PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^4 + z^3 + z, z^2 + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^4 + z^3 + z, z^2 + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^4+z^3+z, z^2+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{18}+z^{17}+z^{15}+z^{14}+z^{13}+z^{10}+z^9+z^8+z^7+z^6+z^4+z^2+z+1,\\
p_1(z) &= z^{22}+z^{19}+z^{14}+z^{13}+z^{12}+z^{11}+z^7+z^3+z^2+z,\\
p_2(z) &= z^{24}+z^{14}+z^{12}+z^{10}+z^6+z^2,\\
p_3(z) &= z^{22}+z^{19}+z^{14}+z^{13}+z^{12}+z^{11}+z^7+z^3+z^2+z,\\
p_4(z) &= z^{20}+z^{18}+z^{17}+z^{15}+z^{13}+z^9+z^7+z^4+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{18}+z^{17}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^9+z^7+z^6+z^2+z,\\
p_1(z) &= z^{22}+z^{19}+z^{14}+z^{13}+z^{12}+z^{11}+z^7+z^3+z^2+z,\\
p_2(z) &= z^{24}+z^{14}+z^{12}+z^{10}+z^6+z^2,\\
p_3(z) &= z^{22}+z^{19}+z^{14}+z^{13}+z^{12}+z^{11}+z^7+z^3+z^2+z,\\
p_4(z) &= z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{13}+z^{12}+z^9+z^8+z^7+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) = {} & x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
& \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) = {} & x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
& \cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] &= M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{4u} M^e[:2u] + x^u M^e[:6u] \\
&\quad + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{3u} M^e[:6u] \\
&\quad + x^{4u} M^e[:5u] + x^{5u} M^e[:4u] + x^{7u} M^e[:2u] + x^{6u} M^e[:6u] \\
&\quad + x^{8u} M^e[:4u] + (x^{6u} + x^{7u} + x^{9u}) R^e, \\
W^e[:12u] &= W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{4u} W^e[:2u] + x^u W^e[:6u] \\
&\quad + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{3u} W^e[:6u]
\end{aligned}$$

$$+ x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u] + x^{6u}W^e[:6u] + x^{8u}W^e[:4u] + (x^{6u} + x^{7u} + x^{9u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^uM^o[:5u] + x^{2u}M^o[:4u] + x^{4u}M^o[:2u] + x^uM^o[:6u] \\ &\quad + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{3u}M^o[:6u] \\ &\quad + x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u] + x^{6u}M^o[:6u] \\ &\quad + x^{8u}M^o[:4u] + (x^{6u} + x^{7u} + x^{9u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^uW^o[:5u] + x^{2u}W^o[:4u] + x^{4u}W^o[:2u] + x^uW^o[:6u] \\ &\quad + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{3u}W^o[:6u] \\ &\quad + x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u] + x^{6u}W^o[:6u] \\ &\quad + x^{8u}W^o[:4u] + (x^{6u} + x^{7u} + x^{9u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] = T[:6u] &+ x^uT[:5u] + x^{2u}T[:4u] + x^{4u}T[:2u] + x^uT[:6u] + x^{2u}T[:5u] \\ &+ x^{3u}T[:4u] + x^{5u}T[:2u] + x^{3u}T[:6u] + x^{4u}T[:5u] + x^{5u}T[:4u] \\ &+ x^{7u}T[:2u] + x^{6u}T[:6u] + x^{8u}T[:4u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{6u}T[:u] + x^{4u}T[2u:3u] + x^{2u}T[4u:5u] + x^uT[5u:6u] + T[6u:7u], \\ 0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] \\ &\quad + x^{3u}T[4u:5u] + T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] \\ &\quad + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + T[9u:10u], \\ 0 &= x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u], \\ 0 &= x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$\begin{aligned} 0 = T[[w]_2] &+ T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &+ T[[111w]_2], \end{aligned} \tag{2.8}$$

$$\begin{aligned} 0 = T[[w]_2] &+ T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &+ T[[101w]_2] + T[[1000w]_2], \end{aligned} \tag{2.9}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[111w]_2], \end{aligned} \tag{2.14}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[101w]_2] + T[[1000w]_2], \end{aligned} \tag{2.15}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1419 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,110)), \end{aligned} \tag{2.19}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.20}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1001)), \tag{2.22}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.23}$$

$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ & + \tau(A(s,110)), \end{aligned} \tag{2.25}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.27}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1001)), \tag{2.28}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.29}$$

$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{34}), E_{34}) = (A(i, 0^{20}), E_{20})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 17$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{4u} M^e[:2u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{4u} W^e[:2u] + x^{6u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^u M^o[:5u] + x^{2u} M^o[:4u] + x^{4u} M^o[:2u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^u W^o[:5u] + x^{2u} W^o[:4u] + x^{4u} W^o[:2u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k} M_{2k} = \big(W^e[:6v] + x^v W^e[:5v] + x^{2v} W^e[:4v] + x^{4v} W^e[:2v] + x^{6u} R^e\big) \times \big($$
$$M^e[:6v] + x^v M^e[:5v] + x^{2v} M^e[:4v] + x^{4v} M^e[:2v] + x^{6u} R^e\big); \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v] M^e[:12v] + x^{2v} W^e[:10v] M^e[:10v] + x^{4v} W^e[:8v] M^e[:8v] + x^{8v} W^e[:4v] M^e[:4v].$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X := x^v,$$
$$a_n := W^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$b_n := M^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$c := R^e.$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned}
\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\
\lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\
\lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\
\lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.
\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^9\\
&\quad + x^7 + x^6,\\
q_1(x) &= x^{23} + x^{22} + x^{21} + x^{17} + x^{13} + x^{12} + x^{11} + x^{10} + x^5 + x^2,\\
q_2(x) &= x^{22} + x^{18} + x^{14} + x^{12} + x^{10} + 1,\\
q_3(x) &= x^{23} + x^{22} + x^{21} + x^{17} + x^{13} + x^{12} + x^{11} + x^{10} + x^5 + x^2,\\
q_4(x) &= x^{23} + x^{20} + x^{17} + x^{15} + x^{11} + x^9 + x^7 + x^6 + x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{116} + x^{110} + x^{108} + x^{102} + x^{96} + x^{94} + x^{92} + x^{82} + x^{76} + x^{74} \\
&\quad + x^{70} + x^{64} + x^{62} + x^{60} + x^{50} + x^{46} + x^{42} + x^{40} + x^{36} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{112} + x^{104} + x^{102} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} \\
&\quad + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{40} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{26} + x^{22} + x^{18}, \\
p_6(x) &= x^{116} + x^{112} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{84} + x^{76} + x^{74} + x^{68} \\
&\quad + x^{62} + x^{60} + x^{52} + x^{50} + x^{46} + x^{44} + x^{32} + x^{24} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{14} + x^{10} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{98} + x^{94} + x^{90} \\
&\quad + x^{88} + x^{86} + x^{82} + x^{80} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} \\
&\quad + x^{50} + x^{46} + x^{34} + x^{32} + x^{26} + x^{22} + x^{18} + x^{10} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{118} + x^{114} + x^{112} + x^{102} + x^{98} + x^{96} + x^{86} + x^{82} + x^{80} + x^{54} + x^{50} \\
&\quad + x^{48} + x^{38} + x^{34} + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} \\
&\quad + x^{113} + x^{110} + x^{108} + x^{106} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} \\
&\quad + x^{92} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{73} + x^{71} + x^{69} + x^{66} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{49} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{30}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} \\
&\quad + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{104} + x^{102} + x^{100} + x^{97} + x^{95} \\
&\quad + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} \\
&\quad + x^{72} + x^{70} + x^{68} + x^{65} + x^{63} + x^{60} + x^{58} + x^{56} + x^{53} + x^{52} + x^{46} \\
&\quad + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} \\
&\quad + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{119} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} \\
&\quad + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{91} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12}, \\
p_9(x) &= x^{122} + x^{121} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{106} + x^{105} + x^{101}
\end{aligned}$$

$$+ x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{45} + x^{43} + x^{40} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{27} + x^{26} + x^{22} + x^{21} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},$$

$$p_{12}(x) = x^{121} + x^{118} + x^{117} + x^{116} + x^{109} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{89} + x^{86} + x^{85} + x^{84} + x^{77} + x^{74} + x^{73} + x^{72} + x^{69} + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{17} + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{126} + x^{125} + x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{108} + x^{106} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{92} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{69} + x^{66} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{49} + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{30},$$

$$p_3(x) = x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{104} + x^{102} + x^{100} + x^{97} + x^{95} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{72} + x^{70} + x^{68} + x^{65} + x^{63} + x^{60} + x^{58} + x^{56} + x^{53} + x^{52} + x^{46} + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18},$$

$$p_6(x) = x^{123} + x^{121} + x^{120} + x^{119} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{91} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12},$$

$$p_9(x) = x^{122} + x^{121} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{106} + x^{105} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{45} + x^{43} + x^{40} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{27} + x^{26} + x^{22} + x^{21} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},$$

$$\begin{aligned}p_{12}(x) = {} & x^{121} + x^{118} + x^{117} + x^{116} + x^{109} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\ & + x^{97} + x^{96} + x^{89} + x^{86} + x^{85} + x^{84} + x^{77} + x^{74} + x^{73} + x^{72} + x^{69} \\ & + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} \\ & + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{17} \\ & + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}p_0(x) = {} & x^{132} + x^{130} + x^{128} + x^{124} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} \\ & + x^{102} + x^{96} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{66} + x^{60} \\ & + x^{58} + x^{52} + x^{48} + x^{42} + x^{38} + x^{36}, \\ p_3(x) = {} & x^{126} + x^{124} + x^{122} + x^{118} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} + x^{94} \\ & + x^{92} + x^{78} + x^{72} + x^{58} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{32} \\ & + x^{30} + x^{24} + x^{20} + x^{16} + x^{12}, \\ p_6(x) = {} & x^{126} + x^{118} + x^{102} + x^{96} + x^{94} + x^{88} + x^{76} + x^{74} + x^{72} + x^{68} + x^{62} \\ & + x^{60} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{26} + x^{20} + x^{18} + x^{16} \\ & + x^{10} + 1, \\ p_9(x) = {} & x^{124} + x^{122} + x^{120} + x^{114} + x^{112} + x^{110} + x^{106} + x^{102} + x^{96} + x^{94} \\ & + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{72} + x^{68} + x^{58} + x^{54} \\ & + x^{50} + x^{48} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{14} + x^{12} \\ & + x^{10} + x^{4}, \\ p_{12}(x) = {} & x^{124} + x^{120} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{68} + x^{64} + x^{60} \\ & + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{20} + x^{16} + x^{12} + x^{8} + x^{4} + 1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}p_0(x) = {} & x^{132} + x^{130} + x^{124} + x^{122} + x^{118} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} \\ & + x^{96} + x^{88} + x^{82} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} \\ & + x^{52} + x^{46} + x^{44} + x^{40} + x^{38} + x^{32} + x^{30} + x^{24}, \\ p_3(x) = {} & x^{126} + x^{124} + x^{120} + x^{114} + x^{112} + x^{106} + x^{102} + x^{98} + x^{86} + x^{84} \\ & + x^{78} + x^{76} + x^{74} + x^{72} + x^{62} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} + x^{34} \\ & + x^{32} + x^{26} + x^{24} + x^{22} + x^{18}, \\ p_6(x) = {} & x^{126} + x^{122} + x^{114} + x^{112} + x^{110} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88} \\ & + x^{86} + x^{76} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{38} \\ & + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{16} + x^{10} + x^{8} + 1, \\ p_9(x) = {} & x^{124} + x^{122} + x^{120} + x^{114} + x^{112} + x^{110} + x^{106} + x^{102} + x^{96} + x^{94} \\ & + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{72} + x^{68} + x^{58} + x^{54}\end{aligned}$$

