PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^3+z,z^3+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^3+z,z^3+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^3+z,z^3+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{12}+z^{11}+z^9+z^8+z^6+z^5+z^3+z^2+z+1,\\
p_1(z)&=z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^3+z^2+z,\\
p_2(z)&=z^{20}+z^{16}+z^{10}+z^2,\\
p_3(z)&=z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^3+z^2+z,\\
p_4(z)&=z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^6+z^5+z^4+z^3+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{12}+z^{11}+z^9+z^8+z^6+z^5+z^3+z^2+z,\\
p_1(z)&=z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^3+z^2+z,\\
p_2(z)&=z^{20}+z^{16}+z^{10}+z^2,\\
p_3(z)&=z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^3+z^2+z,\\
p_4(z)&=z^{14}+z^{12}+z^{11}+z^{10}+z^9+z^6+z^5+z^4+z^3+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 24 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^{2u}M^e[:6u] \\ & + x^{4u}M^e[:4u] + x^{5u}M^e[:3u] + x^{7u}M^e[:u] + x^{3u}M^e[:6u] \\ & + x^{5u}M^e[:4u] + x^{6u}M^e[:3u] + x^{8u}M^e[:u] + x^{4u}M^e[:6u] \\ & + x^{6u}M^e[:4u] + x^{7u}M^e[:3u] + x^{9u}M^e[:u] + x^{5u}M^e[:6u] \\ & + x^{7u}M^e[:4u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] + x^{7u}M^e[:5u] \\ & + x^{9u}M^e[:3u] + (x^{6u}+x^{8u}+x^{9u}+x^{10u}+x^{11u})R^e, \end{aligned}$$

$$\begin{aligned}W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{2u}W^e[:6u]\\ &\quad + x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{7u}W^e[:u] + x^{3u}W^e[:6u]\\ &\quad + x^{5u}W^e[:4u] + x^{6u}W^e[:3u] + x^{8u}W^e[:u] + x^{4u}W^e[:6u]\\ &\quad + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{9u}W^e[:u] + x^{5u}W^e[:6u]\\ &\quad + x^{7u}W^e[:4u] + x^{8u}W^e[:3u] + x^{10u}W^e[:u] + x^{7u}W^e[:5u]\\ &\quad + x^{9u}W^e[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^e.\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{2u}M^o[:6u]\\ &\quad + x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{7u}M^o[:u] + x^{3u}M^o[:6u]\\ &\quad + x^{5u}M^o[:4u] + x^{6u}M^o[:3u] + x^{8u}M^o[:u] + x^{4u}M^o[:6u]\\ &\quad + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{9u}M^o[:u] + x^{5u}M^o[:6u]\\ &\quad + x^{7u}M^o[:4u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u] + x^{7u}M^o[:5u]\\ &\quad + x^{9u}M^o[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o,\\ W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{2u}W^o[:6u]\\ &\quad + x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{7u}W^o[:u] + x^{3u}W^o[:6u]\\ &\quad + x^{5u}W^o[:4u] + x^{6u}W^o[:3u] + x^{8u}W^o[:u] + x^{4u}W^o[:6u]\\ &\quad + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{9u}W^o[:u] + x^{5u}W^o[:6u]\\ &\quad + x^{7u}W^o[:4u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u] + x^{7u}W^o[:5u]\\ &\quad + x^{9u}W^o[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o.\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{5u}T[:u] + x^{2u}T[:6u] + x^{4u}T[:4u]\\ &\quad + x^{5u}T[:3u] + x^{7u}T[:u] + x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u]\\ &\quad + x^{8u}T[:u] + x^{4u}T[:6u] + x^{6u}T[:4u] + x^{7u}T[:3u] + x^{9u}T[:u]\\ &\quad + x^{5u}T[:6u] + x^{7u}T[:4u] + x^{8u}T[:3u] + x^{10u}T[:u] + x^{7u}T[:5u]\\ &\quad + x^{9u}T[:3u].\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}0 &= x^{5u}T[u:2u] + x^{3u}T[3u:4u] + x^{2u}T[4u:5u] + T[6u:7u],\\ 0 &= x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u],\\ 0 &= x^{7u}T[u:2u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u],\\ 0 &= x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u]\\ &\quad + x^{4u}T[5u:6u] + T[9u:10u],\\ 0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{5u}T[5u:6u]\end{aligned}$$

$$+ T[10u:11u],$$
$$0 = x^{9u}T[2u:3u] + x^{7u}T[4u:5u] + T[11u:12u],$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$\begin{aligned} 0 = &T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\ &+ T[[1001w]_2], \end{aligned} \tag{2.10}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$\begin{aligned} 0 = &T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\ &+ T[[1001w]_2], \end{aligned} \tag{2.16}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1243 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.19}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.20}$$
$$0 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.21}$$
$$\begin{aligned} 0 = &\tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &+ \tau(A(s,101)) + \tau(A(s,1001)), \end{aligned} \tag{2.22}$$
$$\begin{aligned} 0 = &\tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ &+ \tau(A(s,1010)), \end{aligned} \tag{2.23}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.25}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.26}$$
$$0 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.27}$$

$$
\begin{aligned}
0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\
&\quad + \tau(A(s,101)) + \tau(A(s,1001)),
\end{aligned} \tag{2.28}
$$

$$
\begin{aligned}
0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\
&\quad + \tau(A(s,1010)),
\end{aligned} \tag{2.29}
$$

$$
0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.30}
$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{20}), E_{20})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$
\begin{aligned}
M_{2k} &= M^e[:6u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{6u} R^e, \\
W_{2k} &= W^e[:6u] + x^{2u} W^e[:4u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^{6u} R^e.
\end{aligned}
$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$
\begin{aligned}
M_{2k+1} &= M^o[:6u] + x^{2u} M^o[:4u] + x^{3u} M^o[:3u] + x^{5u} M^o[:u] + x^{6u} R^o, \\
W_{2k+1} &= W^o[:6u] + x^{2u} W^o[:4u] + x^{3u} W^o[:3u] + x^{5u} W^o[:u] + x^{6u} R^o.
\end{aligned}
$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$
\begin{aligned}
M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\
M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}.
\end{aligned}
$$

We give the proof of

$$
M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},
$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$
\begin{aligned}
W_{2k} M_{2k} &= \big(W^e[:6v] + x^{2v} W^e[:4v] + x^{3v} W^e[:3v] + x^{5v} W^e[:v] + x^{6u} R^e\big) \times \big(\\
&\quad M^e[:6v] + x^{2v} M^e[:4v] + x^{3v} M^e[:3v] + x^{5v} M^e[:v] + x^{6u} R^e\big);
\end{aligned} \tag{2.31}
$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$
\begin{aligned}
W^e[:12v] M^e[:12v] + x^{4v} W^e[:8v] M^e[:8v] + x^{6v} W^e[:6v] M^e[:6v] \\
+ x^{10v} W^e[:2v] M^e[:2v].
\end{aligned}
$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$
\begin{aligned}
X &:= x^v, \\
a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\
b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n,
\end{aligned}
$$

$$c := R^e.$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} + x^{11} + x^9 + x^8 + x^6,$$
$$q_1(x) = x^{19} + x^{18} + x^{17} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^6 + x^5,$$
$$q_2(x) = x^{18} + x^{10} + x^4 + 1,$$
$$q_3(x) = x^{19} + x^{18} + x^{17} + x^{13} + x^{12} + x^{10} + x^9 + x^7 + x^6 + x^5,$$
$$q_4(x) = x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^9 + x^8 + x^6.$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 24 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{76} + x^{72} + x^{68} \\
&\quad + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{102} + x^{100} + x^{90} + x^{88} + x^{76} + x^{74} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} \\
&\quad + x^{52} + x^{50} + x^{40} + x^{38} + x^{30} + x^{26} + x^{22}, \\
p_6(x) &= x^{104} + x^{100} + x^{98} + x^{90} + x^{86} + x^{82} + x^{80} + x^{72} + x^{70} + x^{60} + x^{58} \\
&\quad + x^{56} + x^{54} + x^{48} + x^{46} + x^{38} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{16} + x^{14} + x^{12} + x^{10} + x^{2} + 1, \\
p_9(x) &= x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{66} + x^{64} + x^{60} + x^{56} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{32} + x^{24} + x^{20} + x^{16} + x^{10}, \\
p_{12}(x) &= x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{50} \\
&\quad + x^{48} + x^{42} + x^{40} + x^{26} + x^{24} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{92} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{73} + x^{70} + x^{69} + x^{68} + x^{66} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{36}, \\
p_3(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{60} + x^{59} \\
&\quad + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45}, \\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} \\
&\quad + x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{54} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} \\
&\quad + x^{35} + x^{33} + x^{32} + x^{30}, \\
p_9(x) &= x^{104} + x^{103} + x^{98} + x^{97} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} \\
&\quad + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{58} + x^{57} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38}
\end{aligned}$$

