# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^2 + z, z^4 + z^2 + z + 1)$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the $(a, b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a, b)$ in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^2 + z, z^4 + z^2 + z + 1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a, b)$-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When $(a,b)=(z^2+z, z^4+z^2+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{12}+z^{11}+z^4+z+1,\\
p_1(z)&=z^{16}+z^{15}+z^{14}+z^{11}+z^6+z^5+z^4+z,\\
p_2(z)&=z^{18}+z^{16}+z^{12}+z^8+z^6+z^2,\\
p_3(z)&=z^{16}+z^{15}+z^{14}+z^{11}+z^6+z^5+z^4+z,\\
p_4(z)&=z^{14}+z^{12}+z^{11}+z^{10}+z,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{12}+z^{11}+z^{10}+z^8+z^4+z^2+z,\\
p_1(z)&=z^{16}+z^{15}+z^{14}+z^{11}+z^6+z^5+z^4+z,\\
p_2(z)&=z^{18}+z^{16}+z^{12}+z^8+z^6+z^2,\\
p_3(z)&=z^{16}+z^{15}+z^{14}+z^{11}+z^6+z^5+z^4+z,\\
p_4(z)&=z^{14}+z^{12}+z^{11}+z^8+z^2+z+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2 \times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] &= M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{4u} M^e[:2u] + x^u M^e[:6u] \\
&\quad + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{3u} M^e[:6u] \\
&\quad + x^{4u} M^e[:5u] + x^{5u} M^e[:4u] + x^{7u} M^e[:2u] + x^{8u} M^e[:4u] \\
&\quad + x^{10u} M^e[:2u] + (x^{6u} + x^{7u} + x^{9u}) R^e, \\
W^e[:12u] &= W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{4u} W^e[:2u] + x^u W^e[:6u] \\
&\quad + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{3u} W^e[:6u]
\end{aligned}$$

$$+ x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + (x^{6u} + x^{7u} + x^{9u})R^e.$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^u M^o[:5u] + x^{2u}M^o[:4u] + x^{4u}M^o[:2u] + x^u M^o[:6u] \\ &\quad + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{3u}M^o[:6u] \\ &\quad + x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u] + x^{8u}M^o[:4u] \\ &\quad + x^{10u}M^o[:2u] + (x^{6u} + x^{7u} + x^{9u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^u W^o[:5u] + x^{2u}W^o[:4u] + x^{4u}W^o[:2u] + x^u W^o[:6u] \\ &\quad + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{3u}W^o[:6u] \\ &\quad + x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u] + x^{8u}W^o[:4u] \\ &\quad + x^{10u}W^o[:2u] + (x^{6u} + x^{7u} + x^{9u})R^o. \end{aligned}$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] &= T[:6u] + x^u T[:5u] + x^{2u}T[:4u] + x^{4u}T[:2u] + x^u T[:6u] + x^{2u}T[:5u] \\ &\quad + x^{3u}T[:4u] + x^{5u}T[:2u] + x^{3u}T[:6u] + x^{4u}T[:5u] + x^{5u}T[:4u] \\ &\quad + x^{7u}T[:2u] + x^{8u}T[:4u] + x^{10u}T[:2u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{4u}T[2u:3u] + x^{2u}T[4u:5u] + x^u T[5u:6u] + T[6u:7u], \\ 0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u] \\ &\quad + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + T[9u:10u], \\ 0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + T[10u:11u], \\ 0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$

$$\begin{aligned} 0 &= T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\ &\quad + T[[1000w]_2], \end{aligned} \tag{2.9}$$

$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\ & + T[[1000w]_2], \end{aligned} \tag{2.15}$$

$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 1020 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.19}$$

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.20}$$

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 101)) + \tau(A(s, 1000)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 1001)), \tag{2.22}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 1010)), \tag{2.23}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.25}$$

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 111)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) \\ & + \tau(A(s, 101)) + \tau(A(s, 1000)), \end{aligned} \tag{2.27}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 1001)), \tag{2.28}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 1010)), \tag{2.29}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{26}), E_{26}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 13$, which turns out to be true. $\square$

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{4u} M^e[:2u] + x^{6u} R^e, \\ W_{2k} &= W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{4u} W^e[:2u] + x^{6u} R^e. \end{aligned}$$

*For all integer $k \geq 2$, and $u = 2^{2k}$,*

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^u M^o[:5u] + x^{2u} M^o[:4u] + x^{4u} M^o[:2u] + x^{6u} R^o, \\ W_{2k+1} &= W^o[:6u] + x^u W^o[:5u] + x^{2u} W^o[:4u] + x^{4u} W^o[:2u] + x^{6u} R^o. \end{aligned}$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(W^e[:6v] + x^v W^e[:5v] + x^{2v} W^e[:4v] + x^{4v} W^e[:2v] + x^{6u} R^e\big) \times \big( \\ M^e[:6v] + x^v M^e[:5v] + x^{2v} M^e[:4v] + x^{4v} M^e[:2v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{12v}$, to

$$\begin{aligned} W^e[:12v] M^e[:12v] + x^{2v} W^e[:10v] M^e[:10v] + x^{4v} W^e[:8v] M^e[:8v] \\ + x^{8v} W^e[:4v] M^e[:4v]. \end{aligned}$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.31) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n \to \infty} M_{2n,j,k} = M^e_{j,k},$$

$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{18} + x^{17} + x^{14} + x^7 + x^6,$$
$$q_1(x) = x^{17} + x^{14} + x^{13} + x^{12} + x^7 + x^4 + x^3 + x^2,$$
$$q_2(x) = x^{16} + x^{12} + x^{10} + x^6 + x^2 + 1,$$
$$q_3(x) = x^{17} + x^{14} + x^{13} + x^{12} + x^7 + x^4 + x^3 + x^2,$$
$$q_4(x) = x^{17} + x^8 + x^7 + x^6 + x^4.$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned} p_0(x) = {} & x^{102} + x^{84} + x^{82} + x^{80} + x^{76} + x^{66} + x^{62} + x^{58} + x^{50} + x^{48} + x^{46} \\ & + x^{42} + x^{40} + x^{38} + x^{30} + x^{28} + x^{24}, \\ p_3(x) = {} & x^{98} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{74} + x^{72} + x^{70} \\ & + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{34} + x^{32} \\ & + x^{30} + x^{20}, \\ p_6(x) = {} & x^{96} + x^{94} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{74} + x^{70} + x^{68} + x^{62} \end{aligned}$$