$$+ x^{50} + x^{48} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{14} + x^{12} + x^{10} + x^{4},$$

$$p_{12}(x) = x^{124} + x^{120} + x^{108} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{20} + x^{16} + x^{12} + x^{8} + x^{4} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
& + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{102} + x^{101} + x^{100} + x^{98} + x^{96} \\
& + x^{95} + x^{93} + x^{92} + x^{91} + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{77} + x^{75} \\
& + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\
& + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{50} + x^{46} + x^{42} + x^{35} + x^{32} + x^{30}, \\
p_3(x) = {} & x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} \\
& + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} + x^{104} + x^{102} + x^{100} + x^{97} + x^{95} \\
& + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} \\
& + x^{72} + x^{70} + x^{68} + x^{65} + x^{63} + x^{60} + x^{58} + x^{56} + x^{53} + x^{52} + x^{46} \\
& + x^{45} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{32} + x^{29} + x^{28} + x^{22} + x^{21} \\
& + x^{20} + x^{19} + x^{18}, \\
p_6(x) = {} & x^{123} + x^{121} + x^{120} + x^{119} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} \\
& + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{91} + x^{88} + x^{86} + x^{84} \\
& + x^{82} + x^{80} + x^{78} + x^{77} + x^{76} + x^{71} + x^{68} + x^{67} + x^{66} + x^{63} + x^{60} \\
& + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} \\
& + x^{44} + x^{43} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} \\
& + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12}, \\
p_9(x) = {} & x^{122} + x^{121} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} + x^{106} + x^{105} + x^{101} \\
& + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} \\
& + x^{84} + x^{83} + x^{82} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63} + x^{60} \\
& + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{45} + x^{43} + x^{40} + x^{38} \\
& + x^{35} + x^{34} + x^{32} + x^{31} + x^{27} + x^{26} + x^{22} + x^{21} + x^{18} + x^{16} + x^{13} \\
& + x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) = {} & x^{121} + x^{118} + x^{117} + x^{116} + x^{109} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
& + x^{97} + x^{96} + x^{89} + x^{86} + x^{85} + x^{84} + x^{77} + x^{74} + x^{73} + x^{72} + x^{69} \\
& + x^{66} + x^{65} + x^{64} + x^{61} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} \\
& + x^{41} + x^{38} + x^{37} + x^{36} + x^{29} + x^{26} + x^{25} + x^{24} + x^{21} + x^{18} + x^{17} \\
& + x^{16} + x^{13} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1].

For $\phi(W^e,1,0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126}+x^{124}+x^{123}+x^{121}+x^{118}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}\\
&+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{102}+x^{101}+x^{100}+x^{98}+x^{96}\\
&+x^{95}+x^{93}+x^{92}+x^{91}+x^{88}+x^{86}+x^{85}+x^{84}+x^{81}+x^{77}+x^{75}\\
&+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{66}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}\\
&+x^{58}+x^{57}+x^{56}+x^{55}+x^{51}+x^{50}+x^{46}+x^{42}+x^{35}+x^{32}+x^{30},\\
p_3(x) = {} & x^{124}+x^{123}+x^{122}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{114}+x^{113}\\
&+x^{112}+x^{111}+x^{110}+x^{108}+x^{107}+x^{104}+x^{102}+x^{100}+x^{97}+x^{95}\\
&+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}+x^{82}+x^{81}+x^{80}+x^{79}+x^{77}+x^{74}\\
&+x^{72}+x^{70}+x^{68}+x^{65}+x^{63}+x^{60}+x^{58}+x^{56}+x^{53}+x^{52}+x^{46}\\
&+x^{45}+x^{41}+x^{39}+x^{38}+x^{37}+x^{35}+x^{32}+x^{29}+x^{28}+x^{22}+x^{21}\\
&+x^{20}+x^{19}+x^{18},\\
p_6(x) = {} & x^{123}+x^{121}+x^{120}+x^{119}+x^{116}+x^{114}+x^{112}+x^{111}+x^{110}+x^{107}\\
&+x^{105}+x^{104}+x^{103}+x^{100}+x^{99}+x^{98}+x^{91}+x^{88}+x^{86}+x^{84}\\
&+x^{82}+x^{80}+x^{78}+x^{77}+x^{76}+x^{71}+x^{68}+x^{67}+x^{66}+x^{63}+x^{60}\\
&+x^{59}+x^{58}+x^{57}+x^{55}+x^{54}+x^{53}+x^{50}+x^{49}+x^{48}+x^{47}+x^{45}\\
&+x^{44}+x^{43}+x^{40}+x^{38}+x^{36}+x^{35}+x^{34}+x^{31}+x^{29}+x^{28}+x^{27}\\
&+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{16}+x^{14}+x^{13}+x^{12},\\
p_9(x) = {} & x^{122}+x^{121}+x^{117}+x^{115}+x^{114}+x^{113}+x^{112}+x^{106}+x^{105}+x^{101}\\
&+x^{100}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{90}+x^{89}+x^{88}+x^{87}+x^{85}\\
&+x^{84}+x^{83}+x^{82}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{64}+x^{63}+x^{60}\\
&+x^{57}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}+x^{45}+x^{43}+x^{40}+x^{38}\\
&+x^{35}+x^{34}+x^{32}+x^{31}+x^{27}+x^{26}+x^{22}+x^{21}+x^{18}+x^{16}+x^{13}\\
&+x^{12}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6},\\
p_{12}(x) = {} & x^{121}+x^{118}+x^{117}+x^{116}+x^{109}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}\\
&+x^{97}+x^{96}+x^{89}+x^{86}+x^{85}+x^{84}+x^{77}+x^{74}+x^{73}+x^{72}+x^{69}\\
&+x^{66}+x^{65}+x^{64}+x^{61}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{49}+x^{48}\\
&+x^{41}+x^{38}+x^{37}+x^{36}+x^{29}+x^{26}+x^{25}+x^{24}+x^{21}+x^{18}+x^{17}\\
&+x^{16}+x^{13}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2}+x+1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1].

For $\phi(W^e,1,1)$:

$$\begin{aligned}
p_0(x) = {} & x^{120}+x^{118}+x^{114}+x^{110}+x^{104}+x^{100}+x^{98}+x^{92}+x^{90}+x^{76}\\
&+x^{72}+x^{70}+x^{62}+x^{54}+x^{50}+x^{44}+x^{42}+x^{40}+x^{36},\\
p_3(x) = {} & x^{116}+x^{114}+x^{110}+x^{106}+x^{104}+x^{102}+x^{98}+x^{94}+x^{92}+x^{88}\\
&+x^{86}+x^{82}+x^{78}+x^{66}+x^{60}+x^{54}+x^{52}+x^{42}+x^{40}+x^{34}+x^{32}
\end{aligned}$$

$$
\begin{aligned}
&+x^{22}+x^{20}+x^{18}+x^{14}+x^{12},\\
p_6(x) &= x^{110}+x^{108}+x^{106}+x^{98}+x^{92}+x^{90}+x^{80}+x^{74}+x^{68}+x^{66}+x^{58}\\
&+x^{54}+x^{42}+x^{40}+x^{38}+x^{32}+x^{26}+x^{20}+x^{18}+x^{14}+x^{12}+x^{4}\\
&+x^{2}+1,\\
p_9(x) &= x^{118}+x^{116}+x^{114}+x^{112}+x^{110}+x^{108}+x^{104}+x^{98}+x^{94}+x^{90}\\
&+x^{88}+x^{86}+x^{82}+x^{80}+x^{68}+x^{64}+x^{62}+x^{60}+x^{58}+x^{56}+x^{52}\\
&+x^{50}+x^{46}+x^{34}+x^{32}+x^{26}+x^{22}+x^{18}+x^{10}+x^{8}+x^{6}+x^{4},\\
p_{12}(x) &= x^{118}+x^{114}+x^{112}+x^{102}+x^{98}+x^{96}+x^{86}+x^{82}+x^{80}+x^{54}+x^{50}\\
&+x^{48}+x^{38}+x^{34}+x^{32}+x^{6}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126}+x^{125}+x^{124}+x^{123}+x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{114}\\
&+x^{112}+x^{111}+x^{109}+x^{108}+x^{107}+x^{105}+x^{104}+x^{102}+x^{101}\\
&+x^{98}+x^{95}+x^{91}+x^{87}+x^{81}+x^{80}+x^{79}+x^{77}+x^{74}+x^{71}+x^{70}\\
&+x^{68}+x^{67}+x^{65}+x^{63}+x^{61}+x^{59}+x^{57}+x^{54}+x^{53}+x^{51}+x^{50}\\
&+x^{49}+x^{45}+x^{44}+x^{41}+x^{40}+x^{38}+x^{37}+x^{36}+x^{31}+x^{30}+x^{28}\\
&+x^{26}+x^{25}+x^{24},\\
p_3(x) &= x^{125}+x^{124}+x^{123}+x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{115}+x^{114}\\
&+x^{112}+x^{111}+x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{99}+x^{97}\\
&+x^{95}+x^{94}+x^{92}+x^{89}+x^{88}+x^{87}+x^{84}+x^{83}+x^{81}+x^{79}+x^{76}\\
&+x^{75}+x^{74}+x^{71}+x^{68}+x^{64}+x^{63}+x^{62}+x^{61}+x^{58}+x^{57}+x^{56}\\
&+x^{55}+x^{54}+x^{51}+x^{49}+x^{48}+x^{47}+x^{46}+x^{42}+x^{39}+x^{38}+x^{36}\\
&+x^{34}+x^{32}+x^{31}+x^{28}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{20}+x^{18}\\
&+x^{17}+x^{16},\\
p_6(x) &= x^{125}+x^{124}+x^{123}+x^{122}+x^{117}+x^{115}+x^{113}+x^{112}+x^{111}+x^{109}\\
&+x^{107}+x^{106}+x^{104}+x^{99}+x^{98}+x^{96}+x^{94}+x^{91}+x^{90}+x^{88}+x^{86}\\
&+x^{84}+x^{82}+x^{81}+x^{80}+x^{74}+x^{73}+x^{72}+x^{68}+x^{67}+x^{66}+x^{65}\\
&+x^{64}+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{50}+x^{48}\\
&+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}+x^{32}+x^{28}+x^{27}+x^{20}+x^{18}\\
&+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{9}+x^{6}+x^{4}+x^{2}+x+1,\\
p_9(x) &= x^{125}+x^{122}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}+x^{110}+x^{108}+x^{106}\\
&+x^{105}+x^{104}+x^{93}+x^{90}+x^{88}+x^{86}+x^{84}+x^{82}+x^{80}+x^{78}+x^{77}\\
&+x^{76}+x^{73}+x^{70}+x^{68}+x^{66}+x^{64}+x^{62}+x^{60}+x^{58}+x^{57}+x^{56}\\
&+x^{45}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{29}+x^{28}+x^{25}\\
&+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12}+x^{10}+x^{9}+x^{8},
\end{aligned}
$$