$$\begin{aligned}
&+x^{37}+x^{32}+x^{31}+x^{28}+x^{27}+x^{22}+x^{21}+x^{20}+x^{19}+x^{16}+x^{15},\\
p_{12}(x)&=x^{103}+x^{100}+x^{95}+x^{92}+x^{91}+x^{88}+x^{87}+x^{84}+x^{83}+x^{80}+x^{79}\\
&+x^{76}+x^{75}+x^{72}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}+x^{60}+x^{59}+x^{56}\\
&+x^{51}+x^{48}+x^{39}+x^{36}+x^{31}+x^{28}+x^{27}+x^{24}+x^{23}+x^{20}+x^{19}\\
&+x^{16}+x^{15}+x^{12}+x^{11}+x^{8}+x^{3}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^e,1,0)$:

$$\begin{aligned}
p_0(x)&=x^{108}+x^{107}+x^{106}+x^{105}+x^{103}+x^{92}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{80}+x^{78}+x^{77}+x^{73}+x^{70}+x^{69}+x^{68}+x^{66}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{56}+x^{54}+x^{52}+x^{51}+x^{50}\\
&+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{42}+x^{39}+x^{36},\\
p_3(x)&=x^{106}+x^{105}+x^{104}+x^{103}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}\\
&+x^{85}+x^{84}+x^{83}+x^{78}+x^{77}+x^{76}+x^{75}+x^{70}+x^{69}+x^{60}+x^{59}\\
&+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45},\\
p_6(x)&=x^{105}+x^{103}+x^{102}+x^{100}+x^{99}+x^{96}+x^{95}+x^{93}+x^{92}+x^{90}+x^{89}\\
&+x^{87}+x^{86}+x^{84}+x^{83}+x^{80}+x^{79}+x^{76}+x^{75}+x^{73}+x^{72}+x^{70}\\
&+x^{69}+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}+x^{61}+x^{59}+x^{58}+x^{57}+x^{56}\\
&+x^{54}+x^{49}+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{39}+x^{38}+x^{36}\\
&+x^{35}+x^{33}+x^{32}+x^{30},\\
p_9(x)&=x^{104}+x^{103}+x^{98}+x^{97}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{84}\\
&+x^{83}+x^{80}+x^{79}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}\\
&+x^{58}+x^{57}+x^{50}+x^{49}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{38}\\
&+x^{37}+x^{32}+x^{31}+x^{28}+x^{27}+x^{22}+x^{21}+x^{20}+x^{19}+x^{16}+x^{15},\\
p_{12}(x)&=x^{103}+x^{100}+x^{95}+x^{92}+x^{91}+x^{88}+x^{87}+x^{84}+x^{83}+x^{80}+x^{79}\\
&+x^{76}+x^{75}+x^{72}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}+x^{60}+x^{59}+x^{56}\\
&+x^{51}+x^{48}+x^{39}+x^{36}+x^{31}+x^{28}+x^{27}+x^{24}+x^{23}+x^{20}+x^{19}\\
&+x^{16}+x^{15}+x^{12}+x^{11}+x^{8}+x^{3}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^e,1,1)$:

$$\begin{aligned}
p_0(x)&=x^{108}+x^{102}+x^{100}+x^{98}+x^{92}+x^{88}+x^{86}+x^{78}+x^{76}+x^{74}+x^{68}\\
&+x^{66}+x^{62}+x^{54}+x^{52}+x^{50}+x^{48}+x^{38}+x^{36},\\
p_3(x)&=x^{102}+x^{96}+x^{94}+x^{92}+x^{90}+x^{86}+x^{84}+x^{82}+x^{78}+x^{76}+x^{72}\\
&+x^{70}+x^{68}+x^{66}+x^{64}+x^{50}+x^{48}+x^{42}+x^{36}+x^{34}+x^{28}+x^{22},\\
p_6(x)&=x^{102}+x^{100}+x^{94}+x^{92}+x^{90}+x^{86}+x^{84}+x^{78}+x^{76}+x^{74}+x^{72}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{70} + x^{66} + x^{64} + x^{60} + x^{52} + x^{44} + x^{34} + x^{32} + x^{26} + x^{22} + x^{20}\\
&+ x^{18} + x^{12} + x^{10} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{100} + x^{92} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{68} + x^{66}\\
&+ x^{64} + x^{54} + x^{48} + x^{40} + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{14} + x^{10},\\
p_{12}(x) &= x^{100} + x^{98} + x^{96} + x^{52} + x^{50} + x^{48} + x^{36} + x^{34} + x^{32} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{84} + x^{82} + x^{76} + x^{74}\\
&+ x^{72} + x^{68} + x^{66} + x^{62} + x^{52} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36},\\
p_3(x) &= x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{70}\\
&+ x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{48} + x^{46} + x^{38} + x^{36} + x^{34} + x^{30}\\
&+ x^{28} + x^{22},\\
p_6(x) &= x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{80} + x^{78} + x^{76} + x^{74}\\
&+ x^{70} + x^{68} + x^{62} + x^{56} + x^{42} + x^{38} + x^{34} + x^{18} + x^{12} + x^{10} + x^{4}\\
&+ x^{2} + 1,\\
p_9(x) &= x^{100} + x^{92} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{68} + x^{66}\\
&+ x^{64} + x^{54} + x^{48} + x^{40} + x^{34} + x^{32} + x^{30} + x^{20} + x^{18} + x^{14} + x^{10},\\
p_{12}(x) &= x^{100} + x^{98} + x^{96} + x^{52} + x^{50} + x^{48} + x^{36} + x^{34} + x^{32} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{91}\\
&+ x^{90} + x^{89} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{73}\\
&+ x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{57} + x^{52} + x^{51}\\
&+ x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{36},\\
p_3(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{60} + x^{59}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45},\\
p_6(x) &= x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89}\\
&+ x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70}\\
&+ x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{54} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36}\\
&+ x^{35} + x^{33} + x^{32} + x^{30},\\
p_9(x) &= x^{104} + x^{103} + x^{98} + x^{97} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84}\\
&+ x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}
\end{aligned}
$$

$$+ x^{58} + x^{57} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38}$$
$$+ x^{37} + x^{32} + x^{31} + x^{28} + x^{27} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{15},$$
$$p_{12}(x) = x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79}$$
$$+ x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56}$$
$$+ x^{51} + x^{48} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19}$$
$$+ x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$p_0(x) = x^{108} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{91}$$
$$+ x^{90} + x^{89} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} + x^{73}$$
$$+ x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{57} + x^{52} + x^{51}$$
$$+ x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{36},$$
$$p_3(x) = x^{106} + x^{105} + x^{104} + x^{103} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}$$
$$+ x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{60} + x^{59}$$
$$+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45},$$
$$p_6(x) = x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89}$$
$$+ x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70}$$
$$+ x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56}$$
$$+ x^{54} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36}$$
$$+ x^{35} + x^{33} + x^{32} + x^{30},$$
$$p_9(x) = x^{104} + x^{103} + x^{98} + x^{97} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84}$$
$$+ x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}$$
$$+ x^{58} + x^{57} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38}$$
$$+ x^{37} + x^{32} + x^{31} + x^{28} + x^{27} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{15},$$
$$p_{12}(x) = x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{80} + x^{79}$$
$$+ x^{76} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56}$$
$$+ x^{51} + x^{48} + x^{39} + x^{36} + x^{31} + x^{28} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19}$$
$$+ x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$p_0(x) = x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{84} + x^{76} + x^{72}$$
$$+ x^{64} + x^{60} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36},$$
$$p_3(x) = x^{104} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78}$$
$$+ x^{70} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42}$$

