PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^2 + z, z^4 + z^2 + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^2 + z, z^4 + z^2 + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^2+z, z^4+z^2+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{16}+z^{12}+z^{10}+z^6+z^3+z+1,\\
p_1(z)&=z^{22}+z^{21}+z^{19}+z^{17}+z^8+z^7+z^6+z,\\
p_2(z)&=z^{24}+z^{18}+z^{10}+z^8+z^4+z^2,\\
p_3(z)&=z^{22}+z^{21}+z^{19}+z^{17}+z^8+z^7+z^6+z,\\
p_4(z)&=z^{20}+z^{17}+z^{16}+z^6+z^4+z^3+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{17}+z^{12}+z^{10}+z^6+z^4+z^3+z,\\
p_1(z)&=z^{22}+z^{21}+z^{19}+z^{17}+z^8+z^7+z^6+z,\\
p_2(z)&=z^{24}+z^{18}+z^{10}+z^8+z^4+z^2,\\
p_3(z)&=z^{22}+z^{21}+z^{19}+z^{17}+z^8+z^7+z^6+z,\\
p_4(z)&=z^{20}+z^{17}+z^6+z^3+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^u M^e[:6u] \\ & + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] + x^{4u} M^e[:6u] \\ & + x^{5u} M^e[:5u] + x^{6u} M^e[:4u] + x^{7u} M^e[:3u] + x^{5u} M^e[:6u] \\ & + x^{6u} M^e[:5u] + x^{7u} M^e[:4u] + x^{8u} M^e[:3u] \\ & + (x^{6u} + x^{7u} + x^{10u} + x^{11u}) R^e, \\ W^e[:12u] = {} & W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{3u} W^e[:3u] + x^u W^e[:6u] \end{aligned}$$

$$\begin{aligned}
&+ x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{4u}W^e[:6u]\\
&+ x^{5u}W^e[:5u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{5u}W^e[:6u]\\
&+ x^{6u}W^e[:5u] + x^{7u}W^e[:4u] + x^{8u}W^e[:3u]\\
&+ (x^{6u} + x^{7u} + x^{10u} + x^{11u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] = {}& M^o[:6u] + x^uM^o[:5u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^uM^o[:6u]\\
&+ x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{4u}M^o[:6u]\\
&+ x^{5u}M^o[:5u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{5u}M^o[:6u]\\
&+ x^{6u}M^o[:5u] + x^{7u}M^o[:4u] + x^{8u}M^o[:3u]\\
&+ (x^{6u} + x^{7u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] = {}& W^o[:6u] + x^uW^o[:5u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^uW^o[:6u]\\
&+ x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{4u}W^o[:6u]\\
&+ x^{5u}W^o[:5u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{5u}W^o[:6u]\\
&+ x^{6u}W^o[:5u] + x^{7u}W^o[:4u] + x^{8u}W^o[:3u]\\
&+ (x^{6u} + x^{7u} + x^{10u} + x^{11u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] = {}& T[:6u] + x^uT[:5u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^uT[:6u] + x^{2u}T[:5u]\\
&+ x^{3u}T[:4u] + x^{4u}T[:3u] + x^{4u}T[:6u] + x^{5u}T[:5u] + x^{6u}T[:4u]\\
&+ x^{7u}T[:3u] + x^{5u}T[:6u] + x^{6u}T[:5u] + x^{7u}T[:4u] + x^{8u}T[:3u].
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
0 &= x^{3u}T[3u:4u] + x^{2u}T[4u:5u] + x^uT[5u:6u] + T[6u:7u],\\
0 &= x^{4u}T[3u:4u] + T[7u:8u],\\
0 &= x^{8u}T[:u] + x^{4u}T[4u:5u] + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + x^{4u}T[5u:6u] + T[9u:10u],\\
0 &= x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u]\\
&\quad + T[10u:11u],\\
0 &= T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[11w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.9}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[11w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 554 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.19}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 111)), \tag{2.20}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.21}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$
$$\begin{aligned} 0 = {} & \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1010)), \end{aligned} \tag{2.23}$$
$$0 = \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.25}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 111)), \tag{2.26}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.27}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$
$$\begin{aligned} 0 = {} & \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1010)), \end{aligned} \tag{2.29}$$
$$0 = \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{20}), E_{20})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{3u} W^e[:3u] + x^{6u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^u M^o[:5u] + x^{2u} M^o[:4u] + x^{3u} M^o[:3u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^u W^o[:5u] + x^{2u} W^o[:4u] + x^{3u} W^o[:3u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k} M_{2k} = \big(W^e[:6v] + x^v W^e[:5v] + x^{2v} W^e[:4v] + x^{3v} W^e[:3v] + x^{6u} R^e\big) \times \big($$
$$M^e[:6v] + x^v M^e[:5v] + x^{2v} M^e[:4v] + x^{3v} M^e[:3v] + x^{6u} R^e\big); \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v] M^e[:12v] + x^{2v} W^e[:10v] M^e[:10v] + x^{4v} W^e[:8v] M^e[:8v]$$
$$+ x^{6v} W^e[:6v] M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X := x^v,$$
$$a_n := W^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$b_n := M^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$c := R^e.$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n \to \infty} M_{2n,j,k} = M^e_{j,k},$$