$$\begin{aligned} p_{12}(x) &= x^{125} + x^{122} + x^{121} + x^{120} + x^{117} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108} \\ &+ x^{105} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85} \\ &+ x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62} \\ &+ x^{61} + x^{60} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41} \\ &+ x^{40} + x^{37} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} \\ &+ x^{17} + x^{14} + x^{13} + x^{12} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1, \end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0].
For $\phi(M^o, 0, 1)$:

$$\begin{aligned} p_0(x) &= x^{136} + x^{133} + x^{132} + x^{131} + x^{129} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} \\ &+ x^{116} + x^{114} + x^{113} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\ &+ x^{99} + x^{95} + x^{90} + x^{87} + x^{83} + x^{78} + x^{77} + x^{75} + x^{74} + x^{71} + x^{68} \\ &+ x^{65} + x^{64} + x^{60} + x^{57} + x^{56} + x^{53} + x^{51} + x^{47} + x^{44} + x^{41} + x^{40} \\ &+ x^{39} + x^{38} + x^{34} + x^{31} + x^{30}, \\ p_3(x) &= x^{129} + x^{128} + x^{127} + x^{125} + x^{124} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} \\ &+ x^{111} + x^{110} + x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{93} + x^{92} \\ &+ x^{91} + x^{88} + x^{86} + x^{85} + x^{84} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} \\ &+ x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} \\ &+ x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{42} + x^{41} + x^{40} + x^{37} + x^{35} \\ &+ x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{26} + x^{21} + x^{18}, \\ p_6(x) &= x^{130} + x^{128} + x^{124} + x^{122} + x^{116} + x^{106} + x^{104} + x^{102} + x^{94} + x^{86} \\ &+ x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} \\ &+ x^{46} + x^{40} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\ p_9(x) &= x^{131} + x^{130} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{121} + x^{117} + x^{115} \\ &+ x^{114} + x^{113} + x^{110} + x^{108} + x^{105} + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} \\ &+ x^{92} + x^{91} + x^{90} + x^{89} + x^{85} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} \\ &+ x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{57} + x^{54} + x^{51} + x^{50} \\ &+ x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{41} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30} \\ &+ x^{27} + x^{26} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{14} + x^{12} + x^{9} + x^{6}, \\ p_{12}(x) &= x^{132} + x^{116} + x^{112} + x^{104} + x^{100} + x^{96} + x^{80} + x^{72} + x^{68} + x^{56} + x^{52} \\ &+ x^{48} + x^{32} + x^{24} + x^{20} + x^{8} + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 1].
For $\phi(M^o, 1, 0)$:

$$\begin{aligned} p_0(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} \\ &+ x^{110} + x^{109} + x^{108} + x^{105} + x^{102} + x^{99} + x^{97} + x^{92} + x^{91} + x^{90} \\ &+ x^{89} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{66} + x^{62} + x^{61} \end{aligned}$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{30},\\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{110} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90}\\
&+ x^{89} + x^{86} + x^{85} + x^{82} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{69} + x^{68}\\
&+ x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{46}\\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{32} + x^{30} + x^{29} + x^{27}\\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{18},\\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{108} + x^{104} + x^{98} + x^{94} + x^{90}\\
&+ x^{80} + x^{76} + x^{74} + x^{60} + x^{54} + x^{50} + x^{44} + x^{42} + x^{38} + x^{32} + x^{28}\\
&+ x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12},\\
p_9(x) &= x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114}\\
&+ x^{113} + x^{112} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{102} + x^{95} + x^{94}\\
&+ x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{54} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12}\\
&+ x^{10} + x^{9} + x^{6},\\
p_{12}(x) &= x^{128} + x^{116} + x^{112} + x^{104} + x^{100} + x^{84} + x^{72} + x^{56} + x^{52} + x^{36} + x^{24}\\
&+ x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{120} + x^{118} + x^{117} + x^{111} + x^{110} + x^{108} + x^{107} + x^{103}\\
&+ x^{96} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} + x^{83} + x^{79} + x^{78} + x^{75} + x^{74}\\
&+ x^{70} + x^{69} + x^{68} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{39} + x^{38} + x^{37} + x^{36},\\
p_3(x) &= x^{125} + x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{113} + x^{112} + x^{111} + x^{109}\\
&+ x^{107} + x^{106} + x^{105} + x^{102} + x^{100} + x^{99} + x^{95} + x^{93} + x^{91} + x^{89}\\
&+ x^{87} + x^{86} + x^{85} + x^{83} + x^{79} + x^{78} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63}\\
&+ x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{20},\\
p_6(x) &= x^{125} + x^{124} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{110}\\
&+ x^{109} + x^{106} + x^{105} + x^{103} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92} + x^{90}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{85} + x^{82} + x^{81} + x^{80} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{63}\\
&+ x^{59} + x^{54} + x^{53} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{39} + x^{38} + x^{33}\\
&+ x^{32} + x^{31} + x^{29} + x^{24} + x^{22} + x^{21} + x^{19} + x^{18} + x^{15} + x^{14} + x^{12}\\
&+ x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106}\\
&+ x^{105} + x^{104} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{76} + x^{73} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56}\\
&+ x^{45} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{25}\\
&+ x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{125} + x^{122} + x^{121} + x^{120} + x^{117} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108}\\
&+ x^{105} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62}\\
&+ x^{61} + x^{60} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41}\\
&+ x^{40} + x^{37} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20}\\
&+ x^{17} + x^{14} + x^{13} + x^{12} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{101}\\
&+ x^{98} + x^{95} + x^{91} + x^{87} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{71} + x^{70}\\
&+ x^{68} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50}\\
&+ x^{49} + x^{45} + x^{44} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{28}\\
&+ x^{26} + x^{25} + x^{24},\\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114}\\
&+ x^{112} + x^{111} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97}\\
&+ x^{95} + x^{94} + x^{92} + x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{81} + x^{79} + x^{76}\\
&+ x^{75} + x^{74} + x^{71} + x^{68} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{54} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} + x^{39} + x^{38} + x^{36}\\
&+ x^{34} + x^{32} + x^{31} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18}\\
&+ x^{17} + x^{16},\\
p_6(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109}\\
&+ x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{86}\\
&+ x^{84} + x^{82} + x^{81} + x^{80} + x^{74} + x^{73} + x^{72} + x^{68} + x^{67} + x^{66} + x^{65}\\
&+ x^{64} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48}\\
&+ x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{28} + x^{27} + x^{20} + x^{18}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106}\\
&+ x^{105} + x^{104} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{76} + x^{73} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56}\\
&+ x^{45} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{25}\\
&+ x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{125} + x^{122} + x^{121} + x^{120} + x^{117} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108}\\
&+ x^{105} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62}\\
&+ x^{61} + x^{60} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41}\\
&+ x^{40} + x^{37} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20}\\
&+ x^{17} + x^{14} + x^{13} + x^{12} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112}\\
&+ x^{110} + x^{109} + x^{108} + x^{105} + x^{102} + x^{99} + x^{97} + x^{92} + x^{91} + x^{90}\\
&+ x^{89} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{66} + x^{62} + x^{61}\\
&+ x^{59} + x^{58} + x^{55} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{30},\\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{110} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{101} + x^{100} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90}\\
&+ x^{89} + x^{86} + x^{85} + x^{82} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{69} + x^{68}\\
&+ x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{46}\\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{32} + x^{30} + x^{29} + x^{27}\\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{18},\\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{108} + x^{104} + x^{98} + x^{94} + x^{90}\\
&+ x^{80} + x^{76} + x^{74} + x^{60} + x^{54} + x^{50} + x^{44} + x^{42} + x^{38} + x^{32} + x^{28}\\
&+ x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{12},\\
p_9(x) &= x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114}\\
&+ x^{113} + x^{112} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{102} + x^{95} + x^{94}\\
&+ x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79}\\
&+ x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{54} + x^{47} + x^{46} + x^{44} + x^{43} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{10}+x^{9}+x^{6},\\
p_{12}(x) &= x^{128}+x^{116}+x^{112}+x^{104}+x^{100}+x^{84}+x^{72}+x^{56}+x^{52}+x^{36}+x^{24}\\
&\quad +x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^o,1,0)$:

$$
\begin{aligned}
p_0(x) &= x^{136}+x^{133}+x^{132}+x^{131}+x^{129}+x^{123}+x^{122}+x^{120}+x^{119}+x^{118}\\
&\quad +x^{116}+x^{114}+x^{113}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}\\
&\quad +x^{99}+x^{95}+x^{90}+x^{87}+x^{83}+x^{78}+x^{77}+x^{75}+x^{74}+x^{71}+x^{68}\\
&\quad +x^{65}+x^{64}+x^{60}+x^{57}+x^{56}+x^{53}+x^{51}+x^{47}+x^{44}+x^{41}+x^{40}\\
&\quad +x^{39}+x^{38}+x^{34}+x^{31}+x^{30},\\
p_3(x) &= x^{129}+x^{128}+x^{127}+x^{125}+x^{124}+x^{119}+x^{116}+x^{115}+x^{114}+x^{113}\\
&\quad +x^{111}+x^{110}+x^{105}+x^{102}+x^{101}+x^{100}+x^{99}+x^{95}+x^{93}+x^{92}\\
&\quad +x^{91}+x^{88}+x^{86}+x^{85}+x^{84}+x^{81}+x^{79}+x^{78}+x^{77}+x^{74}+x^{73}\\
&\quad +x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{59}+x^{56}+x^{55}+x^{54}+x^{53}\\
&\quad +x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}+x^{35}\\
&\quad +x^{34}+x^{32}+x^{31}+x^{30}+x^{29}+x^{27}+x^{26}+x^{21}+x^{18},\\
p_6(x) &= x^{130}+x^{128}+x^{124}+x^{122}+x^{116}+x^{106}+x^{104}+x^{102}+x^{94}+x^{86}\\
&\quad +x^{84}+x^{82}+x^{80}+x^{74}+x^{72}+x^{70}+x^{68}+x^{62}+x^{60}+x^{58}+x^{56}\\
&\quad +x^{46}+x^{40}+x^{26}+x^{24}+x^{20}+x^{18}+x^{12},\\
p_9(x) &= x^{131}+x^{130}+x^{128}+x^{127}+x^{124}+x^{123}+x^{122}+x^{121}+x^{117}+x^{115}\\
&\quad +x^{114}+x^{113}+x^{110}+x^{108}+x^{105}+x^{102}+x^{99}+x^{98}+x^{96}+x^{95}\\
&\quad +x^{92}+x^{91}+x^{90}+x^{89}+x^{85}+x^{81}+x^{80}+x^{79}+x^{78}+x^{75}+x^{74}\\
&\quad +x^{70}+x^{69}+x^{67}+x^{66}+x^{65}+x^{62}+x^{60}+x^{57}+x^{54}+x^{51}+x^{50}\\
&\quad +x^{48}+x^{47}+x^{44}+x^{43}+x^{42}+x^{41}+x^{37}+x^{33}+x^{32}+x^{31}+x^{30}\\
&\quad +x^{27}+x^{26}+x^{22}+x^{21}+x^{19}+x^{18}+x^{17}+x^{14}+x^{12}+x^{9}+x^{6},\\
p_{12}(x) &= x^{132}+x^{116}+x^{112}+x^{104}+x^{100}+x^{96}+x^{80}+x^{72}+x^{68}+x^{56}+x^{52}\\
&\quad +x^{48}+x^{32}+x^{24}+x^{20}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 1].

For $\phi(W^o,1,1)$:

$$
\begin{aligned}
p_0(x) &= x^{126}+x^{121}+x^{120}+x^{118}+x^{117}+x^{111}+x^{110}+x^{108}+x^{107}+x^{103}\\
&\quad +x^{96}+x^{91}+x^{90}+x^{88}+x^{86}+x^{85}+x^{83}+x^{79}+x^{78}+x^{75}+x^{74}\\
&\quad +x^{70}+x^{69}+x^{68}+x^{66}+x^{62}+x^{61}+x^{60}+x^{59}+x^{56}+x^{54}+x^{53}\\
&\quad +x^{51}+x^{50}+x^{49}+x^{48}+x^{44}+x^{39}+x^{38}+x^{37}+x^{36},\\
p_3(x) &= x^{125}+x^{124}+x^{122}+x^{119}+x^{118}+x^{117}+x^{113}+x^{112}+x^{111}+x^{109}\\
&\quad +x^{107}+x^{106}+x^{105}+x^{102}+x^{100}+x^{99}+x^{95}+x^{93}+x^{91}+x^{89}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{87} + x^{86} + x^{85} + x^{83} + x^{79} + x^{78} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63}\\
&+ x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{20},\\
p_6(x) &= x^{125} + x^{124} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{110}\\
&+ x^{109} + x^{106} + x^{105} + x^{103} + x^{100} + x^{97} + x^{96} + x^{93} + x^{92} + x^{90}\\
&+ x^{85} + x^{82} + x^{81} + x^{80} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{63}\\
&+ x^{59} + x^{54} + x^{53} + x^{47} + x^{46} + x^{45} + x^{43} + x^{42} + x^{39} + x^{38} + x^{33}\\
&+ x^{32} + x^{31} + x^{29} + x^{24} + x^{22} + x^{21} + x^{19} + x^{18} + x^{15} + x^{14} + x^{12}\\
&+ x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106}\\
&+ x^{105} + x^{104} + x^{93} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{76} + x^{73} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{57} + x^{56}\\
&+ x^{45} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{25}\\
&+ x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{125} + x^{122} + x^{121} + x^{120} + x^{117} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108}\\
&+ x^{105} + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{69} + x^{68} + x^{65} + x^{62}\\
&+ x^{61} + x^{60} + x^{57} + x^{54} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} + x^{41}\\
&+ x^{40} + x^{37} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20}\\
&+ x^{17} + x^{14} + x^{13} + x^{12} + x^{9} + x^{6} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	355	[514, 515]	710	[866, 867]	1065	[311, 572]
1	[3, 4]	356	[452, 447]	711	[136, 390]	1066	[698, 1180]
2	[5, 6]	357	[505, 485]	712	[401, 214]	1067	[1181, 1182]
3	[7, 8]	358	[516, 40]	713	[488, 772]	1068	[1183, 531]
4	[9, 10]	359	[45, 517]	714	[402, 146]	1069	[25, 1184]
5	[11, 12]	360	[167, 518]	715	[868, 869]	1070	[1185, 343]
6	[13, 14]	361	[519, 40]	716	[825, 775]	1071	[575, 614]
7	[15, 16]	362	[210, 518]	717	[870, 784]	1072	[1186, 356]
8	[17, 18]	363	[520, 464]	718	[871, 367]	1073	[1187, 1188]
9	[19, 20]	364	[171, 521]	719	[750, 828]	1074	[673, 1189]
10	[21, 22]	365	[352, 522]	720	[872, 419]	1075	[1190, 1191]
11	[23, 24]	366	[330, 277]	721	[59, 432]	1076	[1192, 927]

12	[25, 26]	367	[335, 523]	722	[873, 444]	1077	[1014, 1154]
13	[27, 28]	368	[524, 525]	723	[195, 366]	1078	[664, 1158]
14	[29, 30]	369	[350, 526]	724	[784, 874]	1079	[1193, 975]
15	[31, 32]	370	[24, 294]	725	[487, 831]	1080	[1194, 267]
16	[33, 34]	371	[527, 528]	726	[875, 410]	1081	[535, 555]
17	[35, 36]	372	[529, 530]	727	[876, 467]	1082	[1157, 224]
18	[37, 38]	373	[531, 267]	728	[877, 640]	1083	[1195, 352]
19	[39, 40]	374	[532, 533]	729	[84, 310]	1084	[1196, 690]
20	[41, 42]	375	[522, 534]	730	[878, 616]	1085	[695, 1197]
21	[43, 44]	376	[535, 536]	731	[305, 87]	1086	[1198, 1199]
22	[45, 46]	377	[115, 537]	732	[879, 880]	1087	[1200, 1201]
23	[47, 48]	378	[538, 539]	733	[881, 882]	1088	[334, 948]
24	[49, 50]	379	[540, 541]	734	[883, 79]	1089	[1195, 1202]
25	[51, 52]	380	[98, 542]	735	[88, 556]	1090	[1203, 1204]
26	[53, 54]	381	[543, 544]	736	[884, 885]	1091	[652, 1205]
27	[55, 36]	382	[545, 546]	737	[260, 886]	1092	[1206, 339]
28	[56, 57]	383	[291, 547]	738	[887, 888]	1093	[1207, 885]
29	[58, 59]	384	[548, 549]	739	[333, 549]	1094	[1208, 943]
30	[60, 61]	385	[550, 551]	740	[889, 890]	1095	[991, 620]
31	[62, 63]	386	[552, 553]	741	[891, 892]	1096	[915, 912]
32	[64, 65]	387	[275, 117]	742	[893, 121]	1097	[1209, 231]
33	[66, 67]	388	[554, 555]	743	[894, 313]	1098	[961, 1210]
34	[68, 69]	389	[109, 556]	744	[895, 97]	1099	[1211, 222]
35	[70, 71]	390	[20, 277]	745	[17, 896]	1100	[1212, 1213]
36	[72, 73]	391	[557, 558]	746	[596, 618]	1101	[1214, 1215]
37	[74, 75]	392	[559, 560]	747	[897, 898]	1102	[279, 609]
38	[76, 77]	393	[561, 113]	748	[899, 288]	1103	[886, 1216]
39	[78, 79]	394	[562, 563]	749	[900, 901]	1104	[273, 266]
40	[80, 81]	395	[555, 564]	750	[902, 687]	1105	[1028, 257]
41	[41, 41]	396	[565, 566]	751	[116, 276]	1106	[1217, 331]
42	[82, 83]	397	[527, 336]	752	[353, 534]	1107	[1008, 1218]
43	[84, 85]	398	[567, 568]	753	[903, 334]	1108	[1219, 1220]
44	[86, 87]	399	[569, 570]	754	[904, 258]	1109	[64, 1221]
45	[88, 89]	400	[571, 572]	755	[905, 906]	1110	[1222, 998]
46	[90, 91]	401	[573, 120]	756	[907, 619]	1111	[1223, 1019]
47	[92, 93]	402	[574, 575]	757	[295, 908]	1112	[987, 1224]
48	[94, 95]	403	[576, 234]	758	[27, 909]	1113	[994, 1225]
49	[96, 97]	404	[577, 578]	759	[910, 911]	1114	[1226, 1186]
50	[98, 99]	405	[579, 580]	760	[601, 707]	1115	[1227, 1228]
51	[100, 101]	406	[581, 66]	761	[912, 665]	1116	[1229, 301]
52	[102, 103]	407	[582, 583]	762	[913, 914]	1117	[1230, 631]
53	[104, 105]	408	[279, 77]	763	[96, 118]	1118	[1231, 880]
54	[106, 107]	409	[584, 585]	764	[915, 560]	1119	[1185, 921]
55	[108, 109]	410	[586, 587]	765	[916, 917]	1120	[1232, 1034]