$$
\begin{aligned}
&\quad + x^{36} + x^{34} + x^{26} + x^{22},\\
p_6(x) &= x^{102} + x^{100} + x^{98} + x^{88} + x^{84} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68}\\
&\quad + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{38} + x^{34} + x^{30} + x^{28} + x^{18}\\
&\quad + x^{16} + x^{14} + x^{12} + x^{10} + x^{2} + 1,\\
p_9(x) &= x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78}\\
&\quad + x^{76} + x^{74} + x^{66} + x^{64} + x^{60} + x^{56} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38}\\
&\quad + x^{34} + x^{32} + x^{24} + x^{20} + x^{16} + x^{10},\\
p_{12}(x) &= x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{50}\\
&\quad + x^{48} + x^{42} + x^{40} + x^{26} + x^{24} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{105} + x^{102} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90}\\
&\quad + x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{69}\\
&\quad + x^{68} + x^{66} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&\quad + x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{36},\\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{92} + x^{90}\\
&\quad + x^{89} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{72} + x^{65}\\
&\quad + x^{62} + x^{60} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41}\\
&\quad + x^{40} + x^{38} + x^{35} + x^{32},\\
p_6(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} + x^{91} + x^{86}\\
&\quad + x^{83} + x^{80} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{65} + x^{64}\\
&\quad + x^{62} + x^{58} + x^{55} + x^{53} + x^{52} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41}\\
&\quad + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{28} + x^{26}\\
&\quad + x^{25} + x^{22} + x^{21} + x^{18} + x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{91} + x^{88} + x^{87} + x^{84} + x^{75} + x^{72} + x^{71}\\
&\quad + x^{68} + x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20},\\
p_{12}(x) &= x^{107} + x^{104} + x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63}\\
&\quad + x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16}\\
&\quad + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{112} + x^{109} + x^{106} + x^{105} + x^{103} + x^{102} + x^{99} + x^{97} + x^{92} + x^{90}\\
&\quad + x^{89} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73}\\
&\quad + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{53}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{52} + x^{51} + x^{44} + x^{43} + x^{42} + x^{40} + x^{36},\\
p_3(x) &= x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{81}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63}\\
&+ x^{62} + x^{61} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46}\\
&+ x^{45},\\
p_6(x) &= x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{88} + x^{86} + x^{74} + x^{72} + x^{68} + x^{64}\\
&+ x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30},\\
p_9(x) &= x^{107} + x^{106} + x^{104} + x^{103} + x^{99} + x^{98} + x^{96} + x^{95} + x^{91} + x^{90} + x^{88}\\
&+ x^{87} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{72} + x^{71} + x^{67} + x^{66}\\
&+ x^{64} + x^{63} + x^{43} + x^{42} + x^{40} + x^{39} + x^{35} + x^{34} + x^{32} + x^{31} + x^{27}\\
&+ x^{26} + x^{24} + x^{23} + x^{19} + x^{18} + x^{16} + x^{15},\\
p_{12}(x) &= x^{108} + x^{104} + x^{96} + x^{92} + x^{84} + x^{76} + x^{68} + x^{56} + x^{52} + x^{48} + x^{44}\\
&+ x^{40} + x^{32} + x^{28} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{100} + x^{95} + x^{89} + x^{88} + x^{86}\\
&+ x^{85} + x^{84} + x^{81} + x^{77} + x^{76} + x^{72} + x^{69} + x^{67} + x^{65} + x^{63} + x^{57}\\
&+ x^{55} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{38}\\
&+ x^{36},\\
p_3(x) &= x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94}\\
&+ x^{93} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{68} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53}\\
&+ x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45},\\
p_6(x) &= x^{108} + x^{104} + x^{102} + x^{98} + x^{84} + x^{78} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56}\\
&+ x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30},\\
p_9(x) &= x^{109} + x^{108} + x^{107} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{81} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{75} + x^{65} + x^{64} + x^{63} + x^{45} + x^{44} + x^{43} + x^{33}\\
&+ x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{17} + x^{16} + x^{15},\\
p_{12}(x) &= x^{110} + x^{108} + x^{102} + x^{100} + x^{94} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78}\\
&+ x^{76} + x^{70} + x^{68} + x^{66} + x^{64} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36}\\
&+ x^{30} + x^{28} + x^{22} + x^{20} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 1)$:

$$p_0(x) = x^{108} + x^{106} + x^{103} + x^{102} + x^{98} + x^{96} + x^{93} + x^{92} + x^{89} + x^{88} + x^{86}$$

$$
\begin{aligned}
&+ x^{85} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{66}\\
&+ x^{65} + x^{64} + x^{61} + x^{59} + x^{56} + x^{55} + x^{53} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{36},\\
p_3(x) &= x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{92} + x^{91} + x^{90}\\
&+ x^{89} + x^{87} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70}\\
&+ x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53}\\
&+ x^{52} + x^{51} + x^{47} + x^{46} + x^{44} + x^{41} + x^{38} + x^{35} + x^{32},\\
p_6(x) &= x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} + x^{93} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{84} + x^{77} + x^{76} + x^{74} + x^{72} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{63} + x^{61} + x^{60} + x^{57} + x^{54} + x^{53} + x^{47} + x^{46} + x^{43} + x^{40}\\
&+ x^{39} + x^{37} + x^{34} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{91} + x^{88} + x^{87} + x^{84} + x^{75} + x^{72} + x^{71}\\
&+ x^{68} + x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20},\\
p_{12}(x) &= x^{107} + x^{104} + x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63}\\
&+ x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16}\\
&+ x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{105} + x^{102} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{91} + x^{90}\\
&+ x^{87} + x^{86} + x^{83} + x^{82} + x^{81} + x^{77} + x^{76} + x^{72} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{66} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{51} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{36},\\
p_3(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{96} + x^{92} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{72} + x^{65}\\
&+ x^{62} + x^{60} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} + x^{43} + x^{41}\\
&+ x^{40} + x^{38} + x^{35} + x^{32},\\
p_6(x) &= x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} + x^{91} + x^{86}\\
&+ x^{83} + x^{80} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{65} + x^{64}\\
&+ x^{62} + x^{58} + x^{55} + x^{53} + x^{52} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{28} + x^{26}\\
&+ x^{25} + x^{22} + x^{21} + x^{18} + x^{11} + x^{8} + x^{3} + 1,\\
p_9(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{91} + x^{88} + x^{87} + x^{84} + x^{75} + x^{72} + x^{71}\\
&+ x^{68} + x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20},
\end{aligned}
$$

$$\begin{aligned} p_{12}(x) &= x^{107} + x^{104} + x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63} \\ &+ x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16} \\ &+ x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1, \end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{100} + x^{95} + x^{89} + x^{88} + x^{86} \\ &+ x^{85} + x^{84} + x^{81} + x^{77} + x^{76} + x^{72} + x^{69} + x^{67} + x^{65} + x^{63} + x^{57} \\ &+ x^{55} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{38} \\ &+ x^{36}, \\ p_3(x) &= x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} \\ &+ x^{93} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} \\ &+ x^{68} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{53} \\ &+ x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{45}, \\ p_6(x) &= x^{108} + x^{104} + x^{102} + x^{98} + x^{84} + x^{78} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56} \\ &+ x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{30}, \\ p_9(x) &= x^{109} + x^{108} + x^{107} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{81} + x^{80} \\ &+ x^{79} + x^{77} + x^{76} + x^{75} + x^{65} + x^{64} + x^{63} + x^{45} + x^{44} + x^{43} + x^{33} \\ &+ x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{17} + x^{16} + x^{15}, \\ p_{12}(x) &= x^{110} + x^{108} + x^{102} + x^{100} + x^{94} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} \\ &+ x^{76} + x^{70} + x^{68} + x^{66} + x^{64} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} \\ &+ x^{30} + x^{28} + x^{22} + x^{20} + x^{18} + x^{16} + x^{2} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned} p_0(x) &= x^{112} + x^{109} + x^{106} + x^{105} + x^{103} + x^{102} + x^{99} + x^{97} + x^{92} + x^{90} \\ &+ x^{89} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} \\ &+ x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} + x^{53} \\ &+ x^{52} + x^{51} + x^{44} + x^{43} + x^{42} + x^{40} + x^{36}, \\ p_3(x) &= x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{81} \\ &+ x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63} \\ &+ x^{62} + x^{61} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} \\ &+ x^{45}, \\ p_6(x) &= x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{88} + x^{86} + x^{74} + x^{72} + x^{68} + x^{64} \\ &+ x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30}, \\ p_9(x) &= x^{107} + x^{106} + x^{104} + x^{103} + x^{99} + x^{98} + x^{96} + x^{95} + x^{91} + x^{90} + x^{88} \end{aligned}$$

$$+ x^{87} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{72} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{43} + x^{42} + x^{40} + x^{39} + x^{35} + x^{34} + x^{32} + x^{31} + x^{27} + x^{26} + x^{24} + x^{23} + x^{19} + x^{18} + x^{16} + x^{15},$$

$$p_{12}(x) = x^{108} + x^{104} + x^{96} + x^{92} + x^{84} + x^{76} + x^{68} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{32} + x^{28} + x^{20} + x^{8} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$p_0(x) = x^{108} + x^{106} + x^{103} + x^{102} + x^{98} + x^{96} + x^{93} + x^{92} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{56} + x^{55} + x^{53} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{36},$$

