PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^5+z^3+z,z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^5+z^3+z,z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 15:46:58 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 15.9 minutes CPU time used.

Theorem 1.1. *When* $(a,b)=(z^5+z^3+z,z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^8+z^5+z+1,\\
p_1(z)&=z^{23}+z^{22}+z^{19}+z^{14}+z^{12}+z^{10}+z^9+z^8+z^7+z^4+z^2+z,\\
p_2(z)&=z^{24}+z^{22}+z^{16}+z^8+z^6+z^2,\\
p_3(z)&=z^{23}+z^{22}+z^{19}+z^{14}+z^{12}+z^{10}+z^9+z^8+z^7+z^4+z^2+z,\\
p_4(z)&=z^{22}+z^{18}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^8+z^6+z^5+z^4+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{12}+z^{11}+z^{10}+z^8+z^5+z,\\
p_1(z)&=z^{23}+z^{22}+z^{19}+z^{14}+z^{12}+z^{10}+z^9+z^8+z^7+z^4+z^2+z,\\
p_2(z)&=z^{24}+z^{22}+z^{16}+z^8+z^6+z^2,\\
p_3(z)&=z^{23}+z^{22}+z^{19}+z^{14}+z^{12}+z^{10}+z^9+z^8+z^7+z^4+z^2+z,\\
p_4(z)&=z^{22}+z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^5+z^4\\
&\quad+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{2u} M^e[:6u] \\ & + x^{3u} M^e[:5u] + x^{3u} M^e[:6u] + x^{4u} M^e[:5u] + x^{4u} M^e[:6u] \\ & + x^{5u} M^e[:5u] + x^{5u} M^e[:6u] + x^{6u} M^e[:5u] + x^{8u} M^e[:4u] \\ & + x^{10u} M^e[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u}) R^e, \\ W^e[:12u] = {} & W^e[:6u] + x^u W^e[:5u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{2u} W^e[:6u] \\ & + x^{3u} W^e[:5u] + x^{3u} W^e[:6u] + x^{4u} W^e[:5u] + x^{4u} W^e[:6u] \end{aligned}$$

$$+ x^{5u}W^e[:5u] + x^{5u}W^e[:6u] + x^{6u}W^e[:5u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^uM^o[:5u] + x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{2u}M^o[:6u] \\ &\quad + x^{3u}M^o[:5u] + x^{3u}M^o[:6u] + x^{4u}M^o[:5u] + x^{4u}M^o[:6u] \\ &\quad + x^{5u}M^o[:5u] + x^{5u}M^o[:6u] + x^{6u}M^o[:5u] + x^{8u}M^o[:4u] \\ &\quad + x^{10u}M^o[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^uW^o[:5u] + x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{2u}W^o[:6u] \\ &\quad + x^{3u}W^o[:5u] + x^{3u}W^o[:6u] + x^{4u}W^o[:5u] + x^{4u}W^o[:6u] \\ &\quad + x^{5u}W^o[:5u] + x^{5u}W^o[:6u] + x^{6u}W^o[:5u] + x^{8u}W^o[:4u] \\ &\quad + x^{10u}W^o[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] &= T[:6u] + x^uT[:5u] + x^uT[:6u] + x^{2u}T[:5u] + x^{2u}T[:6u] + x^{3u}T[:5u] \\ &\quad + x^{3u}T[:6u] + x^{4u}T[:5u] + x^{4u}T[:6u] + x^{5u}T[:5u] + x^{5u}T[:6u] \\ &\quad + x^{6u}T[:5u] + x^{8u}T[:4u] + x^{10u}T[:2u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{6u}T[:u] + x^uT[5u:6u] + T[6u:7u], \\ 0 &= x^{6u}T[u:2u] + x^{2u}T[5u:6u] + T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{6u}T[2u:3u] + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u], \\ 0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] \\ &\quad + T[10u:11u], \\ 0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[1w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$0=T[[1w]_2]+T[[101w]_2]+T[[111w]_2], \tag{2.14}$$
$$0=T[[w]_2]+T[[10w]_2]+T[[101w]_2]+T[[1000w]_2], \tag{2.15}$$
$$0=T[[1w]_2]+T[[11w]_2]+T[[101w]_2]+T[[1001w]_2], \tag{2.16}$$
$$0=T[[w]_2]+T[[10w]_2]+T[[100w]_2]+T[[101w]_2]+T[[1010w]_2], \tag{2.17}$$
$$0=T[[1w]_2]+T[[11w]_2]+T[[1011w]_2], \tag{2.18}$$

for $w=0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 653 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k}=\{A(i,w):|w|=2k, w\neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0=\tau(A(s,\epsilon))+\tau(A(s,101))+\tau(A(s,110)), \tag{2.19}$$
$$0=\tau(A(s,1))+\tau(A(s,101))+\tau(A(s,111)), \tag{2.20}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.21}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1001)), \tag{2.22}$$
$$\begin{aligned}0=\ &\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))\\ &+\tau(A(s,1010)),\end{aligned} \tag{2.23}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,1011)), \tag{2.24}$$

for all $s\in E_{2k}$, and

$$0=\tau(A(s,\epsilon))+\tau(A(s,101))+\tau(A(s,110)), \tag{2.25}$$
$$0=\tau(A(s,1))+\tau(A(s,101))+\tau(A(s,111)), \tag{2.26}$$
$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.27}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1001)), \tag{2.28}$$
$$\begin{aligned}0=\ &\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))\\ &+\tau(A(s,1010)),\end{aligned} \tag{2.29}$$
$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,1011)), \tag{2.30}$$

for $s=A(i,0^{2k})$. We find that $(A(i,0^{28}),E_{28})=(A(i,0^{16}),E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2\le k\le 14$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k\ge 2$, and $u=2^{2k-1}$,*

$$M_{2k}=M^e[:6u]+x^uM^e[:5u]+x^{6u}R^e,$$
$$W_{2k}=W^e[:6u]+x^uW^e[:5u]+x^{6u}R^e.$$

For all integer $k\ge 2$, and $u=2^{2k}$,

$$M_{2k+1}=M^o[:6u]+x^uM^o[:5u]+x^{6u}R^o,$$

$$W_{2k+1} = W^o[:6u] + x^u W^o[:5u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = &\big(W^e[:6v] + x^v W^e[:5v] + x^{6u} R^e\big) \times \big(M^e[:6v] + x^v M^e[:5v] \\ &+ x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{2v} W^e[:10v]M^e[:10v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y)=\mathrm{Res}_t\left(\phi(M^e,0,1)(x,t),y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x)=M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y)=q_4(x)y^4+q_3(x)y^3+q_2(x)y^2+q_1(x)y+q_0(x),$$