$$
\begin{aligned}
&+ x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{44} + x^{42} + x^{34} + x^{30} + x^{28} + x^{22}\\
&+ x^{18} + x^{12} + x^{6} + 1,\\
p_9(x) &= x^{100} + x^{98} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70}\\
&+ x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42}\\
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{16} + x^{12} + x^{6}\\
&+ x^{4},\\
p_{12}(x) &= x^{100} + x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52}\\
&+ x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84} + x^{81} + x^{80}\\
&+ x^{77} + x^{76} + x^{73} + x^{71} + x^{67} + x^{65} + x^{61} + x^{60} + x^{59} + x^{54} + x^{52}\\
&+ x^{50} + x^{48} + x^{47} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{34}\\
&+ x^{32} + x^{30},\\
p_3(x) &= x^{94} + x^{88} + x^{86} + x^{76} + x^{74} + x^{72} + x^{70} + x^{62} + x^{60} + x^{58} + x^{54}\\
&+ x^{48} + x^{46} + x^{36} + x^{34} + x^{32} + x^{30} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{93} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{76}\\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40}\\
&+ x^{39} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64}\\
&+ x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{6},\\
p_{12}(x) &= x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}\\
&+ x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84} + x^{81} + x^{80}\\
&+ x^{77} + x^{76} + x^{73} + x^{71} + x^{67} + x^{65} + x^{61} + x^{60} + x^{59} + x^{54} + x^{52}\\
&+ x^{50} + x^{48} + x^{47} + x^{45} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36} + x^{34}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{32} + x^{30},\\
p_3(x) &= x^{94} + x^{88} + x^{86} + x^{76} + x^{74} + x^{72} + x^{70} + x^{62} + x^{60} + x^{58} + x^{54}\\
&\quad + x^{48} + x^{46} + x^{36} + x^{34} + x^{32} + x^{30} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{93} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{76}\\
&\quad + x^{74} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&\quad + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40}\\
&\quad + x^{39} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21}\\
&\quad + x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&\quad + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{6},\\
p_{12}(x) &= x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73}\\
&\quad + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&\quad + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&\quad + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&\quad + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}\\
&\quad + x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{60} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{36},\\
p_3(x) &= x^{84} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{52} + x^{44}\\
&\quad + x^{38} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{12},\\
p_6(x) &= x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{62} + x^{52} + x^{48} + x^{44} + x^{36}\\
&\quad + x^{32} + x^{30} + x^{22} + x^{12} + x^{8} + x^{2} + 1,\\
p_9(x) &= x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58}\\
&\quad + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30}\\
&\quad + x^{28} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) &= x^{82} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40}\\
&\quad + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{76} + x^{70} + x^{68} + x^{66} + x^{58} + x^{54}\\
&\quad + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24},\\
p_3(x) &= x^{84} + x^{80} + x^{78} + x^{74} + x^{66} + x^{62} + x^{44} + x^{40} + x^{38} + x^{34} + x^{26}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{22},\\
p_6(x) &= x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{60} + x^{56} + x^{54} + x^{50}\\
&+ x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} + x^{10}\\
&+ x^{2} + 1,\\
p_9(x) &= x^{82} + x^{80} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58}\\
&+ x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30}\\
&+ x^{28} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^{8} + x^{4},\\
p_{12}(x) &= x^{82} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40}\\
&+ x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{12} + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{84} + x^{81} + x^{80}\\
&+ x^{76} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55}\\
&+ x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{44} + x^{43} + x^{40} + x^{38} + x^{36} + x^{35}\\
&+ x^{33} + x^{30},\\
p_3(x) &= x^{94} + x^{88} + x^{86} + x^{76} + x^{74} + x^{72} + x^{70} + x^{62} + x^{60} + x^{58} + x^{54}\\
&+ x^{48} + x^{46} + x^{36} + x^{34} + x^{32} + x^{30} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{93} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{76}\\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40}\\
&+ x^{39} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64}\\
&+ x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{6},\\
p_{12}(x) &= x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}\\
&+ x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{84} + x^{81} + x^{80}\\
&+ x^{76} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{44} + x^{43} + x^{40} + x^{38} + x^{36} + x^{35}\\
&+ x^{33} + x^{30},\\
p_3(x) &= x^{94} + x^{88} + x^{86} + x^{76} + x^{74} + x^{72} + x^{70} + x^{62} + x^{60} + x^{58} + x^{54}\\
&+ x^{48} + x^{46} + x^{36} + x^{34} + x^{32} + x^{30} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{93} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{76}\\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{40}\\
&+ x^{39} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21}\\
&+ x^{20} + x^{19} + x^{16} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64}\\
&+ x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&+ x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{6},\\
p_{12}(x) &= x^{91} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15}\\
&+ x^{13} + x^{12} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{84} + x^{82} + x^{78} + x^{74} + x^{70} + x^{66}\\
&+ x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{46} + x^{44} + x^{42} + x^{38}\\
&+ x^{36},\\
p_3(x) &= x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{72} + x^{68} + x^{66} + x^{62}\\
&+ x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{32} + x^{28} + x^{26}\\
&+ x^{22} + x^{18} + x^{14} + x^{12},\\
p_6(x) &= x^{98} + x^{88} + x^{86} + x^{82} + x^{76} + x^{66} + x^{58} + x^{56} + x^{54} + x^{42} + x^{40}\\
&+ x^{36} + x^{26} + x^{16} + x^{14} + x^{8} + x^{6} + 1,\\
p_9(x) &= x^{100} + x^{98} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70}\\
&+ x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42}\\
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{16} + x^{12} + x^{6}\\
&+ x^{4},\\
p_{12}(x) &= x^{100} + x^{96} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52}\\
&+ x^{48} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{87} + x^{86} + x^{82} + x^{80} + x^{78} \\
& + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} \\
& + x^{59} + x^{57} + x^{54} + x^{53} + x^{46} + x^{45} + x^{43} + x^{41} + x^{38} + x^{36} + x^{35} \\
& + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24}, \\
p_3(x) = {} & x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87} + x^{85} + x^{83} + x^{79} + x^{78} + x^{75} \\
& + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{62} + x^{59} + x^{57} + x^{56} \\
& + x^{55} + x^{54} + x^{52} + x^{51} + x^{48} + x^{47} + x^{45} + x^{43} + x^{39} + x^{38} + x^{35} \\
& + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{19} + x^{17} + x^{16}, \\
p_6(x) = {} & x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{81} + x^{77} + x^{73} \\
& + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52} \\
& + x^{50} + x^{47} + x^{43} + x^{40} + x^{37} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} \\
& + x^{24} + x^{21} + x^{19} + x^{16} + x^{14} + x^{11} + x^{9} + x^{8} + x^{3} + x + 1, \\
p_9(x) = {} & x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) = {} & x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} \\
& + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} \\
& + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} \\
& + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
& + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{11} \\
& + x^{9} + x^{8} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{96} + x^{93} + x^{92} + x^{91} + x^{88} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} \\
& + x^{76} + x^{73} + x^{71} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{56} + x^{53} + x^{51} \\
& + x^{49} + x^{48} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{33} + x^{31} + x^{30}, \\
p_3(x) = {} & x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} \\
& + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{49} + x^{46} \\
& + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{25} + x^{24} + x^{21} + x^{19} + x^{18}, \\
p_6(x) = {} & x^{90} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{64} + x^{60} + x^{56} + x^{54} \\
& + x^{52} + x^{50} + x^{44} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{20} + x^{16} \\
& + x^{14} + x^{12}, \\
p_9(x) = {} & x^{91} + x^{88} + x^{87} + x^{86} + x^{11} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) = {} & x^{92} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} \\
& + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{55} + x^{54} + x^{49} + x^{47} + x^{46} + x^{43} \\
&\quad + x^{39} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30}, \\
p_3(x) &= x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{67} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{43} \\
&\quad + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{27} + x^{24} + x^{23} \\
&\quad + x^{22} + x^{21} + x^{19} + x^{18}, \\
p_6(x) &= x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} \\
&\quad + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{50} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} \\
&\quad + x^{30} + x^{26} + x^{24} + x^{18} + x^{14} + x^{12}, \\
p_9(x) &= x^{103} + x^{100} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} + x^{88} + x^{87} + x^{86} + x^{23} \\
&\quad + x^{20} + x^{19} + x^{18} + x^{15} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{104} + x^{92} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} \\
&\quad + x^{40} + x^{36} + x^{32} + x^{28} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{91} + x^{89} + x^{83} + x^{82} + x^{80} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{69} + x^{68} + x^{63} + x^{60} + x^{59} + x^{56} + x^{55} + x^{51} + x^{50} + x^{48} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36}, \\
p_3(x) &= x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{86} + x^{83} + x^{82} + x^{80} + x^{79} \\
&\quad + x^{76} + x^{73} + x^{68} + x^{65} + x^{64} + x^{61} + x^{60} + x^{55} + x^{54} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{50} + x^{46} + x^{43} + x^{42} + x^{40} + x^{39} + x^{36} + x^{33} + x^{28} + x^{25} \\
&\quad + x^{24} + x^{21} + x^{20}, \\
p_6(x) &= x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{85} + x^{84} + x^{83} + x^{80} + x^{75} + x^{74} \\
&\quad + x^{69} + x^{68} + x^{67} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{45} + x^{44} + x^{41} + x^{35} + x^{34} + x^{29} + x^{28} + x^{27} + x^{23} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{15} + x^{3} + x + 1, \\
p_9(x) &= x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}
\end{aligned}$$

$$+ x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{11} + x^9 + x^8 + x^3 + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{87} + x^{86} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{59} + x^{57} + x^{54} + x^{53} + x^{46} + x^{45} + x^{43} + x^{41} + x^{38} + x^{36} + x^{35} \\
&\quad + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24}, \\
p_3(x) &= x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{87} + x^{85} + x^{83} + x^{79} + x^{78} + x^{75} \\
&\quad + x^{74} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{62} + x^{59} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{52} + x^{51} + x^{48} + x^{47} + x^{45} + x^{43} + x^{39} + x^{38} + x^{35} \\
&\quad + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{19} + x^{17} + x^{16}, \\
p_6(x) &= x^{95} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{81} + x^{77} + x^{73} \\
&\quad + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52} \\
&\quad + x^{50} + x^{47} + x^{43} + x^{40} + x^{37} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} \\
&\quad + x^{24} + x^{21} + x^{19} + x^{16} + x^{14} + x^{11} + x^9 + x^8 + x^3 + x + 1, \\
p_9(x) &= x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^8, \\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} \\
&\quad + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29} \\
&\quad + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{11} \\
&\quad + x^9 + x^8 + x^3 + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{92} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{85} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{55} + x^{54} + x^{49} + x^{47} + x^{46} + x^{43} \\
&\quad + x^{39} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30}, \\
p_3(x) &= x^{101} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{67} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{43} \\
&\quad + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{27} + x^{24} + x^{23} \\
&\quad + x^{22} + x^{21} + x^{19} + x^{18},
\end{aligned}$$