56	[110, 111]	411	[588, 589]	766	[918, 363]	1121	[1233, 1234]
57	[112, 113]	412	[218, 575]	767	[919, 358]	1122	[930, 1235]
58	[114, 115]	413	[590, 591]	768	[920, 897]	1123	[4, 948]
59	[116, 117]	414	[218, 592]	769	[342, 921]	1124	[1236, 91]
60	[118, 119]	415	[593, 342]	770	[922, 75]	1125	[1237, 1238]
61	[120, 121]	416	[594, 595]	771	[597, 599]	1126	[1217, 985]
62	[122, 123]	417	[596, 582]	772	[923, 924]	1127	[277, 576]
63	[124, 125]	418	[597, 598]	773	[925, 926]	1128	[1239, 1240]
64	[126, 127]	419	[599, 600]	774	[41, 680]	1129	[1241, 1242]
65	[128, 129]	420	[601, 602]	775	[927, 598]	1130	[1243, 1244]
66	[130, 131]	421	[603, 221]	776	[928, 929]	1131	[1245, 1246]
67	[132, 133]	422	[604, 605]	777	[571, 312]	1132	[1208, 1247]
68	[134, 132]	423	[531, 240]	778	[930, 931]	1133	[1248, 1203]
69	[135, 136]	424	[606, 607]	779	[92, 878]	1134	[1249, 1250]
70	[137, 32]	425	[554, 536]	780	[722, 932]	1135	[1251, 1252]
71	[138, 139]	426	[63, 608]	781	[933, 712]	1136	[1253, 1254]
72	[140, 141]	427	[76, 609]	782	[934, 935]	1137	[1255, 546]
73	[142, 38]	428	[610, 611]	783	[936, 937]	1138	[1194, 240]
74	[143, 144]	429	[612, 362]	784	[938, 532]	1139	[254, 721]
75	[145, 146]	430	[613, 589]	785	[939, 888]	1140	[562, 1256]
76	[147, 148]	431	[592, 614]	786	[940, 613]	1141	[315, 1154]
77	[149, 150]	432	[281, 615]	787	[892, 941]	1142	[662, 323]
78	[151, 127]	433	[309, 85]	788	[942, 943]	1143	[1226, 355]
79	[152, 153]	434	[616, 617]	789	[944, 691]	1144	[1239, 1015]
80	[154, 155]	435	[4, 335]	790	[945, 660]	1145	[1257, 1258]
81	[156, 157]	436	[618, 583]	791	[919, 68]	1146	[1259, 1260]
82	[158, 159]	437	[619, 620]	792	[619, 946]	1147	[1261, 1262]
83	[133, 136]	438	[621, 66]	793	[947, 21]	1148	[1263, 1136]
84	[160, 8]	439	[528, 337]	794	[948, 523]	1149	[1264, 48]
85	[161, 59]	440	[622, 261]	795	[949, 303]	1150	[1265, 436]
86	[162, 42]	441	[623, 624]	796	[950, 697]	1151	[1266, 465]
87	[163, 164]	442	[625, 346]	797	[951, 304]	1152	[1267, 32]
88	[165, 166]	443	[626, 98]	798	[264, 68]	1153	[390, 33]
89	[167, 168]	444	[627, 628]	799	[952, 953]	1154	[9, 57]
90	[169, 170]	445	[28, 629]	800	[624, 954]	1155	[1268, 822]
91	[171, 172]	446	[30, 630]	801	[581, 654]	1156	[826, 129]
92	[173, 174]	447	[631, 618]	802	[955, 601]	1157	[372, 769]
93	[175, 176]	448	[632, 633]	803	[956, 559]	1158	[746, 435]
94	[177, 178]	449	[634, 635]	804	[697, 957]	1159	[1269, 834]
95	[179, 180]	450	[636, 21]	805	[958, 947]	1160	[192, 195]
96	[181, 182]	451	[637, 638]	806	[959, 960]	1161	[1050, 784]
97	[183, 45]	452	[248, 639]	807	[961, 962]	1162	[427, 10]
98	[184, 34]	453	[29, 640]	808	[357, 963]	1163	[1270, 465]
99	[185, 186]	454	[93, 641]	809	[109, 89]	1164	[1271, 211]

100	[187, 188]	455	[642, 643]	810	[964, 965]	1165	[1272, 123]
101	[189, 56]	456	[644, 360]	811	[966, 229]	1166	[1273, 490]
102	[190, 191]	457	[645, 526]	812	[78, 967]	1167	[1274, 1275]
103	[192, 193]	458	[646, 105]	813	[968, 90]	1168	[804, 506]
104	[194, 144]	459	[647, 648]	814	[969, 970]	1169	[1041, 182]
105	[195, 196]	460	[649, 650]	815	[264, 358]	1170	[403, 1139]
106	[197, 198]	461	[347, 232]	816	[971, 725]	1171	[1276, 487]
107	[32, 164]	462	[651, 271]	817	[972, 973]	1172	[1104, 1277]
108	[199, 167]	463	[652, 653]	818	[974, 975]	1173	[1278, 60]
109	[200, 135]	464	[654, 67]	819	[976, 977]	1174	[1279, 830]
110	[201, 166]	465	[588, 655]	820	[978, 86]	1175	[728, 446]
111	[133, 202]	466	[570, 656]	821	[532, 979]	1176	[1280, 39]
112	[141, 203]	467	[657, 233]	822	[963, 615]	1177	[1281, 1136]
113	[204, 205]	468	[3, 334]	823	[980, 981]	1178	[1102, 202]
114	[194, 206]	469	[16, 336]	824	[982, 983]	1179	[1282, 405]
115	[207, 208]	470	[658, 659]	825	[82, 984]	1180	[124, 375]
116	[147, 209]	471	[660, 325]	826	[909, 629]	1181	[1283, 1284]
117	[210, 168]	472	[246, 661]	827	[985, 332]	1182	[1285, 188]
118	[211, 212]	473	[662, 269]	828	[621, 654]	1183	[1286, 867]
119	[182, 213]	474	[663, 664]	829	[573, 893]	1184	[146, 1139]
120	[214, 61]	475	[560, 665]	830	[640, 630]	1185	[165, 198]
121	[215, 216]	476	[666, 667]	831	[877, 30]	1186	[1098, 203]
122	[217, 218]	477	[668, 669]	832	[28, 986]	1187	[807, 14]
123	[219, 220]	478	[670, 671]	833	[987, 988]	1188	[826, 486]
124	[221, 222]	479	[672, 99]	834	[989, 990]	1189	[146, 1071]
125	[223, 224]	480	[673, 674]	835	[991, 946]	1190	[1287, 822]
126	[225, 226]	481	[675, 676]	836	[703, 877]	1191	[504, 366]
127	[227, 228]	482	[677, 282]	837	[992, 285]	1192	[756, 758]
128	[229, 230]	483	[219, 348]	838	[910, 993]	1193	[783, 740]
129	[107, 83]	484	[97, 119]	839	[994, 995]	1194	[776, 738]
130	[231, 232]	485	[678, 679]	840	[252, 553]	1195	[474, 52]
131	[233, 234]	486	[680, 41]	841	[672, 542]	1196	[1126, 847]
132	[235, 236]	487	[104, 681]	842	[996, 997]	1197	[1044, 464]
133	[237, 111]	488	[682, 683]	843	[590, 998]	1198	[1288, 869]
134	[238, 239]	489	[684, 685]	844	[956, 915]	1199	[1289, 193]
135	[232, 20]	490	[641, 617]	845	[999, 1000]	1200	[1290, 487]
136	[240, 241]	491	[686, 687]	846	[553, 651]	1201	[1291, 851]
137	[242, 243]	492	[688, 689]	847	[265, 274]	1202	[791, 434]
138	[86, 235]	493	[690, 691]	848	[1001, 1002]	1203	[463, 208]
139	[67, 237]	494	[692, 693]	849	[1003, 1004]	1204	[1292, 445]
140	[244, 245]	495	[694, 695]	850	[253, 651]	1205	[1293, 490]
141	[246, 247]	496	[696, 697]	851	[1005, 306]	1206	[441, 1065]
142	[248, 249]	497	[698, 688]	852	[958, 636]	1207	[1294, 511]
143	[250, 251]	498	[699, 700]	853	[1006, 1007]	1208	[459, 52]

144	[252, 253]	499	[283, 701]	854	[310, 720]	1209	[1295, 39]
145	[85, 254]	500	[702, 703]	855	[1008, 289]	1210	[1296, 1080]
146	[87, 236]	501	[704, 249]	856	[1009, 1010]	1211	[790, 204]
147	[255, 256]	502	[705, 320]	857	[259, 1011]	1212	[1297, 1136]
148	[257, 258]	503	[231, 19]	858	[1012, 1013]	1213	[202, 439]
149	[259, 260]	504	[706, 707]	859	[1014, 316]	1214	[1298, 156]
150	[261, 262]	505	[708, 554]	860	[1015, 1016]	1215	[466, 1277]
151	[263, 264]	506	[536, 564]	861	[1017, 1018]	1216	[153, 407]
152	[265, 266]	507	[612, 709]	862	[926, 1019]	1217	[1097, 182]
153	[267, 241]	508	[710, 711]	863	[1020, 612]	1218	[1299, 373]
154	[268, 269]	509	[525, 712]	864	[1021, 1022]	1219	[1300, 1145]
155	[270, 271]	510	[713, 714]	865	[1023, 637]	1220	[857, 1301]
156	[74, 272]	511	[715, 716]	866	[1024, 1025]	1221	[1302, 754]
157	[273, 274]	512	[644, 239]	867	[1026, 1027]	1222	[1100, 416]
158	[275, 276]	513	[717, 260]	868	[1028, 904]	1223	[154, 61]
159	[277, 233]	514	[718, 719]	869	[1029, 1030]	1224	[445, 419]
160	[278, 279]	515	[720, 721]	870	[1031, 891]	1225	[1303, 1119]
161	[280, 281]	516	[722, 671]	871	[361, 709]	1226	[1304, 1129]
162	[282, 82]	517	[723, 724]	872	[682, 230]	1227	[1305, 1108]
163	[283, 284]	518	[676, 725]	873	[1032, 1033]	1228	[1306, 858]
164	[285, 286]	519	[726, 727]	874	[1034, 1013]	1229	[1307, 1308]
165	[287, 288]	520	[687, 685]	875	[1035, 1036]	1230	[1309, 60]
166	[289, 290]	521	[568, 703]	876	[970, 533]	1231	[368, 1065]
167	[291, 292]	522	[728, 419]	877	[153, 475]	1232	[1310, 14]
168	[293, 294]	523	[456, 205]	878	[813, 801]	1233	[1311, 495]
169	[295, 296]	524	[729, 401]	879	[1037, 410]	1234	[392, 1277]
170	[297, 237]	525	[730, 366]	880	[520, 208]	1235	[748, 835]
171	[298, 299]	526	[478, 731]	881	[1038, 39]	1236	[1064, 1312]
172	[300, 301]	527	[732, 426]	882	[1039, 202]	1237	[1313, 489]
173	[302, 223]	528	[158, 34]	883	[1040, 399]	1238	[1314, 38]
174	[303, 304]	529	[733, 8]	884	[1041, 736]	1239	[1315, 849]
175	[305, 235]	530	[734, 731]	885	[37, 397]	1240	[1042, 431]
176	[222, 306]	531	[139, 131]	886	[784, 741]	1241	[1316, 1284]
177	[307, 308]	532	[735, 475]	887	[1042, 515]	1242	[1317, 1124]
178	[309, 310]	533	[736, 503]	888	[123, 187]	1243	[1318, 1129]
179	[311, 312]	534	[215, 737]	889	[1043, 405]	1244	[1291, 1058]
180	[313, 81]	535	[732, 164]	890	[461, 742]	1245	[35, 750]
181	[314, 238]	536	[516, 738]	891	[1044, 208]	1246	[1319, 835]
182	[315, 316]	537	[206, 739]	892	[847, 435]	1247	[1303, 1080]
183	[94, 317]	538	[740, 741]	893	[43, 50]	1248	[1320, 1275]
184	[241, 318]	539	[170, 742]	894	[1045, 156]	1249	[1073, 399]
185	[319, 320]	540	[743, 744]	895	[842, 1046]	1250	[179, 785]
186	[321, 322]	541	[745, 746]	896	[129, 1047]	1251	[1321, 1090]
187	[268, 323]	542	[507, 33]	897	[1048, 376]	1252	[366, 434]

