PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^4+z^2+z,z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^4+z^2+z,z^2+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}}\tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)).\tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots\tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}).\tag{1.4}$$

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Theorem 1.1. *When $(a,b)=(z^4+z^2+z, z^2+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{15}+z^{13}+z^{11}+z^6+z^5+z^4+z^3+z+1,\\
p_1(z)&=z^{22}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^4+z,\\
p_2(z)&=z^{24}+z^{20}+z^{12}+z^{10}+z^4+z^2,\\
p_3(z)&=z^{22}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^4+z,\\
p_4(z)&=z^{20}+z^{18}+z^{16}+z^{15}+z^{13}+z^{11}+z^{10}+z^8+z^6+z^5+z^3+z^2+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{13}+z^{11}+z^6+z^5+z^3+z,\\
p_1(z)&=z^{22}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^4+z,\\
p_2(z)&=z^{24}+z^{20}+z^{12}+z^{10}+z^4+z^2,\\
p_3(z)&=z^{22}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^8+z^7+z^4+z,\\
p_4(z)&=z^{20}+z^{18}+z^{15}+z^{13}+z^{11}+z^{10}+z^8+z^6+z^5+z^4+z^3+z^2+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] = {} & M^e[:6u] + x^{3u}M^e[:3u] + x^{5u}M^e[:u] + x^{3u}M^e[:6u] + x^{6u}M^e[:3u] \\
& + x^{8u}M^e[:u] + x^{5u}M^e[:6u] + x^{8u}M^e[:3u] + x^{10u}M^e[:u] \\
& + x^{6u}M^e[:6u] + x^{9u}M^e[:3u] + x^{11u}M^e[:u] + x^{7u}M^e[:5u] \\
& + (x^{6u} + x^{9u} + x^{11u})R^e, \\
W^e[:12u] = {} & W^e[:6u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{3u}W^e[:6u] + x^{6u}W^e[:3u] \\
& + x^{8u}W^e[:u] + x^{5u}W^e[:6u] + x^{8u}W^e[:3u] + x^{10u}W^e[:u]
\end{aligned}$$

$$
\begin{aligned}
&+ x^{6u}W^e[:6u] + x^{9u}W^e[:3u] + x^{11u}W^e[:u] + x^{7u}W^e[:5u]\\
&+ (x^{6u}+x^{9u}+x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] = {}& M^o[:6u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u]\\
&+ x^{8u}M^o[:u] + x^{5u}M^o[:6u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u]\\
&+ x^{6u}M^o[:6u] + x^{9u}M^o[:3u] + x^{11u}M^o[:u] + x^{7u}M^o[:5u]\\
&+ (x^{6u}+x^{9u}+x^{11u})R^o,\\
W^o[:12u] = {}& W^o[:6u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u]\\
&+ x^{8u}W^o[:u] + x^{5u}W^o[:6u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u]\\
&+ x^{6u}W^o[:6u] + x^{9u}W^o[:3u] + x^{11u}W^o[:u] + x^{7u}W^o[:5u]\\
&+ (x^{6u}+x^{9u}+x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] = {}& T[:6u] + x^{3u}T[:3u] + x^{5u}T[:u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{8u}T[:u]\\
&+ x^{5u}T[:6u] + x^{8u}T[:3u] + x^{10u}T[:u] + x^{6u}T[:6u] + x^{9u}T[:3u]\\
&+ x^{11u}T[:u] + x^{7u}T[:5u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
0 &= x^{5u}T[u:2u] + x^{3u}T[3u:4u] + T[6u:7u],\\
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + T[7u:8u],\\
0 &= x^{7u}T[u:2u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] + T[8u:9u],\\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u]\\
&\quad + x^{5u}T[4u:5u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u]\\
&\quad + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u]\\
&\quad + T[11u:12u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$

$$
\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]\\
&\quad + T[[1001w]_2],
\end{aligned} \tag{2.10}
$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]$$

$$+T[[101w]_2]+T[[1010w]_2], \tag{2.11}$$

$$0=T[[w]_2]+T[[10w]_2]+T[[100w]_2]+T[[101w]_2]+T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0=T[[1w]_2]+T[[11w]_2]+T[[110w]_2], \tag{2.13}$$

$$0=T[[w]_2]+T[[10w]_2]+T[[100w]_2]+T[[111w]_2], \tag{2.14}$$

$$0=T[[1w]_2]+T[[11w]_2]+T[[101w]_2]+T[[1000w]_2], \tag{2.15}$$

$$\begin{aligned}0&=T[[w]_2]+T[[1w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]\\&\quad+T[[1001w]_2],\end{aligned} \tag{2.16}$$

$$\begin{aligned}0&=T[[w]_2]+T[[1w]_2]+T[[10w]_2]+T[[11w]_2]+T[[100w]_2]\\&\quad+T[[101w]_2]+T[[1010w]_2],\end{aligned} \tag{2.17}$$

$$0=T[[w]_2]+T[[10w]_2]+T[[100w]_2]+T[[101w]_2]+T[[1011w]_2], \tag{2.18}$$

for $w=0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1576 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k}=\{A(i,w): |w|=2k, w\neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,110)), \tag{2.19}$$

$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,111)), \tag{2.20}$$

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.21}$$

$$\begin{aligned}0&=\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\&\quad+\tau(A(s,1001)),\end{aligned} \tag{2.22}$$

$$\begin{aligned}0&=\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\&\quad+\tau(A(s,101))+\tau(A(s,1010)),\end{aligned} \tag{2.23}$$

$$\begin{aligned}0&=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))\\&\quad+\tau(A(s,1011)),\end{aligned} \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,110)), \tag{2.25}$$

$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,111)), \tag{2.26}$$

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.27}$$

$$\begin{aligned}0&=\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\&\quad+\tau(A(s,1001)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}0&=\tau(A(s,\epsilon))+\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\&\quad+\tau(A(s,101))+\tau(A(s,1010)),\end{aligned} \tag{2.29}$$

$$\begin{aligned}0&=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))\\&\quad+\tau(A(s,1011)),\end{aligned} \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{48}), E_{48}) = (A(i, 0^{20}), E_{20})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 24$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^{6u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u} M^o[:3u] + x^{5u} M^o[:u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u} W^o[:3u] + x^{5u} W^o[:u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^{3v}W^e[:3v] + x^{5v}W^e[:v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ &+ x^{3v}M^e[:3v] + x^{5v}M^e[:v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v] + x^{10v}W^e[:2v]M^e[:2v].$$

For all $n \le 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned}\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.\end{aligned}$$

Let $z=1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}q_0(x) &= x^{24}+x^{23}+x^{21}+x^{20}+x^{19}+x^{18}+x^{13}+x^{11}+x^{9}+x^{6},\\ q_1(x) &= x^{23}+x^{20}+x^{17}+x^{16}+x^{13}+x^{12}+x^{10}+x^{7}+x^{6}+x^{2},\\ q_2(x) &= x^{22}+x^{20}+x^{14}+x^{12}+x^{4}+1,\\ q_3(x) &= x^{23}+x^{20}+x^{17}+x^{16}+x^{13}+x^{12}+x^{10}+x^{7}+x^{6}+x^{2},\\ q_4(x) &= x^{23}+x^{22}+x^{21}+x^{19}+x^{18}+x^{16}+x^{14}+x^{13}+x^{11}+x^{9}+x^{8}+x^{6}\\ &\quad +x^{4}.\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}p_0(x) &= x^{120}+x^{116}+x^{110}+x^{106}+x^{102}+x^{100}+x^{98}+x^{88}+x^{86}+x^{80}\\ &\quad +x^{76}+x^{74}+x^{72}+x^{70}+x^{66}+x^{64}+x^{62}+x^{60}+x^{54}+x^{52}+x^{40}\end{aligned}$$

$$
\begin{aligned}
&+ x^{36} + x^{32} + x^{30} + x^{24},\\
p_3(x) &= x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{94} + x^{92} + x^{90} + x^{86}\\
&+ x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{54} + x^{52} + x^{50} + x^{46} + x^{42}\\
&+ x^{40} + x^{36} + x^{34} + x^{28} + x^{24} + x^{22} + x^{20},\\
p_6(x) &= x^{116} + x^{110} + x^{108} + x^{100} + x^{98} + x^{96} + x^{88} + x^{86} + x^{84} + x^{78} + x^{74}\\
&+ x^{72} + x^{70} + x^{66} + x^{62} + x^{60} + x^{56} + x^{50} + x^{48} + x^{44} + x^{40} + x^{38}\\
&+ x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{22} + x^{18} + x^{10} + x^{6} + 1,\\
p_9(x) &= x^{118} + x^{110} + x^{108} + x^{104} + x^{102} + x^{94} + x^{80} + x^{74} + x^{70} + x^{64} + x^{60}\\
&+ x^{54} + x^{50} + x^{44} + x^{40} + x^{38} + x^{34} + x^{28} + x^{22} + x^{20} + x^{10} + x^{4},\\
p_{12}(x) &= x^{118} + x^{116} + x^{112} + x^{102} + x^{100} + x^{96} + x^{86} + x^{84} + x^{80} + x^{38} + x^{36}\\
&+ x^{32} + x^{22} + x^{20} + x^{16} + x^{6} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{120} + x^{119} + x^{116} + x^{114} + x^{113} + x^{101} + x^{99}\\
&+ x^{96} + x^{93} + x^{92} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} + x^{77} + x^{73}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{63} + x^{62} + x^{60} + x^{55} + x^{53} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{106} + x^{105} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85}\\
&+ x^{82} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{65} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{40}\\
&+ x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{21} + x^{18},\\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{103}\\
&+ x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{72} + x^{70} + x^{67}\\
&+ x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47}\\
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{26} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12},\\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{111} + x^{110} + x^{109} + x^{107}\\
&+ x^{105} + x^{104} + x^{102} + x^{96} + x^{95} + x^{91} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80}\\
&+ x^{79} + x^{78} + x^{75} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59}\\
&+ x^{58} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{44} + x^{30} + x^{29}\\
&+ x^{26} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{9} + x^{6},\\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{108} + x^{101} + x^{100}\\
&+ x^{99} + x^{96} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{79} + x^{76} + x^{73}
\end{aligned}
$$

$$+ x^{72} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{52} + x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} + x^{35} + x^{32} + x^{25} + x^{24} + x^{23} + x^{21} + x^{19} + x^{17} + x^{15} + x^{12} + x^{5} + x^{4} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{120} + x^{119} + x^{116} + x^{114} + x^{113} + x^{101} + x^{99} \\
&\quad + x^{96} + x^{93} + x^{92} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{79} + x^{77} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{63} + x^{62} + x^{60} + x^{55} + x^{53} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{37} + x^{35} + x^{34} + x^{32} + x^{30}, \\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{106} + x^{105} + x^{101} + x^{100} \\
&\quad + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85} \\
&\quad + x^{82} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{65} + x^{61} + x^{60} + x^{59} + x^{57} \\
&\quad + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{40} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{21} + x^{18}, \\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{72} + x^{70} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} \\
&\quad + x^{26} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12}, \\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{111} + x^{110} + x^{109} + x^{107} \\
&\quad + x^{105} + x^{104} + x^{102} + x^{96} + x^{95} + x^{91} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{44} + x^{30} + x^{29} \\
&\quad + x^{26} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{108} + x^{101} + x^{100} \\
&\quad + x^{99} + x^{96} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{79} + x^{76} + x^{73} \\
&\quad + x^{72} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{52} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} + x^{35} + x^{32} + x^{25} + x^{24} + x^{23} \\
&\quad + x^{21} + x^{19} + x^{17} + x^{15} + x^{12} + x^{5} + x^{4} + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{126} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{88} \\
&\quad + x^{82} + x^{76} + x^{70} + x^{66} + x^{58} + x^{48} + x^{42} + x^{40} + x^{36}, \\
p_3(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100}
\end{aligned}$$