$$p_3(x) = x^{107} + x^{106} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87} + x^{83} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{53} + x^{52} + x^{51} + x^{47} + x^{46} + x^{44} + x^{41} + x^{38} + x^{35} + x^{32},$$

$$p_6(x) = x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{84} + x^{77} + x^{76} + x^{74} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63} + x^{61} + x^{60} + x^{57} + x^{54} + x^{53} + x^{47} + x^{46} + x^{43} + x^{40} + x^{39} + x^{37} + x^{34} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{11} + x^{8} + x^{3} + 1,$$

$$p_9(x) = x^{107} + x^{104} + x^{103} + x^{100} + x^{91} + x^{88} + x^{87} + x^{84} + x^{75} + x^{72} + x^{71} + x^{68} + x^{43} + x^{40} + x^{39} + x^{36} + x^{27} + x^{24} + x^{23} + x^{20},$$

$$p_{12}(x) = x^{107} + x^{104} + x^{95} + x^{92} + x^{83} + x^{80} + x^{79} + x^{76} + x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{43} + x^{40} + x^{31} + x^{28} + x^{19} + x^{16} + x^{15} + x^{12} + x^{11} + x^{8} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	311	[472, 473]	622	[794, 514]	933	[150, 152]
1	[3, 4]	312	[474, 475]	623	[209, 413]	934	[527, 517]
2	[5, 6]	313	[36, 128]	624	[780, 168]	935	[833, 847]
3	[7, 8]	314	[186, 170]	625	[741, 41]	936	[831, 844]
4	[9, 10]	315	[51, 476]	626	[138, 437]	937	[794, 406]
5	[11, 12]	316	[477, 146]	627	[389, 388]	938	[776, 135]

6	[13, 14]	317	[478, 479]	628	[462, 212]	939	[1028, 456]
7	[15, 16]	318	[32, 175]	629	[795, 528]	940	[793, 792]
8	[17, 18]	319	[48, 146]	630	[796, 429]	941	[135, 60]
9	[19, 20]	320	[480, 481]	631	[797, 196]	942	[1058, 382]
10	[21, 22]	321	[482, 483]	632	[409, 379]	943	[1059, 507]
11	[23, 24]	322	[13, 389]	633	[798, 160]	944	[451, 496]
12	[25, 26]	323	[434, 484]	634	[799, 198]	945	[492, 148]
13	[27, 28]	324	[485, 192]	635	[391, 203]	946	[1060, 1061]
14	[29, 30]	325	[159, 400]	636	[488, 445]	947	[872, 129]
15	[31, 32]	326	[486, 487]	637	[800, 801]	948	[8, 475]
16	[33, 34]	327	[413, 155]	638	[513, 383]	949	[1062, 505]
17	[35, 36]	328	[488, 215]	639	[802, 145]	950	[1063, 506]
18	[37, 38]	329	[211, 386]	640	[803, 12]	951	[1064, 412]
19	[39, 40]	330	[489, 31]	641	[804, 528]	952	[1047, 215]
20	[41, 42]	331	[490, 476]	642	[436, 398]	953	[807, 40]
21	[43, 44]	332	[178, 123]	643	[42, 442]	954	[1065, 421]
22	[45, 46]	333	[393, 379]	644	[206, 416]	955	[748, 158]
23	[47, 48]	334	[414, 138]	645	[805, 61]	956	[1066, 160]
24	[49, 50]	335	[190, 42]	646	[806, 427]	957	[1067, 433]
25	[51, 52]	336	[45, 491]	647	[807, 34]	958	[475, 148]
26	[53, 54]	337	[492, 382]	648	[808, 123]	959	[1024, 445]
27	[55, 56]	338	[493, 50]	649	[809, 206]	960	[385, 10]
28	[57, 58]	339	[494, 473]	650	[810, 523]	961	[1068, 1069]
29	[59, 60]	340	[495, 496]	651	[811, 467]	962	[1070, 525]
30	[61, 14]	341	[140, 171]	652	[411, 132]	963	[394, 205]
31	[62, 63]	342	[491, 432]	653	[768, 151]	964	[1071, 412]
32	[64, 65]	343	[40, 429]	654	[403, 213]	965	[1072, 487]
33	[66, 67]	344	[497, 498]	655	[812, 206]	966	[811, 57]
34	[68, 69]	345	[499, 500]	656	[804, 517]	967	[825, 153]
35	[70, 71]	346	[501, 502]	657	[485, 428]	968	[1073, 478]
36	[72, 73]	347	[503, 388]	658	[130, 212]	969	[1074, 1053]
37	[74, 75]	348	[444, 210]	659	[469, 463]	970	[12, 1075]
38	[76, 77]	349	[504, 505]	660	[134, 469]	971	[1076, 1077]
39	[78, 79]	350	[506, 507]	661	[402, 467]	972	[486, 61]
40	[80, 81]	351	[508, 509]	662	[769, 401]	973	[1070, 201]
41	[82, 83]	352	[510, 175]	663	[813, 391]	974	[1078, 834]
42	[84, 85]	353	[389, 133]	664	[814, 815]	975	[845, 831]
43	[86, 77]	354	[511, 407]	665	[816, 817]	976	[836, 844]
44	[87, 88]	355	[216, 404]	666	[818, 470]	977	[1031, 835]
45	[89, 90]	356	[212, 404]	667	[806, 140]	978	[833, 845]
46	[91, 92]	357	[123, 158]	668	[819, 820]	979	[843, 833]
47	[93, 94]	358	[479, 203]	669	[447, 212]	980	[847, 836]
48	[95, 96]	359	[512, 52]	670	[405, 514]	981	[846, 847]
49	[97, 98]	360	[513, 149]	671	[487, 14]	982	[844, 847]

50	[99, 100]	361	[157, 514]	672	[813, 479]	983	[1079, 1080]
51	[101, 102]	362	[515, 389]	673	[510, 204]	984	[1081, 851]
52	[103, 104]	363	[152, 386]	674	[161, 180]	985	[1076, 842]
53	[105, 106]	364	[466, 57]	675	[421, 145]	986	[494, 381]
54	[107, 108]	365	[210, 155]	676	[420, 165]	987	[754, 149]
55	[109, 110]	366	[468, 135]	677	[752, 821]	988	[803, 839]
56	[111, 112]	367	[516, 517]	678	[495, 166]	989	[1036, 836]
57	[113, 114]	368	[479, 518]	679	[822, 823]	990	[1078, 835]
58	[115, 116]	369	[519, 484]	680	[413, 208]	991	[832, 844]
59	[117, 118]	370	[520, 518]	681	[515, 14]	992	[831, 833]
60	[119, 120]	371	[393, 177]	682	[824, 134]	993	[1082, 1053]
61	[121, 122]	372	[521, 522]	683	[410, 151]	994	[1083, 1034]
62	[35, 123]	373	[42, 523]	684	[825, 386]	995	[852, 1084]
63	[124, 125]	374	[524, 525]	685	[826, 409]	996	[490, 52]
64	[126, 127]	375	[526, 398]	686	[458, 32]	997	[148, 383]
65	[128, 129]	376	[527, 528]	687	[827, 815]	998	[384, 9]
66	[130, 56]	377	[529, 91]	688	[504, 785]	999	[481, 166]
67	[131, 132]	378	[530, 531]	689	[820, 391]	1000	[1085, 481]
68	[14, 133]	379	[532, 533]	690	[471, 42]	1001	[482, 1061]
69	[134, 135]	380	[534, 344]	691	[828, 31]	1002	[1086, 1087]
70	[136, 46]	381	[535, 536]	692	[829, 182]	1003	[767, 484]
71	[137, 138]	382	[18, 537]	693	[791, 505]	1004	[1088, 194]
72	[139, 140]	383	[538, 232]	694	[200, 525]	1005	[1044, 435]
73	[141, 142]	384	[539, 540]	695	[439, 429]	1006	[204, 395]
74	[143, 144]	385	[541, 542]	696	[797, 433]	1007	[1089, 1090]
75	[145, 146]	386	[266, 543]	697	[830, 831]	1008	[1081, 1091]
76	[8, 147]	387	[327, 544]	698	[832, 833]	1009	[1092, 1084]
77	[148, 149]	388	[275, 545]	699	[834, 835]	1010	[1093, 801]
78	[150, 151]	389	[546, 547]	700	[836, 833]	1011	[869, 817]
79	[152, 153]	390	[548, 549]	701	[837, 838]	1012	[1094, 849]
80	[154, 155]	391	[550, 551]	702	[839, 840]	1013	[775, 1095]
81	[156, 157]	392	[552, 312]	703	[841, 842]	1014	[1096, 1097]
82	[39, 34]	393	[553, 554]	704	[830, 836]	1015	[438, 198]
83	[158, 159]	394	[98, 555]	705	[503, 133]	1016	[202, 518]
84	[160, 142]	395	[556, 557]	706	[790, 469]	1017	[1082, 1090]
85	[161, 32]	396	[558, 559]	707	[197, 397]	1018	[1069, 1098]
86	[162, 163]	397	[560, 561]	708	[526, 437]	1019	[1099, 1100]
87	[164, 165]	398	[562, 563]	709	[432, 196]	1020	[186, 173]
88	[10, 166]	399	[564, 246]	710	[843, 844]	1021	[1101, 949]
89	[167, 36]	400	[227, 565]	711	[844, 845]	1022	[1102, 585]
90	[43, 168]	401	[373, 566]	712	[846, 845]	1023	[1103, 562]
91	[169, 170]	402	[567, 568]	713	[847, 831]	1024	[1104, 1105]
92	[171, 42]	403	[569, 570]	714	[848, 849]	1025	[1106, 87]
93	[172, 173]	404	[571, 261]	715	[850, 851]	1026	[88, 1107]

