]	978	[153, 355]
103	[182, 183]	395	[561, 338]	687	[500, 837]	979	[199, 351]
104	[52, 47]	396	[562, 563]	688	[838, 597]	980	[363, 471]
105	[184, 56]	397	[564, 565]	689	[839, 840]	981	[368, 352]
106	[185, 57]	398	[566, 567]	690	[841, 217]	982	[1066, 415]
107	[186, 187]	399	[568, 70]	691	[651, 842]	983	[668, 320]
108	[188, 189]	400	[569, 570]	692	[843, 235]	984	[529, 504]
109	[190, 191]	401	[238, 571]	693	[596, 844]	985	[323, 648]
110	[192, 185]	402	[572, 573]	694	[694, 694]	986	[1067, 251]
111	[193, 194]	403	[574, 575]	695	[845, 846]	987	[861, 1068]
112	[195, 196]	404	[576, 577]	696	[847, 508]	988	[285, 823]
113	[197, 198]	405	[578, 339]	697	[837, 848]	989	[1069, 540]
114	[12, 199]	406	[573, 579]	698	[836, 849]	990	[1070, 665]
115	[200, 201]	407	[579, 580]	699	[850, 527]	991	[1071, 114]
116	[202, 79]	408	[581, 532]	700	[851, 852]	992	[511, 865]
117	[203, 204]	409	[295, 582]	701	[853, 672]	993	[314, 313]
118	[205, 206]	410	[338, 513]	702	[854, 520]	994	[1072, 606]
119	[207, 208]	411	[583, 584]	703	[341, 855]	995	[632, 296]
120	[209, 210]	412	[585, 586]	704	[856, 857]	996	[1073, 497]
121	[211, 212]	413	[587, 588]	705	[90, 858]	997	[75, 966]
122	[213, 214]	414	[589, 590]	706	[859, 585]	998	[1074, 340]
123	[215, 216]	415	[591, 592]	707	[860, 861]	999	[1075, 316]
124	[217, 218]	416	[593, 30]	708	[862, 863]	1000	[903, 1076]
125	[219, 220]	417	[594, 337]	709	[864, 865]	1001	[1077, 560]
126	[221, 222]	418	[595, 596]	710	[866, 867]	1002	[885, 877]
127	[223, 107]	419	[597, 598]	711	[868, 869]	1003	[1078, 580]
128	[224, 225]	420	[599, 250]	712	[870, 871]	1004	[1079, 1080]
129	[226, 27]	421	[600, 601]	713	[872, 873]	1005	[873, 528]
130	[109, 227]	422	[602, 603]	714	[849, 847]	1006	[1081, 221]
131	[228, 19]	423	[604, 495]	715	[874, 597]	1007	[1082, 64]
132	[229, 230]	424	[24, 258]	716	[875, 82]	1008	[867, 212]
133	[20, 231]	425	[605, 606]	717	[876, 877]	1009	[1083, 1084]
134	[232, 233]	426	[519, 284]	718	[878, 879]	1010	[1085, 509]
135	[234, 235]	427	[88, 607]	719	[880, 881]	1011	[1086, 857]
136	[236, 221]	428	[608, 609]	720	[882, 823]	1012	[301, 1087]
137	[237, 238]	429	[610, 611]	721	[883, 884]	1013	[1088, 551]
138	[239, 240]	430	[283, 272]	722	[598, 885]	1014	[225, 497]
139	[241, 242]	431	[612, 284]	723	[886, 205]	1015	[1089, 565]
140	[92, 213]	432	[553, 19]	724	[281, 87]	1016	[1090, 521]
141	[243, 244]	433	[613, 614]	725	[887, 317]	1017	[1091, 849]

142	[245, 246]	434	[483, 80]	726	[888, 255]	1018	[913, 305]
143	[247, 248]	435	[486, 615]	727	[889, 290]	1019	[1092, 510]
144	[249, 250]	436	[616, 76]	728	[890, 305]	1020	[1093, 1094]
145	[251, 252]	437	[617, 214]	729	[891, 87]	1021	[1095, 94]
146	[253, 254]	438	[310, 549]	730	[892, 836]	1022	[978, 642]