$$
\begin{aligned}
&+x^{98}+x^{96}+x^{90}+x^{88}+x^{86}+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}+x^{68}\\
&+x^{62}+x^{56}+x^{54}+x^{46}+x^{44}+x^{38}+x^{36}+x^{32}+x^{28}+x^{16}+x^{12},\\
p_6(x)&=x^{126}+x^{124}+x^{114}+x^{110}+x^{106}+x^{104}+x^{100}+x^{98}+x^{94}+x^{84}\\
&+x^{82}+x^{74}+x^{72}+x^{70}+x^{68}+x^{64}+x^{60}+x^{56}+x^{54}+x^{52}+x^{50}\\
&+x^{44}+x^{38}+x^{36}+x^{34}+x^{30}+x^{24}+x^{22}+x^{20}+x^{14}+x^{8}+1,\\
p_9(x)&=x^{124}+x^{122}+x^{118}+x^{114}+x^{104}+x^{102}+x^{98}+x^{96}+x^{94}+x^{92}\\
&+x^{90}+x^{88}+x^{86}+x^{82}+x^{80}+x^{76}+x^{70}+x^{68}+x^{66}+x^{62}+x^{60}\\
&+x^{50}+x^{46}+x^{44}+x^{38}+x^{34}+x^{32}+x^{30}+x^{26}+x^{18}+x^{14}+x^{4},\\
p_{12}(x)&=x^{124}+x^{120}+x^{116}+x^{112}+x^{100}+x^{96}+x^{92}+x^{88}+x^{84}+x^{80}+x^{76}\\
&+x^{72}+x^{60}+x^{56}+x^{36}+x^{32}+x^{28}+x^{24}+x^{20}+x^{16}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{132}+x^{130}+x^{128}+x^{126}+x^{124}+x^{122}+x^{120}+x^{114}+x^{108}+x^{98}\\
&+x^{94}+x^{90}+x^{84}+x^{80}+x^{78}+x^{76}+x^{74}+x^{72}+x^{68}+x^{66}+x^{62}\\
&+x^{58}+x^{54}+x^{52}+x^{50}+x^{46}+x^{42}+x^{34}+x^{24},\\
p_3(x)&=x^{126}+x^{120}+x^{118}+x^{114}+x^{112}+x^{108}+x^{102}+x^{98}+x^{92}+x^{88}\\
&+x^{86}+x^{84}+x^{82}+x^{78}+x^{76}+x^{64}+x^{62}+x^{60}+x^{56}+x^{54}+x^{50}\\
&+x^{46}+x^{38}+x^{30}+x^{26}+x^{22},\\
p_6(x)&=x^{126}+x^{124}+x^{122}+x^{120}+x^{112}+x^{110}+x^{104}+x^{102}+x^{100}+x^{98}\\
&+x^{94}+x^{90}+x^{84}+x^{80}+x^{72}+x^{68}+x^{64}+x^{60}+x^{54}+x^{52}+x^{48}\\
&+x^{44}+x^{38}+x^{36}+x^{34}+x^{32}+x^{26}+x^{24}+x^{20}+x^{16}+x^{14}+1,\\
p_9(x)&=x^{124}+x^{122}+x^{118}+x^{114}+x^{104}+x^{102}+x^{98}+x^{96}+x^{94}+x^{92}\\
&+x^{90}+x^{88}+x^{86}+x^{82}+x^{80}+x^{76}+x^{70}+x^{68}+x^{66}+x^{62}+x^{60}\\
&+x^{50}+x^{46}+x^{44}+x^{38}+x^{34}+x^{32}+x^{30}+x^{26}+x^{18}+x^{14}+x^{4},\\
p_{12}(x)&=x^{124}+x^{120}+x^{116}+x^{112}+x^{100}+x^{96}+x^{92}+x^{88}+x^{84}+x^{80}+x^{76}\\
&+x^{72}+x^{60}+x^{56}+x^{36}+x^{32}+x^{28}+x^{24}+x^{20}+x^{16}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{126}+x^{124}+x^{123}+x^{121}+x^{118}+x^{116}+x^{113}+x^{111}+x^{110}+x^{109}\\
&+x^{107}+x^{104}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}+x^{95}+x^{93}\\
&+x^{92}+x^{90}+x^{89}+x^{87}+x^{85}+x^{78}+x^{77}+x^{75}+x^{74}+x^{71}+x^{70}\\
&+x^{68}+x^{65}+x^{63}+x^{61}+x^{60}+x^{59}+x^{56}+x^{54}+x^{51}+x^{50}+x^{48}\\
&+x^{46}+x^{44}+x^{43}+x^{40}+x^{39}+x^{38}+x^{30},\\
p_3(x)&=x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{118}+x^{106}+x^{105}+x^{101}+x^{100}\\
&+x^{99}+x^{98}+x^{96}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{86}+x^{85}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{82} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{65} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{40}\\
&+ x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{21} + x^{18},\\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{103}\\
&+ x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{72} + x^{70} + x^{67}\\
&+ x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47}\\
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{26} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12},\\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{111} + x^{110} + x^{109} + x^{107}\\
&+ x^{105} + x^{104} + x^{102} + x^{96} + x^{95} + x^{91} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80}\\
&+ x^{79} + x^{78} + x^{75} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59}\\
&+ x^{58} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{44} + x^{30} + x^{29}\\
&+ x^{26} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{9} + x^{6},\\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{108} + x^{101} + x^{100}\\
&+ x^{99} + x^{96} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{79} + x^{76} + x^{73}\\
&+ x^{72} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{52}\\
&+ x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} + x^{35} + x^{32} + x^{25} + x^{24} + x^{23}\\
&+ x^{21} + x^{19} + x^{17} + x^{15} + x^{12} + x^{5} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{121} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109}\\
&+ x^{107} + x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{93}\\
&+ x^{92} + x^{90} + x^{89} + x^{87} + x^{85} + x^{78} + x^{77} + x^{75} + x^{74} + x^{71} + x^{70}\\
&+ x^{68} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{51} + x^{50} + x^{48}\\
&+ x^{46} + x^{44} + x^{43} + x^{40} + x^{39} + x^{38} + x^{30},\\
p_3(x) &= x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{118} + x^{106} + x^{105} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{85}\\
&+ x^{82} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{65} + x^{61} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{44} + x^{43} + x^{40}\\
&+ x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{26} + x^{25} + x^{21} + x^{18},\\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} + x^{110} + x^{103}\\
&+ x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86}\\
&+ x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{72} + x^{70} + x^{67}\\
&+ x^{66} + x^{65} + x^{63} + x^{60} + x^{57} + x^{56} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{26} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{12},\\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{111} + x^{110} + x^{109} + x^{107}\\
&+ x^{105} + x^{104} + x^{102} + x^{96} + x^{95} + x^{91} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80}\\
&+ x^{79} + x^{78} + x^{75} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59}\\
&+ x^{58} + x^{57} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{46} + x^{44} + x^{30} + x^{29}\\
&+ x^{26} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{13} + x^{9} + x^{6},\\
p_{12}(x) &= x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{108} + x^{101} + x^{100}\\
&+ x^{99} + x^{96} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{79} + x^{76} + x^{73}\\
&+ x^{72} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{52}\\
&+ x^{49} + x^{48} + x^{47} + x^{44} + x^{37} + x^{36} + x^{35} + x^{32} + x^{25} + x^{24} + x^{23}\\
&+ x^{21} + x^{19} + x^{17} + x^{15} + x^{12} + x^{5} + x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{96}\\
&+ x^{94} + x^{86} + x^{82} + x^{80} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{52}\\
&+ x^{46} + x^{36},\\
p_3(x) &= x^{116} + x^{114} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88}\\
&+ x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{64} + x^{62} + x^{52} + x^{50}\\
&+ x^{48} + x^{46} + x^{42} + x^{40} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18}\\
&+ x^{16} + x^{12},\\
p_6(x) &= x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{90} + x^{88} + x^{86} + x^{78} + x^{74} + x^{72}\\
&+ x^{70} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30}\\
&+ x^{16} + x^{14} + x^{10} + x^{8} + x^{6} + 1,\\
p_9(x) &= x^{118} + x^{110} + x^{108} + x^{104} + x^{102} + x^{94} + x^{80} + x^{74} + x^{70} + x^{64} + x^{60}\\
&+ x^{54} + x^{50} + x^{44} + x^{40} + x^{38} + x^{34} + x^{28} + x^{22} + x^{20} + x^{10} + x^{4},\\
p_{12}(x) &= x^{118} + x^{116} + x^{112} + x^{102} + x^{100} + x^{96} + x^{86} + x^{84} + x^{80} + x^{38} + x^{36}\\
&+ x^{32} + x^{22} + x^{20} + x^{16} + x^{6} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{121} + x^{120} + x^{119} + x^{118} + x^{114} + x^{112} + x^{111} + x^{109}\\
&+ x^{108} + x^{102} + x^{100} + x^{99} + x^{93} + x^{90} + x^{89} + x^{83} + x^{82} + x^{81} + x^{79}\\
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{66} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37}\\
&+ x^{36} + x^{30} + x^{29} + x^{28} + x^{27} + x^{24},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{125} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109}\\
&\quad + x^{102} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84}\\
&\quad + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{66}\\
&\quad + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{41}\\
&\quad + x^{40} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{24} + x^{22}\\
&\quad + x^{21} + x^{20} + x^{19} + x^{16},\\
p_6(x) &= x^{125} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109}\\
&\quad + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{97} + x^{95} + x^{92} + x^{88} + x^{86}\\
&\quad + x^{84} + x^{82} + x^{81} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66}\\
&\quad + x^{64} + x^{61} + x^{60} + x^{58} + x^{56} + x^{50} + x^{48} + x^{45} + x^{44} + x^{30} + x^{29}\\
&\quad + x^{26} + x^{25} + x^{24} + x^{19} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4}\\
&\quad + x^{3} + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{116} + x^{109} + x^{108} + x^{107} + x^{104}\\
&\quad + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{73} + x^{72} + x^{71} + x^{68} + x^{57}\\
&\quad + x^{56} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} + x^{29} + x^{28} + x^{27} + x^{25}\\
&\quad + x^{23} + x^{20} + x^{13} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109}\\
&\quad + x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88}\\
&\quad + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{76} + x^{65} + x^{64} + x^{63} + x^{60} + x^{49}\\
&\quad + x^{48} + x^{47} + x^{45} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28}\\
&\quad + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4}\\
&\quad + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{136} + x^{133} + x^{131} + x^{130} + x^{129} + x^{126} + x^{122} + x^{121} + x^{119} + x^{118}\\
&\quad + x^{117} + x^{113} + x^{110} + x^{109} + x^{106} + x^{103} + x^{101} + x^{96} + x^{95} + x^{94}\\
&\quad + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} + x^{79} + x^{75} + x^{74} + x^{71}\\
&\quad + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{52}\\
&\quad + x^{49} + x^{48} + x^{47} + x^{43} + x^{41} + x^{40} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32}\\
&\quad + x^{30},\\
p_3(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112}\\
&\quad + x^{111} + x^{110} + x^{108} + x^{105} + x^{104} + x^{103} + x^{101} + x^{98} + x^{97} + x^{96}\\
&\quad + x^{94} + x^{91} + x^{90} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{72}\\
&\quad + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57}\\
&\quad + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{39} + x^{36} + x^{35}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{34}+x^{32}+x^{29}+x^{28}+x^{27}+x^{23}+x^{18},\\
p_6(x) &= x^{130}+x^{124}+x^{122}+x^{114}+x^{112}+x^{110}+x^{108}+x^{106}+x^{104}+x^{98}\\
&+x^{94}+x^{90}+x^{84}+x^{80}+x^{66}+x^{62}+x^{60}+x^{58}+x^{56}+x^{54}+x^{48}\\
&+x^{44}+x^{42}+x^{36}+x^{34}+x^{30}+x^{22}+x^{12},\\
p_9(x) &= x^{131}+x^{129}+x^{128}+x^{126}+x^{124}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}\\
&+x^{113}+x^{111}+x^{107}+x^{102}+x^{99}+x^{97}+x^{96}+x^{94}+x^{92}+x^{89}\\
&+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}+x^{80}+x^{79}+x^{78}+x^{76}+x^{75}+x^{73}\\
&+x^{71}+x^{69}+x^{68}+x^{67}+x^{64}+x^{63}+x^{62}+x^{60}+x^{59}+x^{57}+x^{55}\\
&+x^{53}+x^{52}+x^{49}+x^{47}+x^{43}+x^{38}+x^{35}+x^{33}+x^{32}+x^{30}+x^{28}\\
&+x^{25}+x^{23}+x^{22}+x^{21}+x^{20}+x^{17}+x^{15}+x^{11}+x^{6},\\
p_{12}(x) &= x^{132}+x^{128}+x^{124}+x^{120}+x^{100}+x^{92}+x^{88}+x^{84}+x^{80}+x^{76}+x^{72}\\
&+x^{68}+x^{60}+x^{56}+x^{48}+x^{36}+x^{28}+x^{24}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126}+x^{125}+x^{124}+x^{121}+x^{119}+x^{118}+x^{115}+x^{114}+x^{112}+x^{107}\\
&+x^{105}+x^{104}+x^{103}+x^{100}+x^{99}+x^{94}+x^{91}+x^{89}+x^{87}+x^{81}\\
&+x^{77}+x^{76}+x^{75}+x^{71}+x^{70}+x^{69}+x^{68}+x^{65}+x^{63}+x^{60}+x^{55}\\
&+x^{50}+x^{49}+x^{48}+x^{45}+x^{44}+x^{43}+x^{42}+x^{39}+x^{37}+x^{35}+x^{33}\\
&+x^{30},\\
p_3(x) &= x^{125}+x^{123}+x^{122}+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}\\
&+x^{111}+x^{106}+x^{101}+x^{99}+x^{98}+x^{96}+x^{95}+x^{94}+x^{92}+x^{88}+x^{86}\\
&+x^{85}+x^{84}+x^{83}+x^{79}+x^{77}+x^{75}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}\\
&+x^{67}+x^{66}+x^{63}+x^{61}+x^{59}+x^{56}+x^{55}+x^{54}+x^{52}+x^{48}+x^{46}\\
&+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{29}+x^{28}+x^{27}\\
&+x^{23}+x^{18},\\
p_6(x) &= x^{126}+x^{122}+x^{120}+x^{118}+x^{116}+x^{110}+x^{106}+x^{102}+x^{98}+x^{96}\\
&+x^{86}+x^{84}+x^{78}+x^{76}+x^{72}+x^{66}+x^{64}+x^{58}+x^{56}+x^{52}+x^{44}\\
&+x^{42}+x^{40}+x^{32}+x^{30}+x^{24}+x^{22}+x^{12},\\
p_9(x) &= x^{127}+x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{118}+x^{117}+x^{116}+x^{114}\\
&+x^{113}+x^{111}+x^{107}+x^{102}+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}\\
&+x^{86}+x^{85}+x^{84}+x^{82}+x^{81}+x^{77}+x^{76}+x^{74}+x^{73}+x^{69}+x^{68}\\
&+x^{66}+x^{65}+x^{61}+x^{60}+x^{58}+x^{57}+x^{53}+x^{52}+x^{50}+x^{49}+x^{47}\\
&+x^{43}+x^{38}+x^{31}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{22}+x^{21}+x^{20}\\
&+x^{18}+x^{17}+x^{15}+x^{11}+x^{6},\\
p_{12}(x) &= x^{128}+x^{116}+x^{108}+x^{84}+x^{80}+x^{76}+x^{68}+x^{60}+x^{52}+x^{48}+x^{44}
\end{aligned}
$$