$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{21} + x^{18} + x^{14} + x^{12} + x^8 + x^7 + x^6,\\
q_1(x) &= x^{23} + x^{18} + x^{17} + x^{16} + x^7 + x^5 + x^3 + x^2,\\
q_2(x) &= x^{22} + x^{20} + x^{16} + x^{14} + x^6 + 1,\\
q_3(x) &= x^{23} + x^{18} + x^{17} + x^{16} + x^7 + x^5 + x^3 + x^2,\\
q_4(x) &= x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^8 + x^7 + x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{124} + x^{122} + x^{118} + x^{116} + x^{112} + x^{100} + x^{94} + x^{90}\\
&\quad + x^{88} + x^{86} + x^{84} + x^{82} + x^{76} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50}\\
&\quad + x^{44} + x^{36} + x^{32} + x^{30} + x^{26} + x^{24},\\
p_3(x) &= x^{124} + x^{118} + x^{116} + x^{106} + x^{104} + x^{102} + x^{100} + x^{90} + x^{88} + x^{76}\\
&\quad + x^{60} + x^{54} + x^{46} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{28} + x^{20},\\
p_6(x) &= x^{128} + x^{120} + x^{118} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{74} + x^{66} + x^{60} + x^{58} + x^{56} + x^{54} + x^{46} + x^{44} + x^{40} + x^{34} + x^{28}\\
&+ x^{26} + x^{20} + x^{18} + x^{12} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{130} + x^{124} + x^{122} + x^{120} + x^{118} + x^{112} + x^{110} + x^{108} + x^{104} + x^{100}\\
&+ x^{98} + x^{94} + x^{88} + x^{86} + x^{82} + x^{80} + x^{72} + x^{66} + x^{56} + x^{54} + x^{52}\\
&+ x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{20} + x^{16} + x^{12} + x^{4},\\
p_{12}(x) &= x^{130} + x^{128} + x^{114} + x^{112} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109}\\
&+ x^{108} + x^{107} + x^{105} + x^{103} + x^{100} + x^{97} + x^{96} + x^{95} + x^{90} + x^{89}\\
&+ x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67}\\
&+ x^{65} + x^{64} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{44} + x^{40}\\
&+ x^{36} + x^{34} + x^{30},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{113} + x^{109} + x^{108} + x^{107}\\
&+ x^{105} + x^{104} + x^{102} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{73} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60}\\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{41}\\
&+ x^{37} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{18},\\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{103} + x^{102} + x^{97} + x^{96} + x^{91} + x^{90} + x^{85} + x^{84} + x^{79} + x^{78} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60}\\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{27} + x^{26}\\
&+ x^{13} + x^{12},\\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110}\\
&+ x^{108} + x^{104} + x^{103} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87}\\
&+ x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48}\\
&+ x^{44} + x^{43} + x^{39} + x^{36} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{107} + x^{106}\\
&+ x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83}\\
&+ x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{65} + x^{64} + x^{63} + x^{62}\\
&+ x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{41} + x^{40} + x^{39}\\
&+ x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16}
\end{aligned}
$$

$$+ x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {}& x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{117} + x^{116} + x^{113} + x^{111} + x^{109} \\
&+ x^{108} + x^{107} + x^{105} + x^{103} + x^{100} + x^{97} + x^{96} + x^{95} + x^{90} + x^{89} \\
&+ x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67} \\
&+ x^{65} + x^{64} + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{50} + x^{49} + x^{44} + x^{40} \\
&+ x^{36} + x^{34} + x^{30}, \\
p_3(x) = {}& x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{113} + x^{109} + x^{108} + x^{107} \\
&+ x^{105} + x^{104} + x^{102} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} \\
&+ x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{73} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} \\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{41} \\
&+ x^{37} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{18}, \\
p_6(x) = {}& x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
&+ x^{103} + x^{102} + x^{97} + x^{96} + x^{91} + x^{90} + x^{85} + x^{84} + x^{79} + x^{78} + x^{73} \\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} \\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{27} + x^{26} \\
&+ x^{13} + x^{12}, \\
p_9(x) = {}& x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} \\
&+ x^{108} + x^{104} + x^{103} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} \\
&+ x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} \\
&+ x^{44} + x^{43} + x^{39} + x^{36} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} \\
&+ x^{20} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) = {}& x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{107} + x^{106} \\
&+ x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} \\
&+ x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{65} + x^{64} + x^{63} + x^{62} \\
&+ x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{41} + x^{40} + x^{39} \\
&+ x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16} \\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {}& x^{120} + x^{118} + x^{114} + x^{108} + x^{98} + x^{92} + x^{84} + x^{82} + x^{78} + x^{72} + x^{68} \\
&+ x^{64} + x^{56} + x^{52} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{114} + x^{110} + x^{102} + x^{100} + x^{96} + x^{84} + x^{82} + x^{80} + x^{72} + x^{70} + x^{66} \\
&\quad + x^{62} + x^{60} + x^{56} + x^{54} + x^{48} + x^{46} + x^{44} + x^{38} + x^{34} + x^{30} + x^{28} \\
&\quad + x^{20} + x^{18} + x^{14} + x^{12}, \\
p_6(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{92} + x^{90} + x^{86} + x^{82} + x^{80} + x^{72} \\
&\quad + x^{70} + x^{66} + x^{60} + x^{54} + x^{42} + x^{38} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{12} + x^{8} + x^{6} + 1, \\
p_9(x) &= x^{112} + x^{110} + x^{106} + x^{100} + x^{96} + x^{94} + x^{90} + x^{88} + x^{78} + x^{68} + x^{64} \\
&\quad + x^{62} + x^{60} + x^{56} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} \\
&\quad + x^{32} + x^{24} + x^{14} + x^{12} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{112} + x^{108} + x^{100} + x^{96} + x^{80} + x^{76} + x^{68} + x^{60} + x^{52} + x^{48} + x^{32} \\
&\quad + x^{28} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{100} + x^{94} + x^{92} \\
&\quad + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{60} + x^{54} + x^{48} \\
&\quad + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24}, \\
p_3(x) &= x^{114} + x^{102} + x^{98} + x^{90} + x^{86} + x^{84} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} \\
&\quad + x^{62} + x^{60} + x^{56} + x^{54} + x^{48} + x^{44} + x^{36} + x^{32} + x^{24}, \\
p_6(x) &= x^{114} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} + x^{88} + x^{78} + x^{74} + x^{72} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{60} + x^{54} + x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{28} + x^{24} \\
&\quad + x^{22} + x^{14} + x^{6} + 1, \\
p_9(x) &= x^{112} + x^{110} + x^{106} + x^{100} + x^{96} + x^{94} + x^{90} + x^{88} + x^{78} + x^{68} + x^{64} \\
&\quad + x^{62} + x^{60} + x^{56} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} \\
&\quad + x^{32} + x^{24} + x^{14} + x^{12} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{112} + x^{108} + x^{100} + x^{96} + x^{80} + x^{76} + x^{68} + x^{60} + x^{52} + x^{48} + x^{32} \\
&\quad + x^{28} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} \\
&\quad + x^{111} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} \\
&\quad + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{82} + x^{81} + x^{79} + x^{75} + x^{72} + x^{71} \\
&\quad + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{30}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{113} + x^{109} + x^{108} + x^{107}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{105} + x^{104} + x^{102} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} \\
&+ x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{73} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} \\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{41} \\
&+ x^{37} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{18}, \\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
&+ x^{103} + x^{102} + x^{97} + x^{96} + x^{91} + x^{90} + x^{85} + x^{84} + x^{79} + x^{78} + x^{73} \\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} \\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{27} + x^{26} \\
&+ x^{13} + x^{12}, \\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} \\
&+ x^{108} + x^{104} + x^{103} + x^{99} + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} \\
&+ x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} \\
&+ x^{44} + x^{43} + x^{39} + x^{36} + x^{30} + x^{29} + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} \\
&+ x^{20} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{107} + x^{106} \\
&+ x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{89} + x^{88} + x^{87} + x^{86} + x^{83} \\
&+ x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{65} + x^{64} + x^{63} + x^{62} \\
&+ x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} + x^{48} + x^{41} + x^{40} + x^{39} \\
&+ x^{38} + x^{35} + x^{34} + x^{31} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{17} + x^{16} \\
&+ x^{15} + x^{14} + x^{11} + x^{10} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} \\
&+ x^{111} + x^{106} + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93} \\
&+ x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{82} + x^{81} + x^{79} + x^{75} + x^{72} + x^{71} \\
&+ x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{56} + x^{54} + x^{52} \\
&+ x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} + x^{33} \\
&+ x^{32} + x^{30}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{113} + x^{109} + x^{108} + x^{107} \\
&+ x^{105} + x^{104} + x^{102} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} \\
&+ x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{73} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} \\
&+ x^{59} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{41} \\
&+ x^{37} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{18},
\end{aligned}
$$