147	[255, 256]	439	[16, 91]	731	[893, 894]	1023	[1096, 505]
148	[257, 258]	440	[618, 619]	732	[895, 287]	1024	[894, 102]
149	[259, 260]	441	[620, 621]	733	[896, 327]	1025	[1097, 936]
150	[261, 262]	442	[622, 613]	734	[877, 897]	1026	[935, 283]
151	[71, 16]	443	[623, 624]	735	[509, 874]	1027	[1098, 595]
152	[263, 264]	444	[625, 504]	736	[898, 899]	1028	[1099, 828]
153	[265, 266]	445	[336, 626]	737	[250, 237]	1029	[1100, 868]
154	[267, 268]	446	[274, 627]	738	[900, 837]	1030	[1101, 929]
155	[98, 269]	447	[628, 239]	739	[901, 670]	1031	[1102, 27]
156	[270, 92]	448	[629, 630]	740	[902, 903]	1032	[111, 220]
157	[271, 251]	449	[631, 296]	741	[848, 510]	1033	[1103, 315]
158	[272, 273]	450	[30, 632]	742	[897, 845]	1034	[1104, 906]
159	[79, 274]	451	[633, 634]	743	[904, 297]	1035	[1105, 1106]
160	[204, 275]	452	[635, 304]	744	[905, 906]	1036	[1107, 1108]
161	[276, 277]	453	[636, 637]	745	[907, 490]	1037	[1109, 106]
162	[278, 279]	454	[638, 639]	746	[908, 533]	1038	[269, 268]
163	[4, 280]	455	[609, 291]	747	[909, 646]	1039	[1110, 972]
164	[61, 281]	456	[640, 309]	748	[910, 605]	1040	[565, 644]
165	[282, 283]	457	[571, 552]	749	[590, 500]	1041	[551, 609]
166	[284, 285]	458	[641, 642]	750	[911, 847]	1042	[840, 531]
167	[286, 287]	459	[643, 644]	751	[912, 913]	1043	[1111, 316]
168	[288, 286]	460	[645, 646]	752	[914, 320]	1044	[1112, 75]
169	[289, 290]	461	[647, 648]	753	[96, 915]	1045	[1113, 613]
170	[291, 106]	462	[649, 330]	754	[855, 225]	1046	[1114, 596]
171	[292, 293]	463	[650, 651]	755	[884, 835]	1047	[1115, 1104]
172	[294, 295]	464	[652, 653]	756	[916, 275]	1048	[1116, 673]
173	[296, 297]	465	[654, 526]	757	[917, 29]	1049	[1117, 109]
174	[298, 299]	466	[655, 489]	758	[918, 919]	1050	[899, 899]
175	[300, 301]	467	[656, 657]	759	[502, 262]	1051	[1094, 566]
176	[302, 303]	468	[619, 658]	760	[920, 886]	1052	[1118, 1119]
177	[304, 61]	469	[531, 659]	761	[921, 511]	1053	[1120, 868]
178	[305, 306]	470	[660, 254]	762	[922, 923]	1054	[871, 843]
179	[307, 308]	471	[661, 662]	763	[924, 925]	1055	[1121, 232]
180	[309, 310]	472	[663, 322]	764	[825, 926]	1056	[1122, 856]
181	[308, 311]	473	[28, 600]	765	[927, 255]	1057	[1123, 24]
182	[312, 313]	474	[664, 99]	766	[928, 595]	1058	[522, 1100]
183	[299, 314]	475	[214, 82]	767	[846, 573]	1059	[1124, 883]
184	[315, 110]	476	[665, 666]	768	[929, 590]	1060	[1125, 974]
185	[316, 317]	477	[667, 668]	769	[844, 883]	1061	[1126, 973]

186	[318, 319]	478	[669, 664]	770	[930, 299]	1062	[1080, 588]
187	[320, 321]	479	[670, 671]	771	[931, 230]	1063	[532, 309]
188	[322, 64]	480	[672, 673]	772	[932, 933]	1064	[488, 1127]
189	[222, 323]	481	[262, 674]	773	[934, 493]	1065	[1128, 289]