where

$$\begin{aligned}
q_0(x)&=x^{24}+x^{23}+x^{19}+x^{16}+x^{14}+x^{13}+x^9+x^8+x^7+x^6,\\
q_1(x)&=x^{23}+x^{22}+x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}+x^{10}+x^5+x^2+x,\\
q_2(x)&=x^{22}+x^{18}+x^{16}+x^8+x^2+1,\\
q_3(x)&=x^{23}+x^{22}+x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}+x^{10}+x^5+x^2+x,\\
q_4(x)&=x^{23}+x^{22}+x^{20}+x^{19}+x^{18}+x^{16}+x^{14}+x^{13}+x^9+x^8+x^7+x^6+x^2\\
&.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y):=P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x)=&\ x^{114}+x^{112}+x^{108}+x^{106}+x^{104}+x^{100}+x^{98}+x^{90}+x^{84}+x^{76}\\
&+x^{72}+x^{68}+x^{66}+x^{64}+x^{60}+x^{58}+x^{56}+x^{54}+x^{48}+x^{44}+x^{38}\\
&+x^{36}+x^{22}+x^{16}+x^{12},\\
p_3(x)=&\ x^{110}+x^{90}+x^{86}+x^{82}+x^{74}+x^{70}+x^{68}+x^{66}+x^{60}+x^{58}+x^{56}\\
&+x^{54}+x^{52}+x^{48}+x^{40}+x^{38}+x^{36}+x^{34}+x^{26}+x^{24}+x^{22}+x^{20}\\
&+x^{18}+x^{14}+x^{10}+x^8,\\
p_6(x)=&\ x^{108}+x^{102}+x^{96}+x^{94}+x^{92}+x^{88}+x^{84}+x^{80}+x^{76}+x^{74}+x^{66}\\
&+x^{64}+x^{62}+x^{56}+x^{52}+x^{48}+x^{42}+x^{34}+x^{32}+x^{24}+x^{22}+x^{20}\\
&+x^{18}+x^{16}+x^{14}+x^{10}+x^8+x^4+x^2+1,\\
p_9(x)=&\ x^{112}+x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{92}+x^{82}\\
&+x^{76}+x^{74}+x^{70}+x^{68}+x^{64}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{46}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{20} + x^{10} + x^{6} + x^{2},\\
p_{12}(x) &= x^{112} + x^{108} + x^{100} + x^{92} + x^{84} + x^{76} + x^{68} + x^{64} + x^{48} + x^{44} + x^{36}\\
&+ x^{32} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{109}\\
&+ x^{106} + x^{103} + x^{100} + x^{97} + x^{94} + x^{91} + x^{88} + x^{87} + x^{82} + x^{80} + x^{79}\\
&+ x^{77} + x^{76} + x^{75} + x^{72} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62}\\
&+ x^{61} + x^{57} + x^{54} + x^{53} + x^{50} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39}\\
&+ x^{37} + x^{35} + x^{34} + x^{32} + x^{29} + x^{24},\\
p_3(x) &= x^{124} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109}\\
&+ x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{95} + x^{94} + x^{91}\\
&+ x^{89} + x^{88} + x^{85} + x^{83} + x^{80} + x^{78} + x^{77} + x^{73} + x^{36} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{29} + x^{25} + x^{22} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9},\\
p_6(x) &= x^{123} + x^{122} + x^{113} + x^{112} + x^{107} + x^{106} + x^{101} + x^{100} + x^{95} + x^{94}\\
&+ x^{91} + x^{90} + x^{89} + x^{88} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{55} + x^{54}\\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{39} + x^{38} + x^{37}\\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^{7} + x^{6},\\
p_9(x) &= x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{102} + x^{97} + x^{94} + x^{92} + x^{91} + x^{90} + x^{85} + x^{82} + x^{79} + x^{78}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{66} + x^{65} + x^{63} + x^{62}\\
&+ x^{61} + x^{60} + x^{59} + x^{58} + x^{53} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{19} + x^{12} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{107} + x^{106}\\
&+ x^{105} + x^{104} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39}\\
&+ x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}\\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
p_0(x) = x^{126} + x^{125} + x^{123} + x^{122} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{109}
$$