$$+ x^{20} + x^{12} + 1,$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{106} + x^{103}\\
&+ x^{101} + x^{99} + x^{98} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{81} + x^{80}\\
&+ x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{60} + x^{59} + x^{55} + x^{54} + x^{51} + x^{50}\\
&+ x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116} + x^{115} + x^{112} + x^{111}\\
&+ x^{110} + x^{109} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{92}\\
&+ x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74}\\
&+ x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{59} + x^{56}\\
&+ x^{55} + x^{54} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{36}\\
&+ x^{35} + x^{32} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20},\\
p_6(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{109}\\
&+ x^{106} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} + x^{82} + x^{81} + x^{78}\\
&+ x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{65} + x^{64} + x^{63} + x^{61} + x^{58} + x^{56}\\
&+ x^{55} + x^{54} + x^{53} + x^{49} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{30}\\
&+ x^{28} + x^{22} + x^{21} + x^{20} + x^{14} + x^{5} + x^{4} + x^{3} + 1,\\
p_9(x) &= x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{116} + x^{109} + x^{108} + x^{107} + x^{104}\\
&+ x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{73} + x^{72} + x^{71} + x^{68} + x^{57}\\
&+ x^{56} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{23} + x^{20} + x^{13} + x^{12} + x^{11} + x^{8},\\
p_{12}(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109}\\
&+ x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88}\\
&+ x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{76} + x^{65} + x^{64} + x^{63} + x^{60} + x^{49}\\
&+ x^{48} + x^{47} + x^{45} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28}\\
&+ x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4}\\
&+ x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{121} + x^{120} + x^{119} + x^{118} + x^{114} + x^{112} + x^{111} + x^{109}\\
&+ x^{108} + x^{102} + x^{100} + x^{99} + x^{93} + x^{90} + x^{89} + x^{83} + x^{82} + x^{81} + x^{79}\\
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{66} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56}\\
&+ x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37}\\
&+ x^{36} + x^{30} + x^{29} + x^{28} + x^{27} + x^{24},
\end{aligned}$$

$$\begin{aligned}
p_3(x) = {} & x^{125} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109} \\
& + x^{102} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84} \\
& + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{66} \\
& + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{41} \\
& + x^{40} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{24} + x^{22} \\
& + x^{21} + x^{20} + x^{19} + x^{16}, \\
p_6(x) = {} & x^{125} + x^{121} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
& + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{97} + x^{95} + x^{92} + x^{88} + x^{86} \\
& + x^{84} + x^{82} + x^{81} + x^{78} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} \\
& + x^{64} + x^{61} + x^{60} + x^{58} + x^{56} + x^{50} + x^{48} + x^{45} + x^{44} + x^{30} + x^{29} \\
& + x^{26} + x^{25} + x^{24} + x^{19} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4} \\
& + x^{3} + 1, \\
p_9(x) = {} & x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{116} + x^{109} + x^{108} + x^{107} + x^{104} \\
& + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{73} + x^{72} + x^{71} + x^{68} + x^{57} \\
& + x^{56} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} + x^{29} + x^{28} + x^{27} + x^{25} \\
& + x^{23} + x^{20} + x^{13} + x^{12} + x^{11} + x^{8}, \\
p_{12}(x) = {} & x^{125} + x^{124} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
& + x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88} \\
& + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{76} + x^{65} + x^{64} + x^{63} + x^{60} + x^{49} \\
& + x^{48} + x^{47} + x^{45} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} \\
& + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4} \\
& + x^{3} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{125} + x^{124} + x^{121} + x^{119} + x^{118} + x^{115} + x^{114} + x^{112} + x^{107} \\
& + x^{105} + x^{104} + x^{103} + x^{100} + x^{99} + x^{94} + x^{91} + x^{89} + x^{87} + x^{81} \\
& + x^{77} + x^{76} + x^{75} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{63} + x^{60} + x^{55} \\
& + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{33} \\
& + x^{30}, \\
p_3(x) = {} & x^{125} + x^{123} + x^{122} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} \\
& + x^{111} + x^{106} + x^{101} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{92} + x^{88} + x^{86} \\
& + x^{85} + x^{84} + x^{83} + x^{79} + x^{77} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} \\
& + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{56} + x^{55} + x^{54} + x^{52} + x^{48} + x^{46} \\
& + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} \\
& + x^{23} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{110} + x^{106} + x^{102} + x^{98} + x^{96} \\
&\quad + x^{86} + x^{84} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{58} + x^{56} + x^{52} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{32} + x^{30} + x^{24} + x^{22} + x^{12}, \\
p_9(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116} + x^{114} \\
&\quad + x^{113} + x^{111} + x^{107} + x^{102} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} \\
&\quad + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{77} + x^{76} + x^{74} + x^{73} + x^{69} + x^{68} \\
&\quad + x^{66} + x^{65} + x^{61} + x^{60} + x^{58} + x^{57} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} \\
&\quad + x^{43} + x^{38} + x^{31} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{18} + x^{17} + x^{15} + x^{11} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{116} + x^{108} + x^{84} + x^{80} + x^{76} + x^{68} + x^{60} + x^{52} + x^{48} + x^{44} \\
&\quad + x^{20} + x^{12} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{133} + x^{131} + x^{130} + x^{129} + x^{126} + x^{122} + x^{121} + x^{119} + x^{118} \\
&\quad + x^{117} + x^{113} + x^{110} + x^{109} + x^{106} + x^{103} + x^{101} + x^{96} + x^{95} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{82} + x^{79} + x^{75} + x^{74} + x^{71} \\
&\quad + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{52} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{43} + x^{41} + x^{40} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} \\
&\quad + x^{30}, \\
p_3(x) &= x^{129} + x^{127} + x^{126} + x^{125} + x^{124} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} \\
&\quad + x^{111} + x^{110} + x^{108} + x^{105} + x^{104} + x^{103} + x^{101} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{94} + x^{91} + x^{90} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{72} \\
&\quad + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} \\
&\quad + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{41} + x^{39} + x^{36} + x^{35} \\
&\quad + x^{34} + x^{32} + x^{29} + x^{28} + x^{27} + x^{23} + x^{18}, \\
p_6(x) &= x^{130} + x^{124} + x^{122} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} \\
&\quad + x^{94} + x^{90} + x^{84} + x^{80} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{36} + x^{34} + x^{30} + x^{22} + x^{12}, \\
p_9(x) &= x^{131} + x^{129} + x^{128} + x^{126} + x^{124} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} \\
&\quad + x^{113} + x^{111} + x^{107} + x^{102} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{73} \\
&\quad + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{55} \\
&\quad + x^{53} + x^{52} + x^{49} + x^{47} + x^{43} + x^{38} + x^{35} + x^{33} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{17} + x^{15} + x^{11} + x^{6}, \\
p_{12}(x) &= x^{132} + x^{128} + x^{124} + x^{120} + x^{100} + x^{92} + x^{88} + x^{84} + x^{80} + x^{76} + x^{72}
\end{aligned}$$