94	[174, 175]	405	[572, 573]	716	[852, 853]	1027	[1108, 623]
95	[176, 177]	406	[574, 575]	717	[172, 170]	1028	[1109, 557]
96	[123, 128]	407	[576, 577]	718	[751, 151]	1029	[1110, 705]
97	[178, 36]	408	[282, 301]	719	[854, 855]	1030	[1111, 924]
98	[179, 44]	409	[578, 579]	720	[856, 456]	1031	[928, 979]
99	[137, 127]	410	[580, 581]	721	[136, 491]	1032	[1112, 1113]
100	[180, 175]	411	[582, 264]	722	[857, 467]	1033	[1114, 1115]
101	[181, 182]	412	[583, 584]	723	[858, 859]	1034	[1116, 912]
102	[183, 184]	413	[585, 586]	724	[860, 861]	1035	[579, 915]
103	[185, 186]	414	[587, 588]	725	[862, 863]	1036	[711, 989]
104	[187, 158]	415	[336, 589]	726	[455, 498]	1037	[1117, 116]
105	[188, 177]	416	[590, 591]	727	[500, 861]	1038	[974, 990]
106	[189, 41]	417	[592, 593]	728	[207, 155]	1039	[554, 1118]
107	[190, 191]	418	[594, 595]	729	[770, 418]	1040	[1119, 1120]
108	[170, 192]	419	[596, 368]	730	[864, 821]	1041	[1121, 371]
109	[193, 194]	420	[597, 234]	731	[865, 461]	1042	[992, 897]
110	[195, 196]	421	[598, 599]	732	[866, 757]	1043	[913, 1122]
111	[197, 198]	422	[340, 600]	733	[867, 868]	1044	[1123, 1124]
112	[199, 192]	423	[601, 602]	734	[459, 129]	1045	[953, 1125]
113	[200, 201]	424	[603, 604]	735	[157, 406]	1046	[1126, 21]
114	[202, 203]	425	[605, 572]	736	[750, 157]	1047	[1127, 1128]
115	[204, 205]	426	[606, 607]	737	[869, 184]	1048	[1129, 622]
116	[206, 38]	427	[608, 609]	738	[861, 787]	1049	[1130, 723]
117	[207, 208]	428	[304, 304]	739	[870, 871]	1050	[731, 1131]
118	[209, 210]	429	[67, 562]	740	[872, 159]	1051	[936, 981]
119	[211, 153]	430	[610, 590]	741	[873, 84]	1052	[1132, 1133]
120	[212, 213]	431	[611, 612]	742	[310, 874]	1053	[1134, 1135]
121	[214, 215]	432	[90, 313]	743	[875, 533]	1054	[1136, 1137]
122	[216, 213]	433	[613, 614]	744	[876, 877]	1055	[1138, 997]
123	[217, 218]	434	[615, 616]	745	[878, 23]	1056	[1139, 1140]
124	[219, 220]	435	[617, 556]	746	[879, 354]	1057	[1141, 118]
125	[221, 220]	436	[618, 619]	747	[880, 244]	1058	[920, 538]
126	[222, 223]	437	[588, 549]	748	[881, 644]	1059	[1135, 892]
127	[224, 225]	438	[620, 621]	749	[882, 613]	1060	[1000, 998]
128	[226, 227]	439	[622, 623]	750	[883, 575]	1061	[1142, 878]
129	[218, 228]	440	[79, 223]	751	[884, 271]	1062	[1143, 1144]
130	[229, 230]	441	[624, 625]	752	[885, 886]	1063	[982, 712]
131	[231, 232]	442	[335, 626]	753	[887, 888]	1064	[1145, 365]
132	[233, 234]	443	[627, 6]	754	[609, 888]	1065	[1146, 564]
133	[122, 235]	444	[628, 120]	755	[889, 890]	1066	[1107, 1147]
134	[236, 237]	445	[629, 630]	756	[891, 241]	1067	[540, 1148]
135	[238, 239]	446	[631, 632]	757	[892, 893]	1068	[960, 1149]
136	[240, 241]	447	[633, 571]	758	[894, 895]	1069	[1150, 1151]
137	[242, 243]	448	[634, 635]	759	[896, 897]	1070	[1152, 1153]

138	[244, 245]	449	[636, 274]	760	[898, 740]	1071	[1154, 618]
139	[246, 247]	450	[637, 615]	761	[899, 900]	1072	[1155, 29]
140	[248, 249]	451	[531, 529]	762	[102, 901]	1073	[1156, 550]
141	[250, 251]	452	[256, 93]	763	[734, 902]	1074	[1157, 1158]
142	[252, 251]	453	[638, 639]	764	[903, 904]	1075	[948, 1139]
143	[253, 254]	454	[640, 348]	765	[905, 906]	1076	[1137, 899]
144	[255, 256]	455	[641, 642]	766	[877, 686]	1077	[1159, 544]
145	[234, 257]	456	[643, 644]	767	[904, 907]	1078	[713, 1135]
146	[94, 258]	457	[645, 224]	768	[908, 909]	1079	[1160, 1161]
147	[259, 260]	458	[646, 309]	769	[910, 258]	1080	[1162, 697]
148	[261, 262]	459	[647, 352]	770	[911, 596]	1081	[1163, 912]
149	[263, 264]	460	[648, 566]	771	[543, 15]	1082	[1164, 1165]
150	[265, 266]	461	[599, 635]	772	[912, 913]	1083	[1166, 944]
151	[267, 268]	462	[649, 650]	773	[914, 343]	1084	[1167, 938]
152	[269, 270]	463	[545, 363]	774	[915, 916]	1085	[247, 293]
153	[271, 98]	464	[651, 652]	775	[917, 638]	1086	[320, 1168]
154	[272, 273]	465	[653, 654]	776	[918, 235]	1087	[1169, 1170]
155	[274, 275]	466	[655, 115]	777	[919, 680]	1088	[1171, 295]
156	[276, 277]	467	[656, 657]	778	[674, 920]	1089	[1172, 1173]
157	[278, 279]	468	[658, 119]	779	[595, 672]	1090	[1174, 979]
158	[71, 280]	469	[659, 660]	780	[921, 591]	1091	[676, 1175]
159	[281, 282]	470	[661, 662]	781	[314, 107]	1092	[678, 1176]
160	[283, 284]	471	[663, 561]	782	[922, 923]	1093	[1115, 1106]
161	[285, 105]	472	[664, 665]	783	[924, 104]	1094	[1177, 1178]
162	[286, 287]	473	[666, 529]	784	[925, 565]	1095	[1179, 1115]
163	[288, 289]	474	[667, 261]	785	[926, 333]	1096	[533, 905]
164	[290, 95]	475	[16, 263]	786	[927, 928]	1097	[1180, 273]
165	[291, 292]	476	[668, 106]	787	[700, 929]	1098	[1181, 1137]
166	[293, 294]	477	[644, 256]	788	[930, 931]	1099	[1151, 729]
167	[262, 226]	478	[669, 355]	789	[932, 710]	1100	[1182, 362]
168	[295, 296]	479	[670, 671]	790	[933, 545]	1101	[1183, 423]
169	[297, 298]	480	[672, 673]	791	[934, 902]	1102	[1184, 464]
170	[299, 300]	481	[632, 674]	792	[897, 935]	1103	[857, 57]
171	[301, 302]	482	[675, 676]	793	[935, 936]	1104	[1185, 179]
172	[303, 304]	483	[677, 678]	794	[937, 938]	1105	[175, 395]
173	[305, 306]	484	[679, 308]	795	[939, 940]	1106	[1085, 10]
174	[307, 308]	485	[680, 681]	796	[941, 322]	1107	[165, 48]
175	[309, 310]	486	[682, 347]	797	[566, 70]	1108	[1186, 1083]
176	[311, 296]	487	[683, 684]	798	[942, 221]	1109	[396, 198]
177	[312, 75]	488	[685, 631]	799	[665, 943]	1110	[1187, 449]
178	[313, 314]	489	[686, 317]	800	[944, 945]	1111	[1188, 45]
179	[315, 316]	490	[687, 688]	801	[946, 619]	1112	[1032, 849]
180	[317, 318]	491	[589, 342]	802	[874, 947]	1113	[1189, 1098]
181	[319, 320]	492	[689, 690]	803	[4, 948]	1114	[1021, 753]