$$
\begin{aligned}
&+x^{106}+x^{103}+x^{100}+x^{97}+x^{94}+x^{91}+x^{88}+x^{87}+x^{82}+x^{80}+x^{79}\\
&+x^{77}+x^{76}+x^{75}+x^{72}+x^{69}+x^{68}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}\\
&+x^{61}+x^{57}+x^{54}+x^{53}+x^{50}+x^{48}+x^{45}+x^{44}+x^{43}+x^{42}+x^{39}\\
&+x^{37}+x^{35}+x^{34}+x^{32}+x^{29}+x^{24},\\
p_3(x)={}&x^{124}+x^{121}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{112}+x^{111}+x^{109}\\
&+x^{108}+x^{107}+x^{105}+x^{103}+x^{102}+x^{100}+x^{99}+x^{95}+x^{94}+x^{91}\\
&+x^{89}+x^{88}+x^{85}+x^{83}+x^{80}+x^{78}+x^{77}+x^{73}+x^{36}+x^{33}+x^{32}\\
&+x^{31}+x^{30}+x^{29}+x^{25}+x^{22}+x^{19}+x^{17}+x^{16}+x^{14}+x^{13}+x^{9},\\
p_6(x)={}&x^{123}+x^{122}+x^{113}+x^{112}+x^{107}+x^{106}+x^{101}+x^{100}+x^{95}+x^{94}\\
&+x^{91}+x^{90}+x^{89}+x^{88}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{73}\\
&+x^{72}+x^{71}+x^{70}+x^{67}+x^{66}+x^{63}+x^{62}+x^{61}+x^{60}+x^{55}+x^{54}\\
&+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{39}+x^{38}+x^{37}\\
&+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}\\
&+x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{12}+x^{7}+x^{6},\\
p_9(x)={}&x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{111}+x^{110}+x^{108}+x^{107}+x^{106}\\
&+x^{104}+x^{102}+x^{97}+x^{94}+x^{92}+x^{91}+x^{90}+x^{85}+x^{82}+x^{79}+x^{78}\\
&+x^{77}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{66}+x^{65}+x^{63}+x^{62}\\
&+x^{61}+x^{60}+x^{59}+x^{58}+x^{53}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}\\
&+x^{40}+x^{39}+x^{37}+x^{35}+x^{34}+x^{32}+x^{31}+x^{28}+x^{27}+x^{26}+x^{25}\\
&+x^{24}+x^{19}+x^{12}+x^{9}+x^{8}+x^{7}+x^{5}+x^{3},\\
p_{12}(x)={}&x^{121}+x^{120}+x^{119}+x^{118}+x^{115}+x^{114}+x^{111}+x^{110}+x^{107}+x^{106}\\
&+x^{105}+x^{104}+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{90}+x^{89}+x^{88}+x^{81}\\
&+x^{80}+x^{79}+x^{78}+x^{75}+x^{74}+x^{71}+x^{70}+x^{67}+x^{66}+x^{65}+x^{64}\\
&+x^{57}+x^{56}+x^{55}+x^{54}+x^{51}+x^{50}+x^{47}+x^{46}+x^{43}+x^{42}+x^{39}\\
&+x^{38}+x^{35}+x^{34}+x^{33}+x^{32}+x^{25}+x^{24}+x^{23}+x^{22}+x^{19}+x^{18}\\
&+x^{15}+x^{14}+x^{11}+x^{10}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)={}&x^{138}+x^{128}+x^{126}+x^{124}+x^{114}+x^{110}+x^{106}+x^{104}+x^{102}+x^{98}\\
&+x^{96}+x^{94}+x^{90}+x^{86}+x^{84}+x^{76}+x^{70}+x^{66}+x^{60}+x^{56}+x^{50}\\
&+x^{48}+x^{44}+x^{40}+x^{36},\\
p_3(x)={}&x^{132}+x^{128}+x^{116}+x^{110}+x^{108}+x^{104}+x^{96}+x^{94}+x^{92}+x^{90}\\
&+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{78}+x^{74}+x^{70}+x^{68}+x^{66}+x^{64}\\
&+x^{62}+x^{50}+x^{48}+x^{42}+x^{38}+x^{36}+x^{34}+x^{20}+x^{18}+x^{8}+x^{6},\\
p_6(x)={}&x^{132}+x^{130}+x^{128}+x^{126}+x^{124}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{72}\\
&+ x^{70} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38}\\
&+ x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{20} + x^{14} + x^{10} + x^{6} + 1,\\
p_9(x) &= x^{130} + x^{126} + x^{122} + x^{110} + x^{108} + x^{104} + x^{96} + x^{94} + x^{84} + x^{80}\\
&+ x^{78} + x^{76} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50}\\
&+ x^{46} + x^{38} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{12} + x^{10}\\
&+ x^{4} + x^{2},\\
p_{12}(x) &= x^{130} + x^{128} + x^{98} + x^{96} + x^{50} + x^{48} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{134} + x^{132} + x^{130} + x^{128} + x^{124} + x^{120} + x^{118} + x^{110} + x^{106}\\
&+ x^{102} + x^{100} + x^{94} + x^{90} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{64}\\
&+ x^{58} + x^{56} + x^{52} + x^{50} + x^{42} + x^{32} + x^{30} + x^{20} + x^{18} + x^{16} + x^{14}\\
&+ x^{12},\\
p_3(x) &= x^{132} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{104} + x^{102} + x^{100} + x^{98}\\
&+ x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{76} + x^{64} + x^{60} + x^{58} + x^{54} + x^{48}\\
&+ x^{42} + x^{38} + x^{36} + x^{34} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14}\\
&+ x^{10} + x^{8},\\
p_6(x) &= x^{132} + x^{130} + x^{120} + x^{116} + x^{114} + x^{106} + x^{104} + x^{100} + x^{98} + x^{92}\\
&+ x^{90} + x^{88} + x^{84} + x^{78} + x^{74} + x^{72} + x^{70} + x^{66} + x^{64} + x^{62} + x^{56}\\
&+ x^{54} + x^{52} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24}\\
&+ x^{22} + x^{20} + x^{16} + x^{10} + x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{130} + x^{126} + x^{122} + x^{110} + x^{108} + x^{104} + x^{96} + x^{94} + x^{84} + x^{80}\\
&+ x^{78} + x^{76} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50}\\
&+ x^{46} + x^{38} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{12} + x^{10}\\
&+ x^{4} + x^{2},\\
p_{12}(x) &= x^{130} + x^{128} + x^{98} + x^{96} + x^{50} + x^{48} + x^{18} + x^{16} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{119} + x^{116} + x^{115} + x^{114} + x^{111} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{105} + x^{102} + x^{101} + x^{95} + x^{93} + x^{90} + x^{88} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{67} + x^{64}\\
&+ x^{62} + x^{61} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44}\\
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25} + x^{23} + x^{20}\\
&+ x^{19} + x^{18},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{124}+x^{121}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{112}+x^{111}+x^{109}\\
&\quad +x^{108}+x^{107}+x^{105}+x^{103}+x^{102}+x^{100}+x^{99}+x^{95}+x^{94}+x^{91}\\
&\quad +x^{89}+x^{88}+x^{85}+x^{83}+x^{80}+x^{78}+x^{77}+x^{73}+x^{36}+x^{33}+x^{32}\\
&\quad +x^{31}+x^{30}+x^{29}+x^{25}+x^{22}+x^{19}+x^{17}+x^{16}+x^{14}+x^{13}+x^{9},\\
p_6(x) &= x^{123}+x^{122}+x^{113}+x^{112}+x^{107}+x^{106}+x^{101}+x^{100}+x^{95}+x^{94}\\
&\quad +x^{91}+x^{90}+x^{89}+x^{88}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{73}\\
&\quad +x^{72}+x^{71}+x^{70}+x^{67}+x^{66}+x^{63}+x^{62}+x^{61}+x^{60}+x^{55}+x^{54}\\
&\quad +x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{39}+x^{38}+x^{37}\\
&\quad +x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}\\
&\quad +x^{21}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}+x^{13}+x^{12}+x^{7}+x^{6},\\
p_9(x) &= x^{122}+x^{120}+x^{119}+x^{118}+x^{116}+x^{111}+x^{110}+x^{108}+x^{107}+x^{106}\\
&\quad +x^{104}+x^{102}+x^{97}+x^{94}+x^{92}+x^{91}+x^{90}+x^{85}+x^{82}+x^{79}+x^{78}\\
&\quad +x^{77}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{66}+x^{65}+x^{63}+x^{62}\\
&\quad +x^{61}+x^{60}+x^{59}+x^{58}+x^{53}+x^{50}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}\\
&\quad +x^{40}+x^{39}+x^{37}+x^{35}+x^{34}+x^{32}+x^{31}+x^{28}+x^{27}+x^{26}+x^{25}\\
&\quad +x^{24}+x^{19}+x^{12}+x^{9}+x^{8}+x^{7}+x^{5}+x^{3},\\
p_{12}(x) &= x^{121}+x^{120}+x^{119}+x^{118}+x^{115}+x^{114}+x^{111}+x^{110}+x^{107}+x^{106}\\
&\quad +x^{105}+x^{104}+x^{97}+x^{96}+x^{95}+x^{94}+x^{91}+x^{90}+x^{89}+x^{88}+x^{81}\\
&\quad +x^{80}+x^{79}+x^{78}+x^{75}+x^{74}+x^{71}+x^{70}+x^{67}+x^{66}+x^{65}+x^{64}\\
&\quad +x^{57}+x^{56}+x^{55}+x^{54}+x^{51}+x^{50}+x^{47}+x^{46}+x^{43}+x^{42}+x^{39}\\
&\quad +x^{38}+x^{35}+x^{34}+x^{33}+x^{32}+x^{25}+x^{24}+x^{23}+x^{22}+x^{19}+x^{18}\\
&\quad +x^{15}+x^{14}+x^{11}+x^{10}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{123}+x^{119}+x^{116}+x^{115}+x^{114}+x^{111}+x^{110}+x^{109}+x^{108}\\
&\quad +x^{107}+x^{105}+x^{102}+x^{101}+x^{95}+x^{93}+x^{90}+x^{88}+x^{86}+x^{85}\\
&\quad +x^{84}+x^{83}+x^{82}+x^{80}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{67}+x^{64}\\
&\quad +x^{62}+x^{61}+x^{58}+x^{57}+x^{55}+x^{52}+x^{51}+x^{48}+x^{46}+x^{45}+x^{44}\\
&\quad +x^{40}+x^{38}+x^{36}+x^{32}+x^{31}+x^{28}+x^{27}+x^{26}+x^{25}+x^{23}+x^{20}\\
&\quad +x^{19}+x^{18},\\
p_3(x) &= x^{124}+x^{121}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{112}+x^{111}+x^{109}\\
&\quad +x^{108}+x^{107}+x^{105}+x^{103}+x^{102}+x^{100}+x^{99}+x^{95}+x^{94}+x^{91}\\
&\quad +x^{89}+x^{88}+x^{85}+x^{83}+x^{80}+x^{78}+x^{77}+x^{73}+x^{36}+x^{33}+x^{32}\\
&\quad +x^{31}+x^{30}+x^{29}+x^{25}+x^{22}+x^{19}+x^{17}+x^{16}+x^{14}+x^{13}+x^{9},\\
p_6(x) &= x^{123}+x^{122}+x^{113}+x^{112}+x^{107}+x^{106}+x^{101}+x^{100}+x^{95}+x^{94}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{91} + x^{90} + x^{89} + x^{88} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{55} + x^{54}\\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{39} + x^{38} + x^{37}\\
&+ x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26}\\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^{7} + x^{6},\\
p_9(x) = {}& x^{122} + x^{120} + x^{119} + x^{118} + x^{116} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{102} + x^{97} + x^{94} + x^{92} + x^{91} + x^{90} + x^{85} + x^{82} + x^{79} + x^{78}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{66} + x^{65} + x^{63} + x^{62}\\
&+ x^{61} + x^{60} + x^{59} + x^{58} + x^{53} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{28} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{19} + x^{12} + x^{9} + x^{8} + x^{7} + x^{5} + x^{3},\\
p_{12}(x) = {}& x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{107} + x^{106}\\
&+ x^{105} + x^{104} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39}\\
&+ x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}\\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{114} + x^{110} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{86} + x^{76} + x^{72}\\
&+ x^{70} + x^{56} + x^{54} + x^{46} + x^{42} + x^{40} + x^{38} + x^{36} + x^{30} + x^{24},\\
p_3(x) = {}& x^{108} + x^{104} + x^{92} + x^{90} + x^{80} + x^{78} + x^{76} + x^{48} + x^{44} + x^{42} + x^{40}\\
&+ x^{30} + x^{24} + x^{18} + x^{12} + x^{6},\\
p_6(x) = {}& x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{96} + x^{94} + x^{92} + x^{86} + x^{84} + x^{82}\\
&+ x^{80} + x^{78} + x^{72} + x^{70} + x^{62} + x^{58} + x^{54} + x^{40} + x^{38} + x^{30} + x^{26}\\
&+ x^{24} + x^{20} + x^{18} + x^{6} + x^{2} + 1,\\
p_9(x) = {}& x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{92} + x^{82}\\
&+ x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46}\\
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{20} + x^{10} + x^{6} + x^{2},\\
p_{12}(x) = {}& x^{112} + x^{108} + x^{100} + x^{92} + x^{84} + x^{76} + x^{68} + x^{64} + x^{48} + x^{44} + x^{36}\\
&+ x^{32} + x^{16} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{114} + x^{113}\\
&+ x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{86} + x^{85} + x^{81}\\
&+ x^{80} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{63} + x^{57} + x^{54} + x^{52} + x^{46} + x^{42} + x^{39} + x^{37} + x^{35}\\
&+ x^{34} + x^{31} + x^{29} + x^{28} + x^{26} + x^{23} + x^{22} + x^{20} + x^{19} + x^{13} + x^{12},\\
p_3(x) = {}& x^{125} + x^{120} + x^{119} + x^{117} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106}\\
&+ x^{102} + x^{100} + x^{94} + x^{91} + x^{90} + x^{88} + x^{85} + x^{84} + x^{80} + x^{78} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{53} + x^{52} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{22} + x^{19} + x^{18} + x^{17} + x^{14}\\
&+ x^{13} + x^{12} + x^{9} + x^{8},\\
p_6(x) = {}& x^{125} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106}\\
&+ x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68}\\
&+ x^{66} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{44} + x^{42} + x^{37} + x^{36} + x^{34} + x^{33} + x^{25} + x^{24}\\
&+ x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{13} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5}\\
&+ x^{4} + x^{3} + x^{2} + x + 1,\\
p_9(x) = {}& x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110}\\
&+ x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{94}\\
&+ x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77}\\
&+ x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{61} + x^{60} + x^{59} + x^{58}\\
&+ x^{57} + x^{56} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{37} + x^{36} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) = {}& x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35}\\
&+ x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x\\
&+ 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
p_0(x) = x^{140} + x^{137} + x^{135} + x^{133} + x^{132} + x^{131} + x^{128} + x^{127} + x^{124} + x^{123}
$$