$$\begin{aligned}p_6(x) &= x^{123}+x^{122}+x^{121}+x^{120}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}\\&\quad+x^{103}+x^{102}+x^{97}+x^{96}+x^{91}+x^{90}+x^{85}+x^{84}+x^{79}+x^{78}+x^{73}\\&\quad+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{61}+x^{60}\\&\quad+x^{57}+x^{56}+x^{55}+x^{54}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}\\&\quad+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{35}+x^{34}+x^{33}+x^{32}+x^{27}+x^{26}\\&\quad+x^{13}+x^{12},\\p_9(x) &= x^{122}+x^{121}+x^{120}+x^{119}+x^{118}+x^{115}+x^{114}+x^{113}+x^{111}+x^{110}\\&\quad+x^{108}+x^{104}+x^{103}+x^{99}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}+x^{88}+x^{87}\\&\quad+x^{84}+x^{80}+x^{79}+x^{78}+x^{77}+x^{75}+x^{73}+x^{72}+x^{70}+x^{68}+x^{67}\\&\quad+x^{66}+x^{65}+x^{64}+x^{62}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{51}+x^{48}\\&\quad+x^{44}+x^{43}+x^{39}+x^{36}+x^{30}+x^{29}+x^{26}+x^{24}+x^{23}+x^{22}+x^{21}\\&\quad+x^{20}+x^{18}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{6},\\p_{12}(x) &= x^{121}+x^{120}+x^{119}+x^{118}+x^{115}+x^{114}+x^{111}+x^{110}+x^{107}+x^{106}\\&\quad+x^{103}+x^{102}+x^{99}+x^{98}+x^{97}+x^{96}+x^{89}+x^{88}+x^{87}+x^{86}+x^{83}\\&\quad+x^{82}+x^{79}+x^{78}+x^{75}+x^{74}+x^{73}+x^{72}+x^{65}+x^{64}+x^{63}+x^{62}\\&\quad+x^{59}+x^{58}+x^{55}+x^{54}+x^{51}+x^{50}+x^{49}+x^{48}+x^{41}+x^{40}+x^{39}\\&\quad+x^{38}+x^{35}+x^{34}+x^{31}+x^{30}+x^{27}+x^{26}+x^{25}+x^{24}+x^{17}+x^{16}\\&\quad+x^{15}+x^{14}+x^{11}+x^{10}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}p_0(x) &= x^{132}+x^{130}+x^{128}+x^{124}+x^{122}+x^{118}+x^{116}+x^{114}+x^{102}+x^{98}\\&\quad+x^{96}+x^{94}+x^{92}+x^{86}+x^{78}+x^{76}+x^{74}+x^{72}+x^{68}+x^{66}+x^{64}\\&\quad+x^{62}+x^{50}+x^{48}+x^{36},\\p_3(x) &= x^{128}+x^{126}+x^{122}+x^{120}+x^{118}+x^{116}+x^{114}+x^{108}+x^{106}+x^{100}\\&\quad+x^{96}+x^{92}+x^{88}+x^{84}+x^{74}+x^{68}+x^{64}+x^{62}+x^{58}+x^{52}+x^{42}\\&\quad+x^{36}+x^{34}+x^{32}+x^{26}+x^{20}+x^{18}+x^{12},\\p_6(x) &= x^{122}+x^{118}+x^{114}+x^{110}+x^{108}+x^{106}+x^{104}+x^{102}+x^{96}+x^{94}\\&\quad+x^{92}+x^{88}+x^{86}+x^{82}+x^{80}+x^{76}+x^{74}+x^{72}+x^{70}+x^{60}+x^{56}\\&\quad+x^{54}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{26}+x^{24}+x^{22}+x^{14}\\&\quad+x^{12}+x^{10}+x^{8}+x^{4}+x^{2}+1,\\p_9(x) &= x^{130}+x^{124}+x^{122}+x^{120}+x^{118}+x^{112}+x^{110}+x^{108}+x^{104}+x^{100}\\&\quad+x^{98}+x^{94}+x^{88}+x^{86}+x^{82}+x^{80}+x^{72}+x^{66}+x^{56}+x^{54}+x^{52}\\&\quad+x^{40}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{20}+x^{16}+x^{12}+x^{4},\\p_{12}(x) &= x^{130}+x^{128}+x^{114}+x^{112}+x^{2}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{122} + x^{118} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104}\\
&\quad + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{93} + x^{91} + x^{84}\\
&\quad + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{69} + x^{66} + x^{62} + x^{61} + x^{58}\\
&\quad + x^{56} + x^{54} + x^{52} + x^{50} + x^{49} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{34}\\
&\quad + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24},\\
p_3(x) &= x^{125} + x^{122} + x^{118} + x^{115} + x^{113} + x^{112} + x^{111} + x^{108} + x^{106} + x^{105}\\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{93} + x^{89}\\
&\quad + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{70}\\
&\quad + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{47} + x^{45}\\
&\quad + x^{44} + x^{43} + x^{38} + x^{37} + x^{36} + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{19}\\
&\quad + x^{18} + x^{17} + x^{16},\\
p_6(x) &= x^{125} + x^{122} + x^{120} + x^{119} + x^{118} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105}\\
&\quad + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{94} + x^{93} + x^{89} + x^{87} + x^{86}\\
&\quad + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{69} + x^{68} + x^{67} + x^{66} + x^{63}\\
&\quad + x^{60} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{41} + x^{39}\\
&\quad + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{24} + x^{22} + x^{21}\\
&\quad + x^{19} + x^{15} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{109} + x^{108}\\
&\quad + x^{107} + x^{106} + x^{105} + x^{104} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86}\\
&\quad + x^{85} + x^{84} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{61} + x^{60} + x^{59}\\
&\quad + x^{58} + x^{57} + x^{56} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} + x^{36}\\
&\quad + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9}\\
&\quad + x^{8},\\
p_{12}(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114}\\
&\quad + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98}\\
&\quad + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83}\\
&\quad + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64}\\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49}\\
&\quad + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34}\\
&\quad + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15}\\
&\quad + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0].