190	[324, 325]	482	[675, 152]	774	[582, 935]	1066	[1129, 848]
191	[94, 326]	483	[375, 427]	775	[936, 302]	1067	[807, 422]
192	[327, 328]	484	[430, 56]	776	[642, 937]	1068	[454, 201]
193	[83, 329]	485	[367, 14]	777	[938, 835]	1069	[1130, 753]
194	[330, 331]	486	[676, 677]	778	[939, 63]	1070	[1010, 751]
195	[332, 204]	487	[678, 364]	779	[521, 940]	1071	[999, 364]
196	[333, 334]	488	[676, 350]	780	[941, 586]	1072	[470, 364]
197	[335, 336]	489	[371, 427]	781	[942, 846]	1073	[1131, 718]
198	[337, 338]	490	[679, 145]	782	[943, 944]	1074	[723, 152]
199	[339, 244]	491	[47, 680]	783	[945, 639]	1075	[1132, 469]
200	[340, 341]	492	[681, 463]	784	[946, 104]	1076	[711, 375]
201	[208, 342]	493	[429, 374]	785	[865, 928]	1077	[994, 764]
202	[343, 344]	494	[126, 55]	786	[580, 929]	1078	[1018, 748]
203	[345, 346]	495	[185, 158]	787	[947, 568]	1079	[1046, 1020]
204	[347, 348]	496	[682, 176]	788	[948, 266]	1080	[763, 753]
205	[349, 350]	497	[192, 56]	789	[949, 950]	1081	[44, 355]
206	[351, 352]	498	[374, 683]	790	[951, 656]	1082	[1006, 125]
207	[353, 354]	499	[684, 685]	791	[952, 217]	1083	[1065, 366]
208	[355, 356]	500	[686, 687]	792	[953, 954]	1084	[818, 409]
209	[357, 358]	501	[688, 689]	793	[955, 205]	1085	[1042, 1133]
210	[359, 360]	502	[690, 691]	794	[634, 956]	1086	[788, 422]
211	[361, 8]	503	[147, 185]	795	[65, 104]	1087	[14, 473]
212	[362, 163]	504	[445, 51]	796	[523, 483]	1088	[1134, 126]
213	[38, 166]	505	[682, 124]	797	[957, 958]	1089	[1135, 45]
214	[135, 183]	506	[692, 125]	798	[548, 568]	1090	[1136, 1060]
215	[363, 364]	507	[119, 382]	799	[592, 279]	1091	[1137, 2]
216	[365, 198]	508	[693, 694]	800	[959, 960]	1092	[1045, 1009]
217	[366, 367]	509	[695, 686]	801	[961, 862]	1093	[1019, 1047]
218	[368, 369]	510	[696, 697]	802	[962, 963]	1094	[690, 1012]
219	[370, 49]	511	[698, 697]	803	[964, 98]	1095	[452, 171]
220	[371, 372]	512	[699, 165]	804	[965, 966]	1096	[1022, 461]
221	[373, 374]	513	[122, 433]	805	[967, 968]	1097	[1138, 998]
222	[355, 375]	514	[434, 446]	806	[969, 508]	1098	[1139, 1133]
223	[376, 377]	515	[700, 701]	807	[970, 86]	1099	[464, 171]
224	[378, 379]	516	[702, 703]	808	[67, 570]	1100	[461, 159]
225	[380, 374]	517	[425, 346]	809	[971, 972]	1101	[1140, 1025]
226	[381, 382]	518	[704, 682]	810	[329, 973]	1102	[1141, 430]
227	[383, 384]	519	[426, 372]	811	[974, 72]	1103	[1134, 384]
228	[385, 40]	520	[373, 144]	812	[975, 93]	1104	[1048, 1064]
229	[386, 8]	521	[344, 160]	813	[976, 315]	1105	[688, 762]

230	[387, 181]	522	[49, 191]	814	[977, 978]	1106	[1021, 987]
231	[388, 155]	523	[47, 705]	815	[979, 925]	1107	[1142, 481]
232	[389, 390]	524	[350, 454]	816	[980, 525]	1108	[160, 51]
233	[391, 362]	525	[706, 350]	817	[113, 981]	1109	[983, 186]