$$
\begin{aligned}
&+x^{120}+x^{117}+x^{114}+x^{113}+x^{112}+x^{105}+x^{104}+x^{103}+x^{101}\\
&+x^{99}+x^{98}+x^{96}+x^{95}+x^{94}+x^{93}+x^{89}+x^{88}+x^{87}+x^{85}+x^{82}\\
&+x^{81}+x^{77}+x^{76}+x^{75}+x^{73}+x^{65}+x^{63}+x^{59}+x^{57}+x^{55}+x^{54}\\
&+x^{53}+x^{51}+x^{49}+x^{46}+x^{43}+x^{40}+x^{39}+x^{36}+x^{34}+x^{31}+x^{28}\\
&+x^{27}+x^{26}+x^{25}+x^{24},\\
p_3(x)=&\,x^{133}+x^{132}+x^{130}+x^{124}+x^{123}+x^{122}+x^{119}+x^{115}+x^{113}+x^{109}\\
&+x^{107}+x^{103}+x^{101}+x^{97}+x^{95}+x^{91}+x^{89}+x^{84}+x^{83}+x^{82}+x^{79}\\
&+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{45}+x^{44}+x^{42}+x^{37}+x^{35}+x^{31}\\
&+x^{28}+x^{26}+x^{25}+x^{23}+x^{21}+x^{17}+x^{15}+x^{13}+x^{12}+x^{11}+x^{10}\\
&+x^{9},\\
p_6(x)=&\,x^{134}+x^{132}+x^{128}+x^{116}+x^{114}+x^{112}+x^{110}+x^{108}+x^{106}+x^{104}\\
&+x^{98}+x^{88}+x^{80}+x^{76}+x^{72}+x^{66}+x^{64}+x^{58}+x^{56}+x^{54}+x^{52}\\
&+x^{50}+x^{46}+x^{44}+x^{42}+x^{38}+x^{34}+x^{26}+x^{22}+x^{14}+x^{8}+x^{6},\\
p_9(x)=&\,x^{135}+x^{134}+x^{133}+x^{130}+x^{129}+x^{128}+x^{125}+x^{123}+x^{122}+x^{121}\\
&+x^{120}+x^{118}+x^{116}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{99}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}\\
&+x^{85}+x^{83}+x^{82}+x^{81}+x^{80}+x^{78}+x^{76}+x^{70}+x^{69}+x^{68}+x^{67}\\
&+x^{66}+x^{65}+x^{64}+x^{61}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{52}\\
&+x^{50}+x^{49}+x^{48}+x^{46}+x^{44}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}\\
&+x^{32}+x^{29}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{17}\\
&+x^{16}+x^{14}+x^{12}+x^{7}+x^{4}+x^{3},\\
p_{12}(x)=&\,x^{136}+x^{132}+x^{124}+x^{112}+x^{108}+x^{92}+x^{88}+x^{84}+x^{72}+x^{68}+x^{64}\\
&+x^{60}+x^{56}+x^{52}+x^{48}+x^{40}+x^{36}+x^{32}+x^{28}+x^{24}+x^{20}+x^{16}+1\\
&,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1].

For $\phi(M^o,1,0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{122}+x^{121}+x^{120}+x^{117}+x^{116}+x^{114}+x^{112}+x^{111}+x^{110}+x^{109}\\
&+x^{106}+x^{105}+x^{102}+x^{101}+x^{99}+x^{97}+x^{96}+x^{95}+x^{92}+x^{91}\\
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}+x^{73}+x^{72}+x^{70}+x^{69}\\
&+x^{68}+x^{66}+x^{64}+x^{63}+x^{62}+x^{60}+x^{59}+x^{57}+x^{55}+x^{53}+x^{52}\\
&+x^{51}+x^{50}+x^{49}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}\\
&+x^{34}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{22}+x^{21}+x^{20}+x^{18},\\
p_3(x)=&\,x^{121}+x^{120}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{111}+x^{109}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}\\
&+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}+x^{83}+x^{80}+x^{78}+x^{76}+x^{75}+x^{74}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{73}+x^{33}+x^{32}+x^{30}+x^{29}+x^{28}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}\\
&+x^{18}+x^{17}+x^{16}+x^{14}+x^{12}+x^{11}+x^{10}+x^{9},\\
p_6(x)&=x^{122}+x^{120}+x^{118}+x^{114}+x^{110}+x^{102}+x^{98}+x^{88}+x^{80}+x^{76}\\
&+x^{72}+x^{60}+x^{54}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{36}+x^{32}+x^{30}\\
&+x^{26}+x^{20}+x^{16}+x^{12}+x^{10}+x^{8}+x^{6},\\
p_9(x)&=x^{123}+x^{122}+x^{121}+x^{119}+x^{116}+x^{115}+x^{113}+x^{112}+x^{110}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{97}+x^{96}+x^{94}+x^{91}+x^{90}+x^{89}\\
&+x^{88}+x^{87}+x^{81}+x^{80}+x^{78}+x^{72}+x^{68}+x^{67}+x^{59}+x^{58}+x^{57}\\
&+x^{55}+x^{52}+x^{51}+x^{49}+x^{48}+x^{46}+x^{40}+x^{36}+x^{35}+x^{27}+x^{26}\\
&+x^{25}+x^{23}+x^{20}+x^{19}+x^{17}+x^{16}+x^{14}+x^{8}+x^{4}+x^{3},\\
p_{12}(x)&=x^{124}+x^{112}+x^{104}+x^{100}+x^{88}+x^{84}+x^{76}+x^{72}+x^{68}+x^{64}+x^{60}\\
&+x^{48}+x^{44}+x^{40}+x^{36}+x^{32}+x^{28}+x^{16}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0].