182	[321, 322]	493	[691, 543]	804	[949, 950]	1115	[382, 149]
183	[323, 324]	494	[692, 693]	805	[951, 546]	1116	[460, 385]
184	[325, 309]	495	[607, 536]	806	[104, 82]	1117	[778, 45]
185	[326, 327]	496	[20, 570]	807	[952, 356]	1118	[770, 771]
186	[328, 329]	497	[694, 695]	808	[888, 953]	1119	[524, 201]
187	[330, 281]	498	[696, 281]	809	[639, 954]	1120	[199, 428]
188	[331, 88]	499	[697, 698]	810	[371, 955]	1121	[1188, 378]
189	[332, 333]	500	[699, 700]	811	[956, 957]	1122	[1046, 9]
190	[334, 335]	501	[701, 702]	812	[958, 76]	1123	[424, 817]
191	[249, 336]	502	[703, 704]	813	[959, 563]	1124	[787, 506]
192	[329, 298]	503	[705, 706]	814	[612, 960]	1125	[760, 195]
193	[337, 338]	504	[707, 708]	815	[961, 681]	1126	[377, 45]
194	[339, 340]	505	[709, 84]	816	[962, 626]	1127	[1190, 431]
195	[241, 341]	506	[710, 711]	817	[963, 924]	1128	[1191, 763]
196	[342, 343]	507	[712, 713]	818	[964, 937]	1129	[1192, 468]
197	[344, 345]	508	[714, 715]	819	[965, 670]	1130	[1193, 454]
198	[346, 335]	509	[716, 712]	820	[966, 967]	1131	[50, 418]
199	[347, 348]	510	[717, 556]	821	[968, 721]	1132	[1052, 1090]
200	[349, 350]	511	[718, 574]	822	[969, 970]	1133	[1194, 1095]
201	[351, 352]	512	[719, 720]	823	[971, 698]	1134	[841, 1077]
202	[353, 354]	513	[333, 721]	824	[972, 353]	1135	[1051, 845]
203	[355, 356]	514	[277, 574]	825	[973, 108]	1136	[143, 163]
204	[63, 357]	515	[722, 275]	826	[600, 532]	1137	[418, 54]
205	[318, 358]	516	[723, 724]	827	[75, 958]	1138	[164, 421]
206	[359, 360]	517	[725, 613]	828	[96, 64]	1139	[10, 496]
207	[361, 362]	518	[549, 79]	829	[26, 942]	1140	[475, 382]
208	[363, 28]	519	[726, 727]	830	[704, 974]	1141	[774, 1069]
209	[364, 363]	520	[728, 602]	831	[975, 976]	1142	[764, 144]
210	[365, 366]	521	[729, 252]	832	[698, 977]	1143	[519, 435]
211	[367, 368]	522	[730, 731]	833	[978, 927]	1144	[796, 440]
212	[369, 370]	523	[83, 604]	834	[834, 834]	1145	[1195, 468]
213	[230, 371]	524	[720, 732]	835	[979, 980]	1146	[742, 161]
214	[372, 373]	525	[733, 734]	836	[981, 982]	1147	[1196, 770]
215	[374, 375]	526	[735, 736]	837	[983, 984]	1148	[1030, 180]
216	[376, 294]	527	[737, 738]	838	[985, 974]	1149	[1037, 421]
217	[377, 378]	528	[739, 740]	839	[986, 987]	1150	[864, 753]
218	[176, 379]	529	[741, 171]	840	[24, 319]	1151	[783, 419]
219	[380, 381]	530	[742, 31]	841	[284, 552]	1152	[773, 859]
220	[382, 383]	531	[743, 38]	842	[988, 706]	1153	[757, 500]
221	[384, 385]	532	[744, 50]	843	[929, 697]	1154	[1071, 449]
222	[55, 212]	533	[481, 496]	844	[989, 975]	1155	[1072, 61]
223	[151, 386]	534	[745, 746]	845	[977, 978]	1156	[1064, 449]
224	[387, 388]	535	[747, 414]	846	[990, 991]	1157	[1197, 1198]
225	[61, 389]	536	[748, 128]	847	[976, 992]	1158	[850, 1091]

226	[390, 391]	537	[36, 158]	848	[993, 994]	1159	[1068, 775]
227	[41, 191]	538	[749, 432]	849	[995, 989]	1160	[837, 849]
228	[378, 46]	539	[139, 427]	850	[996, 997]	1161	[1061, 1091]
229	[392, 393]	540	[743, 416]	851	[998, 999]	1162	[1096, 1100]
230	[394, 395]	541	[465, 170]	852	[591, 1000]	1163	[472, 381]
231	[396, 397]	542	[46, 432]	853	[1001, 573]	1164	[1199, 1200]
232	[138, 398]	543	[431, 416]	854	[251, 675]	1165	[12, 840]
233	[399, 400]	544	[750, 407]	855	[1002, 736]	1166	[512, 476]
234	[194, 168]	545	[467, 132]	856	[1003, 358]	1167	[1186, 1087]
235	[215, 401]	546	[407, 514]	857	[1004, 233]	1168	[493, 770]
236	[402, 57]	547	[412, 210]	858	[1005, 614]	1169	[782, 473]
237	[403, 404]	548	[751, 152]	859	[1006, 607]	1170	[865, 146]
238	[405, 406]	549	[445, 401]	860	[980, 704]	1171	[421, 48]
239	[156, 407]	550	[469, 60]	861	[893, 699]	1172	[1094, 838]
240	[408, 140]	551	[449, 210]	862	[1007, 1008]	1173	[1087, 1055]
241	[409, 177]	552	[47, 145]	863	[1009, 892]	1174	[1092, 853]
242	[410, 152]	553	[752, 753]	864	[1010, 1011]	1175	[145, 461]
243	[411, 58]	554	[754, 383]	865	[73, 289]	1176	[50, 771]
244	[210, 208]	555	[426, 491]	866	[991, 710]	1177	[1201, 1202]
245	[412, 413]	556	[749, 195]	867	[1012, 1013]	1178	[839, 1075]
246	[189, 171]	557	[170, 428]	868	[1014, 928]	1179	[744, 770]
247	[414, 127]	558	[755, 756]	869	[1015, 1016]	1180	[1060, 483]
248	[415, 45]	559	[507, 757]	870	[1017, 1018]	1181	[474, 147]
249	[37, 416]	560	[758, 759]	871	[1019, 990]	1182	[1086, 1083]
250	[417, 52]	561	[760, 432]	872	[1020, 734]	1183	[77, 1203]
251	[418, 419]	562	[56, 213]	873	[828, 161]	1184	[1204, 278]
252	[420, 421]	563	[57, 132]	874	[470, 34]	1185	[1131, 1205]
253	[422, 423]	564	[141, 125]	875	[1021, 821]	1186	[1206, 1207]
254	[424, 184]	565	[391, 518]	876	[779, 140]	1187	[1208, 735]
255	[425, 33]	566	[522, 142]	877	[188, 379]	1188	[955, 689]
256	[426, 46]	567	[761, 393]	878	[477, 461]	1189	[1209, 1175]
257	[427, 171]	568	[191, 442]	879	[11, 839]	1190	[1210, 1211]
258	[173, 428]	569	[762, 525]	880	[1022, 412]	1191	[116, 1212]
259	[169, 173]	570	[40, 440]	881	[808, 36]	1192	[1213, 728]
260	[128, 159]	571	[159, 763]	882	[1023, 127]	1193	[999, 1214]
261	[160, 125]	572	[135, 463]	883	[1024, 215]	1194	[1215, 999]
262	[140, 41]	573	[511, 157]	884	[1025, 179]	1195	[1216, 238]
263	[32, 204]	574	[56, 404]	885	[1026, 1027]	1196	[1217, 594]
264	[34, 429]	575	[151, 153]	886	[1028, 498]	1197	[1218, 1219]
265	[430, 431]	576	[154, 208]	887	[1029, 465]	1198	[1220, 991]
266	[432, 433]	577	[487, 389]	888	[1030, 32]	1199	[1221, 1222]
267	[434, 435]	578	[764, 163]	889	[795, 517]	1200	[1223, 977]
268	[436, 437]	579	[53, 419]	890	[520, 203]	1201	[1224, 1225]
269	[438, 397]	580	[765, 522]	891	[399, 763]	1202	[1226, 981]