234	[392, 393]	526	[55, 189]	818	[533, 619]	1110	[985, 196]
235	[394, 193]	527	[707, 118]	819	[982, 937]	1111	[775, 463]
236	[395, 396]	528	[708, 55]	820	[313, 536]	1112	[1143, 159]
237	[129, 183]	529	[144, 683]	821	[677, 200]	1113	[793, 1064]
238	[397, 33]	530	[156, 35]	822	[983, 346]	1114	[1062, 1144]
239	[180, 398]	531	[457, 134]	823	[150, 193]	1115	[761, 689]
240	[35, 399]	532	[40, 155]	824	[984, 422]	1116	[1032, 453]
241	[400, 162]	533	[462, 151]	825	[366, 169]	1117	[692, 383]
242	[401, 10]	534	[48, 190]	826	[179, 439]	1118	[1138, 706]
243	[402, 403]	535	[709, 710]	827	[169, 167]	1119	[1014, 368]
244	[384, 177]	536	[711, 356]	828	[472, 424]	1120	[459, 142]
245	[404, 405]	537	[712, 713]	829	[985, 797]	1121	[1145, 691]
246	[406, 407]	538	[714, 715]	830	[757, 367]	1122	[988, 998]
247	[408, 162]	539	[716, 132]	831	[132, 42]	1123	[395, 441]
248	[399, 390]	540	[439, 398]	832	[986, 987]	1124	[1034, 760]
249	[165, 39]	541	[717, 718]	833	[988, 706]	1125	[1030, 1035]
250	[128, 164]	542	[719, 720]	834	[820, 167]	1126	[1146, 40]
251	[197, 173]	543	[721, 722]	835	[755, 698]	1127	[200, 455]
252	[409, 410]	544	[723, 58]	836	[769, 769]	1128	[1043, 430]
253	[411, 152]	545	[126, 177]	837	[697, 767]	1129	[1140, 1144]
254	[412, 57]	546	[724, 153]	838	[989, 2]	1130	[1147, 486]
255	[413, 47]	547	[725, 129]	839	[750, 990]	1131	[1148, 694]
256	[384, 55]	548	[726, 166]	840	[991, 794]	1132	[973, 277]
257	[414, 415]	549	[726, 39]	841	[796, 196]	1133	[1149, 971]
258	[45, 375]	550	[727, 393]	842	[467, 410]	1134	[1150, 653]
259	[416, 417]	551	[728, 432]	843	[383, 126]	1135	[1151, 601]
260	[418, 419]	552	[729, 164]	844	[696, 686]	1136	[1152, 844]
261	[420, 391]	553	[154, 41]	845	[687, 749]	1137	[879, 248]
262	[42, 10]	554	[132, 9]	846	[992, 749]	1138	[1153, 903]
263	[421, 422]	555	[161, 391]	847	[766, 687]	1139	[1154, 611]
264	[423, 424]	556	[730, 731]	848	[687, 698]	1140	[1155, 968]
265	[425, 186]	557	[732, 733]	849	[698, 686]	1141	[1156, 514]
266	[426, 427]	558	[734, 735]	850	[993, 383]	1142	[570, 548]
267	[428, 382]	559	[736, 43]	851	[994, 753]	1143	[1157, 1108]
268	[429, 144]	560	[737, 134]	852	[399, 138]	1144	[1158, 1159]
269	[377, 430]	561	[738, 710]	853	[995, 365]	1145	[1160, 302]
270	[431, 38]	562	[739, 740]	854	[996, 149]	1146	[1161, 626]
271	[198, 409]	563	[741, 742]	855	[125, 384]	1147	[170, 453]
272	[45, 356]	564	[743, 441]	856	[997, 409]	1148	[989, 1133]
273	[432, 433]	565	[744, 193]	857	[998, 677]	1149	[795, 168]

274	[344, 434]	566	[439, 181]	858	[162, 362]	1150	[681, 682]
275	[346, 435]	567	[162, 737]	859	[370, 190]	1151	[480, 58]
276	[436, 165]	568	[388, 41]	860	[999, 471]	1152	[1162, 1144]
277	[136, 183]	569	[745, 32]	861	[168, 367]	1153	[1143, 344]