For $\phi(M^o, 0, 1)$:

$$p_0(x) = x^{128} + x^{125} + x^{123} + x^{119} + x^{117} + x^{116} + x^{113} + x^{108} + x^{106} + x^{105}$$

$$
\begin{aligned}
& + x^{103} + x^{101} + x^{99} + x^{98} + x^{94} + x^{90} + x^{89} + x^{88} + x^{86} + x^{82} + x^{81} \\
& + x^{80} + x^{79} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} \\
& + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{50} + x^{49} + x^{47} + x^{42} \\
& + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{31} + x^{30}, \\
p_3(x) = {} & x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{110} + x^{108} + x^{105} + x^{104} + x^{103} \\
& + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} \\
& + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{70} + x^{68} + x^{64} \\
& + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} \\
& + x^{44} + x^{43} + x^{38} + x^{36} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} \\
& + x^{19} + x^{18}, \\
p_6(x) = {} & x^{122} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{80} \\
& + x^{76} + x^{70} + x^{68} + x^{64} + x^{60} + x^{58} + x^{52} + x^{46} + x^{44} + x^{40} + x^{38} \\
& + x^{36} + x^{30} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{123} + x^{121} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{104} \\
& + x^{103} + x^{102} + x^{91} + x^{89} + x^{86} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} \\
& + x^{75} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{56} \\
& + x^{55} + x^{54} + x^{43} + x^{41} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} \\
& + x^{27} + x^{25} + x^{24} + x^{23} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{8} \\
& + x^{7} + x^{6}, \\
p_{12}(x) = {} & x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{96} + x^{92} + x^{88} + x^{84} + x^{68} + x^{64} \\
& + x^{60} + x^{56} + x^{48} + x^{44} + x^{40} + x^{36} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{126} + x^{122} + x^{121} + x^{119} + x^{117} \\
& + x^{112} + x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{84} + x^{81} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{59} + x^{54} \\
& + x^{50} + x^{46} + x^{44} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30}, \\
p_3(x) = {} & x^{133} + x^{131} + x^{128} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{114} \\
& + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{98} + x^{96} + x^{92} \\
& + x^{90} + x^{86} + x^{84} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{69} + x^{67} + x^{64} \\
& + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{37} + x^{35} \\
& + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = {} & x^{134} + x^{130} + x^{126} + x^{124} + x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
& + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{68}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{66} + x^{60} + x^{58} + x^{52} + x^{44} + x^{42} + x^{40} + x^{38} + x^{28} + x^{26} + x^{24}\\
&+ x^{22} + x^{18} + x^{14} + x^{12},\\
p_9(x) &= x^{135} + x^{133} + x^{131} + x^{130} + x^{129} + x^{127} + x^{124} + x^{121} + x^{120} + x^{118}\\
&+ x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{92} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73}\\
&+ x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56}\\
&+ x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38}\\
&+ x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23}\\
&+ x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{136} + x^{124} + x^{100} + x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{52}\\
&+ x^{48} + x^{44} + x^{40} + x^{28} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{119} + x^{116} + x^{115} + x^{110} + x^{109} + x^{103} + x^{96} + x^{92}\\
&+ x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75}\\
&+ x^{74} + x^{72} + x^{69} + x^{66} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{49}\\
&+ x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36},\\
p_3(x) &= x^{125} + x^{123} + x^{118} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} + x^{106} + x^{103}\\
&+ x^{102} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{88} + x^{86} + x^{84} + x^{82} + x^{79}\\
&+ x^{77} + x^{76} + x^{75} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59}\\
&+ x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{42} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{25}\\
&+ x^{24} + x^{22} + x^{21} + x^{20},\\
p_6(x) &= x^{125} + x^{123} + x^{120} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{107} + x^{104}\\
&+ x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{86}\\
&+ x^{85} + x^{82} + x^{81} + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{63} + x^{61} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{47} + x^{46}\\
&+ x^{41} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25}\\
&+ x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7}\\
&+ x^{6} + x^{3} + x^{2} + x + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{109} + x^{108}\\
&+ x^{107} + x^{106} + x^{105} + x^{104} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{61} + x^{60} + x^{59}\\
&+ x^{58} + x^{57} + x^{56} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} + x^{36}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9}\\
&+x^{8},\\
p_{12}(x) = {}& x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{117}+x^{116}+x^{115}+x^{114}\\
&+x^{111}+x^{110}+x^{109}+x^{108}+x^{105}+x^{104}+x^{103}+x^{102}+x^{99}+x^{98}\\
&+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{79}+x^{78}+x^{75}+x^{74}+x^{71}+x^{70}+x^{69}+x^{68}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}+x^{51}+x^{50}+x^{49}\\
&+x^{48}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36}+x^{35}+x^{34}\\
&+x^{31}+x^{30}+x^{27}+x^{26}+x^{23}+x^{22}+x^{21}+x^{20}+x^{17}+x^{16}+x^{15}\\
&+x^{14}+x^{13}+x^{12}+x^{9}+x^{8}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{126}+x^{125}+x^{122}+x^{118}+x^{111}+x^{110}+x^{109}+x^{107}+x^{106}+x^{104}\\
&+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{93}+x^{91}+x^{84}\\
&+x^{81}+x^{78}+x^{77}+x^{76}+x^{75}+x^{72}+x^{69}+x^{66}+x^{62}+x^{61}+x^{58}\\
&+x^{56}+x^{54}+x^{52}+x^{50}+x^{49}+x^{44}+x^{43}+x^{40}+x^{38}+x^{37}+x^{34}\\
&+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24},\\
p_3(x) = {}& x^{125}+x^{122}+x^{118}+x^{115}+x^{113}+x^{112}+x^{111}+x^{108}+x^{106}+x^{105}\\
&+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{97}+x^{96}+x^{94}+x^{93}+x^{89}\\
&+x^{88}+x^{87}+x^{85}+x^{82}+x^{81}+x^{80}+x^{79}+x^{77}+x^{76}+x^{74}+x^{70}\\
&+x^{62}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{54}+x^{52}+x^{50}+x^{47}+x^{45}\\
&+x^{44}+x^{43}+x^{38}+x^{37}+x^{36}+x^{30}+x^{28}+x^{25}+x^{24}+x^{23}+x^{19}\\
&+x^{18}+x^{17}+x^{16},\\
p_6(x) = {}& x^{125}+x^{122}+x^{120}+x^{119}+x^{118}+x^{112}+x^{111}+x^{109}+x^{106}+x^{105}\\
&+x^{103}+x^{102}+x^{100}+x^{99}+x^{98}+x^{97}+x^{94}+x^{93}+x^{89}+x^{87}+x^{86}\\
&+x^{84}+x^{82}+x^{81}+x^{79}+x^{78}+x^{77}+x^{69}+x^{68}+x^{67}+x^{66}+x^{63}\\
&+x^{60}+x^{56}+x^{55}+x^{54}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{41}+x^{39}\\
&+x^{38}+x^{37}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{26}+x^{24}+x^{22}+x^{21}\\
&+x^{19}+x^{15}+x^{13}+x^{12}+x^{9}+x^{8}+x^{7}+x^{6}+x^{3}+x^{2}+x+1,\\
p_9(x) = {}& x^{125}+x^{124}+x^{123}+x^{122}+x^{119}+x^{118}+x^{117}+x^{116}+x^{109}+x^{108}\\
&+x^{107}+x^{106}+x^{105}+x^{104}+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}\\
&+x^{85}+x^{84}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{61}+x^{60}+x^{59}\\
&+x^{58}+x^{57}+x^{56}+x^{45}+x^{44}+x^{43}+x^{42}+x^{39}+x^{38}+x^{37}+x^{36}\\
&+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9}\\
&+x^{8},
\end{aligned}
$$