$$+ x^{68} + x^{60} + x^{56} + x^{48} + x^{36} + x^{28} + x^{24} + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{106} + x^{103} \\
& + x^{101} + x^{99} + x^{98} + x^{94} + x^{91} + x^{90} + x^{87} + x^{86} + x^{83} + x^{81} + x^{80} \\
& + x^{72} + x^{71} + x^{69} + x^{67} + x^{66} + x^{60} + x^{59} + x^{55} + x^{54} + x^{51} + x^{50} \\
& + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{36}, \\
p_3(x) = {} & x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116} + x^{115} + x^{112} + x^{111} \\
& + x^{110} + x^{109} + x^{105} + x^{104} + x^{103} + x^{102} + x^{99} + x^{98} + x^{97} + x^{92} \\
& + x^{91} + x^{90} + x^{88} + x^{87} + x^{85} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} \\
& + x^{73} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{59} + x^{56} \\
& + x^{55} + x^{54} + x^{51} + x^{49} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{36} \\
& + x^{35} + x^{32} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20}, \\
p_6(x) = {} & x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{109} \\
& + x^{106} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} + x^{82} + x^{81} + x^{78} \\
& + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{65} + x^{64} + x^{63} + x^{61} + x^{58} + x^{56} \\
& + x^{55} + x^{54} + x^{53} + x^{49} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{30} \\
& + x^{28} + x^{22} + x^{21} + x^{20} + x^{14} + x^{5} + x^{4} + x^{3} + 1, \\
p_9(x) = {} & x^{125} + x^{124} + x^{123} + x^{121} + x^{119} + x^{116} + x^{109} + x^{108} + x^{107} + x^{104} \\
& + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{73} + x^{72} + x^{71} + x^{68} + x^{57} \\
& + x^{56} + x^{55} + x^{52} + x^{45} + x^{44} + x^{43} + x^{40} + x^{29} + x^{28} + x^{27} + x^{25} \\
& + x^{23} + x^{20} + x^{13} + x^{12} + x^{11} + x^{8}, \\
p_{12}(x) = {} & x^{125} + x^{124} + x^{123} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
& + x^{107} + x^{104} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} + x^{91} + x^{88} \\
& + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{76} + x^{65} + x^{64} + x^{63} + x^{60} + x^{49} \\
& + x^{48} + x^{47} + x^{45} + x^{43} + x^{40} + x^{37} + x^{36} + x^{35} + x^{32} + x^{29} + x^{28} \\
& + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{11} + x^{8} + x^{5} + x^{4} \\
& + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	395	[212, 620]	790	[600, 1015]	1185	[1325, 1025]
1	[3, 4]	396	[160, 32]	791	[606, 557]	1186	[1326, 512]
2	[5, 6]	397	[186, 231]	792	[624, 203]	1187	[1327, 951]
3	[7, 8]	398	[621, 580]	793	[1016, 513]	1188	[1328, 1329]

4	[9, 10]	399	[524, 182]	794	[228, 919]	1189	[1330, 968]
5	[11, 12]	400	[622, 623]	795	[1017, 1018]	1190	[1258, 477]
6	[13, 14]	401	[207, 606]	796	[545, 209]	1191	[433, 163]
7	[15, 16]	402	[520, 518]	797	[507, 1019]	1192	[1331, 1332]
8	[17, 18]	403	[523, 516]	798	[963, 213]	1193	[470, 150]
9	[19, 20]	404	[624, 625]	799	[1020, 1021]	1194	[167, 173]
10	[21, 22]	405	[530, 626]	800	[1022, 54]	1195	[1333, 1334]
11	[23, 24]	406	[537, 627]	801	[486, 605]	1196	[957, 915]
12	[25, 26]	407	[164, 628]	802	[1023, 911]	1197	[1335, 917]
13	[27, 28]	408	[629, 627]	803	[1024, 478]	1198	[936, 440]
14	[29, 30]	409	[191, 630]	804	[188, 968]	1199	[954, 516]
15	[31, 32]	410	[631, 632]	805	[229, 920]	1200	[1278, 1336]
16	[33, 34]	411	[633, 462]	806	[1025, 466]	1201	[550, 123]
17	[35, 36]	412	[634, 220]	807	[56, 588]	1202	[1337, 1035]
18	[37, 38]	413	[124, 573]	808	[493, 568]	1203	[56, 518]
19	[39, 40]	414	[210, 548]	809	[949, 570]	1204	[1266, 1294]
20	[41, 42]	415	[598, 509]	810	[904, 1026]	1205	[1271, 38]
21	[43, 44]	416	[33, 178]	811	[1027, 185]	1206	[1337, 1278]
22	[45, 46]	417	[553, 439]	812	[230, 187]	1207	[1338, 1045]
23	[47, 48]	418	[635, 636]	813	[1028, 493]	1208	[1311, 527]
24	[49, 50]	419	[637, 638]	814	[418, 225]	1209	[623, 147]
25	[51, 52]	420	[639, 365]	815	[1029, 1030]	1210	[1339, 524]
26	[53, 54]	421	[299, 640]	816	[430, 597]	1211	[501, 559]
27	[55, 56]	422	[112, 641]	817	[967, 617]	1212	[633, 509]
28	[57, 58]	423	[642, 643]	818	[970, 514]	1213	[1288, 420]
29	[59, 60]	424	[644, 645]	819	[552, 133]	1214	[451, 920]
30	[61, 46]	425	[380, 385]	820	[1031, 633]	1215	[915, 993]
31	[62, 63]	426	[646, 647]	821	[443, 575]	1216	[984, 995]
32	[64, 65]	427	[648, 649]	822	[918, 186]	1217	[1061, 958]
33	[66, 67]	428	[650, 645]	823	[1032, 1033]	1218	[916, 1329]
34	[68, 69]	429	[651, 652]	824	[614, 429]	1219	[1340, 1272]
35	[70, 71]	430	[314, 653]	825	[1034, 1035]	1220	[1341, 36]
36	[72, 73]	431	[654, 655]	826	[1036, 1037]	1221	[994, 985]
37	[74, 75]	432	[656, 657]	827	[938, 170]	1222	[1054, 42]
38	[76, 77]	433	[658, 395]	828	[139, 198]	1223	[1342, 1051]
39	[78, 79]	434	[659, 660]	829	[599, 466]	1224	[1052, 1060]
40	[80, 81]	435	[661, 643]	830	[1038, 599]	1225	[996, 948]
41	[82, 83]	436	[662, 663]	831	[576, 445]	1226	[1343, 508]
42	[84, 85]	437	[664, 665]	832	[1039, 452]	1227	[489, 527]
43	[86, 87]	438	[324, 666]	833	[550, 207]	1228	[1344, 970]
44	[88, 89]	439	[347, 68]	834	[1040, 616]	1229	[963, 620]
45	[90, 91]	440	[67, 360]	835	[1041, 550]	1230	[1314, 921]
46	[92, 93]	441	[667, 668]	836	[952, 454]	1231	[132, 419]
47	[94, 95]	442	[669, 670]	837	[1025, 600]	1232	[975, 973]

48	[96, 97]	443	[671, 672]	838	[445, 601]	1233	[563, 561]
49	[98, 99]	444	[673, 674]	839	[1040, 592]	1234	[1259, 187]
50	[100, 77]	445	[675, 371]	840	[1042, 630]	1235	[7, 476]
51	[101, 102]	446	[676, 677]	841	[1004, 952]	1236	[969, 455]
52	[103, 104]	447	[348, 678]	842	[1043, 37]	1237	[499, 1063]
53	[105, 106]	448	[679, 680]	843	[1044, 546]	1238	[1345, 1346]
54	[107, 73]	449	[414, 681]	844	[589, 1014]	1239	[1262, 155]
55	[108, 109]	450	[682, 683]	845	[1045, 206]	1240	[1322, 1270]
56	[110, 111]	451	[684, 685]	846	[144, 1019]	1241	[1347, 935]
57	[112, 113]	452	[686, 687]	847	[1046, 485]	1242	[523, 587]
58	[114, 115]	453	[688, 689]	848	[1012, 931]	1243	[1348, 45]
59	[116, 117]	454	[690, 69]	849	[192, 140]	1244	[1349, 167]
60	[118, 119]	455	[691, 692]	850	[1047, 142]	1245	[1350, 1351]
61	[120, 121]	456	[693, 694]	851	[1046, 568]	1246	[1329, 971]
62	[122, 123]	457	[695, 696]	852	[591, 217]	1247	[1309, 1300]
63	[124, 125]	458	[697, 698]	853	[534, 935]	1248	[941, 478]
64	[126, 127]	459	[699, 700]	854	[1048, 622]	1249	[138, 443]
65	[128, 129]	460	[701, 294]	855	[564, 139]	1250	[1048, 191]
66	[130, 131]	461	[702, 703]	856	[1049, 627]	1251	[1352, 1353]
67	[34, 132]	462	[704, 705]	857	[991, 1050]	1252	[917, 1304]
68	[133, 134]	463	[236, 706]	858	[594, 429]	1253	[1354, 1355]
69	[131, 135]	464	[707, 254]	859	[459, 612]	1254	[420, 187]
70	[136, 137]	465	[708, 96]	860	[506, 620]	1255	[1055, 219]
71	[138, 139]	466	[709, 710]	861	[965, 923]	1256	[870, 1356]
72	[140, 141]	467	[711, 712]	862	[980, 42]	1257	[342, 1233]
73	[142, 143]	468	[713, 401]	863	[997, 1051]	1258	[1357, 1358]
74	[144, 145]	469	[699, 714]	864	[447, 437]	1259	[822, 1359]
75	[146, 147]	470	[715, 716]	865	[32, 227]	1260	[783, 1360]
76	[148, 149]	471	[717, 718]	866	[1052, 1053]	1261	[1361, 847]
77	[150, 151]	472	[719, 720]	867	[1054, 150]	1262	[390, 1362]
78	[152, 153]	473	[25, 646]	868	[1007, 188]	1263	[116, 1363]
79	[154, 155]	474	[402, 638]	869	[519, 56]	1264	[1364, 883]
80	[156, 157]	475	[721, 722]	870	[1011, 419]	1265	[797, 749]
81	[158, 159]	476	[723, 306]	871	[585, 515]	1266	[275, 1365]
82	[160, 161]	477	[724, 725]	872	[1055, 503]	1267	[643, 1366]
83	[162, 163]	478	[726, 89]	873	[1015, 908]	1268	[1367, 1368]
84	[164, 165]	479	[661, 727]	874	[484, 568]	1269	[1162, 1369]
85	[166, 167]	480	[728, 729]	875	[12, 205]	1270	[1370, 867]
86	[168, 169]	481	[730, 731]	876	[544, 139]	1271	[300, 1371]
87	[53, 170]	482	[732, 730]	877	[1056, 195]	1272	[1211, 1368]
88	[171, 172]	483	[733, 6]	878	[1057, 1047]	1273	[1372, 1373]
89	[173, 174]	484	[337, 734]	879	[606, 464]	1274	[1374, 806]
90	[175, 176]	485	[104, 735]	880	[1058, 203]	1275	[1375, 1376]
91	[177, 178]	486	[736, 737]	881	[948, 203]	1276	[1158, 1377]