270	[439, 440]	581	[766, 205]	892	[1031, 834]	1203	[147, 382]
271	[441, 442]	582	[767, 435]	893	[845, 836]	1204	[1022, 449]
272	[443, 60]	583	[768, 152]	894	[1032, 838]	1205	[1227, 475]
273	[444, 413]	584	[769, 446]	895	[1033, 1034]	1206	[165, 145]
274	[131, 58]	585	[387, 133]	896	[1035, 2]	1207	[1227, 147]
275	[445, 446]	586	[464, 157]	897	[1036, 831]	1208	[777, 487]
276	[447, 56]	587	[214, 445]	898	[820, 479]	1209	[417, 476]
277	[448, 401]	588	[448, 446]	899	[147, 148]	1210	[385, 481]
278	[14, 388]	589	[33, 40]	900	[1037, 165]	1211	[1228, 385]
279	[449, 413]	590	[49, 770]	901	[1038, 793]	1212	[187, 128]
280	[46, 195]	591	[771, 54]	902	[173, 192]	1213	[1187, 412]
281	[194, 44]	592	[772, 746]	903	[1039, 1027]	1214	[781, 418]
282	[31, 180]	593	[773, 756]	904	[1040, 756]	1215	[380, 473]
283	[450, 144]	594	[124, 142]	905	[480, 10]	1216	[1229, 156]
284	[451, 166]	595	[378, 491]	906	[1041, 147]	1217	[415, 378]
285	[452, 161]	596	[158, 129]	907	[1042, 792]	1218	[1074, 1090]
286	[453, 454]	597	[408, 427]	908	[1043, 409]	1219	[483, 851]
287	[455, 456]	598	[478, 391]	909	[129, 400]	1220	[1099, 1097]
288	[457, 137]	599	[491, 195]	910	[1044, 484]	1221	[1230, 1202]
289	[458, 180]	600	[9, 481]	911	[390, 479]	1222	[1033, 1055]
290	[167, 123]	601	[774, 775]	912	[771, 419]	1223	[1056, 2]
291	[126, 138]	602	[776, 469]	913	[1045, 771]	1224	[1231, 1200]
292	[171, 191]	603	[516, 528]	914	[799, 397]	1225	[1054, 1034]
293	[459, 159]	604	[34, 440]	915	[802, 48]	1226	[1035, 1057]
294	[127, 437]	605	[777, 61]	916	[1046, 385]	1227	[1232, 624]
295	[460, 9]	606	[778, 378]	917	[450, 163]	1228	[1233, 632]
296	[48, 461]	607	[522, 125]	918	[1047, 445]	1229	[1234, 576]
297	[462, 56]	608	[779, 427]	919	[1048, 156]	1230	[1235, 1236]
298	[215, 446]	609	[780, 44]	920	[1023, 138]	1231	[1237, 1238]
299	[59, 463]	610	[781, 771]	921	[1049, 753]	1232	[452, 31]
300	[464, 407]	611	[782, 381]	922	[1050, 855]	1233	[489, 161]
301	[465, 173]	612	[783, 54]	923	[755, 859]	1234	[805, 487]
302	[180, 204]	613	[174, 204]	924	[179, 168]	1235	[848, 838]
303	[466, 467]	614	[127, 398]	925	[762, 201]	1236	[1239, 1075]
304	[467, 58]	615	[784, 785]	926	[441, 523]	1237	[1089, 1053]
305	[407, 406]	616	[786, 787]	927	[835, 835]	1238	[1240, 840]
306	[468, 469]	617	[788, 789]	928	[1051, 847]	1239	[1241, 319]
307	[470, 40]	618	[443, 463]	929	[835, 834]	1240	[1242, 1139]
308	[471, 191]	619	[790, 135]	930	[1052, 1053]	1241	[1049, 821]
309	[431, 38]	620	[791, 785]	931	[1054, 1055]	1242	[162, 144]
310	[427, 41]	621	[792, 793]	932	[1056, 1057]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	208	0	416	1	624	0	832	0	1040	0
1	0	209	0	417	1	625	0	833	1	1041	1
2	0	210	1	418	0	626	1	834	0	1042	0
3	0	211	0	419	1	627	1	835	1	1043	1
4	0	212	1	420	0	628	1	836	1	1044	0
5	0	213	0	421	0	629	1	837	1	1045	1
6	1	214	0	422	1	630	1	838	0	1046	0
7	0	215	0	423	1	631	1	839	1	1047	1
8	0	216	0	424	0	632	0	840	1	1048	1
9	0	217	0	425	1	633	0	841	0	1049	0
10	1	218	0	426	1	634	1	842	0	1050	1
11	0	219	0	427	1	635	0	843	1	1051	0
12	0	220	1	428	1	636	0	844	0	1052	1
13	1	221	1	429	1	637	0	845	0	1053	0
14	0	222	1	430	0	638	1	846	1	1054	1
15	0	223	1	431	1	639	1	847	1	1055	0
16	0	224	0	432	1	640	0	848	0	1056	1
17	0	225	0	433	1	641	1	849	1	1057	1
18	1	226	0	434	1	642	1	850	1	1058	0
19	0	227	0	435	1	643	0	851	1	1059	0
20	0	228	1	436	1	644	0	852	1	1060	0
21	1	229	0	437	1	645	0	853	0	1061	0
22	0	230	0	438	0	646	0	854	0	1062	1
23	0	231	1	439	0	647	1	855	1	1063	0
24	1	232	1	440	0	648	1	856	0	1064	0
25	0	233	1	441	0	649	1	857	0	1065	1
26	1	234	1	442	0	650	1	858	0	1066	1
27	1	235	0	443	1	651	1	859	0	1067	0
28	0	236	1	444	1	652	0	860	1	1068	1
29	0	237	1	445	1	653	1	861	0	1069	1
30	0	238	1	446	1	654	1	862	1	1070	1
31	0	239	0	447	0	655	0	863	1	1071	1
32	1	240	0	448	1	656	1	864	1	1072	0
33	0	241	0	449	0	657	0	865	1	1073	0
34	0	242	1	450	0	658	0	866	0	1074	0
35	0	243	0	451	0	659	0	867	1	1075	0
36	1	244	1	452	1	660	1	868	0	1076	0
37	1	245	1	453	1	661	1	869	0	1077	1
38	0	246	1	454	0	662	0	870	0	1078	1
39	0	247	0	455	1	663	1	871	0	1079	1
40	1	248	0	456	0	664	0	872	0	1080	0
41	0	249	1	457	0	665	1	873	0	1081	0
42	0	250	1	458	0	666	1	874	1	1082	0

43	1	251	0	459	1	667	0	875	0	1083	0
44	0	252	0	460	1	668	0	876	1	1084	1
45	0	253	1	461	0	669	0	877	1	1085	0
46	1	254	0	462	1	670	1	878	0	1086	1
47	0	255	1	463	1	671	1	879	0	1087	1
48	0	256	1	464	1	672	1	880	0	1088	0
49	1	257	1	465	1	673	0	881	1	1089	1
50	1	258	1	466	0	674	1	882	0	1090	1
51	0	259	1	467	1	675	0	883	1	1091	0
52	1	260	0	468	0	676	0	884	0	1092	1
53	1	261	0	469	0	677	1	885	1	1093	1
54	0	262	0	470	1	678	1	886	1	1094	1
55	1	263	1	471	1	679	0	887	1	1095	1
56	0	264	0	472	0	680	0	888	1	1096	0
57	0	265	0	473	1	681	0	889	1	1097	0
58	0	266	1	474	0	682	1	890	0	1098	0
59	0	267	1	475	0	683	1	891	1	1099	0
60	0	268	1	476	0	684	1	892	0	1100	1
61	0	269	0	477	0	685	0	893	0	1101	0
62	0	270	0	478	0	686	0	894	0	1102	0
63	0	271	0	479	1	687	1	895	0	1103	0
64	1	272	1	480	1	688	0	896	1	1104	1
65	0	273	1	481	0	689	1	897	0	1105	1
66	0	274	1	482	0	690	1	898	1	1106	0
67	1	275	1	483	1	691	0	899	1	1107	1
68	0	276	0	484	0	692	1	900	1	1108	1
69	1	277	1	485	0	693	0	901	1	1109	1
70	0	278	0	486	1	694	0	902	1	1110	1
71	1	279	0	487	1	695	0	903	0	1111	1
72	1	280	1	488	0	696	1	904	0	1112	0
73	1	281	1	489	0	697	1	905	1	1113	1
74	1	282	0	490	1	698	0	906	1	1114	0
75	1	283	0	491	0	699	0	907	0	1115	1
76	0	284	0	492	1	700	1	908	1	1116	1
77	0	285	1	493	0	701	1	909	0	1117	1
78	0	286	1	494	1	702	1	910	0	1118	0
79	0	287	1	495	1	703	0	911	0	1119	0
80	1	288	0	496	0	704	1	912	1	1120	1
81	0	289	0	497	0	705	1	913	1	1121	1
82	0	290	0	498	1	706	0	914	1	1122	0
83	1	291	1	499	1	707	0	915	1	1123	0
84	0	292	1	500	0	708	0	916	0	1124	1
85	1	293	1	501	1	709	1	917	0	1125	1
86	1	294	0	502	0	710	1	918	1	1126	0