$$
\begin{aligned}
p_{12}(x) = {} & x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} \\
& + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} \\
& + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} \\
& + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} \\
& + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} \\
& + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} \\
& + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} \\
& + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{126} + x^{122} + x^{121} + x^{119} + x^{117} \\
& + x^{112} + x^{109} + x^{108} + x^{107} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
& + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{84} + x^{81} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{59} + x^{54} \\
& + x^{50} + x^{46} + x^{44} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30}, \\
p_3(x) = {} & x^{133} + x^{131} + x^{128} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{114} \\
& + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{102} + x^{98} + x^{96} + x^{92} \\
& + x^{90} + x^{86} + x^{84} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{69} + x^{67} + x^{64} \\
& + x^{60} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{37} + x^{35} \\
& + x^{32} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = {} & x^{134} + x^{130} + x^{126} + x^{124} + x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
& + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{68} \\
& + x^{66} + x^{60} + x^{58} + x^{52} + x^{44} + x^{42} + x^{40} + x^{38} + x^{28} + x^{26} + x^{24} \\
& + x^{22} + x^{18} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{135} + x^{133} + x^{131} + x^{130} + x^{129} + x^{127} + x^{124} + x^{121} + x^{120} + x^{118} \\
& + x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} \\
& + x^{104} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{95} + x^{92} + x^{89} + x^{88} + x^{87} \\
& + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} \\
& + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} \\
& + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} \\
& + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{23} \\
& + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) = {} & x^{136} + x^{124} + x^{100} + x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{52} \\
& + x^{48} + x^{44} + x^{40} + x^{28} + x^{24} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{128} + x^{125} + x^{123} + x^{119} + x^{117} + x^{116} + x^{113} + x^{108} + x^{106} + x^{105} \\
&\quad + x^{103} + x^{101} + x^{99} + x^{98} + x^{94} + x^{90} + x^{89} + x^{88} + x^{86} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{79} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{64} + x^{62} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{50} + x^{49} + x^{47} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{31} + x^{30}, \\
p_3(x) &= x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{110} + x^{108} + x^{105} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{90} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{75} + x^{70} + x^{68} + x^{64} \\
&\quad + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{38} + x^{36} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} \\
&\quad + x^{19} + x^{18}, \\
p_6(x) &= x^{122} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{100} + x^{92} + x^{88} + x^{80} \\
&\quad + x^{76} + x^{70} + x^{68} + x^{64} + x^{60} + x^{58} + x^{52} + x^{46} + x^{44} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{30} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{123} + x^{121} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{91} + x^{89} + x^{86} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{43} + x^{41} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{27} + x^{25} + x^{24} + x^{23} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{8} \\
&\quad + x^{7} + x^{6}, \\
p_{12}(x) &= x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{96} + x^{92} + x^{88} + x^{84} + x^{68} + x^{64} \\
&\quad + x^{60} + x^{56} + x^{48} + x^{44} + x^{40} + x^{36} + x^{20} + x^{16} + x^{12} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{119} + x^{116} + x^{115} + x^{110} + x^{109} + x^{103} + x^{96} + x^{92} \\
&\quad + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{69} + x^{66} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{49} \\
&\quad + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36}, \\
p_3(x) &= x^{125} + x^{123} + x^{118} + x^{114} + x^{113} + x^{112} + x^{110} + x^{108} + x^{106} + x^{103} \\
&\quad + x^{102} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{88} + x^{86} + x^{84} + x^{82} + x^{79} \\
&\quad + x^{77} + x^{76} + x^{75} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} \\
&\quad + x^{42} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{25} \\
&\quad + x^{24} + x^{22} + x^{21} + x^{20},
\end{aligned}$$