188	[324, 325]	543	[747, 435]	898	[1049, 851]	1253	[1322, 744]
189	[316, 112]	544	[748, 54]	899	[1050, 740]	1254	[1323, 1142]
190	[326, 327]	545	[749, 750]	900	[1051, 149]	1255	[476, 1086]
191	[328, 329]	546	[751, 752]	901	[202, 390]	1256	[1324, 387]
192	[232, 330]	547	[753, 754]	902	[181, 736]	1257	[1325, 1115]
193	[331, 332]	548	[12, 370]	903	[1052, 460]	1258	[1306, 1301]
194	[6, 333]	549	[14, 445]	904	[49, 44]	1259	[1326, 1090]
195	[334, 335]	550	[755, 206]	905	[1053, 465]	1260	[464, 1082]
196	[336, 337]	551	[450, 9]	906	[1054, 56]	1261	[1327, 1078]
197	[338, 339]	552	[756, 141]	907	[1055, 149]	1262	[506, 462]
198	[340, 341]	553	[161, 376]	908	[827, 761]	1263	[1328, 1162]
199	[23, 293]	554	[33, 159]	909	[843, 204]	1264	[543, 1329]
200	[342, 343]	555	[134, 131]	910	[1056, 174]	1265	[93, 616]
201	[344, 345]	556	[389, 439]	911	[464, 752]	1266	[920, 570]
202	[236, 346]	557	[757, 60]	912	[793, 515]	1267	[1330, 1331]
203	[282, 107]	558	[170, 462]	913	[1057, 123]	1268	[637, 1332]
204	[347, 19]	559	[467, 176]	914	[850, 1058]	1269	[1233, 1333]
205	[348, 349]	560	[758, 203]	915	[1059, 801]	1270	[529, 299]
206	[350, 351]	561	[430, 515]	916	[1060, 822]	1271	[1334, 638]
207	[352, 353]	562	[759, 156]	917	[1061, 153]	1272	[892, 1335]
208	[354, 318]	563	[760, 761]	918	[398, 171]	1273	[1216, 1157]
209	[355, 356]	564	[159, 139]	919	[197, 166]	1274	[1154, 561]
210	[357, 281]	565	[762, 167]	920	[1062, 36]	1275	[701, 1336]
211	[358, 69]	566	[439, 184]	921	[185, 150]	1276	[966, 682]
212	[304, 359]	567	[763, 401]	922	[1063, 444]	1277	[1337, 22]
213	[238, 360]	568	[764, 153]	923	[1064, 492]	1278	[697, 1338]
214	[310, 254]	569	[765, 477]	924	[1065, 800]	1279	[106, 82]
215	[361, 362]	570	[735, 407]	925	[1066, 1067]	1280	[1339, 887]
216	[363, 364]	571	[749, 36]	926	[480, 50]	1281	[1340, 1341]
217	[365, 174]	572	[512, 742]	927	[513, 767]	1282	[1182, 1342]
218	[366, 367]	573	[766, 215]	928	[1068, 449]	1283	[1343, 652]
219	[368, 178]	574	[766, 767]	929	[207, 464]	1284	[1344, 1345]
220	[369, 370]	575	[768, 769]	930	[1069, 739]	1285	[118, 1346]
221	[371, 57]	576	[770, 209]	931	[1070, 1071]	1286	[1343, 1347]
222	[372, 373]	577	[771, 772]	932	[753, 1072]	1287	[303, 1175]
223	[374, 375]	578	[764, 442]	933	[796, 816]	1288	[1348, 1349]
224	[376, 377]	579	[773, 191]	934	[1073, 171]	1289	[237, 1350]
225	[378, 379]	580	[774, 775]	935	[876, 211]	1290	[1351, 1352]
226	[380, 139]	581	[776, 40]	936	[1074, 401]	1291	[1353, 957]
227	[381, 382]	582	[777, 44]	937	[775, 129]	1292	[229, 683]
228	[383, 384]	583	[456, 471]	938	[1075, 511]	1293	[1202, 353]
229	[385, 45]	584	[778, 379]	939	[1076, 203]	1294	[1354, 1355]
230	[386, 387]	585	[439, 33]	940	[1077, 1078]	1295	[894, 80]
231	[162, 41]	586	[779, 382]	941	[1079, 1080]	1296	[1356, 65]

232	[130, 132]	587	[780, 781]	942	[733, 468]	1297	[537, 1357]
233	[42, 41]	588	[782, 59]	943	[1081, 1082]	1298	[1237, 1358]
234	[135, 202]	589	[457, 425]	944	[1083, 492]	1299	[1169, 1359]
235	[388, 159]	590	[783, 784]	945	[1084, 206]	1300	[1243, 1360]
236	[42, 42]	591	[433, 785]	946	[149, 186]	1301	[1361, 892]
237	[131, 135]	592	[174, 375]	947	[175, 212]	1302	[1362, 18]
238	[184, 159]	593	[786, 379]	948	[872, 446]	1303	[1363, 284]
239	[389, 390]	594	[787, 788]	949	[1052, 52]	1304	[925, 1223]
240	[388, 34]	595	[383, 789]	950	[1085, 1086]	1305	[1364, 1365]
241	[131, 133]	596	[790, 456]	951	[1059, 45]	1306	[319, 1366]
242	[391, 48]	597	[791, 367]	952	[1087, 39]	1307	[1367, 1246]
243	[392, 393]	598	[758, 792]	953	[136, 439]	1308	[1368, 322]
244	[394, 167]	599	[793, 431]	954	[465, 392]	1309	[553, 270]
245	[380, 395]	600	[767, 737]	955	[1088, 489]	1310	[1192, 597]
246	[396, 8]	601	[442, 407]	956	[482, 758]	1311	[1348, 1369]
247	[142, 397]	602	[489, 794]	957	[750, 809]	1312	[988, 1370]
248	[398, 399]	603	[795, 741]	958	[1089, 1090]	1313	[706, 602]
249	[193, 366]	604	[11, 796]	959	[1091, 174]	1314	[1371, 256]
250	[400, 401]	605	[1, 746]	960	[1092, 835]	1315	[980, 340]
251	[402, 403]	606	[143, 206]	961	[819, 182]	1316	[1024, 1372]
252	[404, 405]	607	[797, 467]	962	[751, 1082]	1317	[985, 1373]
253	[406, 407]	608	[123, 414]	963	[1093, 860]	1318	[326, 1374]
254	[8, 408]	609	[798, 184]	964	[1094, 1078]	1319	[1249, 345]
255	[409, 410]	610	[799, 141]	965	[1095, 830]	1320	[947, 1337]
256	[411, 9]	611	[734, 479]	966	[1096, 435]	1321	[686, 684]
257	[412, 155]	612	[395, 132]	967	[1093, 862]	1322	[1149, 1375]
258	[413, 414]	613	[442, 475]	968	[1040, 171]	1323	[884, 1376]
259	[415, 416]	614	[767, 216]	969	[1097, 736]	1324	[1377, 26]
260	[417, 153]	615	[178, 800]	970	[1048, 59]	1325	[692, 1378]
261	[418, 419]	616	[768, 373]	971	[1098, 792]	1326	[1179, 1379]
262	[48, 420]	617	[801, 46]	972	[745, 847]	1327	[1207, 1376]
263	[421, 422]	618	[60, 155]	973	[1099, 467]	1328	[1380, 1078]
264	[423, 135]	619	[395, 131]	974	[1100, 449]	1329	[1070, 1139]
265	[424, 215]	620	[798, 33]	975	[797, 211]	1330	[1381, 487]
266	[386, 425]	621	[139, 132]	976	[1101, 149]	1331	[1382, 794]
267	[163, 426]	622	[802, 48]	977	[1102, 136]	1332	[1293, 794]
268	[427, 57]	623	[803, 465]	978	[1103, 867]	1333	[1383, 213]
269	[428, 429]	624	[804, 170]	979	[1104, 393]	1334	[424, 767]
270	[430, 431]	625	[770, 148]	980	[1105, 495]	1335	[208, 1082]
271	[376, 432]	626	[805, 379]	981	[1106, 469]	1336	[144, 493]
272	[433, 180]	627	[806, 807]	982	[1107, 1108]	1337	[214, 155]
273	[196, 434]	628	[808, 384]	983	[1109, 500]	1338	[1384, 1058]
274	[435, 436]	629	[36, 809]	984	[1110, 1111]	1339	[1385, 123]
275	[437, 41]	630	[59, 377]	985	[871, 434]	1340	[1386, 741]