$$
\begin{aligned}
p_6(x) &= x^{102} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70}\\
&\quad + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{50} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36}\\
&\quad + x^{30} + x^{26} + x^{24} + x^{18} + x^{14} + x^{12},\\
p_9(x) &= x^{103} + x^{100} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} + x^{88} + x^{87} + x^{86} + x^{23}\\
&\quad + x^{20} + x^{19} + x^{18} + x^{15} + x^{12} + x^{10} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{104} + x^{92} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44}\\
&\quad + x^{40} + x^{36} + x^{32} + x^{28} + x^{20} + x^{16} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0].
For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{93} + x^{92} + x^{91} + x^{88} + x^{85} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77}\\
&\quad + x^{76} + x^{73} + x^{71} + x^{67} + x^{64} + x^{63} + x^{61} + x^{60} + x^{56} + x^{53} + x^{51}\\
&\quad + x^{49} + x^{48} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{33} + x^{31} + x^{30},\\
p_3(x) &= x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73}\\
&\quad + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{49} + x^{46}\\
&\quad + x^{45} + x^{44} + x^{43} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{33} + x^{30} + x^{29}\\
&\quad + x^{28} + x^{27} + x^{25} + x^{24} + x^{21} + x^{19} + x^{18},\\
p_6(x) &= x^{90} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{64} + x^{60} + x^{56} + x^{54}\\
&\quad + x^{52} + x^{50} + x^{44} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{20} + x^{16}\\
&\quad + x^{14} + x^{12},\\
p_9(x) &= x^{91} + x^{88} + x^{87} + x^{86} + x^{11} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{92} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52} + x^{48} + x^{44}\\
&\quad + x^{40} + x^{36} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1].
For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{91} + x^{89} + x^{83} + x^{82} + x^{80} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72}\\
&\quad + x^{69} + x^{68} + x^{63} + x^{60} + x^{59} + x^{56} + x^{55} + x^{51} + x^{50} + x^{48} + x^{44}\\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36},\\
p_3(x) &= x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{86} + x^{83} + x^{82} + x^{80} + x^{79}\\
&\quad + x^{76} + x^{73} + x^{68} + x^{65} + x^{64} + x^{61} + x^{60} + x^{55} + x^{54} + x^{53} + x^{52}\\
&\quad + x^{51} + x^{50} + x^{46} + x^{43} + x^{42} + x^{40} + x^{39} + x^{36} + x^{33} + x^{28} + x^{25}\\
&\quad + x^{24} + x^{21} + x^{20},\\
p_6(x) &= x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{85} + x^{84} + x^{83} + x^{80} + x^{75} + x^{74}\\
&\quad + x^{69} + x^{68} + x^{67} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52}\\
&\quad + x^{51} + x^{45} + x^{44} + x^{41} + x^{35} + x^{34} + x^{29} + x^{28} + x^{27} + x^{23} + x^{20}\\
&\quad + x^{19} + x^{17} + x^{16} + x^{15} + x^{3} + x + 1,
\end{aligned}
$$