$$\begin{aligned}
p_6(x) = {} & x^{125} + x^{123} + x^{120} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{107} + x^{104} \\
& + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{86} \\
& + x^{85} + x^{82} + x^{81} + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} \\
& + x^{66} + x^{63} + x^{61} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{47} + x^{46} \\
& + x^{41} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} \\
& + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{10} + x^{7} \\
& + x^{6} + x^{3} + x^{2} + x + 1, \\
p_9(x) = {} & x^{125} + x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{117} + x^{116} + x^{109} + x^{108} \\
& + x^{107} + x^{106} + x^{105} + x^{104} + x^{93} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} \\
& + x^{85} + x^{84} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{61} + x^{60} + x^{59} \\
& + x^{58} + x^{57} + x^{56} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} + x^{36} \\
& + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} \\
& + x^{8}, \\
p_{12}(x) = {} & x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} \\
& + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} \\
& + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{85} + x^{84} + x^{83} \\
& + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} \\
& + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{49} \\
& + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} \\
& + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} \\
& + x^{14} + x^{13} + x^{12} + x^{9} + x^{8} + x^{7} + x^{6} + x^{3} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	111	[176, 151]	222	[123, 115]	333	[407, 16]	444	[479, 198]
1	[3, 4]	112	[12, 177]	223	[307, 135]	334	[408, 86]	445	[480, 474]
2	[5, 6]	113	[178, 179]	224	[308, 122]	335	[409, 203]	446	[481, 342]
3	[7, 8]	114	[180, 181]	225	[309, 48]	336	[410, 254]	447	[482, 246]
4	[9, 10]	115	[182, 183]	226	[52, 310]	337	[355, 411]	448	[88, 483]
5	[11, 12]	116	[116, 116]	227	[311, 8]	338	[87, 405]	449	[484, 230]
6	[13, 14]	117	[184, 185]	228	[32, 312]	339	[208, 412]	450	[485, 486]
7	[15, 16]	118	[78, 186]	229	[8, 169]	340	[354, 413]	451	[487, 243]
8	[17, 18]	119	[63, 185]	230	[165, 40]	341	[414, 57]	452	[408, 184]
9	[19, 20]	120	[185, 187]	231	[48, 42]	342	[415, 57]	453	[488, 356]
10	[21, 22]	121	[188, 189]	232	[313, 8]	343	[334, 44]	454	[489, 76]
11	[23, 24]	122	[74, 190]	233	[310, 305]	344	[8, 269]	455	[490, 491]

12	[25, 26]	123	[191, 182]	234	[314, 276]	345	[416, 144]	456	[492, 107]
13	[27, 28]	124	[189, 192]	235	[315, 316]	346	[339, 312]	457	[493, 286]
14	[29, 30]	125	[179, 193]	236	[169, 301]	347	[417, 167]	458	[494, 157]
15	[31, 32]	126	[194, 75]	237	[317, 299]	348	[418, 36]	459	[495, 48]
16	[33, 34]	127	[195, 80]	238	[318, 319]	349	[39, 128]	460	[496, 51]
17	[35, 36]	128	[79, 196]	239	[287, 47]	350	[419, 10]	461	[497, 47]
18	[37, 38]	129	[197, 198]	240	[320, 321]	351	[420, 421]	462	[146, 128]
19	[39, 40]	130	[199, 200]	241	[162, 312]	352	[422, 420]	463	[310, 150]
20	[41, 42]	131	[201, 63]	242	[322, 310]	353	[423, 424]	464	[498, 47]
21	[43, 36]	132	[202, 203]	243	[313, 55]	354	[169, 271]	465	[125, 295]
22	[44, 45]	133	[204, 205]	244	[323, 143]	355	[425, 151]	466	[499, 144]
23	[46, 47]	134	[206, 207]	245	[324, 122]	356	[45, 175]	467	[500, 269]
24	[48, 49]	135	[208, 209]	246	[136, 115]	357	[273, 426]	468	[501, 280]
25	[50, 51]	136	[210, 211]	247	[325, 326]	358	[281, 123]	469	[502, 152]
26	[52, 38]	137	[212, 213]	248	[176, 327]	359	[427, 12]	470	[425, 327]
27	[53, 54]	138	[193, 109]	249	[328, 273]	360	[150, 327]	471	[503, 420]
28	[7, 55]	139	[214, 187]	250	[329, 330]	361	[58, 280]	472	[419, 424]
29	[56, 57]	140	[215, 216]	251	[331, 286]	362	[428, 135]	473	[504, 157]
30	[58, 59]	141	[187, 217]	252	[297, 140]	363	[282, 58]	474	[322, 38]
31	[60, 61]	142	[217, 207]	253	[332, 321]	364	[429, 273]	475	[423, 10]
32	[62, 63]	143	[218, 219]	254	[33, 312]	365	[274, 175]	476	[420, 330]
33	[64, 65]	144	[110, 220]	255	[333, 51]	366	[269, 327]	477	[505, 287]
34	[61, 66]	145	[221, 222]	256	[160, 174]	367	[430, 114]	478	[506, 137]
35	[67, 68]	146	[223, 92]	257	[334, 171]	368	[431, 287]	479	[507, 164]
36	[69, 70]	147	[224, 76]	258	[32, 34]	369	[432, 122]	480	[508, 421]
37	[71, 72]	148	[225, 101]	259	[335, 32]	370	[433, 152]	481	[509, 58]
38	[3, 73]	149	[226, 227]	260	[336, 310]	371	[288, 280]	482	[510, 137]
39	[74, 75]	150	[99, 228]	261	[311, 55]	372	[434, 287]	483	[9, 424]
40	[76, 77]	151	[73, 229]	262	[337, 276]	373	[273, 319]	484	[511, 164]
41	[78, 79]	152	[230, 231]	263	[277, 115]	374	[435, 295]	485	[512, 167]
42	[80, 81]	153	[232, 233]	264	[338, 175]	375	[303, 141]	486	[168, 269]
43	[82, 83]	154	[234, 216]	265	[305, 271]	376	[436, 144]	487	[513, 426]
44	[84, 85]	155	[80, 191]	266	[44, 174]	377	[437, 167]	488	[514, 452]
45	[86, 87]	156	[235, 236]	267	[339, 34]	378	[418, 149]	489	[515, 137]
46	[5, 88]	157	[237, 238]	268	[173, 340]	379	[37, 310]	490	[55, 305]
47	[4, 89]	158	[63, 239]	269	[341, 29]	380	[36, 152]	491	[338, 153]
48	[90, 91]	159	[240, 241]	270	[342, 342]	381	[56, 158]	492	[516, 114]
49	[92, 93]	160	[242, 243]	271	[343, 344]	382	[438, 305]	493	[517, 191]
50	[94, 95]	161	[244, 210]	272	[345, 346]	383	[173, 276]	494	[518, 219]
51	[96, 97]	162	[245, 246]	273	[347, 348]	384	[439, 440]	495	[519, 80]
52	[98, 99]	163	[247, 248]	274	[24, 349]	385	[441, 143]	496	[520, 341]
53	[100, 101]	164	[249, 250]	275	[111, 4]	386	[442, 56]	497	[521, 248]
54	[102, 103]	165	[251, 252]	276	[350, 351]	387	[143, 337]	498	[249, 522]
55	[104, 105]	166	[253, 254]	277	[92, 189]	388	[443, 56]	499	[523, 475]