276	[438, 439]	631	[371, 10]	986	[1049, 1058]	1341	[1279, 1047]
277	[132, 135]	632	[810, 422]	987	[1112, 1067]	1342	[1099, 211]
278	[440, 149]	633	[138, 395]	988	[1113, 408]	1343	[1387, 739]
279	[437, 42]	634	[811, 807]	989	[1114, 1115]	1344	[1388, 1310]
280	[441, 178]	635	[812, 781]	990	[1109, 1116]	1345	[1127, 145]
281	[152, 442]	636	[813, 45]	991	[519, 738]	1346	[1389, 1277]
282	[423, 133]	637	[196, 367]	992	[1117, 156]	1347	[841, 176]
283	[443, 444]	638	[814, 815]	993	[1118, 1119]	1348	[1390, 1067]
284	[417, 442]	639	[369, 816]	994	[799, 758]	1349	[1324, 425]
285	[445, 446]	640	[777, 50]	995	[1081, 752]	1350	[1110, 1391]
286	[156, 447]	641	[174, 125]	996	[414, 188]	1351	[1392, 489]
287	[448, 449]	642	[817, 171]	997	[846, 170]	1352	[506, 742]
288	[450, 56]	643	[411, 56]	998	[145, 403]	1353	[211, 176]
289	[451, 125]	644	[818, 164]	999	[1120, 784]	1354	[1393, 1275]
290	[452, 157]	645	[819, 736]	1000	[840, 816]	1355	[1054, 9]
291	[448, 416]	646	[820, 444]	1001	[1121, 1067]	1356	[817, 399]
292	[453, 129]	647	[821, 822]	1002	[466, 393]	1357	[208, 752]
293	[412, 61]	648	[473, 403]	1003	[1122, 853]	1358	[1095, 1047]
294	[48, 454]	649	[823, 824]	1004	[1123, 1124]	1359	[1319, 54]
295	[455, 456]	650	[825, 826]	1005	[374, 125]	1360	[1394, 809]
296	[457, 387]	651	[141, 792]	1006	[1125, 1126]	1361	[1395, 1396]
297	[458, 198]	652	[385, 801]	1007	[1127, 484]	1362	[1397, 477]
298	[459, 460]	653	[814, 429]	1008	[1128, 1129]	1363	[1084, 144]
299	[461, 462]	654	[39, 738]	1009	[1130, 1129]	1364	[1398, 849]
300	[463, 464]	655	[827, 436]	1010	[1131, 429]	1365	[1399, 815]
301	[465, 466]	656	[36, 828]	1011	[859, 862]	1366	[1302, 1072]
302	[58, 376]	657	[458, 166]	1012	[1132, 860]	1367	[1397, 1086]
303	[467, 212]	658	[415, 449]	1013	[14, 418]	1368	[1299, 769]
304	[468, 469]	659	[829, 170]	1014	[1133, 405]	1369	[760, 436]
305	[470, 148]	660	[829, 506]	1015	[1076, 792]	1370	[1134, 831]
306	[204, 471]	661	[486, 830]	1016	[845, 420]	1371	[864, 1065]
307	[472, 14]	662	[455, 204]	1017	[765, 1086]	1372	[1092, 54]
308	[473, 146]	663	[831, 772]	1018	[502, 767]	1373	[1389, 393]
309	[474, 460]	664	[730, 196]	1019	[1134, 488]	1374	[1265, 761]
310	[406, 475]	665	[801, 517]	1020	[1135, 1136]	1375	[496, 425]
311	[476, 477]	666	[832, 772]	1021	[839, 847]	1376	[129, 830]
312	[478, 479]	667	[775, 486]	1022	[189, 9]	1377	[396, 468]
313	[480, 44]	668	[833, 834]	1023	[144, 739]	1378	[1400, 828]
314	[481, 167]	669	[774, 826]	1024	[1137, 1090]	1379	[486, 1047]
315	[482, 141]	670	[755, 144]	1025	[1138, 1139]	1380	[1017, 718]
316	[483, 204]	671	[453, 486]	1026	[1140, 796]	1381	[959, 1401]
317	[484, 146]	672	[818, 426]	1027	[1141, 1142]	1382	[1347, 653]
318	[34, 395]	673	[140, 758]	1028	[1143, 495]	1383	[1402, 1018]
319	[409, 485]	674	[53, 835]	1029	[1144, 1145]	1384	[1252, 893]

320	[128, 486]	675	[836, 487]	1030	[499, 1116]	1385	[1403, 268]
321	[451, 375]	676	[514, 431]	1031	[1, 847]	1386	[1367, 1404]
322	[487, 488]	677	[837, 32]	1032	[1146, 422]	1387	[950, 1353]
323	[489, 490]	678	[838, 839]	1033	[390, 184]	1388	[1351, 1405]
324	[491, 492]	679	[808, 789]	1034	[1061, 442]	1389	[631, 582]
325	[206, 493]	680	[501, 209]	1035	[1147, 867]	1390	[1009, 1406]
326	[494, 495]	681	[840, 370]	1036	[1039, 136]	1391	[1240, 1016]
327	[496, 387]	682	[841, 212]	1037	[1148, 275]	1392	[331, 1373]
328	[497, 498]	683	[435, 761]	1038	[1149, 1150]	1393	[718, 1407]
329	[499, 500]	684	[56, 10]	1039	[16, 528]	1394	[545, 1408]
330	[501, 148]	685	[736, 213]	1040	[1151, 300]	1395	[949, 951]
331	[502, 215]	686	[842, 511]	1041	[1152, 1153]	1396	[1409, 262]
332	[182, 503]	687	[843, 456]	1042	[1154, 112]	1397	[1410, 1411]
333	[504, 196]	688	[844, 769]	1043	[1155, 1156]	1398	[1364, 1412]
334	[483, 456]	689	[845, 454]	1044	[302, 1157]	1399	[1413, 951]
335	[8, 469]	690	[846, 506]	1045	[1158, 273]	1400	[1414, 352]
336	[32, 426]	691	[847, 827]	1046	[1159, 1160]	1401	[1138, 1071]
337	[34, 139]	692	[848, 849]	1047	[598, 600]	1402	[1062, 750]
338	[505, 410]	693	[850, 851]	1048	[1161, 292]	1403	[1415, 489]
339	[169, 506]	694	[852, 853]	1049	[1162, 296]	1404	[403, 1071]
340	[418, 446]	695	[854, 188]	1050	[1163, 1164]	1405	[1416, 479]
341	[413, 187]	696	[55, 750]	1051	[1165, 1166]	1406	[1382, 490]
342	[200, 133]	697	[183, 801]	1052	[1167, 1168]	1407	[1384, 851]
343	[507, 184]	698	[855, 849]	1053	[1169, 1170]	1408	[1296, 1119]
344	[508, 127]	699	[856, 498]	1054	[1018, 719]	1409	[1106, 408]
345	[484, 403]	700	[857, 858]	1055	[1171, 321]	1410	[1417, 1275]
346	[159, 395]	701	[859, 860]	1056	[687, 1172]	1411	[1271, 467]
347	[470, 209]	702	[861, 862]	1057	[1173, 1174]	1412	[1273, 794]
348	[193, 196]	703	[401, 60]	1058	[1175, 359]	1413	[7, 468]
349	[178, 509]	704	[863, 127]	1059	[999, 723]	1414	[160, 468]
350	[510, 511]	705	[864, 178]	1060	[221, 1005]	1415	[878, 641]
351	[512, 462]	706	[782, 376]	1061	[897, 656]	1416	[1418, 727]
352	[513, 215]	707	[428, 815]	1062	[1176, 621]	1417	[613, 655]
353	[468, 408]	708	[865, 422]	1063	[1177, 1178]	1418	[177, 1065]
354	[201, 198]	709	[438, 390]	1064	[1179, 73]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	237	1	474	1	711	0	948	0	1185	0
1	0	238	1	475	0	712	0	949	1	1186	1
2	0	239	1	476	0	713	0	950	1	1187	1
3	0	240	0	477	1	714	0	951	0	1188	1
4	1	241	1	478	1	715	1	952	1	1189	1
5	0	242	1	479	0	716	1	953	0	1190	1

6	0	243	1	480	1	717	0	954	1	1191	1
7	0	244	1	481	0	718	0	955	0	1192	0
8	1	245	1	482	0	719	0	956	0	1193	1
9	1	246	1	483	0	720	0	957	0	1194	1
10	0	247	0	484	1	721	0	958	1	1195	1
11	0	248	0	485	1	722	0	959	1	1196	0
12	0	249	1	486	1	723	0	960	0	1197	0
13	0	250	1	487	0	724	1	961	1	1198	0
14	0	251	0	488	0	725	0	962	0	1199	1
15	0	252	1	489	0	726	0	963	0	1200	1
16	0	253	1	490	1	727	1	964	0	1201	0
17	1	254	1	491	1	728	1	965	0	1202	1
18	1	255	0	492	0	729	0	966	0	1203	1
19	1	256	1	493	0	730	0	967	0	1204	1
20	0	257	1	494	0	731	1	968	0	1205	1
21	0	258	1	495	1	732	1	969	0	1206	0
22	0	259	1	496	0	733	1	970	1	1207	0
23	0	260	0	497	1	734	0	971	1	1208	0
24	0	261	1	498	0	735	0	972	1	1209	0
25	0	262	1	499	1	736	1	973	0	1210	1
26	0	263	1	500	1	737	0	974	0	1211	0
27	0	264	0	501	1	738	1	975	0	1212	0
28	0	265	1	502	1	739	1	976	1	1213	1
29	0	266	0	503	0	740	0	977	1	1214	1
30	0	267	1	504	1	741	0	978	1	1215	0
31	0	268	0	505	1	742	0	979	0	1216	1
32	0	269	1	506	0	743	0	980	1	1217	0
33	0	270	0	507	1	744	1	981	0	1218	1
34	1	271	1	508	0	745	1	982	1	1219	0
35	1	272	1	509	0	746	0	983	0	1220	1
36	1	273	0	510	0	747	1	984	0	1221	0
37	1	274	1	511	1	748	1	985	0	1222	0
38	0	275	1	512	0	749	1	986	0	1223	0
39	1	276	0	513	0	750	0	987	1	1224	0
40	0	277	0	514	0	751	0	988	1	1225	0
41	0	278	0	515	0	752	0	989	1	1226	1
42	1	279	1	516	0	753	1	990	0	1227	0
43	0	280	0	517	0	754	0	991	0	1228	0
44	0	281	1	518	0	755	1	992	0	1229	0
45	0	282	0	519	0	756	0	993	0	1230	0
46	1	283	1	520	1	757	1	994	1	1231	0
47	0	284	0	521	0	758	0	995	1	1232	0
48	1	285	0	522	1	759	1	996	1	1233	0
49	0	286	1	523	0	760	0	997	0	1234	1