For $\phi(M^o,1,1)$:

$$
\begin{aligned}
p_0(x)&=x^{126}+x^{124}+x^{121}+x^{119}+x^{117}+x^{116}+x^{113}+x^{112}+x^{108}+x^{107}\\
&+x^{106}+x^{100}+x^{99}+x^{97}+x^{93}+x^{92}+x^{91}+x^{89}+x^{86}+x^{84}+x^{79}\\
&+x^{78}+x^{74}+x^{73}+x^{70}+x^{67}+x^{66}+x^{65}+x^{62}+x^{61}+x^{60}+x^{59}\\
&+x^{57}+x^{52}+x^{49}+x^{48}+x^{46}+x^{43}+x^{42}+x^{41}+x^{39}+x^{37}+x^{35}\\
&+x^{34}+x^{33}+x^{32}+x^{30},\\
p_3(x)&=x^{125}+x^{123}+x^{122}+x^{121}+x^{119}+x^{117}+x^{115}+x^{114}+x^{113}+x^{111}\\
&+x^{110}+x^{108}+x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{101}+x^{95}+x^{93}\\
&+x^{92}+x^{89}+x^{87}+x^{86}+x^{83}+x^{82}+x^{80}+x^{79}+x^{71}+x^{70}+x^{65}\\
&+x^{64}+x^{63}+x^{62}+x^{59}+x^{58}+x^{57}+x^{56}+x^{53}+x^{52}+x^{47}+x^{46}\\
&+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}+x^{32}+x^{30}+x^{29}+x^{28}\\
&+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{16}+x^{15}+x^{13}+x^{12},\\
p_6(x)&=x^{125}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{117}+x^{114}+x^{111}+x^{109}\\
&+x^{108}+x^{107}+x^{104}+x^{102}+x^{98}+x^{97}+x^{96}+x^{94}+x^{90}+x^{81}\\
&+x^{80}+x^{79}+x^{77}+x^{76}+x^{73}+x^{72}+x^{70}+x^{67}+x^{63}+x^{56}+x^{49}\\
&+x^{48}+x^{46}+x^{45}+x^{42}+x^{41}+x^{40}+x^{34}+x^{32}+x^{27}+x^{26}+x^{25}\\
&+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12}\\
&+x^{10}+x^{9}+x^{8}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,\\
p_9(x)&=x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{113}+x^{112}+x^{111}+x^{110}\\
&+x^{107}+x^{106}+x^{103}+x^{102}+x^{101}+x^{100}+x^{97}+x^{96}+x^{95}+x^{94}\\
&+x^{91}+x^{90}+x^{87}+x^{86}+x^{85}+x^{84}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}\\
&+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{61}+x^{60}+x^{59}+x^{58}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{57} + x^{56} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{37} + x^{36} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) = {}& x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35}\\
&+ x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{114} + x^{113}\\
&+ x^{112} + x^{111} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{86} + x^{85} + x^{81}\\
&+ x^{80} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{63} + x^{57} + x^{54} + x^{52} + x^{46} + x^{42} + x^{39} + x^{37} + x^{35}\\
&+ x^{34} + x^{31} + x^{29} + x^{28} + x^{26} + x^{23} + x^{22} + x^{20} + x^{19} + x^{13} + x^{12},\\
p_3(x) = {}& x^{125} + x^{120} + x^{119} + x^{117} + x^{113} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106}\\
&+ x^{102} + x^{100} + x^{94} + x^{91} + x^{90} + x^{88} + x^{85} + x^{84} + x^{80} + x^{78} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{53} + x^{52} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{22} + x^{19} + x^{18} + x^{17} + x^{14}\\
&+ x^{13} + x^{12} + x^{9} + x^{8},\\
p_6(x) = {}& x^{125} + x^{118} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{106}\\
&+ x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68}\\
&+ x^{66} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{44} + x^{42} + x^{37} + x^{36} + x^{34} + x^{33} + x^{25} + x^{24}\\
&+ x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{13} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5}\\
&+ x^{4} + x^{3} + x^{2} + x + 1,\\
p_9(x) = {}& x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110}\\
&+ x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{94}\\
&+ x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{76}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{61}+x^{60}+x^{59}+x^{58}\\
&+x^{57}+x^{56}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{41}+x^{40}+x^{39}\\
&+x^{38}+x^{37}+x^{36}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{17}+x^{16}\\
&+x^{15}+x^{14}+x^{13}+x^{12}+x^{9}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4},\\
p_{12}(x)=\;&x^{125}+x^{124}+x^{123}+x^{122}+x^{119}+x^{118}+x^{115}+x^{114}+x^{113}+x^{112}\\
&+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{89}+x^{88}+x^{87}+x^{86}\\
&+x^{85}+x^{84}+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{69}+x^{68}+x^{67}\\
&+x^{66}+x^{65}+x^{64}+x^{61}+x^{60}+x^{59}+x^{58}+x^{55}+x^{54}+x^{51}+x^{50}\\
&+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36}+x^{35}\\
&+x^{34}+x^{33}+x^{32}+x^{29}+x^{28}+x^{27}+x^{26}+x^{23}+x^{22}+x^{19}+x^{18}\\
&+x^{17}+x^{16}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{5}+x^{4}+x^{3}+x^{2}+x\\
&+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_{0}(x)=\;&x^{122}+x^{121}+x^{120}+x^{117}+x^{116}+x^{114}+x^{112}+x^{111}+x^{110}+x^{109}\\
&+x^{106}+x^{105}+x^{102}+x^{101}+x^{99}+x^{97}+x^{96}+x^{95}+x^{92}+x^{91}\\
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}+x^{73}+x^{72}+x^{70}+x^{69}\\
&+x^{68}+x^{66}+x^{64}+x^{63}+x^{62}+x^{60}+x^{59}+x^{57}+x^{55}+x^{53}+x^{52}\\
&+x^{51}+x^{50}+x^{49}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}\\
&+x^{34}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{22}+x^{21}+x^{20}+x^{18},\\
p_{3}(x)=\;&x^{121}+x^{120}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}+x^{111}+x^{109}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}\\
&+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}+x^{83}+x^{80}+x^{78}+x^{76}+x^{75}+x^{74}\\
&+x^{73}+x^{33}+x^{32}+x^{30}+x^{29}+x^{28}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}\\
&+x^{18}+x^{17}+x^{16}+x^{14}+x^{12}+x^{11}+x^{10}+x^{9},\\
p_{6}(x)=\;&x^{122}+x^{120}+x^{118}+x^{114}+x^{110}+x^{102}+x^{98}+x^{88}+x^{80}+x^{76}\\
&+x^{72}+x^{60}+x^{54}+x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{36}+x^{32}+x^{30}\\
&+x^{26}+x^{20}+x^{16}+x^{12}+x^{10}+x^{8}+x^{6},\\
p_{9}(x)=\;&x^{123}+x^{122}+x^{121}+x^{119}+x^{116}+x^{115}+x^{113}+x^{112}+x^{110}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{97}+x^{96}+x^{94}+x^{91}+x^{90}+x^{89}\\
&+x^{88}+x^{87}+x^{81}+x^{80}+x^{78}+x^{72}+x^{68}+x^{67}+x^{59}+x^{58}+x^{57}\\
&+x^{55}+x^{52}+x^{51}+x^{49}+x^{48}+x^{46}+x^{40}+x^{36}+x^{35}+x^{27}+x^{26}\\
&+x^{25}+x^{23}+x^{20}+x^{19}+x^{17}+x^{16}+x^{14}+x^{8}+x^{4}+x^{3},\\
p_{12}(x)=\;&x^{124}+x^{112}+x^{104}+x^{100}+x^{88}+x^{84}+x^{76}+x^{72}+x^{68}+x^{64}+x^{60}\\
&+x^{48}+x^{44}+x^{40}+x^{36}+x^{32}+x^{28}+x^{16}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{140} + x^{137} + x^{135} + x^{133} + x^{132} + x^{131} + x^{128} + x^{127} + x^{124} + x^{123} \\
&\quad + x^{120} + x^{117} + x^{114} + x^{113} + x^{112} + x^{105} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} + x^{88} + x^{87} + x^{85} + x^{82} \\
&\quad + x^{81} + x^{77} + x^{76} + x^{75} + x^{73} + x^{65} + x^{63} + x^{59} + x^{57} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{51} + x^{49} + x^{46} + x^{43} + x^{40} + x^{39} + x^{36} + x^{34} + x^{31} + x^{28} \\
&\quad + x^{27} + x^{26} + x^{25} + x^{24}, \\
p_3(x) &= x^{133} + x^{132} + x^{130} + x^{124} + x^{123} + x^{122} + x^{119} + x^{115} + x^{113} + x^{109} \\
&\quad + x^{107} + x^{103} + x^{101} + x^{97} + x^{95} + x^{91} + x^{89} + x^{84} + x^{83} + x^{82} + x^{79} \\
&\quad + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{45} + x^{44} + x^{42} + x^{37} + x^{35} + x^{31} \\
&\quad + x^{28} + x^{26} + x^{25} + x^{23} + x^{21} + x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} \\
&\quad + x^{9}, \\
p_6(x) &= x^{134} + x^{132} + x^{128} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} \\
&\quad + x^{98} + x^{88} + x^{80} + x^{76} + x^{72} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{46} + x^{44} + x^{42} + x^{38} + x^{34} + x^{26} + x^{22} + x^{14} + x^{8} + x^{6}, \\
p_9(x) &= x^{135} + x^{134} + x^{133} + x^{130} + x^{129} + x^{128} + x^{125} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{120} + x^{118} + x^{116} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{99} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} \\
&\quad + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{76} + x^{70} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} \\
&\quad + x^{16} + x^{14} + x^{12} + x^{7} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{136} + x^{132} + x^{124} + x^{112} + x^{108} + x^{92} + x^{88} + x^{84} + x^{72} + x^{68} + x^{64} \\
&\quad + x^{60} + x^{56} + x^{52} + x^{48} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + 1 \\
&\quad ,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{121} + x^{119} + x^{117} + x^{116} + x^{113} + x^{112} + x^{108} + x^{107} \\
&\quad + x^{106} + x^{100} + x^{99} + x^{97} + x^{93} + x^{92} + x^{91} + x^{89} + x^{86} + x^{84} + x^{79} \\
&\quad + x^{78} + x^{74} + x^{73} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{57} + x^{52} + x^{49} + x^{48} + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{35} \\
&\quad + x^{34} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{111}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{95} + x^{93}\\
&+ x^{92} + x^{89} + x^{87} + x^{86} + x^{83} + x^{82} + x^{80} + x^{79} + x^{71} + x^{70} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{59} + x^{58} + x^{57} + x^{56} + x^{53} + x^{52} + x^{47} + x^{46}\\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{32} + x^{30} + x^{29} + x^{28}\\
&+ x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_6(x) = &\; x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{114} + x^{111} + x^{109}\\
&+ x^{108} + x^{107} + x^{104} + x^{102} + x^{98} + x^{97} + x^{96} + x^{94} + x^{90} + x^{81}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{70} + x^{67} + x^{63} + x^{56} + x^{49}\\
&+ x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{34} + x^{32} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12}\\
&+ x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,\\
p_9(x) = &\; x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{113} + x^{112} + x^{111} + x^{110}\\
&+ x^{107} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{94}\\
&+ x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77}\\
&+ x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{61} + x^{60} + x^{59} + x^{58}\\
&+ x^{57} + x^{56} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{37} + x^{36} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16}\\
&+ x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) = &\; x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35}\\
&+ x^{34} + x^{33} + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{19} + x^{18}\\
&+ x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{5} + x^{4} + x^{3} + x^{2} + x\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	131	[221, 222]	262	[381, 187]	393	[497, 498]	524	[44, 169]
1	[3, 4]	132	[223, 224]	263	[382, 361]	394	[499, 500]	525	[563, 196]
2	[5, 6]	133	[225, 226]	264	[132, 383]	395	[501, 502]	526	[157, 182]
3	[7, 8]	134	[227, 228]	265	[367, 123]	396	[200, 503]	527	[575, 348]
4	[9, 10]	135	[229, 230]	266	[384, 371]	397	[504, 505]	528	[579, 348]
5	[11, 12]	136	[231, 232]	267	[194, 346]	398	[226, 506]	529	[140, 396]