$$\begin{aligned}
p_9(x) &= x^{95} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) &= x^{95} + x^{93} + x^{92} + x^{87} + x^{85} + x^{84} + x^{79} + x^{77} + x^{76} + x^{75} + x^{73}\\
&\quad + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59}\\
&\quad + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44}\\
&\quad + x^{43} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{29}\\
&\quad + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{11}\\
&\quad + x^{9} + x^{8} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 256 | [425, 426] | 512 | [678, 263] | 768 | [861, 598] |
| 1 | [3, 4] | 257 | [427, 428] | 513 | [134, 343] | 769 | [875, 552] |
| 2 | [5, 6] | 258 | [429, 427] | 514 | [679, 680] | 770 | [829, 564] |
| 3 | [7, 8] | 259 | [430, 198] | 515 | [681, 511] | 771 | [852, 559] |
| 4 | [9, 10] | 260 | [431, 432] | 516 | [455, 384] | 772 | [876, 593] |
| 5 | [11, 12] | 261 | [433, 434] | 517 | [463, 682] | 773 | [877, 671] |
| 6 | [13, 14] | 262 | [435, 436] | 518 | [682, 683] | 774 | [878, 618] |
| 7 | [15, 16] | 263 | [59, 437] | 519 | [684, 8] | 775 | [211, 554] |
| 8 | [17, 18] | 264 | [438, 439] | 520 | [685, 497] | 776 | [879, 858] |
| 9 | [19, 20] | 265 | [440, 46] | 521 | [686, 133] | 777 | [880, 302] |
| 10 | [21, 22] | 266 | [441, 384] | 522 | [687, 123] | 778 | [310, 82] |
| 11 | [23, 24] | 267 | [442, 160] | 523 | [688, 689] | 779 | [284, 845] |
| 12 | [25, 26] | 268 | [443, 377] | 524 | [690, 364] | 780 | [555, 103] |
| 13 | [27, 28] | 269 | [444, 445] | 525 | [691, 692] | 781 | [881, 325] |
| 14 | [29, 30] | 270 | [446, 194] | 526 | [693, 694] | 782 | [673, 576] |
| 15 | [31, 32] | 271 | [447, 448] | 527 | [695, 496] | 783 | [294, 558] |
| 16 | [33, 34] | 272 | [449, 125] | 528 | [696, 694] | 784 | [882, 17] |
| 17 | [35, 36] | 273 | [44, 179] | 529 | [697, 698] | 785 | [883, 636] |
| 18 | [37, 38] | 274 | [450, 451] | 530 | [699, 373] | 786 | [884, 838] |
| 19 | [39, 40] | 275 | [452, 348] | 531 | [700, 688] | 787 | [588, 334] |
| 20 | [41, 42] | 276 | [383, 154] | 532 | [191, 389] | 788 | [885, 310] |
| 21 | [43, 44] | 277 | [453, 374] | 533 | [701, 115] | 789 | [886, 304] |
| 22 | [45, 46] | 278 | [454, 455] | 534 | [702, 393] | 790 | [887, 550] |
| 23 | [47, 48] | 279 | [12, 445] | 535 | [703, 704] | 791 | [827, 220] |
| 24 | [49, 50] | 280 | [456, 457] | 536 | [705, 426] | 792 | [90, 262] |
| 25 | [51, 39] | 281 | [120, 166] | 537 | [32, 36] | 793 | [888, 281] |
| 26 | [52, 53] | 282 | [118, 141] | 538 | [706, 55] | 794 | [889, 262] |
| 27 | [54, 55] | 283 | [151, 439] | 539 | [436, 14] | 795 | [890, 105] |
| 28 | [56, 57] | 284 | [168, 458] | 540 | [202, 707] | 796 | [891, 340] |
| 29 | [58, 59] | 285 | [459, 460] | 541 | [437, 708] | 797 | [241, 650] |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 30 | [60, 49] | 286 | [461, 42] | 542 | [709, 392] | 798 | [634, 524] |
| 31 | [61, 62] | 287 | [462, 463] | 543 | [710, 711] | 799 | [892, 323] |
| 32 | [63, 64] | 288 | [451, 464] | 544 | [405, 712] | 800 | [893, 845] |
| 33 | [65, 66] | 289 | [356, 465] | 545 | [713, 370] | 801 | [894, 590] |
| 34 | [67, 68] | 290 | [2, 466] | 546 | [352, 39] | 802 | [895, 591] |
| 35 | [69, 70] | 291 | [467, 355] | 547 | [714, 197] | 803 | [319, 636] |
| 36 | [62, 71] | 292 | [468, 459] | 548 | [715, 460] | 804 | [896, 326] |
| 37 | [72, 73] | 293 | [176, 193] | 549 | [716, 200] | 805 | [897, 558] |
| 38 | [74, 75] | 294 | [434, 57] | 550 | [717, 682] | 806 | [609, 568] |
| 39 | [76, 77] | 295 | [469, 54] | 551 | [414, 718] | 807 | [898, 598] |
| 40 | [78, 79] | 296 | [470, 138] | 552 | [719, 44] | 808 | [899, 634] |
| 41 | [80, 81] | 297 | [471, 352] | 553 | [10, 41] | 809 | [550, 305] |
| 42 | [82, 83] | 298 | [472, 2] | 554 | [720, 383] | 810 | [900, 549] |
| 43 | [84, 85] | 299 | [473, 425] | 555 | [348, 127] | 811 | [901, 226] |
| 44 | [86, 87] | 300 | [474, 202] | 556 | [721, 722] | 812 | [632, 214] |
| 45 | [88, 3] | 301 | [475, 188] | 557 | [723, 405] | 813 | [902, 31] |
| 46 | [89, 90] | 302 | [411, 364] | 558 | [724, 163] | 814 | [903, 168] |
| 47 | [91, 92] | 303 | [476, 414] | 559 | [725, 455] | 815 | [904, 365] |
| 48 | [93, 94] | 304 | [477, 428] | 560 | [488, 692] | 816 | [765, 721] |
| 49 | [75, 95] | 305 | [478, 182] | 561 | [726, 161] | 817 | [905, 385] |
| 50 | [96, 97] | 306 | [200, 409] | 562 | [727, 463] | 818 | [906, 187] |
| 51 | [98, 99] | 307 | [479, 141] | 563 | [728, 431] | 819 | [174, 716] |
| 52 | [100, 101] | 308 | [480, 156] | 564 | [729, 464] | 820 | [907, 802] |
| 53 | [79, 102] | 309 | [481, 45] | 565 | [439, 382] | 821 | [908, 909] |
| 54 | [103, 104] | 310 | [482, 155] | 566 | [127, 698] | 822 | [910, 713] |
| 55 | [105, 106] | 311 | [483, 431] | 567 | [147, 131] | 823 | [464, 911] |
| 56 | [107, 6] | 312 | [372, 484] | 568 | [730, 458] | 824 | [465, 433] |
| 57 | [108, 109] | 313 | [344, 115] | 569 | [731, 732] | 825 | [912, 693] |
| 58 | [110, 111] | 314 | [485, 360] | 570 | [733, 734] | 826 | [913, 722] |
| 59 | [112, 113] | 315 | [486, 456] | 571 | [735, 354] | 827 | [711, 404] |
| 60 | [114, 96] | 316 | [487, 488] | 572 | [36, 160] | 828 | [149, 177] |
| 61 | [115, 116] | 317 | [489, 490] | 573 | [393, 362] | 829 | [914, 344] |
| 62 | [117, 118] | 318 | [491, 492] | 574 | [736, 394] | 830 | [915, 40] |
| 63 | [119, 120] | 319 | [493, 186] | 575 | [53, 119] | 831 | [916, 503] |
| 64 | [121, 122] | 320 | [494, 487] | 576 | [737, 504] | 832 | [445, 387] |
| 65 | [122, 123] | 321 | [495, 166] | 577 | [368, 128] | 833 | [917, 349] |
| 66 | [124, 125] | 322 | [492, 496] | 578 | [712, 738] | 834 | [704, 499] |
| 67 | [126, 127] | 323 | [497, 377] | 579 | [738, 713] | 835 | [508, 918] |
| 68 | [128, 129] | 324 | [498, 10] | 580 | [739, 489] | 836 | [919, 353] |
| 69 | [130, 131] | 325 | [499, 483] | 581 | [740, 437] | 837 | [920, 351] |
| 70 | [132, 133] | 326 | [500, 118] | 582 | [38, 60] | 838 | [370, 711] |
| 71 | [116, 134] | 327 | [501, 34] | 583 | [741, 196] | 839 | [921, 404] |
| 72 | [135, 136] | 328 | [502, 184] | 584 | [742, 378] | 840 | [756, 756] |
| 73 | [137, 138] | 329 | [503, 504] | 585 | [683, 376] | 841 | [922, 447] |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 74 | [139, 140] | 330 | [505, 451] | 586 | [743, 744] | 842 | [923, 417] |
| 75 | [141, 142] | 331 | [396, 505] | 587 | [745, 400] | 843 | [924, 432] |
| 76 | [136, 143] | 332 | [416, 470] | 588 | [746, 50] | 844 | [925, 482] |
| 77 | [144, 145] | 333 | [506, 7] | 589 | [747, 170] | 845 | [926, 457] |
| 78 | [146, 147] | 334 | [507, 483] | 590 | [422, 424] | 846 | [399, 785] |
| 79 | [148, 149] | 335 | [131, 194] | 591 | [748, 749] | 847 | [927, 501] |
| 80 | [150, 151] | 336 | [123, 508] | 592 | [145, 425] | 848 | [182, 459] |
| 81 | [152, 124] | 337 | [409, 436] | 593 | [749, 422] | 849 | [362, 346] |
| 82 | [153, 154] | 338 | [378, 413] | 594 | [734, 749] | 850 | [490, 727] |
| 83 | [155, 156] | 339 | [133, 509] | 595 | [424, 145] | 851 | [928, 718] |
| 84 | [157, 151] | 340 | [510, 162] | 596 | [750, 744] | 852 | [929, 488] |
| 85 | [158, 159] | 341 | [511, 456] | 597 | [751, 434] | 853 | [129, 172] |
| 86 | [160, 161] | 342 | [512, 401] | 598 | [374, 176] | 854 | [930, 911] |
| 87 | [42, 162] | 343 | [513, 205] | 599 | [752, 487] | 855 | [143, 366] |
| 88 | [48, 163] | 344 | [239, 313] | 600 | [753, 754] | 856 | [931, 679] |
| 89 | [164, 165] | 345 | [514, 420] | 601 | [755, 756] | 857 | [932, 689] |
| 90 | [166, 59] | 346 | [515, 249] | 602 | [757, 758] | 858 | [933, 505] |
| 91 | [167, 168] | 347 | [516, 227] | 603 | [759, 165] | 859 | [460, 811] |
| 92 | [169, 170] | 348 | [288, 517] | 604 | [760, 433] | 860 | [154, 357] |
| 93 | [171, 172] | 349 | [518, 519] | 605 | [761, 363] | 861 | [934, 490] |
| 94 | [173, 142] | 350 | [520, 521] | 606 | [692, 399] | 862 | [732, 12] |
| 95 | [140, 174] | 351 | [522, 265] | 607 | [694, 355] | 863 | [935, 403] |
| 96 | [175, 176] | 352 | [517, 327] | 608 | [762, 470] | 864 | [428, 196] |
| 97 | [177, 140] | 353 | [523, 524] | 609 | [138, 394] | 865 | [57, 794] |
| 98 | [178, 179] | 354 | [296, 27] | 610 | [376, 719] | 866 | [936, 785] |
| 99 | [180, 129] | 355 | [525, 526] | 611 | [763, 157] | 867 | [811, 351] |
| 100 | [181, 182] | 356 | [527, 528] | 612 | [764, 358] | 868 | [937, 164] |
| 101 | [183, 158] | 357 | [529, 270] | 613 | [360, 765] | 869 | [938, 721] |
| 102 | [184, 124] | 358 | [530, 221] | 614 | [766, 197] | 870 | [172, 789] |
| 103 | [185, 186] | 359 | [531, 532] | 615 | [397, 402] | 871 | [162, 489] |
| 104 | [187, 188] | 360 | [308, 100] | 616 | [767, 155] | 872 | [939, 680] |
| 105 | [189, 38] | 361 | [533, 534] | 617 | [718, 768] | 873 | [156, 498] |
| 106 | [190, 191] | 362 | [535, 536] | 618 | [769, 134] | 874 | [940, 343] |
| 107 | [192, 193] | 363 | [16, 537] | 619 | [770, 719] | 875 | [802, 367] |
| 108 | [194, 195] | 364 | [538, 539] | 620 | [771, 191] | 876 | [941, 167] |
| 109 | [196, 169] | 365 | [282, 261] | 621 | [407, 396] | 877 | [942, 508] |
| 110 | [197, 198] | 366 | [540, 538] | 622 | [772, 367] | 878 | [943, 766] |
| 111 | [199, 200] | 367 | [541, 111] | 623 | [758, 187] | 879 | [944, 716] |
| 112 | [201, 202] | 368 | [542, 231] | 624 | [773, 149] | 880 | [509, 136] |
| 113 | [203, 190] | 369 | [220, 543] | 625 | [466, 372] | 881 | [945, 500] |
| 114 | [204, 177] | 370 | [544, 545] | 626 | [774, 167] | 882 | [946, 37] |
| 115 | [205, 206] | 371 | [546, 338] | 627 | [775, 776] | 883 | [50, 773] |
| 116 | [207, 208] | 372 | [6, 279] | 628 | [777, 465] | 884 | [947, 796] |
| 117 | [209, 210] | 373 | [547, 305] | 629 | [448, 482] | 885 | [948, 795] |