278	[437, 43]	570	[746, 39]	862	[460, 432]	1154	[1039, 817]
279	[438, 398]	571	[746, 166]	863	[1000, 434]	1155	[1163, 794]
280	[8, 439]	572	[747, 748]	864	[1001, 685]	1156	[815, 346]
281	[117, 159]	573	[749, 697]	865	[1002, 769]	1157	[809, 190]
282	[440, 441]	574	[750, 751]	866	[1003, 1004]	1158	[995, 790]
283	[442, 432]	575	[752, 753]	867	[816, 987]	1159	[997, 467]
284	[443, 187]	576	[754, 444]	868	[744, 151]	1160	[678, 471]
285	[444, 355]	577	[707, 59]	869	[1005, 356]	1161	[1013, 117]
286	[445, 446]	578	[392, 379]	870	[1006, 383]	1162	[1164, 90]
287	[186, 435]	579	[755, 749]	871	[443, 435]	1163	[1165, 584]
288	[345, 186]	580	[697, 687]	872	[381, 120]	1164	[1145, 1012]
289	[394, 151]	581	[756, 458]	873	[380, 144]	1165	[421, 789]
290	[430, 185]	582	[190, 145]	874	[694, 769]		
291	[432, 178]	583	[173, 467]	875	[1007, 128]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	195	0	390	0	585	1	780	1	975	0
1	0	196	1	391	0	586	1	781	0	976	1
2	0	197	1	392	0	587	1	782	1	977	1
3	0	198	0	393	1	588	1	783	1	978	1
4	1	199	1	394	1	589	0	784	1	979	1
5	0	200	0	395	0	590	0	785	1	980	1
6	0	201	1	396	0	591	1	786	1	981	0
7	0	202	0	397	1	592	1	787	0	982	1
8	1	203	0	398	0	593	0	788	1	983	0
9	1	204	1	399	1	594	0	789	0	984	0
10	0	205	1	400	1	595	1	790	0	985	1
11	0	206	1	401	1	596	0	791	0	986	0
12	0	207	0	402	1	597	0	792	1	987	0
13	0	208	1	403	1	598	1	793	0	988	0
14	0	209	0	404	1	599	0	794	1	989	0
15	0	210	0	405	0	600	1	795	1	990	0
16	0	211	1	406	0	601	1	796	0	991	0
17	1	212	0	407	0	602	1	797	0	992	1
18	0	213	1	408	1	603	0	798	0	993	1
19	1	214	0	409	0	604	0	799	1	994	1
20	0	215	1	410	0	605	1	800	1	995	0
21	0	216	1	411	1	606	1	801	1	996	0
22	0	217	1	412	1	607	1	802	0	997	1

23	0	218	0	413	1	608	0	803	0	998	0
24	0	219	1	414	0	609	0	804	1	999	0
25	0	220	1	415	1	610	1	805	1	1000	1
26	0	221	0	416	0	611	0	806	1	1001	1
27	0	222	1	417	0	612	0	807	0	1002	1
28	0	223	1	418	1	613	1	808	1	1003	0
29	0	224	1	419	0	614	1	809	1	1004	1
30	0	225	0	420	0	615	0	810	1	1005	0
31	0	226	1	421	1	616	0	811	0	1006	0
32	0	227	0	422	1	617	1	812	0	1007	0
33	1	228	0	423	0	618	0	813	1	1008	1
34	1	229	1	424	0	619	1	814	1	1009	0
35	0	230	0	425	1	620	1	815	1	1010	0
36	0	231	1	426	0	621	1	816	1	1011	1
37	1	232	0	427	0	622	0	817	1	1012	0
38	1	233	0	428	0	623	1	818	0	1013	1
39	0	234	0	429	1	624	0	819	1	1014	0
40	0	235	1	430	0	625	0	820	1	1015	1
41	1	236	0	431	0	626	1	821	0	1016	0
42	0	237	1	432	0	627	0	822	0	1017	1
43	0	238	1	433	1	628	0	823	0	1018	0
44	0	239	1	434	0	629	1	824	0	1019	0
45	0	240	0	435	0	630	1	825	1	1020	0
46	0	241	1	436	0	631	0	826	0	1021	0
47	0	242	1	437	1	632	0	827	1	1022	1
48	0	243	1	438	1	633	0	828	1	1023	1
49	0	244	1	439	0	634	1	829	1	1024	0