87	0	295	1	503	1	711	0	919	1	1127	1
88	1	296	0	504	0	712	1	920	0	1128	0
89	0	297	1	505	1	713	1	921	0	1129	1
90	1	298	0	506	1	714	0	922	1	1130	0
91	1	299	0	507	1	715	1	923	1	1131	1
92	1	300	1	508	0	716	1	924	0	1132	1
93	0	301	1	509	1	717	0	925	1	1133	0
94	1	302	0	510	0	718	0	926	0	1134	0
95	0	303	0	511	0	719	0	927	1	1135	0
96	0	304	1	512	0	720	0	928	0	1136	1
97	1	305	1	513	1	721	0	929	1	1137	0
98	0	306	0	514	1	722	0	930	1	1138	0
99	1	307	1	515	0	723	0	931	1	1139	1
100	0	308	1	516	0	724	1	932	1	1140	0
101	0	309	1	517	1	725	1	933	0	1141	1
102	1	310	1	518	1	726	1	934	0	1142	0
103	1	311	0	519	1	727	0	935	1	1143	1
104	0	312	0	520	0	728	0	936	0	1144	1
105	1	313	1	521	0	729	0	937	0	1145	0
106	1	314	0	522	1	730	1	938	1	1146	1
107	0	315	0	523	1	731	1	939	1	1147	0
108	0	316	0	524	0	732	0	940	1	1148	1
109	1	317	0	525	1	733	1	941	1	1149	1
110	0	318	1	526	0	734	1	942	0	1150	1
111	0	319	0	527	0	735	0	943	0	1151	0
112	1	320	1	528	0	736	1	944	0	1152	1
113	0	321	0	529	0	737	0	945	1	1153	0
114	1	322	1	530	1	738	0	946	0	1154	1
115	0	323	1	531	0	739	0	947	0	1155	0
116	0	324	0	532	1	740	0	948	0	1156	0
117	0	325	1	533	0	741	0	949	1	1157	0
118	0	326	1	534	0	742	1	950	0	1158	1
119	0	327	0	535	0	743	0	951	0	1159	1
120	1	328	0	536	1	744	1	952	1	1160	1
121	0	329	0	537	1	745	0	953	1	1161	0
122	0	330	0	538	0	746	0	954	1	1162	0
123	0	331	1	539	1	747	0	955	1	1163	0
124	0	332	1	540	0	748	1	956	1	1164	0
125	1	333	1	541	1	749	0	957	0	1165	0
126	1	334	0	542	1	750	1	958	0	1166	0
127	0	335	0	543	1	751	0	959	1	1167	1
128	0	336	0	544	1	752	1	960	1	1168	0
129	0	337	1	545	1	753	1	961	1	1169	1
130	0	338	0	546	1	754	0	962	1	1170	1

131	1	339	1	547	1	755	1	963	0	1171	0
132	1	340	1	548	0	756	1	964	1	1172	1
133	0	341	0	549	1	757	0	965	0	1173	1
134	1	342	0	550	0	758	0	966	1	1174	1
135	1	343	1	551	0	759	1	967	1	1175	1
136	0	344	0	552	0	760	1	968	0	1176	1
137	1	345	1	553	1	761	1	969	0	1177	1
138	1	346	1	554	0	762	1	970	0	1178	1
139	1	347	1	555	1	763	1	971	0	1179	1
140	0	348	1	556	0	764	0	972	1	1180	0
141	1	349	0	557	0	765	1	973	1	1181	0
142	0	350	1	558	1	766	1	974	1	1182	1
143	1	351	0	559	1	767	0	975	0	1183	0
144	1	352	0	560	0	768	1	976	1	1184	0
145	1	353	1	561	1	769	0	977	0	1185	1
146	1	354	0	562	0	770	0	978	1	1186	1
147	1	355	0	563	0	771	1	979	1	1187	1
148	0	356	1	564	1	772	1	980	1	1188	1
149	1	357	0	565	0	773	1	981	1	1189	1
150	0	358	1	566	1	774	1	982	0	1190	1
151	1	359	0	567	1	775	0	983	1	1191	0
152	0	360	1	568	1	776	1	984	0	1192	1
153	0	361	0	569	1	777	1	985	0	1193	0
154	1	362	0	570	1	778	1	986	1	1194	0
155	1	363	0	571	1	779	1	987	0	1195	0
156	0	364	0	572	1	780	0	988	0	1196	0
157	0	365	1	573	0	781	0	989	0	1197	0
158	1	366	0	574	0	782	1	990	1	1198	0
159	1	367	0	575	1	783	0	991	0	1199	0
160	0	368	1	576	1	784	1	992	0	1200	1
161	1	369	1	577	1	785	0	993	0	1201	1
162	1	370	0	578	0	786	1	994	0	1202	1
163	0	371	1	579	1	787	1	995	1	1203	1
164	0	372	0	580	1	788	1	996	1	1204	0
165	1	373	0	581	1	789	1	997	0	1205	1
166	1	374	0	582	0	790	0	998	1	1206	1
167	0	375	0	583	1	791	0	999	0	1207	1
168	1	376	0	584	0	792	0	1000	0	1208	1
169	1	377	0	585	0	793	1	1001	0	1209	1
170	0	378	1	586	1	794	0	1002	1	1210	1
171	1	379	1	587	0	795	1	1003	0	1211	0
172	0	380	0	588	1	796	1	1004	0	1212	0
173	1	381	0	589	0	797	1	1005	0	1213	1
174	1	382	1	590	1	798	0	1006	0	1214	0

175	1	383	0	591	1	799	1	1007	1	1215	0
176	0	384	1	592	1	800	0	1008	0	1216	0
177	0	385	1	593	1	801	0	1009	1	1217	0
178	1	386	1	594	0	802	1	1010	1	1218	0
179	0	387	0	595	1	803	0	1011	0	1219	1
180	0	388	1	596	1	804	1	1012	1	1220	0
181	0	389	1	597	0	805	0	1013	0	1221	0
182	0	390	0	598	0	806	0	1014	0	1222	0
183	1	391	0	599	0	807	1	1015	0	1223	1
184	1	392	0	600	0	808	1	1016	1	1224	1
185	1	393	1	601	1	809	1	1017	0	1225	1
186	0	394	0	602	1	810	1	1018	1	1226	1
187	0	395	0	603	0	811	1	1019	0	1227	1
188	1	396	1	604	0	812	0	1020	0	1228	0
189	1	397	0	605	1	813	1	1021	0	1229	0
190	0	398	0	606	1	814	0	1022	0	1230	0
191	1	399	1	607	1	815	1	1023	0	1231	1
192	0	400	0	608	1	816	1	1024	1	1232	1
193	1	401	0	609	0	817	0	1025	0	1233	0
194	1	402	1	610	0	818	1	1026	1	1234	0
195	0	403	1	611	1	819	0	1027	1	1235	0
196	0	404	1	612	0	820	1	1028	1	1236	0
197	0	405	1	613	1	821	0	1029	1	1237	1
198	1	406	0	614	0	822	0	1030	1	1238	1
199	1	407	1	615	1	823	0	1031	0	1239	0
200	0	408	0	616	1	824	1	1032	0	1240	1
201	0	409	0	617	1	825	1	1033	0	1241	0
202	1	410	1	618	1	826	0	1034	1	1242	1
203	0	411	0	619	0	827	1	1035	1		
204	0	412	1	620	0	828	0	1036	0		
205	1	413	0	621	0	829	1	1037	1		
206	0	414	0	622	0	830	1	1038	1		
207	0	415	0	623	0	831	0	1039	0		

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