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|---|---|---|---|---|---|---|---|
| 118 | [113, 211] | 374 | [331, 329] | 630 | [778, 41] | 886 | [949, 752] |
| 119 | [212, 101] | 375 | [548, 206] | 631 | [779, 768] | 887 | [754, 493] |
| 120 | [213, 214] | 376 | [20, 86] | 632 | [385, 391] | 888 | [950, 116] |
| 121 | [215, 216] | 377 | [248, 549] | 633 | [698, 157] | 889 | [387, 774] |
| 122 | [217, 218] | 378 | [232, 234] | 634 | [457, 54] | 890 | [951, 784] |
| 123 | [219, 220] | 379 | [342, 268] | 635 | [780, 765] | 891 | [796, 201] |
| 124 | [221, 222] | 380 | [4, 244] | 636 | [781, 190] | 892 | [952, 498] |
| 125 | [223, 224] | 381 | [66, 550] | 637 | [504, 411] | 893 | [391, 447] |
| 126 | [225, 226] | 382 | [551, 552] | 638 | [782, 195] | 894 | [496, 732] |
| 127 | [227, 228] | 383 | [238, 316] | 639 | [458, 750] | 895 | [689, 417] |
| 128 | [229, 230] | 384 | [553, 554] | 640 | [161, 766] | 896 | [179, 164] |
| 129 | [231, 232] | 385 | [532, 555] | 641 | [426, 734] | 897 | [776, 500] |
| 130 | [233, 234] | 386 | [252, 269] | 642 | [783, 484] | 898 | [953, 60] |
| 131 | [235, 99] | 387 | [556, 557] | 643 | [784, 7] | 899 | [354, 693] |
| 132 | [236, 237] | 388 | [558, 277] | 644 | [785, 373] | 900 | [413, 752] |
| 133 | [95, 238] | 389 | [559, 560] | 645 | [786, 774] | 901 | [954, 485] |
| 134 | [206, 239] | 390 | [561, 231] | 646 | [787, 509] | 902 | [611, 525] |
| 135 | [240, 241] | 391 | [106, 562] | 647 | [788, 448] | 903 | [955, 539] |
| 136 | [242, 223] | 392 | [563, 564] | 648 | [165, 452] | 904 | [263, 291] |
| 137 | [243, 244] | 393 | [565, 210] | 649 | [789, 790] | 905 | [956, 106] |
| 138 | [245, 92] | 394 | [244, 337] | 650 | [791, 712] | 906 | [957, 3] |
| 139 | [246, 224] | 395 | [566, 280] | 651 | [401, 492] | 907 | [83, 16] |
| 140 | [247, 24] | 396 | [567, 568] | 652 | [792, 773] | 908 | [958, 211] |
| 141 | [210, 248] | 397 | [569, 570] | 653 | [793, 350] | 909 | [959, 308] |
| 142 | [249, 250] | 398 | [534, 17] | 654 | [794, 416] | 910 | [650, 601] |
| 143 | [251, 252] | 399 | [571, 572] | 655 | [707, 795] | 911 | [960, 277] |
| 144 | [253, 254] | 400 | [573, 247] | 656 | [795, 796] | 912 | [524, 26] |
| 145 | [255, 256] | 401 | [574, 70] | 657 | [797, 738] | 913 | [961, 654] |
| 146 | [257, 227] | 402 | [575, 576] | 658 | [163, 688] | 914 | [962, 250] |
| 147 | [258, 259] | 403 | [577, 67] | 659 | [798, 356] | 915 | [963, 641] |
| 148 | [260, 261] | 404 | [545, 578] | 660 | [799, 203] | 916 | [605, 671] |
| 149 | [262, 263] | 405 | [557, 579] | 661 | [389, 800] | 917 | [324, 307] |
| 150 | [264, 265] | 406 | [278, 295] | 662 | [800, 203] | 918 | [964, 574] |
| 151 | [266, 267] | 407 | [580, 268] | 663 | [801, 789] | 919 | [560, 670] |
| 152 | [268, 223] | 408 | [73, 581] | 664 | [34, 802] | 920 | [965, 517] |
| 153 | [269, 270] | 409 | [582, 29] | 665 | [803, 146] | 921 | [578, 557] |
| 154 | [271, 108] | 410 | [337, 538] | 666 | [804, 805] | 922 | [851, 616] |
| 155 | [272, 273] | 411 | [583, 584] | 667 | [806, 750] | 923 | [966, 89] |
| 156 | [274, 275] | 412 | [214, 259] | 668 | [708, 497] | 924 | [967, 544] |
| 157 | [228, 276] | 413 | [549, 551] | 669 | [484, 679] | 925 | [552, 272] |
| 158 | [277, 278] | 414 | [519, 585] | 670 | [807, 169] | 926 | [968, 100] |
| 159 | [279, 280] | 415 | [586, 97] | 671 | [808, 49] | 927 | [969, 632] |
| 160 | [64, 281] | 416 | [109, 245] | 672 | [14, 427] | 928 | [222, 861] |
| 161 | [71, 282] | 417 | [587, 588] | 673 | [805, 158] | 929 | [970, 314] |