50	1	287	0	524	0	761	1	998	0	1235	1
51	0	288	0	525	0	762	1	999	0	1236	0
52	0	289	0	526	1	763	0	1000	1	1237	1
53	0	290	0	527	1	764	0	1001	0	1238	1
54	1	291	0	528	1	765	1	1002	0	1239	1
55	0	292	0	529	1	766	0	1003	0	1240	1
56	0	293	1	530	0	767	1	1004	1	1241	0
57	1	294	1	531	0	768	0	1005	0	1242	0
58	0	295	1	532	0	769	1	1006	1	1243	0
59	0	296	1	533	1	770	0	1007	0	1244	0
60	0	297	1	534	0	771	1	1008	1	1245	1
61	1	298	0	535	1	772	0	1009	0	1246	1
62	0	299	1	536	0	773	1	1010	1	1247	0
63	1	300	1	537	0	774	0	1011	1	1248	1
64	0	301	1	538	0	775	0	1012	0	1249	1
65	1	302	0	539	1	776	1	1013	0	1250	0
66	0	303	1	540	0	777	1	1014	1	1251	1
67	0	304	0	541	1	778	0	1015	0	1252	1
68	1	305	1	542	1	779	0	1016	0	1253	1
69	0	306	1	543	1	780	0	1017	1	1254	1
70	1	307	1	544	1	781	1	1018	1	1255	0
71	0	308	1	545	1	782	1	1019	1	1256	1
72	1	309	1	546	0	783	1	1020	1	1257	0
73	0	310	1	547	1	784	1	1021	1	1258	0
74	1	311	0	548	0	785	0	1022	0	1259	0
75	0	312	1	549	0	786	1	1023	1	1260	1
76	0	313	1	550	1	787	1	1024	0	1261	0
77	1	314	0	551	0	788	1	1025	1	1262	0
78	1	315	0	552	0	789	1	1026	1	1263	1
79	1	316	0	553	0	790	0	1027	0	1264	1
80	0	317	1	554	0	791	1	1028	1	1265	1
81	1	318	1	555	1	792	1	1029	1	1266	0
82	1	319	0	556	1	793	1	1030	1	1267	1
83	1	320	1	557	1	794	0	1031	0	1268	0
84	0	321	0	558	1	795	1	1032	0	1269	0
85	0	322	0	559	1	796	1	1033	0	1270	1
86	0	323	0	560	0	797	0	1034	1	1271	1
87	1	324	1	561	0	798	0	1035	0	1272	1
88	0	325	0	562	1	799	1	1036	0	1273	1
89	0	326	0	563	0	800	1	1037	1	1274	1
90	1	327	0	564	0	801	1	1038	1	1275	1
91	0	328	1	565	1	802	0	1039	0	1276	0
92	0	329	1	566	1	803	0	1040	0	1277	1
93	1	330	1	567	0	804	1	1041	1	1278	1

94	1	331	1	568	0	805	1	1042	1	1279	1
95	0	332	0	569	1	806	1	1043	0	1280	0
96	0	333	1	570	0	807	1	1044	0	1281	0
97	1	334	0	571	1	808	1	1045	0	1282	0
98	1	335	1	572	0	809	1	1046	0	1283	1
99	0	336	0	573	0	810	0	1047	0	1284	1
100	0	337	1	574	0	811	0	1048	1	1285	0
101	0	338	1	575	0	812	1	1049	0	1286	1
102	0	339	1	576	0	813	0	1050	1	1287	1
103	0	340	1	577	1	814	0	1051	1	1288	0
104	0	341	1	578	0	815	0	1052	1	1289	1
105	0	342	1	579	1	816	1	1053	1	1290	1
106	1	343	1	580	0	817	1	1054	1	1291	0
107	0	344	0	581	1	818	0	1055	0	1292	1
108	0	345	1	582	1	819	1	1056	1	1293	1
109	1	346	0	583	0	820	1	1057	1	1294	0
110	0	347	1	584	0	821	0	1058	1	1295	0
111	1	348	1	585	1	822	0	1059	0	1296	1
112	1	349	1	586	0	823	1	1060	1	1297	0
113	1	350	0	587	0	824	1	1061	1	1298	1
114	0	351	0	588	1	825	1	1062	0	1299	1
115	0	352	0	589	1	826	1	1063	0	1300	0
116	0	353	0	590	1	827	0	1064	0	1301	1
117	1	354	0	591	1	828	0	1065	0	1302	0
118	0	355	0	592	1	829	0	1066	1	1303	0
119	0	356	0	593	1	830	1	1067	1	1304	1
120	1	357	1	594	1	831	1	1068	1	1305	0
121	0	358	0	595	0	832	0	1069	0	1306	0
122	0	359	0	596	0	833	1	1070	0	1307	0
123	0	360	0	597	1	834	1	1071	0	1308	1
124	1	361	0	598	0	835	0	1072	1	1309	0
125	0	362	1	599	1	836	0	1073	1	1310	0
126	0	363	1	600	1	837	0	1074	1	1311	0
127	1	364	0	601	0	838	1	1075	1	1312	1
128	1	365	0	602	0	839	1	1076	0	1313	1
129	0	366	1	603	1	840	1	1077	1	1314	1
130	0	367	1	604	0	841	0	1078	0	1315	1
131	1	368	0	605	0	842	1	1079	1	1316	0
132	0	369	0	606	1	843	1	1080	1	1317	0
133	1	370	0	607	0	844	0	1081	1	1318	0
134	1	371	1	608	0	845	0	1082	1	1319	1
135	0	372	1	609	0	846	0	1083	1	1320	1
136	0	373	0	610	1	847	1	1084	0	1321	1
137	1	374	0	611	0	848	0	1085	1	1322	1

138	0	375	1	612	1	849	0	1086	0	1323	1
139	0	376	1	613	0	850	1	1087	1	1324	1
140	1	377	0	614	1	851	0	1088	0	1325	0
141	1	378	0	615	1	852	1	1089	1	1326	0
142	0	379	0	616	0	853	1	1090	1	1327	0
143	1	380	1	617	1	854	1	1091	1	1328	1
144	1	381	1	618	0	855	1	1092	0	1329	0
145	0	382	1	619	1	856	0	1093	0	1330	1
146	1	383	0	620	0	857	1	1094	0	1331	0
147	0	384	0	621	0	858	0	1095	0	1332	1
148	1	385	1	622	0	859	1	1096	0	1333	0
149	1	386	0	623	0	860	0	1097	0	1334	1
150	1	387	1	624	1	861	1	1098	1	1335	0
151	1	388	0	625	0	862	1	1099	0	1336	1
152	1	389	1	626	1	863	1	1100	0	1337	1
153	1	390	0	627	1	864	1	1101	1	1338	1
154	0	391	1	628	1	865	1	1102	1	1339	0
155	0	392	1	629	1	866	0	1103	1	1340	0
156	1	393	0	630	0	867	1	1104	0	1341	1
157	0	394	1	631	1	868	1	1105	1	1342	0
158	1	395	1	632	0	869	1	1106	0	1343	1
159	0	396	1	633	0	870	0	1107	1	1344	1
160	0	397	1	634	0	871	0	1108	0	1345	0
161	0	398	0	635	1	872	0	1109	0	1346	1
162	0	399	1	636	0	873	0	1110	0	1347	0
163	1	400	1	637	0	874	1	1111	0	1348	0
164	0	401	0	638	0	875	0	1112	1	1349	1
165	0	402	0	639	0	876	1	1113	1	1350	0
166	0	403	0	640	1	877	1	1114	1	1351	1
167	0	404	1	641	1	878	0	1115	0	1352	0
168	1	405	1	642	1	879	1	1116	0	1353	0
169	1	406	1	643	1	880	1	1117	0	1354	0
170	1	407	1	644	0	881	1	1118	0	1355	1
171	0	408	1	645	1	882	0	1119	0	1356	1
172	1	409	0	646	1	883	0	1120	0	1357	0
173	0	410	0	647	0	884	1	1121	0	1358	0
174	1	411	1	648	1	885	1	1122	0	1359	1
175	1	412	1	649	1	886	1	1123	1	1360	1
176	1	413	1	650	1	887	1	1124	0	1361	1
177	1	414	1	651	1	888	0	1125	1	1362	0
178	1	415	1	652	1	889	0	1126	0	1363	0
179	0	416	1	653	0	890	1	1127	0	1364	0
180	1	417	0	654	1	891	0	1128	1	1365	0
181	0	418	1	655	0	892	1	1129	0	1366	0

182	0	419	1	656	1	893	0	1130	0	1367	0
183	1	420	0	657	1	894	0	1131	1	1368	1
184	1	421	1	658	1	895	1	1132	0	1369	0
185	0	422	0	659	0	896	0	1133	1	1370	1
186	0	423	0	660	0	897	1	1134	1	1371	1
187	0	424	1	661	1	898	0	1135	1	1372	0
188	1	425	0	662	1	899	1	1136	1	1373	1
189	0	426	1	663	1	900	1	1137	0	1374	1
190	0	427	0	664	0	901	1	1138	1	1375	0
191	1	428	1	665	1	902	0	1139	1	1376	0
192	0	429	1	666	0	903	1	1140	1	1377	1
193	1	430	0	667	0	904	0	1141	0	1378	0
194	0	431	1	668	1	905	1	1142	1	1379	1
195	0	432	1	669	0	906	1	1143	1	1380	1
196	0	433	1	670	1	907	0	1144	1	1381	1
197	1	434	0	671	0	908	0	1145	0	1382	0
198	1	435	1	672	0	909	1	1146	0	1383	0
199	0	436	0	673	1	910	1	1147	0	1384	1
200	1	437	1	674	0	911	1	1148	1	1385	0
201	0	438	0	675	0	912	1	1149	1	1386	0
202	1	439	1	676	0	913	1	1150	1	1387	1
203	0	440	0	677	0	914	1	1151	0	1388	1
204	1	441	0	678	1	915	0	1152	1	1389	1
205	1	442	0	679	1	916	1	1153	0	1390	0
206	0	443	1	680	1	917	1	1154	1	1391	1
207	0	444	1	681	1	918	0	1155	0	1392	1
208	0	445	0	682	0	919	1	1156	1	1393	0
209	0	446	0	683	1	920	0	1157	1	1394	1
210	1	447	1	684	0	921	0	1158	0	1395	1
211	0	448	0	685	1	922	0	1159	0	1396	0
212	0	449	0	686	1	923	0	1160	0	1397	0
213	1	450	0	687	1	924	0	1161	1	1398	0
214	1	451	0	688	0	925	1	1162	0	1399	0
215	0	452	0	689	0	926	1	1163	1	1400	0
216	1	453	0	690	0	927	0	1164	1	1401	1
217	0	454	1	691	1	928	1	1165	1	1402	0
218	1	455	1	692	0	929	0	1166	1	1403	0
219	0	456	0	693	1	930	0	1167	1	1404	0
220	0	457	1	694	1	931	0	1168	1	1405	1
221	1	458	1	695	1	932	1	1169	1	1406	0
222	1	459	0	696	0	933	1	1170	0	1407	1
223	0	460	1	697	1	934	1	1171	0	1408	1
224	1	461	1	698	1	935	1	1172	0	1409	0
225	0	462	1	699	0	936	1	1173	1	1410	0

226	1	463	1	700	1	937	0	1174	1	1411	1
227	1	464	1	701	1	938	1	1175	1	1412	1
228	0	465	1	702	1	939	0	1176	0	1413	0
229	1	466	0	703	0	940	1	1177	0	1414	0
230	0	467	1	704	1	941	1	1178	1	1415	0
231	0	468	0	705	1	942	1	1179	0	1416	1
232	0	469	0	706	1	943	1	1180	1	1417	0
233	1	470	1	707	1	944	1	1181	1	1418	1
234	0	471	0	708	1	945	0	1182	0		
235	0	472	1	709	0	946	1	1183	1		
236	1	473	1	710	0	947	1	1184	1		

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