92	[179, 180]	487	[738, 346]	882	[1059, 1060]	1277	[74, 1378]
93	[181, 182]	488	[397, 375]	883	[1061, 915]	1278	[1379, 1380]
94	[183, 172]	489	[399, 739]	884	[1062, 1063]	1279	[103, 331]
95	[184, 185]	490	[724, 740]	885	[1064, 1065]	1280	[1381, 289]
96	[186, 187]	491	[741, 742]	886	[1027, 182]	1281	[1187, 1382]
97	[188, 189]	492	[743, 744]	887	[1066, 917]	1282	[121, 805]
98	[190, 191]	493	[102, 745]	888	[1059, 1053]	1283	[90, 1383]
99	[192, 193]	494	[746, 747]	889	[13, 537]	1284	[1384, 1385]
100	[194, 195]	495	[748, 749]	890	[1067, 1068]	1285	[1386, 280]
101	[196, 61]	496	[750, 751]	891	[470, 42]	1286	[376, 1387]
102	[197, 198]	497	[752, 753]	892	[981, 44]	1287	[824, 1388]
103	[31, 161]	498	[754, 755]	893	[621, 608]	1288	[16, 417]
104	[199, 159]	499	[756, 757]	894	[912, 443]	1289	[879, 1389]
105	[200, 201]	500	[758, 759]	895	[952, 527]	1290	[1390, 1391]
106	[202, 203]	501	[760, 761]	896	[1069, 476]	1291	[1392, 1393]
107	[204, 205]	502	[762, 763]	897	[473, 50]	1292	[893, 1394]
108	[206, 207]	503	[764, 403]	898	[1024, 1070]	1293	[1087, 1395]
109	[208, 209]	504	[262, 765]	899	[986, 982]	1294	[1174, 680]
110	[210, 211]	505	[766, 767]	900	[475, 8]	1295	[1232, 1203]
111	[212, 213]	506	[768, 769]	901	[1003, 460]	1296	[1225, 1396]
112	[214, 215]	507	[770, 771]	902	[1041, 206]	1297	[1397, 1398]
113	[216, 217]	508	[772, 773]	903	[171, 541]	1298	[844, 1399]
114	[218, 219]	509	[774, 355]	904	[775, 778]	1299	[1152, 1400]
115	[220, 221]	510	[775, 776]	905	[97, 248]	1300	[1401, 889]
116	[222, 223]	511	[777, 778]	906	[1071, 402]	1301	[1105, 1402]
117	[224, 225]	512	[779, 780]	907	[779, 1072]	1302	[1403, 404]
118	[226, 227]	513	[684, 286]	908	[95, 1073]	1303	[709, 1404]
119	[133, 33]	514	[781, 323]	909	[24, 640]	1304	[1405, 1406]
120	[228, 229]	515	[782, 783]	910	[803, 865]	1305	[1407, 862]
121	[230, 231]	516	[784, 785]	911	[1074, 1075]	1306	[1408, 679]
122	[232, 233]	517	[786, 787]	912	[769, 1076]	1307	[881, 1409]
123	[234, 235]	518	[788, 789]	913	[1077, 1078]	1308	[1410, 792]
124	[236, 237]	519	[790, 788]	914	[1079, 354]	1309	[1411, 1412]
125	[238, 239]	520	[343, 763]	915	[1080, 753]	1310	[1413, 1414]
126	[240, 241]	521	[121, 249]	916	[659, 1081]	1311	[1415, 1416]
127	[242, 65]	522	[756, 784]	917	[1082, 115]	1312	[29, 1417]
128	[243, 244]	523	[791, 720]	918	[651, 1083]	1313	[1418, 266]
129	[245, 246]	524	[786, 792]	919	[1084, 902]	1314	[1419, 1420]
130	[96, 247]	525	[793, 723]	920	[413, 1085]	1315	[1411, 1421]
131	[247, 248]	526	[794, 651]	921	[885, 1086]	1316	[1422, 374]
132	[249, 250]	527	[375, 334]	922	[1087, 1088]	1317	[1135, 1176]
133	[251, 252]	528	[795, 796]	923	[1089, 83]	1318	[353, 1423]
134	[253, 254]	529	[797, 798]	924	[772, 1090]	1319	[1132, 760]
135	[135, 135]	530	[799, 800]	925	[377, 1091]	1320	[1424, 663]

136	[255, 256]	531	[801, 789]	926	[1092, 1093]	1321	[871, 777]
137	[257, 258]	532	[83, 235]	927	[1094, 62]	1322	[1177, 386]
138	[249, 259]	533	[802, 644]	928	[1095, 1096]	1323	[1425, 655]
139	[260, 261]	534	[15, 803]	929	[702, 1097]	1324	[864, 1426]
140	[262, 263]	535	[416, 804]	930	[639, 1098]	1325	[1427, 871]
141	[264, 68]	536	[805, 250]	931	[804, 252]	1326	[897, 1122]
142	[265, 22]	537	[407, 806]	932	[1099, 415]	1327	[1428, 1429]
143	[266, 267]	538	[28, 807]	933	[1100, 1101]	1328	[1430, 1405]
144	[268, 269]	539	[64, 808]	934	[1102, 1076]	1329	[94, 305]
145	[270, 271]	540	[809, 636]	935	[315, 1103]	1330	[662, 1431]
146	[272, 273]	541	[810, 683]	936	[311, 637]	1331	[1432, 1433]
147	[274, 97]	542	[321, 383]	937	[883, 1104]	1332	[1434, 258]
148	[275, 276]	543	[287, 811]	938	[295, 1105]	1333	[1435, 1436]
149	[277, 278]	544	[812, 364]	939	[1106, 1107]	1334	[1437, 1102]
150	[279, 280]	545	[355, 813]	940	[1108, 1109]	1335	[1140, 1438]
151	[281, 282]	546	[687, 814]	941	[1110, 860]	1336	[1439, 745]
152	[283, 284]	547	[815, 108]	942	[894, 1111]	1337	[101, 1439]
153	[285, 286]	548	[816, 692]	943	[380, 1112]	1338	[1440, 1441]
154	[287, 288]	549	[82, 234]	944	[1113, 301]	1339	[20, 345]
155	[289, 290]	550	[817, 818]	945	[1114, 1115]	1340	[1147, 1442]
156	[291, 292]	551	[334, 119]	946	[1116, 381]	1341	[1443, 1444]
157	[293, 294]	552	[234, 819]	947	[697, 1085]	1342	[335, 1445]
158	[295, 296]	553	[381, 66]	948	[1117, 696]	1343	[1446, 1447]
159	[297, 298]	554	[820, 821]	949	[341, 1118]	1344	[1448, 682]
160	[23, 299]	555	[396, 374]	950	[1119, 652]	1345	[1449, 240]
161	[300, 301]	556	[822, 807]	951	[1120, 1121]	1346	[1450, 880]
162	[302, 303]	557	[282, 67]	952	[1122, 239]	1347	[721, 1451]
163	[304, 305]	558	[823, 824]	953	[1123, 746]	1348	[1452, 92]
164	[306, 307]	559	[825, 111]	954	[265, 1124]	1349	[1236, 324]
165	[308, 309]	560	[826, 827]	955	[1125, 1126]	1350	[1453, 1454]
166	[310, 311]	561	[828, 829]	956	[1127, 1097]	1351	[1455, 1145]
167	[312, 313]	562	[830, 828]	957	[768, 1102]	1352	[408, 1456]
168	[314, 315]	563	[831, 81]	958	[1128, 1129]	1353	[1457, 851]
169	[316, 317]	564	[67, 832]	959	[1130, 873]	1354	[1238, 1458]
170	[318, 319]	565	[284, 833]	960	[362, 1131]	1355	[1459, 1225]
171	[320, 321]	566	[834, 835]	961	[691, 1132]	1356	[1460, 33]
172	[322, 323]	567	[836, 837]	962	[1133, 1134]	1357	[462, 556]
173	[324, 267]	568	[838, 805]	963	[1135, 812]	1358	[1044, 209]
174	[325, 326]	569	[320, 382]	964	[1136, 1137]	1359	[1460, 134]
175	[327, 328]	570	[232, 687]	965	[291, 1138]	1360	[1008, 481]
176	[329, 330]	571	[839, 113]	966	[1139, 716]	1361	[1461, 991]
177	[331, 332]	572	[233, 840]	967	[1140, 1141]	1362	[1462, 606]
178	[333, 334]	573	[841, 739]	968	[1142, 278]	1363	[622, 630]
179	[335, 336]	574	[842, 17]	969	[1143, 350]	1364	[1463, 1280]

180	[337, 319]	575	[259, 347]	970	[1144, 1145]	1365	[556, 140]
181	[338, 283]	576	[754, 843]	971	[1146, 1147]	1366	[989, 1009]
182	[339, 340]	577	[717, 835]	972	[305, 1073]	1367	[1464, 480]
183	[341, 342]	578	[844, 845]	973	[1148, 1149]	1368	[1058, 625]
184	[343, 344]	579	[846, 847]	974	[736, 1150]	1369	[1006, 1050]
185	[345, 346]	580	[848, 746]	975	[105, 271]	1370	[1465, 148]
186	[250, 347]	581	[849, 850]	976	[812, 639]	1371	[958, 993]
187	[119, 309]	582	[366, 851]	977	[743, 1151]	1372	[1466, 1313]
188	[348, 349]	583	[846, 852]	978	[1152, 1153]	1373	[137, 958]
189	[350, 288]	584	[853, 664]	979	[1154, 1155]	1374	[1062, 959]
190	[351, 352]	585	[854, 855]	980	[1156, 1157]	1375	[1467, 1468]
191	[353, 354]	586	[828, 856]	981	[878, 1158]	1376	[1469, 523]
192	[250, 264]	587	[855, 400]	982	[1159, 1160]	1377	[1470, 1257]
193	[355, 356]	588	[109, 238]	983	[1161, 1159]	1378	[1045, 550]
194	[357, 358]	589	[857, 858]	984	[1162, 1163]	1379	[922, 966]
195	[359, 360]	590	[859, 860]	985	[1164, 317]	1380	[1317, 958]
196	[361, 362]	591	[758, 779]	986	[831, 1165]	1381	[51, 492]
197	[87, 363]	592	[861, 862]	987	[1166, 1167]	1382	[1276, 198]
198	[364, 365]	593	[863, 373]	988	[1146, 1168]	1383	[941, 1070]
199	[366, 367]	594	[88, 864]	989	[1169, 1170]	1384	[1471, 1472]
200	[368, 369]	595	[16, 865]	990	[1166, 1171]	1385	[41, 150]
201	[370, 371]	596	[802, 650]	991	[1172, 1173]	1386	[1342, 998]
202	[372, 373]	597	[866, 867]	992	[1174, 1175]	1387	[1312, 225]
203	[374, 375]	598	[868, 416]	993	[1176, 364]	1388	[926, 573]
204	[98, 376]	599	[869, 870]	994	[1177, 1178]	1389	[1279, 960]
205	[377, 378]	600	[871, 775]	995	[1179, 803]	1390	[1289, 950]
206	[379, 380]	601	[872, 873]	996	[1120, 1180]	1391	[1473, 428]
207	[381, 282]	602	[711, 874]	997	[70, 1181]	1392	[1474, 1029]
208	[382, 383]	603	[875, 783]	998	[1182, 1149]	1393	[479, 1295]
209	[384, 385]	604	[799, 876]	999	[1183, 1184]	1394	[1475, 908]
210	[386, 387]	605	[877, 340]	1000	[1185, 836]	1395	[1476, 1285]
211	[388, 349]	606	[233, 814]	1001	[1186, 686]	1396	[1280, 149]
212	[257, 389]	607	[878, 849]	1002	[653, 1103]	1397	[1057, 563]
213	[390, 391]	608	[879, 742]	1003	[1187, 1188]	1398	[1477, 1324]
214	[392, 393]	609	[410, 880]	1004	[251, 1189]	1399	[146, 1295]
215	[394, 395]	610	[675, 881]	1005	[1190, 1191]	1400	[460, 613]
216	[396, 397]	611	[417, 707]	1006	[1192, 1193]	1401	[1463, 148]
217	[398, 399]	612	[882, 883]	1007	[1194, 1142]	1402	[483, 1009]
218	[357, 400]	613	[884, 85]	1008	[1195, 1196]	1403	[990, 496]
219	[401, 326]	614	[885, 886]	1009	[1197, 900]	1404	[1478, 908]
220	[311, 402]	615	[820, 887]	1010	[794, 1119]	1405	[422, 223]
221	[313, 403]	616	[888, 889]	1011	[769, 1198]	1406	[52, 567]
222	[302, 404]	617	[272, 406]	1012	[382, 1199]	1407	[498, 1479]
223	[405, 406]	618	[890, 891]	1013	[1200, 1201]	1408	[1037, 14]