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|---|---|---|---|---|---|---|---|
| 162 | [81, 283] | 418 | [216, 589] | 674 | [809, 397] | 930 | [603, 96] |
| 163 | [92, 284] | 419 | [254, 536] | 675 | [125, 349] | 931 | [971, 840] |
| 164 | [285, 286] | 420 | [420, 420] | 676 | [810, 811] | 932 | [972, 545] |
| 165 | [287, 288] | 421 | [590, 591] | 677 | [365, 503] | 933 | [973, 625] |
| 166 | [101, 289] | 422 | [77, 592] | 678 | [812, 707] | 934 | [974, 275] |
| 167 | [290, 291] | 423 | [592, 255] | 679 | [813, 559] | 935 | [975, 257] |
| 168 | [292, 107] | 424 | [593, 594] | 680 | [814, 815] | 936 | [18, 851] |
| 169 | [293, 294] | 425 | [594, 590] | 681 | [628, 81] | 937 | [330, 287] |
| 170 | [295, 296] | 426 | [256, 595] | 682 | [572, 20] | 938 | [976, 594] |
| 171 | [87, 297] | 427 | [596, 597] | 683 | [816, 817] | 939 | [977, 324] |
| 172 | [298, 299] | 428 | [597, 293] | 684 | [818, 68] | 940 | [286, 230] |
| 173 | [300, 301] | 429 | [598, 597] | 685 | [819, 248] | 941 | [978, 633] |
| 174 | [224, 302] | 430 | [599, 238] | 686 | [820, 75] | 942 | [979, 579] |
| 175 | [303, 304] | 431 | [600, 601] | 687 | [589, 521] | 943 | [647, 584] |
| 176 | [305, 110] | 432 | [602, 603] | 688 | [821, 822] | 944 | [638, 528] |
| 177 | [306, 307] | 433 | [604, 605] | 689 | [823, 647] | 945 | [980, 113] |
| 178 | [230, 308] | 434 | [606, 607] | 690 | [824, 282] | 946 | [307, 74] |
| 179 | [309, 310] | 435 | [608, 560] | 691 | [537, 571] | 947 | [981, 624] |
| 180 | [311, 239] | 436 | [581, 609] | 692 | [314, 673] | 948 | [982, 592] |
| 181 | [312, 82] | 437 | [289, 85] | 693 | [825, 525] | 949 | [304, 267] |
| 182 | [313, 314] | 438 | [610, 551] | 694 | [26, 217] | 950 | [102, 817] |
| 183 | [315, 279] | 439 | [265, 565] | 695 | [826, 827] | 951 | [624, 205] |
| 184 | [316, 317] | 440 | [334, 321] | 696 | [828, 90] | 952 | [250, 21] |
| 185 | [318, 288] | 441 | [562, 225] | 697 | [267, 829] | 953 | [261, 237] |
| 186 | [273, 319] | 442 | [281, 611] | 698 | [830, 829] | 954 | [983, 87] |
| 187 | [320, 269] | 443 | [612, 232] | 699 | [831, 331] | 955 | [984, 499] |
| 188 | [321, 89] | 444 | [613, 27] | 700 | [815, 221] | 956 | [985, 727] |
| 189 | [322, 109] | 445 | [24, 88] | 701 | [564, 98] | 957 | [382, 454] |
| 190 | [323, 324] | 446 | [614, 327] | 702 | [832, 86] | 958 | [142, 784] |
| 191 | [325, 326] | 447 | [615, 287] | 703 | [833, 585] | 959 | [986, 181] |
| 192 | [327, 328] | 448 | [616, 617] | 704 | [834, 28] | 960 | [987, 454] |
| 193 | [329, 330] | 449 | [218, 62] | 705 | [536, 299] | 961 | [988, 147] |
| 194 | [259, 257] | 450 | [270, 618] | 706 | [68, 303] | 962 | [989, 452] |
| 195 | [99, 331] | 451 | [568, 335] | 707 | [835, 255] | 963 | [990, 918] |
| 196 | [332, 295] | 452 | [619, 620] | 708 | [836, 301] | 964 | [991, 132] |
| 197 | [333, 334] | 453 | [621, 319] | 709 | [837, 565] | 965 | [403, 501] |
| 198 | [335, 271] | 454 | [622, 616] | 710 | [822, 822] | 966 | [992, 800] |
| 199 | [336, 337] | 455 | [208, 532] | 711 | [543, 838] | 967 | [346, 909] |
| 200 | [302, 338] | 456 | [623, 103] | 712 | [839, 840] | 968 | [993, 32] |
| 201 | [339, 272] | 457 | [104, 624] | 713 | [840, 241] | 969 | [994, 201] |
| 202 | [340, 112] | 458 | [291, 625] | 714 | [841, 335] | 970 | [911, 805] |
| 203 | [341, 342] | 459 | [626, 253] | 715 | [842, 256] | 971 | [995, 368] |
| 204 | [30, 247] | 460 | [627, 628] | 716 | [843, 74] | 972 | [40, 754] |
| 205 | [1, 159] | 461 | [629, 330] | 717 | [276, 67] | 973 | [996, 120] |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 206 | [343, 344] | 462 | [630, 572] | 718 | [521, 323] | 974 | [997, 402] |
| 207 | [345, 346] | 463 | [631, 632] | 719 | [844, 309] | 975 | [998, 708] |
| 208 | [347, 348] | 464 | [633, 634] | 720 | [845, 603] | 976 | [744, 704] |
| 209 | [349, 350] | 465 | [526, 606] | 721 | [846, 571] | 977 | [768, 407] |
| 210 | [351, 352] | 466 | [635, 581] | 722 | [847, 98] | 978 | [380, 758] |
| 211 | [353, 354] | 467 | [338, 252] | 723 | [601, 839] | 979 | [999, 790] |
| 212 | [355, 356] | 468 | [226, 300] | 724 | [848, 606] | 980 | [170, 353] |
| 213 | [357, 358] | 469 | [636, 105] | 725 | [849, 562] | 981 | [193, 511] |
| 214 | [359, 360] | 470 | [637, 638] | 726 | [625, 519] | 982 | [1000, 143] |
| 215 | [361, 362] | 471 | [639, 276] | 727 | [850, 851] | 983 | [790, 392] |
| 216 | [363, 53] | 472 | [640, 271] | 728 | [539, 342] | 984 | [1001, 609] |
| 217 | [364, 365] | 473 | [641, 253] | 729 | [70, 852] | 985 | [817, 631] |
| 218 | [366, 122] | 474 | [642, 617] | 730 | [853, 245] | 986 | [1002, 313] |
| 219 | [367, 368] | 475 | [643, 88] | 731 | [854, 71] | 987 | [620, 208] |
| 220 | [369, 370] | 476 | [644, 218] | 732 | [855, 589] | 988 | [111, 620] |
| 221 | [371, 372] | 477 | [645, 30] | 733 | [591, 570] | 989 | [275, 561] |
| 222 | [373, 374] | 478 | [646, 22] | 734 | [299, 641] | 990 | [1003, 278] |
| 223 | [375, 376] | 479 | [301, 249] | 735 | [856, 526] | 991 | [1004, 95] |
| 224 | [377, 378] | 480 | [22, 533] | 736 | [94, 588] | 992 | [317, 29] |
| 225 | [379, 380] | 481 | [326, 647] | 737 | [857, 296] | 993 | [1005, 222] |
| 226 | [381, 382] | 482 | [648, 274] | 738 | [579, 544] | 994 | [1006, 83] |
| 227 | [198, 383] | 483 | [28, 325] | 739 | [85, 286] | 995 | [1007, 289] |
| 228 | [384, 385] | 484 | [649, 650] | 740 | [297, 858] | 996 | [1008, 537] |
| 229 | [386, 387] | 485 | [651, 225] | 741 | [97, 665] | 997 | [1009, 595] |
| 230 | [388, 389] | 486 | [652, 104] | 742 | [858, 628] | 998 | [576, 309] |
| 231 | [390, 391] | 487 | [653, 300] | 743 | [859, 578] | 999 | [1010, 329] |
| 232 | [392, 393] | 488 | [328, 654] | 744 | [860, 861] | 1000 | [1011, 317] |
| 233 | [394, 37] | 489 | [655, 21] | 745 | [862, 294] | 1001 | [1012, 466] |
| 234 | [358, 184] | 490 | [283, 66] | 746 | [665, 815] | 1002 | [909, 485] |
| 235 | [395, 396] | 491 | [607, 64] | 747 | [237, 574] | 1003 | [1013, 357] |
| 236 | [186, 397] | 492 | [656, 657] | 748 | [595, 77] | 1004 | [918, 400] |
| 237 | [398, 399] | 493 | [658, 561] | 749 | [570, 593] | 1005 | [1014, 683] |
| 238 | [400, 401] | 494 | [659, 110] | 750 | [863, 839] | 1006 | [1015, 128] |
| 239 | [159, 380] | 495 | [660, 112] | 751 | [864, 108] | 1007 | [1016, 119] |
| 240 | [402, 403] | 496 | [661, 611] | 752 | [865, 328] | 1008 | [1017, 794] |
| 241 | [404, 405] | 497 | [662, 297] | 753 | [866, 18] | 1009 | [432, 2] |
| 242 | [406, 407] | 498 | [663, 264] | 754 | [867, 533] | 1010 | [55, 776] |
| 243 | [408, 409] | 499 | [664, 665] | 755 | [838, 657] | 1011 | [46, 31] |
| 244 | [8, 363] | 500 | [666, 534] | 756 | [657, 827] | 1012 | [1018, 607] |
| 245 | [195, 146] | 501 | [667, 631] | 757 | [585, 638] | 1013 | [618, 605] |
| 246 | [410, 411] | 502 | [668, 284] | 758 | [868, 321] | 1014 | [554, 264] |
| 247 | [412, 413] | 503 | [669, 670] | 759 | [869, 283] | 1015 | [1019, 555] |
| 248 | [350, 414] | 504 | [671, 216] | 760 | [670, 94] | 1016 | [654, 274] |
| 249 | [415, 416] | 505 | [280, 633] | 761 | [234, 79] | 1017 | [584, 73] |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 250 | [188, 45] | 506 | [672, 673] | 762 | [528, 107] | 1018 | [680, 132] |
| 251 | [417, 48] | 507 | [674, 293] | 763 | [870, 228] | 1019 | [722, 181] |
| 252 | [418, 174] | 508 | [675, 4] | 764 | [871, 316] | | |
| 253 | [419, 420] | 509 | [676, 543] | 765 | [872, 254] | | |
| 254 | [421, 422] | 510 | [617, 340] | 766 | [873, 852] | | |
| 255 | [423, 424] | 511 | [677, 303] | 767 | [874, 102] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 171 | 1 | 342 | 1 | 513 | 0 | 684 | 1 | 855 | 1 |