56	[106, 107]	167	[255, 256]	278	[352, 353]	389	[270, 301]	500	[70, 242]
57	[65, 108]	168	[227, 257]	279	[354, 355]	390	[43, 149]	501	[475, 491]
58	[109, 110]	169	[16, 258]	280	[356, 357]	391	[275, 340]	502	[73, 390]
59	[111, 112]	170	[259, 241]	281	[61, 358]	392	[415, 158]	503	[108, 349]
60	[113, 114]	171	[260, 261]	282	[359, 360]	393	[444, 54]	504	[524, 472]
61	[115, 116]	172	[262, 263]	283	[65, 215]	394	[144, 440]	505	[486, 389]
62	[117, 118]	173	[264, 265]	284	[344, 220]	395	[445, 326]	506	[525, 264]
63	[119, 120]	174	[198, 24]	285	[361, 116]	396	[446, 273]	507	[526, 365]
64	[121, 122]	175	[70, 266]	286	[362, 363]	397	[329, 421]	508	[96, 527]
65	[123, 124]	176	[258, 267]	287	[364, 365]	398	[447, 56]	509	[396, 356]
66	[114, 125]	177	[19, 268]	288	[265, 366]	399	[414, 158]	510	[256, 470]
67	[126, 48]	178	[38, 269]	289	[367, 92]	400	[149, 174]	511	[528, 88]
68	[127, 128]	179	[270, 271]	290	[368, 79]	401	[448, 440]	512	[529, 99]
69	[129, 44]	180	[272, 273]	291	[369, 222]	402	[279, 59]	513	[102, 530]
70	[130, 36]	181	[274, 153]	292	[370, 4]	403	[36, 423]	514	[531, 21]
71	[131, 56]	182	[142, 142]	293	[371, 358]	404	[449, 450]	515	[532, 193]
72	[132, 57]	183	[142, 140]	294	[372, 91]	405	[41, 49]	516	[533, 483]
73	[133, 44]	184	[275, 276]	295	[365, 373]	406	[451, 452]	517	[138, 440]
74	[134, 135]	185	[277, 124]	296	[374, 375]	407	[453, 321]	518	[534, 316]
75	[115, 123]	186	[278, 173]	297	[79, 263]	408	[454, 146]	519	[535, 143]
76	[136, 124]	187	[120, 120]	298	[376, 228]	409	[455, 143]	520	[536, 157]
77	[137, 138]	188	[279, 280]	299	[377, 378]	410	[456, 32]	521	[537, 12]
78	[139, 118]	189	[281, 116]	300	[267, 346]	411	[174, 424]	522	[45, 153]
79	[140, 120]	190	[282, 125]	301	[379, 380]	412	[140, 141]	523	[538, 420]
80	[141, 142]	191	[283, 142]	302	[350, 264]	413	[299, 14]	524	[539, 540]
81	[143, 144]	192	[116, 123]	303	[76, 375]	414	[457, 210]	525	[541, 305]
82	[145, 146]	193	[269, 151]	304	[81, 267]	415	[362, 61]	526	[542, 452]
83	[147, 42]	194	[284, 114]	305	[72, 381]	416	[237, 236]	527	[336, 38]
84	[148, 44]	195	[285, 286]	306	[382, 89]	417	[458, 72]	528	[543, 326]
85	[130, 149]	196	[287, 288]	307	[383, 217]	418	[459, 230]	529	[544, 321]
86	[147, 49]	197	[289, 48]	308	[384, 182]	419	[405, 19]	530	[35, 149]
87	[146, 40]	198	[127, 40]	309	[385, 252]	420	[460, 226]	531	[545, 177]
88	[150, 151]	199	[290, 146]	310	[348, 379]	421	[87, 383]	532	[546, 174]
89	[152, 153]	200	[291, 49]	311	[386, 95]	422	[461, 231]	533	[171, 152]
90	[154, 118]	201	[292, 287]	312	[210, 387]	423	[200, 462]	534	[357, 232]
91	[119, 141]	202	[293, 286]	313	[388, 341]	424	[227, 463]	535	[233, 110]
92	[155, 116]	203	[141, 140]	314	[365, 389]	425	[29, 392]	536	[547, 475]
93	[114, 58]	204	[294, 146]	315	[112, 390]	426	[342, 361]	537	[186, 462]
94	[156, 157]	205	[291, 42]	316	[82, 391]	427	[464, 20]	538	[239, 405]
95	[132, 158]	206	[47, 295]	317	[93, 392]	428	[465, 215]	539	[413, 257]
96	[159, 51]	207	[120, 141]	318	[393, 3]	429	[466, 258]	540	[388, 188]
97	[160, 45]	208	[296, 118]	319	[258, 394]	430	[380, 109]	541	[232, 343]
98	[161, 32]	209	[278, 138]	320	[395, 360]	431	[467, 265]	542	[548, 227]
99	[162, 34]	210	[297, 142]	321	[396, 397]	432	[468, 192]	543	[397, 379]