224	[407, 408]	619	[892, 837]	1014	[1202, 725]	1409	[597, 428]
225	[99, 409]	620	[893, 671]	1015	[870, 1203]	1410	[1480, 1481]
226	[410, 411]	621	[86, 894]	1016	[889, 781]	1411	[1482, 1483]
227	[63, 378]	622	[708, 274]	1017	[1204, 1205]	1412	[1484, 1050]
228	[412, 413]	623	[895, 731]	1018	[331, 735]	1413	[1485, 1298]
229	[414, 415]	624	[877, 896]	1019	[1206, 330]	1414	[939, 515]
230	[416, 251]	625	[759, 780]	1020	[1207, 1208]	1415	[222, 423]
231	[417, 253]	626	[897, 238]	1021	[243, 1155]	1416	[1486, 1257]
232	[49, 418]	627	[898, 804]	1022	[329, 1209]	1417	[1487, 910]
233	[184, 182]	628	[865, 253]	1023	[1210, 312]	1418	[1488, 1489]
234	[419, 420]	629	[100, 899]	1024	[1211, 1212]	1419	[554, 1021]
235	[161, 421]	630	[352, 900]	1025	[1190, 1213]	1420	[1490, 223]
236	[422, 423]	631	[4, 754]	1026	[1214, 135]	1421	[149, 1263]
237	[169, 424]	632	[901, 902]	1027	[110, 1215]	1422	[1269, 1323]
238	[207, 425]	633	[27, 822]	1028	[120, 838]	1423	[134, 34]
239	[50, 426]	634	[903, 325]	1029	[1216, 1217]	1424	[1491, 1492]
240	[427, 428]	635	[904, 905]	1030	[1218, 829]	1425	[1493, 1479]
241	[8, 429]	636	[142, 906]	1031	[1219, 1220]	1426	[172, 542]
242	[430, 431]	637	[907, 33]	1032	[1221, 1222]	1427	[1494, 609]
243	[432, 433]	638	[178, 132]	1033	[1223, 30]	1428	[1495, 602]
244	[134, 178]	639	[907, 134]	1034	[1185, 892]	1429	[1297, 36]
245	[434, 435]	640	[48, 908]	1035	[1224, 1225]	1430	[1000, 52]
246	[436, 437]	641	[567, 485]	1036	[1226, 1227]	1431	[442, 620]
247	[438, 132]	642	[204, 909]	1037	[868, 898]	1432	[1496, 1497]
248	[187, 439]	643	[424, 910]	1038	[1228, 709]	1433	[149, 155]
249	[130, 135]	644	[139, 575]	1039	[1144, 1229]	1434	[1293, 157]
250	[231, 133]	645	[911, 167]	1040	[1207, 1230]	1435	[1498, 1499]
251	[438, 440]	646	[912, 139]	1041	[715, 1231]	1436	[1303, 971]
252	[178, 440]	647	[52, 493]	1042	[869, 1232]	1437	[1354, 1489]
253	[34, 440]	648	[913, 914]	1043	[867, 76]	1438	[1022, 170]
254	[439, 33]	649	[137, 915]	1044	[1118, 1233]	1439	[1335, 1329]
255	[441, 442]	650	[916, 917]	1045	[1156, 1234]	1440	[1500, 582]
256	[443, 198]	651	[918, 420]	1046	[72, 1235]	1441	[1341, 180]
257	[444, 445]	652	[919, 920]	1047	[1236, 266]	1442	[1501, 149]
258	[446, 163]	653	[43, 921]	1048	[732, 895]	1443	[168, 534]
259	[226, 421]	654	[922, 923]	1049	[635, 3]	1444	[1282, 1056]
260	[447, 448]	655	[924, 14]	1050	[1237, 1129]	1445	[471, 603]
261	[418, 426]	656	[925, 535]	1051	[1204, 298]	1446	[1502, 1265]
262	[449, 450]	657	[926, 125]	1052	[1238, 1239]	1447	[1469, 515]
263	[50, 225]	658	[927, 127]	1053	[1240, 104]	1448	[1010, 424]
264	[451, 452]	659	[928, 929]	1054	[1116, 281]	1449	[1503, 906]
265	[453, 454]	660	[930, 140]	1055	[1131, 1241]	1450	[1491, 1029]
266	[162, 58]	661	[140, 931]	1056	[1131, 1242]	1451	[176, 973]
267	[455, 189]	662	[631, 932]	1057	[1243, 265]	1452	[1504, 181]

268	[456, 457]	663	[933, 934]	1058	[738, 1244]	1453	[1505, 532]
269	[458, 207]	664	[560, 154]	1059	[1245, 1246]	1454	[427, 1002]
270	[459, 460]	665	[935, 910]	1060	[1247, 744]	1455	[1480, 1032]
271	[461, 462]	666	[936, 132]	1061	[669, 1248]	1456	[1473, 1002]
272	[463, 464]	667	[937, 938]	1062	[854, 357]	1457	[1506, 1032]
273	[465, 466]	668	[939, 523]	1063	[903, 401]	1458	[991, 497]
274	[467, 468]	669	[940, 941]	1064	[826, 1249]	1459	[1465, 1280]
275	[469, 470]	670	[177, 34]	1065	[1100, 665]	1460	[397, 333]
276	[471, 472]	671	[513, 450]	1066	[1250, 1146]	1461	[255, 1507]
277	[473, 418]	672	[608, 942]	1067	[1251, 1252]	1462	[119, 1508]
278	[173, 221]	673	[213, 912]	1068	[1253, 898]	1463	[1143, 287]
279	[59, 474]	674	[943, 573]	1069	[316, 1254]	1464	[351, 1197]
280	[475, 476]	675	[927, 944]	1070	[1255, 1086]	1465	[1194, 277]
281	[477, 478]	676	[945, 632]	1071	[477, 1070]	1466	[1509, 1510]
282	[229, 452]	677	[946, 206]	1072	[1256, 131]	1467	[765, 1511]
283	[479, 147]	678	[202, 625]	1073	[172, 1257]	1468	[676, 1393]
284	[480, 481]	679	[569, 947]	1074	[1258, 941]	1469	[1512, 1513]
285	[482, 483]	680	[14, 627]	1075	[1259, 231]	1470	[84, 1453]
286	[484, 485]	681	[948, 625]	1076	[455, 968]	1471	[1514, 1515]
287	[486, 487]	682	[949, 947]	1077	[1260, 1261]	1472	[1516, 274]
288	[488, 489]	683	[424, 950]	1078	[1262, 1263]	1473	[1517, 1518]
289	[490, 491]	684	[558, 502]	1079	[156, 1264]	1474	[830, 1218]
290	[492, 493]	685	[951, 217]	1080	[221, 474]	1475	[135, 1214]
291	[494, 495]	686	[504, 546]	1081	[1265, 472]	1476	[1519, 1520]
292	[496, 497]	687	[512, 952]	1082	[1266, 1065]	1477	[637, 1521]
293	[498, 499]	688	[490, 953]	1083	[1256, 135]	1478	[247, 1522]
294	[500, 202]	689	[954, 587]	1084	[565, 201]	1479	[1106, 1523]
295	[501, 502]	690	[517, 588]	1085	[220, 174]	1480	[1095, 1524]
296	[442, 213]	691	[126, 944]	1086	[488, 952]	1481	[1525, 1526]
297	[218, 503]	692	[549, 206]	1087	[1267, 1268]	1482	[286, 1527]
298	[123, 425]	693	[955, 956]	1088	[917, 971]	1483	[1528, 1105]
299	[504, 209]	694	[957, 958]	1089	[1269, 1270]	1484	[1529, 1408]
300	[505, 215]	695	[499, 959]	1090	[176, 159]	1485	[1170, 1530]
301	[506, 213]	696	[45, 960]	1091	[1271, 540]	1486	[279, 1531]
302	[507, 145]	697	[604, 487]	1092	[1272, 953]	1487	[848, 1532]
303	[508, 509]	698	[961, 437]	1093	[1047, 561]	1488	[1245, 1114]
304	[510, 511]	699	[962, 963]	1094	[1273, 1274]	1489	[1533, 1380]
305	[512, 489]	700	[964, 44]	1095	[1275, 946]	1490	[658, 1534]
306	[513, 514]	701	[965, 966]	1096	[1276, 575]	1491	[1243, 21]
307	[515, 516]	702	[444, 577]	1097	[529, 601]	1492	[1535, 1536]
308	[517, 518]	703	[420, 231]	1098	[1277, 421]	1493	[673, 1537]
309	[439, 134]	704	[967, 435]	1099	[1034, 1278]	1494	[1538, 809]
310	[519, 520]	705	[181, 185]	1100	[1056, 536]	1495	[1373, 1539]
311	[521, 178]	706	[586, 516]	1101	[1279, 46]	1496	[1534, 1540]