| 1 | 0 | 172 | 0 | 343 | 0 | 514 | 0 | 685 | 0 | 856 | 0 |
| 2 | 0 | 173 | 1 | 344 | 0 | 515 | 0 | 686 | 0 | 857 | 0 |
| 3 | 0 | 174 | 0 | 345 | 0 | 516 | 0 | 687 | 0 | 858 | 1 |
| 4 | 0 | 175 | 1 | 346 | 0 | 517 | 1 | 688 | 1 | 859 | 1 |
| 5 | 0 | 176 | 1 | 347 | 0 | 518 | 0 | 689 | 1 | 860 | 0 |
| 6 | 0 | 177 | 1 | 348 | 0 | 519 | 1 | 690 | 0 | 861 | 1 |
| 7 | 0 | 178 | 0 | 349 | 0 | 520 | 0 | 691 | 0 | 862 | 1 |
| 8 | 1 | 179 | 1 | 350 | 0 | 521 | 0 | 692 | 1 | 863 | 0 |
| 9 | 0 | 180 | 0 | 351 | 0 | 522 | 0 | 693 | 0 | 864 | 0 |
| 10 | 1 | 181 | 0 | 352 | 1 | 523 | 1 | 694 | 0 | 865 | 0 |
| 11 | 0 | 182 | 0 | 353 | 1 | 524 | 0 | 695 | 1 | 866 | 1 |
| 12 | 0 | 183 | 0 | 354 | 1 | 525 | 0 | 696 | 1 | 867 | 1 |
| 13 | 0 | 184 | 1 | 355 | 0 | 526 | 0 | 697 | 1 | 868 | 0 |
| 14 | 1 | 185 | 0 | 356 | 1 | 527 | 1 | 698 | 1 | 869 | 0 |
| 15 | 0 | 186 | 1 | 357 | 1 | 528 | 1 | 699 | 0 | 870 | 0 |
| 16 | 1 | 187 | 1 | 358 | 0 | 529 | 1 | 700 | 0 | 871 | 0 |
| 17 | 1 | 188 | 0 | 359 | 0 | 530 | 0 | 701 | 1 | 872 | 1 |
| 18 | 1 | 189 | 0 | 360 | 1 | 531 | 0 | 702 | 0 | 873 | 1 |
| 19 | 0 | 190 | 1 | 361 | 1 | 532 | 1 | 703 | 1 | 874 | 1 |
| 20 | 0 | 191 | 1 | 362 | 1 | 533 | 1 | 704 | 0 | 875 | 1 |
| 21 | 1 | 192 | 0 | 363 | 1 | 534 | 0 | 705 | 0 | 876 | 0 |
| 22 | 1 | 193 | 1 | 364 | 1 | 535 | 1 | 706 | 1 | 877 | 0 |
| 23 | 0 | 194 | 0 | 365 | 1 | 536 | 0 | 707 | 0 | 878 | 1 |
| 24 | 0 | 195 | 0 | 366 | 0 | 537 | 0 | 708 | 1 | 879 | 1 |
| 25 | 0 | 196 | 0 | 367 | 1 | 538 | 1 | 709 | 0 | 880 | 0 |
| 26 | 0 | 197 | 1 | 368 | 0 | 539 | 1 | 710 | 1 | 881 | 0 |
| 27 | 0 | 198 | 0 | 369 | 0 | 540 | 0 | 711 | 1 | 882 | 0 |
| 28 | 0 | 199 | 1 | 370 | 0 | 541 | 1 | 712 | 0 | 883 | 1 |
| 29 | 1 | 200 | 1 | 371 | 1 | 542 | 0 | 713 | 1 | 884 | 1 |
| 30 | 0 | 201 | 0 | 372 | 0 | 543 | 1 | 714 | 0 | 885 | 1 |
| 31 | 0 | 202 | 0 | 373 | 0 | 544 | 0 | 715 | 0 | 886 | 1 |
| 32 | 0 | 203 | 1 | 374 | 0 | 545 | 1 | 716 | 0 | 887 | 1 |
| 33 | 1 | 204 | 0 | 375 | 0 | 546 | 1 | 717 | 0 | 888 | 1 |
| 34 | 1 | 205 | 0 | 376 | 0 | 547 | 0 | 718 | 0 | 889 | 0 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 35 | 1 | 206 | 0 | 377 | 0 | 548 | 0 | 719 | 0 | 890 | 0 |
| 36 | 0 | 207 | 0 | 378 | 1 | 549 | 0 | 720 | 1 | 891 | 0 |
| 37 | 1 | 208 | 0 | 379 | 1 | 550 | 0 | 721 | 0 | 892 | 0 |
| 38 | 0 | 209 | 0 | 380 | 0 | 551 | 1 | 722 | 1 | 893 | 1 |
| 39 | 0 | 210 | 0 | 381 | 1 | 552 | 0 | 723 | 0 | 894 | 1 |
| 40 | 0 | 211 | 1 | 382 | 1 | 553 | 1 | 724 | 0 | 895 | 1 |
| 41 | 0 | 212 | 0 | 383 | 0 | 554 | 1 | 725 | 1 | 896 | 1 |
| 42 | 1 | 213 | 1 | 384 | 1 | 555 | 0 | 726 | 0 | 897 | 1 |
| 43 | 1 | 214 | 0 | 385 | 1 | 556 | 0 | 727 | 1 | 898 | 1 |
| 44 | 1 | 215 | 1 | 386 | 1 | 557 | 0 | 728 | 1 | 899 | 1 |
| 45 | 1 | 216 | 1 | 387 | 0 | 558 | 0 | 729 | 1 | 900 | 0 |
| 46 | 1 | 217 | 1 | 388 | 0 | 559 | 1 | 730 | 0 | 901 | 1 |
| 47 | 0 | 218 | 0 | 389 | 1 | 560 | 1 | 731 | 0 | 902 | 0 |
| 48 | 1 | 219 | 1 | 390 | 0 | 561 | 0 | 732 | 1 | 903 | 1 |
| 49 | 0 | 220 | 0 | 391 | 1 | 562 | 1 | 733 | 0 | 904 | 0 |
| 50 | 1 | 221 | 1 | 392 | 1 | 563 | 1 | 734 | 1 | 905 | 0 |
| 51 | 0 | 222 | 0 | 393 | 0 | 564 | 1 | 735 | 0 | 906 | 1 |
| 52 | 0 | 223 | 0 | 394 | 1 | 565 | 0 | 736 | 1 | 907 | 0 |
| 53 | 1 | 224 | 0 | 395 | 0 | 566 | 0 | 737 | 0 | 908 | 1 |
| 54 | 0 | 225 | 1 | 396 | 0 | 567 | 0 | 738 | 0 | 909 | 0 |
| 55 | 0 | 226 | 1 | 397 | 0 | 568 | 0 | 739 | 1 | 910 | 1 |
| 56 | 0 | 227 | 0 | 398 | 0 | 569 | 0 | 740 | 1 | 911 | 0 |
| 57 | 0 | 228 | 1 | 399 | 0 | 570 | 0 | 741 | 1 | 912 | 0 |
| 58 | 1 | 229 | 1 | 400 | 0 | 571 | 0 | 742 | 1 | 913 | 1 |
| 59 | 0 | 230 | 0 | 401 | 1 | 572 | 0 | 743 | 1 | 914 | 1 |
| 60 | 0 | 231 | 0 | 402 | 1 | 573 | 0 | 744 | 0 | 915 | 1 |
| 61 | 0 | 232 | 1 | 403 | 0 | 574 | 1 | 745 | 1 | 916 | 0 |
| 62 | 0 | 233 | 1 | 404 | 1 | 575 | 1 | 746 | 1 | 917 | 1 |
| 63 | 0 | 234 | 0 | 405 | 0 | 576 | 0 | 747 | 0 | 918 | 0 |
| 64 | 1 | 235 | 0 | 406 | 0 | 577 | 0 | 748 | 0 | 919 | 1 |
| 65 | 1 | 236 | 1 | 407 | 1 | 578 | 0 | 749 | 0 | 920 | 0 |
| 66 | 1 | 237 | 0 | 408 | 0 | 579 | 0 | 750 | 0 | 921 | 0 |
| 67 | 1 | 238 | 0 | 409 | 0 | 580 | 1 | 751 | 0 | 922 | 0 |
| 68 | 1 | 239 | 0 | 410 | 0 | 581 | 1 | 752 | 0 | 923 | 0 |
| 69 | 1 | 240 | 1 | 411 | 1 | 582 | 0 | 753 | 1 | 924 | 0 |
| 70 | 1 | 241 | 1 | 412 | 0 | 583 | 1 | 754 | 1 | 925 | 0 |
| 71 | 0 | 242 | 0 | 413 | 0 | 584 | 1 | 755 | 0 | 926 | 1 |
| 72 | 1 | 243 | 0 | 414 | 1 | 585 | 1 | 756 | 1 | 927 | 1 |
| 73 | 0 | 244 | 1 | 415 | 1 | 586 | 1 | 757 | 1 | 928 | 0 |
| 74 | 0 | 245 | 0 | 416 | 0 | 587 | 1 | 758 | 0 | 929 | 0 |
| 75 | 0 | 246 | 0 | 417 | 1 | 588 | 1 | 759 | 0 | 930 | 0 |
| 76 | 0 | 247 | 0 | 418 | 1 | 589 | 0 | 760 | 1 | 931 | 0 |
| 77 | 1 | 248 | 0 | 419 | 1 | 590 | 1 | 761 | 0 | 932 | 0 |
| 78 | 0 | 249 | 1 | 420 | 0 | 591 | 0 | 762 | 1 | 933 | 1 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 79 | 1 | 250 | 0 | 421 | 1 | 592 | 0 | 763 | 0 | 934 | 1 |
| 80 | 0 | 251 | 1 | 422 | 1 | 593 | 0 | 764 | 0 | 935 | 0 |
| 81 | 0 | 252 | 1 | 423 | 0 | 594 | 1 | 765 | 1 | 936 | 1 |
| 82 | 1 | 253 | 1 | 424 | 0 | 595 | 0 | 766 | 1 | 937 | 0 |
| 83 | 0 | 254 | 1 | 425 | 1 | 596 | 0 | 767 | 1 | 938 | 0 |
| 84 | 1 | 255 | 0 | 426 | 1 | 597 | 0 | 768 | 1 | 939 | 1 |
| 85 | 1 | 256 | 1 | 427 | 0 | 598 | 0 | 769 | 1 | 940 | 1 |
| 86 | 1 | 257 | 0 | 428 | 0 | 599 | 0 | 770 | 1 | 941 | 0 |
| 87 | 1 | 258 | 0 | 429 | 0 | 600 | 1 | 771 | 0 | 942 | 0 |
| 88 | 1 | 259 | 0 | 430 | 0 | 601 | 0 | 772 | 0 | 943 | 1 |
| 89 | 1 | 260 | 1 | 431 | 1 | 602 | 1 | 773 | 0 | 944 | 1 |
| 90 | 0 | 261 | 1 | 432 | 1 | 603 | 0 | 774 | 1 | 945 | 0 |
| 91 | 0 | 262 | 1 | 433 | 1 | 604 | 1 | 775 | 1 | 946 | 0 |
| 92 | 1 | 263 | 0 | 434 | 1 | 605 | 0 | 776 | 1 | 947 | 1 |
| 93 | 1 | 264 | 0 | 435 | 1 | 606 | 1 | 777 | 0 | 948 | 1 |
| 94 | 1 | 265 | 0 | 436 | 1 | 607 | 0 | 778 | 0 | 949 | 1 |
| 95 | 0 | 266 | 1 | 437 | 1 | 608 | 1 | 779 | 1 | 950 | 1 |
| 96 | 1 | 267 | 1 | 438 | 0 | 609 | 0 | 780 | 0 | 951 | 0 |
| 97 | 1 | 268 | 0 | 439 | 0 | 610 | 0 | 781 | 0 | 952 | 0 |
| 98 | 0 | 269 | 1 | 440 | 0 | 611 | 0 | 782 | 1 | 953 | 1 |
| 99 | 0 | 270 | 1 | 441 | 1 | 612 | 0 | 783 | 1 | 954 | 1 |
| 100 | 0 | 271 | 0 | 442 | 1 | 613 | 1 | 784 | 0 | 955 | 1 |
| 101 | 0 | 272 | 0 | 443 | 0 | 614 | 1 | 785 | 1 | 956 | 0 |
| 102 | 1 | 273 | 1 | 444 | 1 | 615 | 0 | 786 | 1 | 957 | 1 |
| 103 | 0 | 274 | 1 | 445 | 0 | 616 | 1 | 787 | 1 | 958 | 1 |
| 104 | 1 | 275 | 1 | 446 | 1 | 617 | 0 | 788 | 1 | 959 | 0 |
| 105 | 0 | 276 | 0 | 447 | 0 | 618 | 1 | 789 | 1 | 960 | 0 |
| 106 | 1 | 277 | 1 | 448 | 1 | 619 | 1 | 790 | 1 | 961 | 1 |
| 107 | 0 | 278 | 0 | 449 | 0 | 620 | 0 | 791 | 1 | 962 | 1 |
| 108 | 0 | 279 | 0 | 450 | 1 | 621 | 1 | 792 | 0 | 963 | 1 |
| 109 | 0 | 280 | 0 | 451 | 0 | 622 | 0 | 793 | 1 | 964 | 0 |
| 110 | 1 | 281 | 1 | 452 | 1 | 623 | 0 | 794 | 0 | 965 | 0 |
| 111 | 1 | 282 | 1 | 453 | 1 | 624 | 0 | 795 | 0 | 966 | 0 |
| 112 | 0 | 283 | 1 | 454 | 0 | 625 | 0 | 796 | 0 | 967 | 0 |
| 113 | 1 | 284 | 1 | 455 | 0 | 626 | 1 | 797 | 1 | 968 | 1 |
| 114 | 0 | 285 | 1 | 456 | 0 | 627 | 1 | 798 | 1 | 969 | 1 |
| 115 | 0 | 286 | 1 | 457 | 1 | 628 | 0 | 799 | 0 | 970 | 0 |
| 116 | 0 | 287 | 0 | 458 | 1 | 629 | 1 | 800 | 1 | 971 | 0 |
| 117 | 0 | 288 | 0 | 459 | 1 | 630 | 0 | 801 | 1 | 972 | 0 |
| 118 | 1 | 289 | 1 | 460 | 1 | 631 | 1 | 802 | 1 | 973 | 1 |
| 119 | 0 | 290 | 0 | 461 | 1 | 632 | 1 | 803 | 1 | 974 | 1 |
| 120 | 1 | 291 | 1 | 462 | 0 | 633 | 1 | 804 | 1 | 975 | 0 |
| 121 | 1 | 292 | 1 | 463 | 1 | 634 | 1 | 805 | 1 | 976 | 0 |
| 122 | 1 | 293 | 1 | 464 | 1 | 635 | 0 | 806 | 0 | 977 | 1 |