100	[163, 164]	211	[283, 140]	322	[398, 399]	433	[379, 232]	544	[549, 111]
101	[165, 128]	212	[298, 299]	323	[400, 355]	434	[469, 411]	545	[255, 260]
102	[166, 167]	213	[300, 301]	324	[401, 211]	435	[219, 470]	546	[343, 21]
103	[168, 169]	214	[302, 295]	325	[6, 105]	436	[471, 472]	547	[550, 540]
104	[170, 8]	215	[303, 120]	326	[398, 402]	437	[473, 399]	548	[318, 426]
105	[133, 171]	216	[124, 115]	327	[242, 403]	438	[21, 474]	549	[551, 326]
106	[172, 135]	217	[124, 124]	328	[363, 381]	439	[356, 179]	550	[351, 242]
107	[155, 123]	218	[304, 299]	329	[404, 69]	440	[475, 476]	551	[552, 343]
108	[137, 173]	219	[300, 271]	330	[405, 206]	441	[266, 366]	552	[553, 14]
109	[174, 10]	220	[305, 301]	331	[394, 183]	442	[477, 412]	553	[485, 17]
110	[152, 175]	221	[306, 137]	332	[406, 181]	443	[478, 65]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	80	0	160	1	240	0	320	0	400	0	480	0
1	0	81	0	161	0	241	0	321	0	401	0	481	0
2	0	82	1	162	0	242	1	322	1	402	1	482	1
3	0	83	1	163	0	243	0	323	0	403	0	483	0
4	0	84	1	164	1	244	0	324	0	404	1	484	1
5	0	85	0	165	0	245	0	325	0	405	1	485	1
6	0	86	1	166	0	246	1	326	1	406	0	486	1
7	0	87	1	167	1	247	0	327	1	407	1	487	0
8	1	88	0	168	1	248	1	328	1	408	0	488	1
9	0	89	0	169	1	249	1	329	1	409	1	489	0
10	1	90	0	170	1	250	1	330	1	410	1	490	1
11	0	91	0	171	1	251	0	331	0	411	1	491	1
12	0	92	0	172	0	252	1	332	0	412	1	492	1
13	0	93	1	173	1	253	0	333	1	413	0	493	1
14	0	94	0	174	1	254	1	334	0	414	1	494	1
15	0	95	0	175	0	255	1	335	1	415	1	495	0
16	1	96	0	176	1	256	1	336	1	416	1	496	1
17	1	97	1	177	0	257	0	337	0	417	1	497	1
18	0	98	0	178	0	258	1	338	1	418	0	498	1
19	0	99	0	179	1	259	1	339	0	419	1	499	0
20	1	100	0	180	1	260	1	340	1	420	1	500	0
21	1	101	0	181	0	261	1	341	1	421	1	501	0
22	1	102	0	182	1	262	0	342	1	422	1	502	1
23	0	103	1	183	1	263	0	343	0	423	0	503	1
24	0	104	1	184	1	264	1	344	1	424	1	504	0
25	0	105	1	185	0	265	0	345	1	425	0	505	1
26	0	106	0	186	1	266	1	346	0	426	1	506	0
27	0	107	0	187	0	267	0	347	1	427	1	507	1
28	0	108	1	188	1	268	1	348	0	428	1	508	0
29	0	109	1	189	1	269	1	349	0	429	0	509	0

30	1	110	0	190	1	270	1	350	1	430	0	510	1
31	0	111	1	191	0	271	0	351	1	431	0	511	1
32	1	112	0	192	0	272	1	352	1	432	0	512	1
33	1	113	0	193	1	273	1	353	0	433	0	513	0
34	1	114	1	194	1	274	0	354	1	434	1	514	1
35	1	115	1	195	1	275	1	355	0	435	0	515	0
36	0	116	0	196	0	276	1	356	1	436	1	516	1
37	0	117	1	197	0	277	0	357	1	437	0	517	1
38	0	118	1	198	1	278	1	358	1	438	1	518	1
39	0	119	0	199	0	279	1	359	1	439	1	519	0
40	1	120	0	200	0	280	1	360	0	440	0	520	1
41	1	121	1	201	0	281	1	361	1	441	1	521	1
42	0	122	0	202	0	282	1	362	1	442	1	522	1
43	1	123	0	203	0	283	0	363	1	443	0	523	0
44	1	124	1	204	1	284	1	364	0	444	1	524	0
45	1	125	1	205	0	285	1	365	0	445	0	525	0
46	0	126	1	206	0	286	1	366	1	446	0	526	1
47	0	127	1	207	0	287	0	367	0	447	1	527	1
48	0	128	1	208	0	288	0	368	0	448	0	528	1
49	0	129	0	209	1	289	0	369	0	449	1	529	1
50	0	130	0	210	1	290	0	370	0	450	1	530	1
51	0	131	0	211	0	291	0	371	0	451	0	531	1
52	0	132	0	212	1	292	0	372	1	452	0	532	0
53	0	133	1	213	0	293	0	373	1	453	1	533	1
54	0	134	0	214	1	294	1	374	0	454	0	534	1
55	1	135	0	215	1	295	0	375	1	455	1	535	0
56	0	136	1	216	1	296	0	376	1	456	1	536	1
57	0	137	1	217	1	297	1	377	0	457	1	537	1
58	1	138	1	218	0	298	1	378	0	458	1	538	0
59	1	139	1	219	0	299	0	379	0	459	0	539	0
60	0	140	1	220	0	300	0	380	0	460	1	540	0
61	1	141	0	221	1	301	0	381	0	461	1	541	0
62	1	142	1	222	0	302	1	382	1	462	1	542	1
63	0	143	0	223	1	303	1	383	1	463	0	543	1
64	1	144	0	224	1	304	0	384	1	464	1	544	1
65	0	145	1	225	1	305	0	385	1	465	1	545	1
66	1	146	1	226	0	306	1	386	1	466	0	546	0
67	1	147	1	227	1	307	1	387	0	467	0	547	1
68	1	148	1	228	1	308	1	388	0	468	0	548	1
69	0	149	0	229	1	309	1	389	1	469	1	549	1
70	0	150	0	230	0	310	0	390	1	470	0	550	1
71	0	151	1	231	0	311	1	391	1	471	1	551	1
72	0	152	0	232	0	312	1	392	1	472	1	552	1
73	1	153	0	233	0	313	0	393	1	473	0	553	1

74	0	154	0	234	0	314	0	394	0	474	1		
75	1	155	0	235	0	315	0	395	0	475	0		
76	1	156	0	236	1	316	1	396	0	476	1		
77	1	157	1	237	1	317	1	397	1	477	1		
78	1	158	0	238	1	318	1	398	1	478	0		
79	1	159	0	239	0	319	1	399	1	479	1		

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