312	[522, 523]	707	[583, 968]	1102	[199, 973]	1497	[1541, 1399]
313	[524, 185]	708	[969, 188]	1103	[526, 476]	1498	[808, 1542]
314	[525, 526]	709	[179, 36]	1104	[1280, 154]	1499	[1543, 874]
315	[453, 527]	710	[970, 450]	1105	[933, 993]	1500	[1544, 1545]
316	[528, 433]	711	[945, 932]	1106	[1281, 470]	1501	[1546, 116]
317	[529, 468]	712	[951, 129]	1107	[1018, 1056]	1502	[290, 1547]
318	[530, 531]	713	[971, 972]	1108	[46, 1282]	1503	[782, 1548]
319	[150, 532]	714	[489, 454]	1109	[943, 125]	1504	[1093, 339]
320	[533, 534]	715	[432, 57]	1110	[200, 566]	1505	[810, 1549]
321	[535, 140]	716	[158, 973]	1111	[169, 935]	1506	[1226, 1550]
322	[194, 536]	717	[214, 571]	1112	[1283, 493]	1507	[1066, 1329]
323	[537, 538]	718	[630, 147]	1113	[615, 592]	1508	[1551, 588]
324	[419, 186]	719	[974, 531]	1114	[1284, 1285]	1509	[1552, 1553]
325	[539, 540]	720	[561, 906]	1115	[122, 207]	1510	[1321, 971]
326	[541, 542]	721	[175, 975]	1116	[1001, 229]	1511	[1554, 570]
327	[543, 544]	722	[976, 419]	1117	[1286, 628]	1512	[928, 1037]
328	[545, 546]	723	[522, 515]	1118	[582, 170]	1513	[1555, 1324]
329	[547, 548]	724	[547, 211]	1119	[57, 163]	1514	[1556, 1557]
330	[549, 550]	725	[128, 217]	1120	[1287, 975]	1515	[1329, 1304]
331	[440, 551]	726	[971, 511]	1121	[1288, 186]	1516	[1488, 1355]
332	[552, 439]	727	[977, 46]	1122	[539, 38]	1517	[1067, 956]
333	[553, 133]	728	[978, 979]	1123	[12, 909]	1518	[555, 202]
334	[231, 439]	729	[980, 150]	1124	[1289, 910]	1519	[955, 1068]
335	[554, 555]	730	[981, 921]	1125	[1290, 1291]	1520	[929, 14]
336	[556, 193]	731	[166, 982]	1126	[1292, 1263]	1521	[1330, 189]
337	[463, 557]	732	[983, 166]	1127	[1293, 1264]	1522	[1277, 227]
338	[558, 559]	733	[984, 985]	1128	[1286, 165]	1523	[1558, 953]
339	[560, 149]	734	[986, 167]	1129	[526, 8]	1524	[1328, 917]
340	[561, 143]	735	[161, 227]	1130	[1064, 1294]	1525	[1315, 979]
341	[562, 563]	736	[987, 957]	1131	[579, 437]	1526	[924, 537]
342	[564, 443]	737	[930, 193]	1132	[623, 1295]	1527	[483, 481]
343	[565, 566]	738	[988, 905]	1133	[1296, 222]	1528	[1464, 483]
344	[567, 568]	739	[580, 942]	1134	[1297, 180]	1529	[1466, 40]
345	[569, 570]	740	[493, 485]	1135	[55, 520]	1530	[1559, 570]
346	[42, 532]	741	[974, 626]	1136	[136, 957]	1531	[1560, 143]
347	[135, 131]	742	[935, 950]	1137	[1298, 491]	1532	[1318, 600]
348	[505, 571]	743	[528, 57]	1138	[1292, 155]	1533	[1493, 499]
349	[461, 509]	744	[446, 58]	1139	[1299, 1300]	1534	[915, 934]
350	[572, 573]	745	[61, 960]	1140	[1281, 41]	1535	[978, 1316]
351	[574, 475]	746	[14, 538]	1141	[544, 443]	1536	[500, 948]
352	[197, 575]	747	[989, 481]	1142	[208, 546]	1537	[1561, 487]
353	[576, 577]	748	[990, 991]	1143	[217, 488]	1538	[1562, 142]
354	[578, 478]	749	[992, 993]	1144	[1013, 590]	1539	[1559, 947]
355	[579, 448]	750	[994, 995]	1145	[992, 934]	1540	[431, 428]

356	[580, 581]	751	[996, 202]	1146	[1272, 491]	1541	[1506, 1481]
357	[582, 54]	752	[997, 998]	1147	[191, 623]	1542	[431, 1002]
358	[224, 426]	753	[911, 982]	1148	[510, 972]	1543	[1474, 1492]
359	[583, 189]	754	[8, 595]	1149	[48, 904]	1544	[1283, 567]
360	[131, 131]	755	[999, 418]	1150	[1265, 603]	1545	[1554, 947]
361	[584, 579]	756	[1000, 492]	1151	[551, 186]	1546	[190, 622]
362	[585, 523]	757	[610, 603]	1152	[1301, 1302]	1547	[1501, 154]
363	[586, 587]	758	[1001, 919]	1153	[1303, 1304]	1548	[1349, 982]
364	[520, 588]	759	[975, 159]	1154	[1299, 1305]	1549	[977, 960]
365	[187, 133]	760	[597, 1002]	1155	[578, 1070]	1550	[964, 921]
366	[589, 590]	761	[206, 123]	1156	[1287, 176]	1551	[338, 1563]
367	[591, 129]	762	[1003, 612]	1157	[577, 468]	1552	[1368, 1564]
368	[592, 593]	763	[216, 129]	1158	[602, 472]	1553	[1565, 75]
369	[594, 595]	764	[1004, 489]	1159	[35, 180]	1554	[1566, 1567]
370	[596, 597]	765	[1005, 630]	1160	[436, 448]	1555	[334, 308]
371	[598, 462]	766	[1006, 497]	1161	[1306, 436]	1556	[73, 770]
372	[221, 60]	767	[458, 123]	1162	[1307, 1308]	1557	[1568, 283]
373	[599, 600]	768	[1007, 455]	1163	[154, 1263]	1558	[1110, 1569]
374	[467, 601]	769	[440, 419]	1164	[1309, 1305]	1559	[1570, 1571]
375	[32, 421]	770	[1008, 1009]	1165	[132, 551]	1560	[791, 1572]
376	[602, 603]	771	[476, 595]	1166	[1310, 591]	1561	[1573, 1574]
377	[604, 605]	772	[940, 477]	1167	[1311, 454]	1562	[310, 1071]
378	[123, 606]	773	[976, 551]	1168	[1312, 426]	1563	[946, 550]
379	[607, 608]	774	[607, 580]	1169	[39, 1313]	1564	[480, 1009]
380	[609, 38]	775	[609, 540]	1170	[1021, 202]	1565	[1461, 496]
381	[521, 34]	776	[1010, 935]	1171	[1314, 44]	1566	[987, 137]
382	[610, 472]	777	[611, 606]	1172	[1315, 1316]	1567	[1468, 223]
383	[534, 424]	778	[515, 587]	1173	[1317, 915]	1568	[482, 480]
384	[611, 425]	779	[1011, 551]	1174	[988, 1026]	1569	[958, 934]
385	[608, 581]	780	[919, 452]	1175	[1318, 466]	1570	[1020, 555]
386	[612, 613]	781	[148, 154]	1176	[445, 468]	1571	[1558, 491]
387	[614, 595]	782	[1012, 141]	1177	[1319, 1320]	1572	[1069, 8]
388	[615, 616]	783	[42, 151]	1178	[1321, 1304]	1573	[562, 1047]
389	[551, 420]	784	[535, 193]	1179	[1322, 1323]	1574	[1575, 1056]
390	[434, 617]	785	[492, 567]	1180	[433, 58]	1575	[66, 359]
391	[37, 540]	786	[1013, 1014]	1181	[938, 54]		
392	[618, 619]	787	[508, 462]	1182	[183, 541]		
393	[476, 429]	788	[572, 125]	1183	[469, 41]		
394	[596, 431]	789	[1015, 904]	1184	[595, 1324]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	263	0	526	0	789	1	1052	1	1315	0
1	0	264	1	527	1	790	0	1053	1	1316	1

2	0	265	1	528	1	791	0	1054	0	1317	0
3	0	266	0	529	0	792	0	1055	0	1318	1
4	1	267	1	530	1	793	0	1056	0	1319	1
5	0	268	0	531	1	794	0	1057	0	1320	0
6	0	269	1	532	0	795	1	1058	1	1321	0
7	0	270	1	533	0	796	0	1059	1	1322	1
8	1	271	1	534	0	797	0	1060	0	1323	1
9	1	272	0	535	0	798	0	1061	1	1324	0
10	0	273	0	536	1	799	1	1062	0	1325	0
11	0	274	1	537	1	800	1	1063	0	1326	0
12	0	275	1	538	0	801	1	1064	1	1327	1
13	0	276	0	539	1	802	0	1065	0	1328	0
14	0	277	0	540	0	803	1	1066	0	1329	0
15	0	278	1	541	1	804	0	1067	0	1330	1
16	0	279	0	542	0	805	1	1068	1	1331	0
17	1	280	1	543	1	806	1	1069	1	1332	0
18	0	281	1	544	0	807	1	1070	0	1333	0
19	1	282	1	545	0	808	0	1071	1	1334	1
20	0	283	1	546	1	809	0	1072	1	1335	1
21	0	284	0	547	1	810	1	1073	0	1336	1
22	1	285	0	548	0	811	1	1074	1	1337	0
23	0	286	0	549	0	812	0	1075	1	1338	1
24	0	287	1	550	1	813	0	1076	1	1339	0
25	0	288	0	551	0	814	1	1077	1	1340	1
26	0	289	1	552	1	815	1	1078	1	1341	0
27	0	290	1	553	0	816	0	1079	0	1342	1
28	0	291	0	554	1	817	1	1080	0	1343	1
29	0	292	1	555	0	818	0	1081	0	1344	0
30	0	293	1	556	1	819	1	1082	1	1345	1
31	0	294	0	557	1	820	1	1083	1	1346	0
32	1	295	1	558	1	821	1	1084	0	1347	1
33	0	296	1	559	0	822	1	1085	0	1348	0
34	1	297	0	560	1	823	1	1086	0	1349	0
35	1	298	1	561	0	824	1	1087	0	1350	1
36	1	299	1	562	0	825	0	1088	1	1351	1
37	0	300	0	563	1	826	1	1089	1	1352	0
38	1	301	0	564	1	827	1	1090	1	1353	1
39	1	302	0	565	0	828	0	1091	0	1354	1
40	0	303	1	566	1	829	0	1092	1	1355	0
41	0	304	1	567	1	830	0	1093	0	1356	0
42	1	305	1	568	1	831	1	1094	1	1357	1
43	0	306	1	569	0	832	0	1095	1	1358	0
44	0	307	1	570	0	833	1	1096	1	1359	0
45	1	308	0	571	1	834	1	1097	0	1360	0

46	1	309	0	572	0	835	1	1098	0	1361	1
47	0	310	0	573	1	836	1	1099	0	1362	1
48	0	311	0	574	0	837	1	1100	0	1363	0
49	0	312	0	575	1	838	1	1101	0	1364	0
50	0	313	0	576	1	839	1	1102	0	1365	1
51	0	314	0	577	0	840	0	1103	0	1366	1
52	0	315	1	578	0	841	1	1104	0	1367	0
53	0	316	1	579	0	842	0	1105	0	1368	1
54	0	317	0	580	1	843	0	1106	1	1369	0
55	0	318	1	581	0	844	0	1107	1	1370	0
56	1	319	0	582	0	845	1	1108	1	1371	1
57	0	320	0	583	0	846	0	1109	1	1372	1
58	0	321	0	584	0	847	1	1110	0	1373	1
59	0	322	0	585	0	848	1	1111	1	1374	0
60	1	323	1	586	0	849	0	1112	1	1375	1
61	0	324	1	587	1	850	0	1113	1	1376	1
62	0	325	1	588	1	851	1	1114	1	1377	1
63	0	326	1	589	0	852	0	1115	0	1378	1
64	1	327	1	590	1	853	0	1116	0	1379	0
65	1	328	0	591	0	854	0	1117	1	1380	0
66	0	329	1	592	0	855	1	1118	0	1381	0
67	1	330	0	593	1	856	1	1119	0	1382	1
68	1	331	1	594	0	857	0	1120	1	1383	0
69	1	332	1	595	0	858	0	1121	0	1384	1
70	1	333	0	596	0	859	1	1122	1	1385	0
71	0	334	0	597	1	860	0	1123	0	1386	1
72	1	335	1	598	0	861	0	1124	0	1387	0
73	1	336	1	599	0	862	1	1125	0	1388	1
74	0	337	0	600	0	863	1	1126	0	1389	0
75	0	338	1	601	0	864	0	1127	0	1390	0
76	1	339	1	602	1	865	1	1128	1	1391	0
77	0	340	0	603	0	866	1	1129	0	1392	0
78	1	341	0	604	1	867	0	1130	1	1393	1
79	1	342	1	605	0	868	0	1131	0	1394	0
80	0	343	0	606	0	869	0	1132	1	1395	0
81	1	344	1	607	0	870	1	1133	1	1396	0
82	0	345	0	608	0	871	0	1134	0	1397	0
83	0	346	1	609	1	872	0	1135	0	1398	1
84	1	347	0	610	1	873	1	1136	1	1399	0
85	0	348	0	611	0	874	0	1137	0	1400	0
86	0	349	1	612	1	875	0	1138	0	1401	0
87	0	350	0	613	0	876	0	1139	0	1402	1
88	0	351	0	614	1	877	0	1140	1	1403	1
89	1	352	0	615	1	878	0	1141	0	1404	1