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|---|---|---|---|---|---|---|---|---|---|---|---|
| 123 | 1 | 294 | 1 | 465 | 0 | 636 | 0 | 807 | 1 | 978 | 0 |
| 124 | 1 | 295 | 0 | 466 | 0 | 637 | 1 | 808 | 1 | 979 | 0 |
| 125 | 0 | 296 | 1 | 467 | 1 | 638 | 1 | 809 | 0 | 980 | 0 |
| 126 | 1 | 297 | 1 | 468 | 1 | 639 | 1 | 810 | 0 | 981 | 1 |
| 127 | 0 | 298 | 0 | 469 | 0 | 640 | 0 | 811 | 1 | 982 | 1 |
| 128 | 1 | 299 | 1 | 470 | 1 | 641 | 1 | 812 | 1 | 983 | 1 |
| 129 | 0 | 300 | 1 | 471 | 1 | 642 | 1 | 813 | 0 | 984 | 1 |
| 130 | 1 | 301 | 0 | 472 | 0 | 643 | 0 | 814 | 1 | 985 | 0 |
| 131 | 0 | 302 | 1 | 473 | 1 | 644 | 1 | 815 | 0 | 986 | 0 |
| 132 | 1 | 303 | 1 | 474 | 1 | 645 | 1 | 816 | 1 | 987 | 0 |
| 133 | 0 | 304 | 1 | 475 | 0 | 646 | 1 | 817 | 0 | 988 | 1 |
| 134 | 0 | 305 | 1 | 476 | 1 | 647 | 1 | 818 | 1 | 989 | 1 |
| 135 | 1 | 306 | 1 | 477 | 1 | 648 | 0 | 819 | 0 | 990 | 1 |
| 136 | 0 | 307 | 0 | 478 | 1 | 649 | 1 | 820 | 0 | 991 | 0 |
| 137 | 0 | 308 | 1 | 479 | 0 | 650 | 1 | 821 | 1 | 992 | 0 |
| 138 | 0 | 309 | 1 | 480 | 1 | 651 | 1 | 822 | 1 | 993 | 1 |
| 139 | 0 | 310 | 0 | 481 | 1 | 652 | 0 | 823 | 1 | 994 | 1 |
| 140 | 0 | 311 | 0 | 482 | 0 | 653 | 1 | 824 | 0 | 995 | 0 |
| 141 | 0 | 312 | 0 | 483 | 0 | 654 | 0 | 825 | 0 | 996 | 1 |
| 142 | 1 | 313 | 0 | 484 | 1 | 655 | 0 | 826 | 1 | 997 | 1 |
| 143 | 1 | 314 | 1 | 485 | 1 | 656 | 0 | 827 | 1 | 998 | 0 |
| 144 | 1 | 315 | 0 | 486 | 0 | 657 | 1 | 828 | 1 | 999 | 0 |
| 145 | 0 | 316 | 1 | 487 | 1 | 658 | 1 | 829 | 1 | 1000 | 1 |
| 146 | 0 | 317 | 0 | 488 | 1 | 659 | 1 | 830 | 1 | 1001 | 1 |
| 147 | 0 | 318 | 0 | 489 | 0 | 660 | 0 | 831 | 0 | 1002 | 0 |
| 148 | 1 | 319 | 1 | 490 | 1 | 661 | 1 | 832 | 0 | 1003 | 1 |
| 149 | 1 | 320 | 1 | 491 | 0 | 662 | 1 | 833 | 1 | 1004 | 0 |
| 150 | 0 | 321 | 0 | 492 | 0 | 663 | 1 | 834 | 0 | 1005 | 1 |
| 151 | 1 | 322 | 0 | 493 | 1 | 664 | 1 | 835 | 0 | 1006 | 1 |
| 152 | 0 | 323 | 1 | 494 | 1 | 665 | 1 | 836 | 1 | 1007 | 0 |
| 153 | 1 | 324 | 1 | 495 | 0 | 666 | 1 | 837 | 0 | 1008 | 1 |
| 154 | 0 | 325 | 1 | 496 | 1 | 667 | 0 | 838 | 0 | 1009 | 1 |
| 155 | 0 | 326 | 1 | 497 | 1 | 668 | 1 | 839 | 0 | 1010 | 0 |
| 156 | 1 | 327 | 0 | 498 | 1 | 669 | 1 | 840 | 1 | 1011 | 1 |
| 157 | 1 | 328 | 1 | 499 | 1 | 670 | 1 | 841 | 0 | 1012 | 1 |
| 158 | 1 | 329 | 1 | 500 | 1 | 671 | 1 | 842 | 0 | 1013 | 1 |
| 159 | 0 | 330 | 0 | 501 | 0 | 672 | 1 | 843 | 0 | 1014 | 1 |
| 160 | 1 | 331 | 0 | 502 | 1 | 673 | 1 | 844 | 0 | 1015 | 1 |
| 161 | 0 | 332 | 0 | 503 | 1 | 674 | 0 | 845 | 1 | 1016 | 0 |
| 162 | 0 | 333 | 1 | 504 | 1 | 675 | 0 | 846 | 0 | 1017 | 1 |
| 163 | 1 | 334 | 0 | 505 | 0 | 676 | 0 | 847 | 1 | 1018 | 1 |
| 164 | 1 | 335 | 0 | 506 | 1 | 677 | 1 | 848 | 0 | 1019 | 1 |
| 165 | 0 | 336 | 1 | 507 | 0 | 678 | 1 | 849 | 1 | | |
| 166 | 0 | 337 | 0 | 508 | 0 | 679 | 0 | 850 | 1 | | |

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|---|---|---|---|---|---|---|---|---|---|
| 167 | 0 | 338 | 1 | 509 | 0 | 680 | 1 | 851 | 0 |
| 168 | 1 | 339 | 0 | 510 | 0 | 681 | 0 | 852 | 0 |
| 169 | 1 | 340 | 0 | 511 | 1 | 682 | 0 | 853 | 0 |
| 170 | 0 | 341 | 1 | 512 | 1 | 683 | 1 | 854 | 0 |

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