6	[13, 14]	137	[233, 234]	268	[385, 33]	399	[471, 272]	530	[140, 363]
7	[15, 16]	138	[235, 236]	269	[371, 14]	400	[507, 508]	531	[359, 36]
8	[17, 18]	139	[237, 238]	270	[386, 43]	401	[509, 510]	532	[139, 355]
9	[19, 20]	140	[239, 240]	271	[8, 56]	402	[511, 512]	533	[580, 380]
10	[21, 22]	141	[241, 242]	272	[349, 182]	403	[513, 514]	534	[132, 38]
11	[23, 24]	142	[243, 244]	273	[198, 198]	404	[515, 516]	535	[386, 46]
12	[25, 26]	143	[245, 246]	274	[387, 138]	405	[517, 518]	536	[581, 341]
13	[27, 28]	144	[247, 248]	275	[37, 42]	406	[519, 520]	537	[53, 343]
14	[29, 30]	145	[249, 250]	276	[57, 362]	407	[521, 306]	538	[387, 541]
15	[31, 32]	146	[251, 252]	277	[157, 157]	408	[325, 522]	539	[545, 345]
16	[33, 34]	147	[253, 254]	278	[184, 378]	409	[523, 115]	540	[582, 215]
17	[35, 36]	148	[255, 256]	279	[388, 136]	410	[524, 525]	541	[583, 584]
18	[37, 38]	149	[257, 258]	280	[43, 190]	411	[323, 526]	542	[585, 586]
19	[39, 40]	150	[259, 260]	281	[389, 171]	412	[527, 528]	543	[587, 588]
20	[41, 42]	151	[261, 262]	282	[390, 185]	413	[529, 530]	544	[589, 501]
21	[43, 44]	152	[263, 264]	283	[41, 38]	414	[531, 532]	545	[590, 591]
22	[45, 46]	153	[265, 266]	284	[188, 371]	415	[533, 534]	546	[592, 337]
23	[47, 48]	154	[267, 268]	285	[32, 391]	416	[295, 535]	547	[593, 594]
24	[49, 50]	155	[269, 270]	286	[392, 2]	417	[536, 537]	548	[230, 90]
25	[51, 52]	156	[96, 271]	287	[134, 57]	418	[538, 539]	549	[68, 595]
26	[53, 34]	157	[272, 273]	288	[393, 394]	419	[397, 164]	550	[596, 597]
27	[13, 54]	158	[158, 158]	289	[395, 44]	420	[540, 119]	551	[598, 599]
28	[55, 56]	159	[89, 221]	290	[126, 396]	421	[180, 346]	552	[600, 601]
29	[57, 58]	160	[274, 275]	291	[182, 157]	422	[418, 10]	553	[602, 603]
30	[59, 60]	161	[276, 277]	292	[342, 189]	423	[137, 541]	554	[604, 605]
31	[61, 62]	162	[278, 279]	293	[119, 396]	424	[361, 154]	555	[518, 330]
32	[63, 64]	163	[280, 281]	294	[397, 352]	425	[50, 58]	556	[606, 432]
33	[65, 66]	164	[282, 283]	295	[398, 129]	426	[143, 156]	557	[492, 607]
34	[67, 68]	165	[284, 285]	296	[399, 182]	427	[182, 182]	558	[608, 284]
35	[69, 70]	166	[286, 287]	297	[8, 130]	428	[542, 36]	559	[609, 610]
36	[71, 72]	167	[288, 289]	298	[400, 401]	429	[543, 187]	560	[525, 611]
37	[73, 74]	168	[254, 91]	299	[377, 350]	430	[33, 180]	561	[612, 420]
38	[75, 76]	169	[290, 291]	300	[402, 9]	431	[189, 46]	562	[613, 280]
39	[77, 78]	170	[292, 109]	301	[368, 343]	432	[544, 157]	563	[614, 615]
40	[79, 80]	171	[62, 293]	302	[403, 364]	433	[59, 121]	564	[616, 617]
41	[81, 82]	172	[294, 295]	303	[404, 143]	434	[45, 43]	565	[618, 619]
42	[83, 84]	173	[296, 290]	304	[405, 406]	435	[35, 360]	566	[424, 464]
43	[85, 86]	174	[4, 297]	305	[188, 190]	436	[545, 413]	567	[620, 621]
44	[87, 4]	175	[298, 299]	306	[407, 60]	437	[131, 175]	568	[622, 623]
45	[88, 89]	176	[300, 301]	307	[408, 363]	438	[414, 345]	569	[624, 625]
46	[90, 29]	177	[302, 303]	308	[123, 351]	439	[546, 8]	570	[626, 627]
47	[91, 92]	178	[304, 305]	309	[409, 121]	440	[168, 411]	571	[611, 628]
48	[93, 94]	179	[306, 307]	310	[197, 60]	441	[385, 372]	572	[599, 309]
49	[95, 96]	180	[64, 308]	311	[121, 59]	442	[547, 140]	573	[514, 629]