90	1	353	1	616	1	879	0	1142	1	1405	0
91	1	354	0	617	0	880	1	1143	0	1406	0
92	1	355	0	618	0	881	1	1144	0	1407	1
93	1	356	1	619	0	882	1	1145	1	1408	0
94	0	357	0	620	0	883	1	1146	1	1409	1
95	0	358	1	621	0	884	0	1147	1	1410	1
96	0	359	0	622	0	885	1	1148	1	1411	0
97	0	360	1	623	1	886	1	1149	0	1412	1
98	0	361	0	624	0	887	0	1150	0	1413	1
99	0	362	0	625	0	888	1	1151	0	1414	1
100	0	363	0	626	0	889	0	1152	0	1415	0
101	0	364	0	627	1	890	0	1153	1	1416	0
102	0	365	1	628	1	891	1	1154	0	1417	1
103	0	366	0	629	0	892	0	1155	0	1418	1
104	0	367	0	630	0	893	0	1156	1	1419	1
105	0	368	0	631	1	894	1	1157	0	1420	1
106	0	369	0	632	1	895	1	1158	1	1421	0
107	0	370	0	633	0	896	1	1159	1	1422	1
108	0	371	0	634	0	897	0	1160	1	1423	1
109	1	372	0	635	1	898	1	1161	0	1424	0
110	1	373	0	636	1	899	1	1162	1	1425	1
111	1	374	1	637	1	900	1	1163	1	1426	0
112	0	375	1	638	0	901	1	1164	0	1427	0
113	0	376	1	639	1	902	1	1165	0	1428	1
114	0	377	1	640	0	903	0	1166	1	1429	0
115	0	378	1	641	1	904	1	1167	0	1430	0
116	0	379	0	642	0	905	0	1168	0	1431	1
117	1	380	1	643	0	906	1	1169	1	1432	0
118	1	381	0	644	0	907	1	1170	1	1433	0
119	1	382	1	645	1	908	0	1171	1	1434	0
120	0	383	0	646	1	909	0	1172	0	1435	0
121	0	384	0	647	0	910	1	1173	0	1436	1
122	0	385	0	648	1	911	1	1174	1	1437	1
123	1	386	1	649	1	912	1	1175	1	1438	1
124	0	387	1	650	1	913	1	1176	1	1439	1
125	0	388	1	651	1	914	0	1177	1	1440	1
126	1	389	0	652	0	915	0	1178	0	1441	0
127	0	390	1	653	0	916	1	1179	1	1442	0
128	1	391	0	654	0	917	1	1180	1	1443	0
129	1	392	0	655	1	918	1	1181	1	1444	0
130	0	393	0	656	1	919	0	1182	0	1445	0
131	1	394	0	657	1	920	0	1183	1	1446	1
132	0	395	1	658	1	921	1	1184	0	1447	1
133	1	396	0	659	1	922	0	1185	0	1448	0

134	1	397	0	660	1	923	1	1186	0	1449	1
135	0	398	0	661	1	924	1	1187	1	1450	0
136	1	399	0	662	1	925	1	1188	0	1451	1
137	1	400	0	663	0	926	1	1189	1	1452	0
138	0	401	0	664	1	927	1	1190	1	1453	1
139	0	402	0	665	1	928	1	1191	1	1454	1
140	1	403	0	666	0	929	1	1192	0	1455	1
141	1	404	0	667	1	930	1	1193	1	1456	0
142	1	405	1	668	1	931	0	1194	0	1457	1
143	0	406	1	669	1	932	0	1195	0	1458	0
144	0	407	1	670	1	933	0	1196	0	1459	0
145	1	408	0	671	1	934	0	1197	1	1460	0
146	0	409	1	672	0	935	1	1198	0	1461	1
147	1	410	1	673	1	936	0	1199	1	1462	1
148	1	411	0	674	1	937	1	1200	0	1463	0
149	0	412	0	675	1	938	1	1201	1	1464	0
150	0	413	0	676	1	939	1	1202	0	1465	0
151	1	414	1	677	0	940	1	1203	1	1466	1
152	1	415	0	678	0	941	0	1204	1	1467	1
153	0	416	0	679	0	942	1	1205	0	1468	1
154	1	417	0	680	0	943	1	1206	0	1469	1
155	1	418	1	681	1	944	1	1207	1	1470	1
156	0	419	1	682	0	945	1	1208	0	1471	1
157	1	420	1	683	0	946	0	1209	1	1472	1
158	1	421	1	684	1	947	1	1210	0	1473	0
159	0	422	0	685	1	948	1	1211	1	1474	0
160	0	423	0	686	1	949	0	1212	0	1475	0
161	0	424	0	687	1	950	0	1213	0	1476	0
162	0	425	1	688	1	951	1	1214	1	1477	1
163	1	426	1	689	1	952	1	1215	0	1478	1
164	1	427	1	690	0	953	0	1216	1	1479	1
165	0	428	1	691	1	954	1	1217	1	1480	1
166	0	429	1	692	0	955	0	1218	1	1481	0
167	0	430	0	693	0	956	0	1219	1	1482	0
168	0	431	0	694	0	957	0	1220	0	1483	0
169	1	432	1	695	0	958	1	1221	1	1484	1
170	1	433	1	696	1	959	1	1222	0	1485	1
171	0	434	1	697	1	960	0	1223	1	1486	0
172	0	435	1	698	1	961	1	1224	1	1487	1
173	1	436	1	699	1	962	1	1225	1	1488	1
174	1	437	1	700	1	963	0	1226	1	1489	1
175	1	438	1	701	0	964	1	1227	0	1490	1
176	1	439	0	702	1	965	0	1228	0	1491	0
177	1	440	1	703	1	966	0	1229	0	1492	0

178	0	441	1	704	1	967	1	1230	1	1493	1
179	1	442	1	705	1	968	1	1231	0	1494	0
180	0	443	1	706	0	969	0	1232	0	1495	1
181	1	444	1	707	0	970	0	1233	1	1496	0
182	1	445	1	708	0	971	1	1234	1	1497	1
183	0	446	1	709	1	972	1	1235	0	1498	0
184	0	447	0	710	0	973	1	1236	0	1499	0
185	0	448	0	711	1	974	1	1237	0	1500	1
186	0	449	1	712	1	975	0	1238	1	1501	0
187	1	450	0	713	1	976	0	1239	1	1502	1
188	0	451	1	714	0	977	1	1240	1	1503	1
189	0	452	1	715	1	978	0	1241	1	1504	0
190	0	453	1	716	1	979	0	1242	0	1505	1
191	1	454	0	717	0	980	1	1243	0	1506	1
192	0	455	1	718	0	981	0	1244	0	1507	0
193	0	456	0	719	1	982	1	1245	1	1508	1
194	0	457	0	720	0	983	0	1246	0	1509	1
195	0	458	1	721	1	984	1	1247	0	1510	0
196	0	459	1	722	0	985	0	1248	0	1511	1
197	0	460	0	723	0	986	1	1249	0	1512	1
198	0	461	1	724	1	987	1	1250	0	1513	0
199	0	462	1	725	1	988	1	1251	0	1514	1
200	0	463	0	726	1	989	1	1252	1	1515	0
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202	0	465	0	728	0	991	0	1254	1	1517	0
203	1	466	1	729	1	992	1	1255	0	1518	0
204	0	467	1	730	0	993	1	1256	1	1519	0
205	1	468	1	731	0	994	1	1257	1	1520	1
206	0	469	1	732	0	995	1	1258	1	1521	1
207	0	470	1	733	1	996	1	1259	1	1522	0
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209	0	472	1	735	0	998	0	1261	1	1524	0
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211	1	474	0	737	1	1000	0	1263	0	1526	1
212	1	475	1	738	1	1001	0	1264	0	1527	1
213	1	476	0	739	1	1002	0	1265	0	1528	0
214	0	477	1	740	0	1003	1	1266	1	1529	1
215	0	478	1	741	1	1004	1	1267	0	1530	1
216	0	479	1	742	1	1005	1	1268	0	1531	0
217	0	480	0	743	1	1006	0	1269	1	1532	1
218	0	481	0	744	1	1007	0	1270	0	1533	1
219	0	482	0	745	0	1008	0	1271	0	1534	0
220	0	483	1	746	0	1009	1	1272	1	1535	0
221	0	484	0	747	1	1010	0	1273	1	1536	0

222	0	485	0	748	1	1011	1	1274	0	1537	0
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225	0	488	0	751	1	1014	0	1277	0	1540	0
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227	0	490	1	753	1	1016	0	1279	0	1542	0
228	0	491	1	754	1	1017	1	1280	0	1543	0
229	1	492	1	755	1	1018	1	1281	1	1544	1
230	0	493	0	756	0	1019	0	1282	0	1545	1
231	0	494	0	757	1	1020	1	1283	1	1546	0
232	0	495	1	758	0	1021	1	1284	1	1547	0
233	0	496	1	759	0	1022	1	1285	1	1548	0
234	1	497	1	760	1	1023	0	1286	1	1549	1
235	0	498	1	761	0	1024	1	1287	1	1550	1
236	0	499	0	762	1	1025	1	1288	0	1551	1
237	1	500	0	763	0	1026	1	1289	0	1552	1
238	0	501	1	764	1	1027	1	1290	0	1553	1
239	0	502	1	765	1	1028	0	1291	0	1554	1
240	1	503	1	766	0	1029	1	1292	0	1555	0
241	1	504	1	767	1	1030	1	1293	0	1556	1
242	0	505	0	768	0	1031	1	1294	1	1557	0
243	1	506	0	769	1	1032	1	1295	0	1558	0
244	1	507	0	770	0	1033	1	1296	1	1559	1
245	1	508	1	771	0	1034	0	1297	0	1560	0
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247	1	510	1	773	0	1036	1	1299	0	1562	0
248	1	511	0	774	0	1037	0	1300	0	1563	0
249	0	512	1	775	1	1038	0	1301	0	1564	0
250	0	513	1	776	0	1039	0	1302	1	1565	1
251	1	514	1	777	0	1040	1	1303	1	1566	1
252	0	515	1	778	1	1041	1	1304	0	1567	1
253	1	516	0	779	1	1042	0	1305	1	1568	0
254	0	517	0	780	0	1043	0	1306	0	1569	1
255	1	518	0	781	1	1044	0	1307	1	1570	1
256	1	519	0	782	1	1045	1	1308	1	1571	0
257	1	520	0	783	1	1046	1	1309	0	1572	1
258	1	521	0	784	0	1047	0	1310	1	1573	0
259	1	522	0	785	1	1048	0	1311	0	1574	0
260	0	523	0	786	0	1049	1	1312	0	1575	0
261	1	524	0	787	1	1050	0	1313	1		
262	1	525	0	788	0	1051	1	1314	1		

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