50	0	245	1	440	0	635	0	830	0	1025	1
51	0	246	0	441	1	636	1	831	1	1026	0
52	0	247	1	442	0	637	1	832	0	1027	1
53	1	248	1	443	1	638	1	833	0	1028	1
54	0	249	0	444	0	639	0	834	1	1029	1
55	1	250	1	445	1	640	0	835	0	1030	1
56	0	251	1	446	1	641	1	836	1	1031	1
57	1	252	0	447	0	642	1	837	1	1032	1
58	0	253	1	448	1	643	1	838	0	1033	1
59	0	254	1	449	0	644	0	839	1	1034	1
60	0	255	1	450	0	645	1	840	0	1035	0
61	0	256	1	451	0	646	1	841	0	1036	1
62	0	257	0	452	0	647	1	842	1	1037	0
63	1	258	0	453	1	648	0	843	0	1038	0
64	1	259	0	454	1	649	0	844	1	1039	1
65	1	260	1	455	0	650	0	845	0	1040	1
66	1	261	0	456	0	651	1	846	1	1041	0

67	1	262	0	457	1	652	1	847	1	1042	0
68	0	263	1	458	1	653	0	848	0	1043	0
69	0	264	0	459	1	654	1	849	1	1044	1
70	0	265	1	460	1	655	1	850	1	1045	0
71	0	266	0	461	1	656	1	851	1	1046	1
72	1	267	0	462	0	657	0	852	1	1047	1
73	1	268	1	463	0	658	0	853	0	1048	1
74	1	269	0	464	1	659	0	854	0	1049	0
75	1	270	0	465	1	660	1	855	1	1050	1
76	0	271	0	466	1	661	0	856	1	1051	0
77	1	272	0	467	1	662	1	857	0	1052	1
78	0	273	0	468	1	663	1	858	1	1053	1
79	0	274	1	469	1	664	0	859	1	1054	1
80	1	275	0	470	1	665	1	860	0	1055	0
81	1	276	0	471	0	666	0	861	0	1056	0
82	0	277	0	472	1	667	0	862	1	1057	0
83	1	278	1	473	0	668	0	863	1	1058	0
84	0	279	1	474	0	669	1	864	1	1059	1
85	0	280	1	475	0	670	0	865	1	1060	1
86	0	281	0	476	1	671	1	866	0	1061	1
87	0	282	0	477	0	672	1	867	1	1062	1
88	0	283	0	478	1	673	1	868	1	1063	0
89	0	284	1	479	0	674	0	869	0	1064	0
90	0	285	0	480	1	675	0	870	0	1065	0
91	0	286	1	481	0	676	0	871	1	1066	1
92	0	287	1	482	0	677	0	872	1	1067	0
93	0	288	0	483	0	678	0	873	0	1068	1
94	0	289	1	484	0	679	0	874	0	1069	0
95	0	290	0	485	0	680	1	875	0	1070	0
96	0	291	0	486	0	681	1	876	1	1071	0
97	0	292	0	487	0	682	1	877	0	1072	1
98	0	293	0	488	0	683	0	878	0	1073	0
99	0	294	0	489	1	684	0	879	1	1074	0
100	1	295	0	490	0	685	0	880	1	1075	0
101	0	296	1	491	0	686	0	881	1	1076	1
102	1	297	1	492	1	687	0	882	0	1077	1
103	1	298	0	493	1	688	0	883	1	1078	0
104	0	299	0	494	0	689	1	884	0	1079	1
105	0	300	0	495	1	690	0	885	1	1080	1
106	1	301	0	496	1	691	1	886	0	1081	0
107	1	302	0	497	1	692	0	887	1	1082	0
108	0	303	1	498	1	693	0	888	0	1083	0
109	1	304	0	499	0	694	0	889	1	1084	0
110	1	305	0	500	0	695	0	890	0	1085	0

111	1	306	0	501	0	696	1	891	0	1086	1
112	0	307	0	502	0	697	1	892	0	1087	0
113	1	308	1	503	1	698	1	893	0	1088	1
114	0	309	1	504	1	699	1	894	0	1089	1
115	0	310	1	505	1	700	1	895	1	1090	0
116	0	311	0	506	0	701	0	896	1	1091	1
117	0	312	1	507	0	702	0	897	1	1092	0
118	1	313	1	508	0	703	1	898	1	1093	0
119	0	314	1	509	0	704	1	899	1	1094	0
120	0	315	0	510	1	705	0	900	0	1095	0
121	1	316	1	511	1	706	1	901	0	1096	1
122	1	317	0	512	1	707	0	902	1	1097	1
123	1	318	1	513	1	708	1	903	1	1098	1