50	[97, 98]	181	[309, 310]	312	[410, 411]	443	[548, 189]	574	[532, 203]
51	[99, 100]	182	[310, 311]	313	[370, 46]	444	[408, 142]	575	[630, 631]
52	[101, 102]	183	[312, 313]	314	[412, 2]	445	[549, 550]	576	[632, 633]
53	[103, 104]	184	[314, 315]	315	[149, 413]	446	[149, 345]	577	[334, 249]
54	[105, 106]	185	[316, 317]	316	[382, 153]	447	[366, 354]	578	[634, 635]
55	[107, 108]	186	[318, 319]	317	[414, 150]	448	[414, 187]	579	[636, 637]
56	[109, 110]	187	[320, 321]	318	[415, 185]	449	[546, 195]	580	[496, 527]
57	[111, 112]	188	[322, 323]	319	[375, 143]	450	[164, 348]	581	[455, 523]
58	[113, 114]	189	[324, 325]	320	[155, 144]	451	[352, 348]	582	[562, 164]
59	[115, 116]	190	[326, 327]	321	[50, 144]	452	[551, 552]	583	[133, 176]
60	[116, 117]	191	[328, 329]	322	[181, 157]	453	[369, 346]	584	[149, 187]
61	[118, 119]	192	[330, 331]	323	[416, 363]	454	[553, 138]	585	[580, 568]
62	[120, 121]	193	[212, 332]	324	[417, 8]	455	[554, 50]	586	[414, 413]
63	[122, 123]	194	[333, 334]	325	[410, 54]	456	[555, 172]	587	[569, 360]
64	[124, 54]	195	[335, 336]	326	[180, 350]	457	[556, 191]	588	[540, 126]
65	[59, 125]	196	[337, 338]	327	[123, 165]	458	[53, 194]	589	[125, 125]
66	[126, 127]	197	[339, 211]	328	[7, 418]	459	[557, 144]	590	[151, 40]
67	[128, 129]	198	[311, 340]	329	[124, 411]	460	[13, 411]	591	[354, 140]
68	[9, 130]	199	[170, 195]	330	[120, 60]	461	[347, 163]	592	[638, 389]
69	[131, 48]	200	[173, 190]	331	[154, 362]	462	[558, 123]	593	[402, 418]
70	[132, 42]	201	[190, 14]	332	[416, 142]	463	[418, 130]	594	[384, 159]
71	[133, 134]	202	[341, 33]	333	[159, 169]	464	[157, 198]	595	[164, 351]
72	[135, 136]	203	[125, 59]	334	[160, 165]	465	[154, 58]	596	[551, 394]
73	[137, 138]	204	[198, 158]	335	[390, 378]	466	[559, 152]	597	[168, 14]
74	[139, 140]	205	[342, 45]	336	[344, 413]	467	[388, 383]	598	[639, 570]
75	[141, 142]	206	[179, 343]	337	[179, 194]	468	[560, 175]	599	[404, 50]
76	[143, 144]	207	[39, 152]	338	[389, 391]	469	[561, 383]	600	[393, 552]
77	[145, 2]	208	[344, 345]	339	[409, 60]	470	[562, 352]	601	[53, 180]
78	[146, 136]	209	[343, 346]	340	[60, 125]	471	[182, 198]	602	[640, 550]
79	[147, 148]	210	[347, 348]	341	[419, 312]	472	[377, 161]	603	[146, 383]
80	[149, 150]	211	[349, 157]	342	[420, 85]	473	[563, 56]	604	[641, 418]
81	[151, 152]	212	[60, 59]	343	[421, 422]	474	[147, 547]	605	[391, 180]
82	[153, 154]	213	[194, 350]	344	[423, 424]	475	[135, 383]	606	[547, 355]
83	[155, 156]	214	[172, 192]	345	[425, 426]	476	[564, 56]	607	[642, 171]
84	[140, 142]	215	[171, 180]	346	[104, 427]	477	[400, 406]	608	[638, 32]
85	[157, 158]	216	[164, 163]	347	[428, 429]	478	[13, 169]	609	[640, 570]
86	[119, 127]	217	[44, 14]	348	[430, 431]	479	[374, 152]	610	[365, 150]
87	[159, 14]	218	[162, 348]	349	[432, 322]	480	[565, 119]	611	[195, 10]
88	[122, 160]	219	[125, 60]	350	[332, 433]	481	[566, 341]	612	[553, 541]
89	[128, 161]	220	[32, 171]	351	[206, 434]	482	[140, 127]	613	[31, 389]
90	[59, 59]	221	[160, 351]	352	[435, 436]	483	[50, 156]	614	[542, 360]
91	[162, 163]	222	[352, 351]	353	[437, 438]	484	[7, 9]	615	[643, 136]
92	[164, 165]	223	[353, 185]	354	[439, 243]	485	[567, 195]	616	[644, 138]
93	[166, 167]	224	[354, 355]	355	[440, 441]	486	[371, 169]	617	[543, 150]

94	[168, 169]	225	[356, 32]	356	[442, 339]	487	[379, 568]	618	[641, 9]
95	[170, 8]	226	[343, 350]	357	[443, 430]	488	[561, 136]	619	[192, 44]
96	[171, 34]	227	[357, 164]	358	[273, 444]	489	[569, 36]	620	[556, 172]
97	[168, 54]	228	[358, 159]	359	[445, 446]	490	[554, 143]	621	[395, 371]
98	[172, 173]	229	[359, 360]	360	[447, 448]	491	[417, 195]	622	[166, 52]
99	[174, 175]	230	[361, 57]	361	[449, 113]	492	[398, 161]	623	[369, 350]
100	[176, 154]	231	[50, 362]	362	[228, 450]	493	[549, 570]	624	[412, 364]
101	[177, 178]	232	[355, 363]	363	[256, 451]	494	[135, 42]	625	[135, 38]
102	[179, 180]	233	[145, 364]	364	[452, 453]	495	[399, 157]	626	[11, 178]
103	[181, 182]	234	[365, 187]	365	[454, 455]	496	[564, 196]	627	[377, 346]
104	[183, 156]	235	[366, 139]	366	[456, 213]	497	[571, 380]	628	[579, 163]
105	[183, 144]	236	[132, 136]	367	[457, 458]	498	[49, 143]	629	[373, 196]
106	[125, 121]	237	[367, 160]	368	[204, 459]	499	[177, 12]	630	[559, 40]
107	[184, 185]	238	[368, 194]	369	[460, 461]	500	[188, 44]	631	[186, 150]
108	[186, 187]	239	[369, 161]	370	[462, 326]	501	[407, 121]	632	[581, 385]
109	[188, 159]	240	[191, 173]	371	[329, 463]	502	[557, 156]	633	[395, 159]
110	[189, 43]	241	[34, 346]	372	[464, 465]	503	[195, 56]	634	[567, 8]
111	[190, 169]	242	[370, 43]	373	[466, 467]	504	[572, 172]	635	[34, 350]
112	[191, 192]	243	[358, 371]	374	[468, 469]	505	[372, 34]	636	[644, 541]
113	[193, 194]	244	[8, 196]	375	[470, 247]	506	[55, 196]	637	[643, 383]
114	[195, 196]	245	[369, 129]	376	[74, 471]	507	[573, 568]	638	[287, 495]
115	[197, 121]	246	[341, 372]	377	[472, 473]	508	[176, 57]	639	[473, 513]
116	[158, 198]	247	[173, 44]	378	[474, 475]	509	[51, 167]	640	[218, 645]
117	[121, 125]	248	[352, 163]	379	[222, 476]	510	[395, 190]	641	[646, 647]
118	[199, 200]	249	[195, 130]	380	[477, 478]	511	[574, 385]	642	[648, 333]
119	[201, 202]	250	[373, 56]	381	[479, 480]	512	[372, 180]	643	[649, 650]
120	[203, 103]	251	[374, 40]	382	[481, 245]	513	[352, 165]	644	[651, 652]
121	[117, 204]	252	[375, 50]	383	[482, 483]	514	[575, 163]	645	[8, 10]
122	[205, 206]	253	[376, 45]	384	[340, 105]	515	[576, 160]	646	[572, 191]
123	[207, 208]	254	[377, 129]	385	[484, 67]	516	[192, 190]	647	[46, 190]
124	[209, 210]	255	[357, 352]	386	[485, 486]	517	[577, 550]	648	[558, 160]
125	[211, 212]	256	[193, 343]	387	[487, 488]	518	[134, 154]	649	[415, 378]
126	[213, 214]	257	[353, 378]	388	[489, 490]	519	[405, 401]	650	[565, 126]
127	[215, 216]	258	[153, 57]	389	[491, 492]	520	[179, 34]	651	[560, 48]
128	[217, 218]	259	[141, 363]	390	[493, 494]	521	[544, 182]	652	[381, 150]
129	[66, 219]	260	[355, 142]	391	[495, 276]	522	[578, 171]		
130	[16, 220]	261	[379, 380]	392	[476, 496]	523	[182, 158]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	94	1	188	0	282	1	376	1	470	0	564	0
1	0	95	0	189	1	283	0	377	1	471	0	565	1
2	1	96	1	190	0	284	0	378	1	472	1	566	0
3	0	97	1	191	0	285	0	379	0	473	0	567	1

4	1	98	1	192	1	286	0	380	1	474	1	568	0
5	1	99	1	193	0	287	0	381	0	475	1	569	1
6	0	100	1	194	1	288	1	382	0	476	0	570	1
7	0	101	0	195	1	289	1	383	0	477	1	571	1
8	0	102	1	196	1	290	1	384	0	478	0	572	1
9	1	103	0	197	0	291	0	385	0	479	0	573	1
10	1	104	1	198	1	292	0	386	1	480	1	574	1
11	1	105	1	199	0	293	0	387	0	481	0	575	1
12	1	106	1	200	0	294	1	388	1	482	0	576	1
13	0	107	1	201	0	295	0	389	1	483	1	577	0
14	0	108	0	202	1	296	0	390	1	484	0	578	1
15	0	109	0	203	1	297	0	391	0	485	1	579	0
16	0	110	1	204	1	298	1	392	0	486	0	580	0
17	0	111	0	205	0	299	1	393	1	487	0	581	1
18	1	112	0	206	1	300	1	394	0	488	1	582	0
19	1	113	0	207	1	301	1	395	1	489	1	583	1
20	0	114	1	208	1	302	0	396	0	490	1	584	0
21	1	115	0	209	0	303	1	397	1	491	1	585	0
22	0	116	0	210	0	304	0	398	0	492	0	586	1
23	1	117	1	211	1	305	0	399	0	493	1	587	1
24	0	118	0	212	0	306	1	400	1	494	1	588	0
25	1	119	0	213	1	307	1	401	1	495	0	589	1
26	0	120	1	214	1	308	1	402	1	496	0	590	0
27	0	121	1	215	1	309	0	403	0	497	1	591	0
28	1	122	0	216	1	310	0	404	1	498	0	592	0
29	0	123	1	217	1	311	1	405	0	499	0	593	1
30	0	124	0	218	1	312	1	406	0	500	0	594	0
31	0	125	1	219	1	313	0	407	1	501	1	595	1
32	0	126	1	220	0	314	1	408	1	502	0	596	0
33	0	127	1	221	0	315	0	409	0	503	1	597	1
34	1	128	1	222	0	316	0	410	1	504	1	598	0
35	0	129	1	223	0	317	1	411	0	505	1	599	1
36	1	130	0	224	0	318	0	412	1	506	1	600	1
37	1	131	0	225	1	319	0	413	0	507	1	601	0
38	1	132	0	226	0	320	0	414	1	508	1	602	1
39	1	133	1	227	0	321	1	415	0	509	1	603	0
40	1	134	0	228	1	322	0	416	0	510	1	604	1
41	0	135	1	229	1	323	0	417	1	511	1	605	0
42	0	136	1	230	0	324	1	418	0	512	1	606	1
43	1	137	1	231	1	325	1	419	1	513	0	607	0
44	1	138	0	232	1	326	0	420	0	514	1	608	0
45	0	139	1	233	1	327	1	421	0	515	1	609	1
46	0	140	0	234	1	328	0	422	0	516	1	610	1
47	1	141	1	235	0	329	0	423	1	517	0	611	1

48	0	142	1	236	0	330	1	424	0	518	0	612	1
49	0	143	0	237	1	331	1	425	1	519	0	613	0
50	1	144	0	238	1	332	0	426	0	520	1	614	0
51	1	145	1	239	0	333	1	427	0	521	1	615	0
52	0	146	0	240	0	334	0	428	0	522	1	616	0
53	0	147	1	241	1	335	1	429	1	523	0	617	1
54	1	148	0	242	0	336	1	430	0	524	1	618	1
55	1	149	0	243	1	337	1	431	1	525	0	619	1
56	0	150	1	244	0	338	1	432	1	526	1	620	1
57	0	151	0	245	0	339	0	433	0	527	1	621	1
58	0	152	0	246	1	340	0	434	0	528	0	622	0
59	0	153	1	247	0	341	1	435	0	529	0	623	0
60	0	154	1	248	0	342	0	436	0	530	0	624	1
61	0	155	0	249	1	343	0	437	0	531	1	625	1
62	1	156	1	250	1	344	1	438	1	532	1	626	1
63	0	157	1	251	0	345	1	439	0	533	0	627	1
64	0	158	0	252	0	346	1	440	1	534	0	628	0
65	0	159	1	253	1	347	0	441	0	535	1	629	1
66	1	160	0	254	1	348	0	442	1	536	1	630	1
67	1	161	0	255	0	349	1	443	0	537	0	631	0
68	1	162	1	256	0	350	0	444	1	538	0	632	1
69	0	163	1	257	0	351	1	445	1	539	0	633	1
70	0	164	1	258	1	352	0	446	0	540	0	634	1
71	1	165	0	259	1	353	0	447	0	541	1	635	1
72	1	166	0	260	1	354	0	448	1	542	0	636	0
73	1	167	1	261	0	355	1	449	0	543	1	637	0
74	1	168	1	262	0	356	1	450	1	544	1	638	0
75	1	169	1	263	0	357	0	451	0	545	0	639	0
76	0	170	0	264	0	358	1	452	0	546	0	640	1
77	1	171	1	265	1	359	1	453	0	547	1	641	1
78	0	172	1	266	0	360	0	454	1	548	0	642	0
79	1	173	0	267	1	361	0	455	1	549	1	643	0
80	0	174	1	268	0	362	1	456	0	550	0	644	0
81	0	175	1	269	0	363	0	457	1	551	0	645	0
82	1	176	1	270	1	364	0	458	0	552	1	646	1
83	0	177	0	271	0	365	1	459	0	553	1	647	0
84	0	178	0	272	1	366	0	460	0	554	1	648	0
85	1	179	1	273	1	367	1	461	0	555	0	649	0
86	0	180	0	274	0	368	1	462	0	556	1	650	1
87	1	181	0	275	1	369	0	463	0	557	0	651	0
88	0	182	0	276	0	370	0	464	1	558	0	652	0
89	1	183	1	277	1	371	0	465	1	559	1		
90	0	184	1	278	1	372	1	466	1	560	0		
91	1	185	0	279	1	373	1	467	1	561	1		

92	1	186	0	280	1	374	0	468	0	562	0		
93	0	187	0	281	1	375	0	469	1	563	0		

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