124	1	319	1	514	0	709	1	904	1	1099	1
125	1	320	1	515	1	710	0	905	1	1100	1
126	0	321	1	516	0	711	1	906	0	1101	1
127	1	322	0	517	1	712	0	907	1	1102	1
128	1	323	1	518	1	713	1	908	1	1103	1
129	1	324	1	519	0	714	1	909	1	1104	1
130	1	325	1	520	0	715	0	910	1	1105	0
131	0	326	1	521	1	716	0	911	1	1106	0
132	1	327	1	522	0	717	1	912	1	1107	1
133	0	328	0	523	0	718	0	913	0	1108	1
134	0	329	1	524	1	719	1	914	0	1109	0
135	0	330	1	525	1	720	0	915	1	1110	1
136	0	331	1	526	1	721	1	916	1	1111	0
137	1	332	0	527	0	722	1	917	0	1112	1
138	1	333	1	528	1	723	0	918	1	1113	0
139	1	334	1	529	0	724	0	919	1	1114	1
140	0	335	1	530	0	725	1	920	1	1115	1
141	1	336	1	531	1	726	0	921	1	1116	1
142	1	337	0	532	0	727	1	922	0	1117	0
143	1	338	0	533	0	728	0	923	1	1118	1
144	0	339	1	534	0	729	0	924	1	1119	0
145	1	340	0	535	1	730	0	925	0	1120	1
146	1	341	1	536	1	731	0	926	0	1121	0
147	1	342	1	537	0	732	1	927	0	1122	0
148	0	343	0	538	1	733	1	928	1	1123	0
149	0	344	1	539	0	734	0	929	0	1124	1
150	0	345	0	540	0	735	0	930	0	1125	1
151	0	346	0	541	1	736	1	931	1	1126	1
152	1	347	1	542	1	737	1	932	1	1127	0
153	1	348	0	543	1	738	0	933	1	1128	0
154	0	349	1	544	0	739	0	934	0	1129	1

155	0	350	1	545	0	740	1	935	0	1130	0
156	0	351	1	546	0	741	0	936	0	1131	0
157	0	352	1	547	1	742	1	937	1	1132	0
158	0	353	0	548	0	743	1	938	0	1133	1
159	0	354	0	549	0	744	1	939	0	1134	1
160	1	355	1	550	1	745	1	940	0	1135	1
161	0	356	1	551	0	746	1	941	1	1136	0
162	1	357	0	552	0	747	1	942	0	1137	1
163	1	358	0	553	0	748	1	943	1	1138	1
164	0	359	0	554	1	749	0	944	1	1139	1
165	0	360	1	555	0	750	1	945	1	1140	1
166	1	361	1	556	0	751	1	946	1	1141	1
167	1	362	0	557	1	752	0	947	0	1142	1
168	0	363	1	558	0	753	0	948	1	1143	1
169	1	364	1	559	1	754	1	949	0	1144	0
170	0	365	1	560	1	755	0	950	1	1145	0
171	0	366	1	561	0	756	1	951	0	1146	1
172	0	367	0	562	0	757	0	952	0	1147	0
173	1	368	0	563	0	758	1	953	1	1148	0
174	0	369	0	564	1	759	0	954	0	1149	1
175	0	370	1	565	1	760	1	955	0	1150	1
176	0	371	1	566	0	761	1	956	1	1151	1
177	0	372	1	567	1	762	0	957	0	1152	0
178	0	373	0	568	1	763	1	958	0	1153	1
179	0	374	1	569	1	764	1	959	1	1154	1
180	1	375	0	570	1	765	0	960	0	1155	1
181	1	376	1	571	1	766	1	961	1	1156	1
182	1	377	0	572	1	767	1	962	0	1157	1
183	0	378	1	573	0	768	0	963	0	1158	0
184	0	379	0	574	1	769	1	964	0	1159	1
185	1	380	0	575	0	770	0	965	1	1160	0
186	1	381	1	576	1	771	1	966	0	1161	1
187	1	382	1	577	0	772	1	967	1	1162	0
188	0	383	0	578	0	773	0	968	0	1163	1
189	1	384	1	579	0	774	1	969	1	1164	0
190	1	385	0	580	1	775	0	970	0	1165	1
191	0	386	1	581	1	776	1	971	1		
192	1	387	0	582	1	777	0	972	0		
193	1	388	1	583	1	778	0	973	0		
194	1	389	0	584	0	779	1	974	0		

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