PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^5+z^4+z^2+z,z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^5+z^4+z^2+z,z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z^5+z^4+z^2+z, z+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z) &= z^{14} + z^{11} + z^5 + z^4 + z^2 + z + 1,\\
p_1(z) &= z^{19} + z^{18} + z^{15} + z^{14} + z^{12} + z^{11} + z^7 + z^4 + z^3 + z,\\
p_2(z) &= z^{20} + z^{18} + z^{10} + z^6 + z^4 + z^2,\\
p_3(z) &= z^{19} + z^{18} + z^{15} + z^{14} + z^{12} + z^{11} + z^7 + z^4 + z^3 + z,\\
p_4(z) &= z^{18} + z^{14} + z^{11} + z^{10} + z^8 + z^6 + z^5 + z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z) &= z^{16} + z^{14} + z^{11} + z^5 + z^2 + z,\\
p_1(z) &= z^{19} + z^{18} + z^{15} + z^{14} + z^{12} + z^{11} + z^7 + z^4 + z^3 + z,\\
p_2(z) &= z^{20} + z^{18} + z^{10} + z^6 + z^4 + z^2,\\
p_3(z) &= z^{19} + z^{18} + z^{15} + z^{14} + z^{12} + z^{11} + z^7 + z^4 + z^3 + z,\\
p_4(z) &= z^{18} + z^{16} + z^{14} + z^{11} + z^{10} + z^8 + z^6 + z^5 + z^4 + z + 1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) = {} & x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix} \times\\
& \cdots \times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) = {} & x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix} \times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix} \times\\
& \cdots \times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$
$$M^o = W^e \cdot M^e, \tag{2.4}$$
$$W^e = M^o \cdot W^o, \tag{2.5}$$
$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{3u} M^e[:6u] + x^{6u} M^e[:3u] \\ & + x^{7u} M^e[:2u] + x^{4u} M^e[:6u] + x^{7u} M^e[:3u] + x^{8u} M^e[:2u] \\ & + x^{6u} M^e[:6u] + x^{7u} M^e[:5u] + x^{11u} M^e[:u] \\ & + (x^{6u} + x^{9u} + x^{10u}) R^e, \\ W^e[:12u] = {} & W^e[:6u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{3u} W^e[:6u] + x^{6u} W^e[:3u] \\ & + x^{7u} W^e[:2u] + x^{4u} W^e[:6u] + x^{7u} W^e[:3u] + x^{8u} W^e[:2u] \end{aligned}$$

$$+ x^{6u}W^e[:6u] + x^{7u}W^e[:5u] + x^{11u}W^e[:u]$$
$$+ (x^{6u} + x^{9u} + x^{10u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned} M^o[:12u] &= M^o[:6u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u] \\ &+ x^{7u}M^o[:2u] + x^{4u}M^o[:6u] + x^{7u}M^o[:3u] + x^{8u}M^o[:2u] \\ &+ x^{6u}M^o[:6u] + x^{7u}M^o[:5u] + x^{11u}M^o[:u] \\ &+ (x^{6u} + x^{9u} + x^{10u})R^o, \\ W^o[:12u] &= W^o[:6u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u] \\ &+ x^{7u}W^o[:2u] + x^{4u}W^o[:6u] + x^{7u}W^o[:3u] + x^{8u}W^o[:2u] \\ &+ x^{6u}W^o[:6u] + x^{7u}W^o[:5u] + x^{11u}W^o[:u] \\ &+ (x^{6u} + x^{9u} + x^{10u})R^o. \end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned} T[:12u] &= T[:6u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{7u}T[:2u] \\ &+ x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u] + x^{6u}T[:6u] + x^{7u}T[:5u] \\ &+ x^{11u}T[:u]. \end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned} 0 &= x^{4u}T[2u:3u] + x^{3u}T[3u:4u] + T[6u:7u], \\ 0 &= x^{7u}T[:u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + T[7u:8u], \\ 0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u], \\ 0 &= x^{8u}T[u:2u] + x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u], \\ 0 &= x^{7u}T[3u:4u] + x^{6u}T[4u:5u] + T[10u:11u], \\ 0 &= x^{11u}T[:u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u] + T[11u:12u], \end{aligned}$$

which can be rewritten as

$$0 = T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[11w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[11w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1128 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.19}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.20}$$
$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1000)), \end{aligned} \tag{2.21}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.23}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.25}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.26}$$
$$\begin{aligned} 0 = {} & \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ & + \tau(A(s, 1000)), \end{aligned} \tag{2.27}$$
$$0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$
$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.29}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{6u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u} M^o[:3u] + x^{4u} M^o[:2u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u} W^o[:3u] + x^{4u} W^o[:2u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = & \big(W^e[:6v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{6u}R^e\big) \times \big(M^e[:6v] \\ & + x^{3v}M^e[:3v] + x^{4v}M^e[:2v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v] + x^{8v}W^e[:4v]M^e[:4v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^9 + x^6,\\
q_1(x) &= x^{19} + x^{17} + x^{16} + x^{13} + x^9 + x^8 + x^6 + x^5 + x^2 + x,\\
q_2(x) &= x^{18} + x^{16} + x^{14} + x^{10} + x^2 + 1,\\
q_3(x) &= x^{19} + x^{17} + x^{16} + x^{13} + x^9 + x^8 + x^6 + x^5 + x^2 + x,\\
q_4(x) &= x^{19} + x^{15} + x^{14} + x^{12} + x^{10} + x^9 + x^6 + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{90} + x^{88} + x^{86} + x^{78} + x^{76} + x^{74} + x^{72} + x^{64} + x^{60} + x^{56}\\
&\quad + x^{48} + x^{40} + x^{36} + x^{20} + x^{18} + x^{12},\\
p_3(x) &= x^{92} + x^{88} + x^{86} + x^{84} + x^{72} + x^{70} + x^{58} + x^{54} + x^{52} + x^{50} + x^{46}\\
&\quad + x^{42} + x^{38} + x^{34} + x^{32} + x^{28} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^8,\\
p_6(x) &= x^{86} + x^{84} + x^{82} + x^{74} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{50}\\
&\quad + x^{48} + x^{40} + x^{32} + x^{26} + x^{24} + x^{22} + x^{20} + x^{10} + x^8 + x^2 + 1,\\
p_9(x) &= x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{58} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} + x^{36} + x^{26} + x^{22} + x^{18}\\
&\quad + x^{16} + x^{12} + x^8 + x^2,\\
p_{12}(x) &= x^{94} + x^{88} + x^{86} + x^{80} + x^{62} + x^{56} + x^{54} + x^{48} + x^{14} + x^8 + x^6 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{104} + x^{101} + x^{98} + x^{97} + x^{95} + x^{90} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{85} + x^{80} + x^{79} + x^{78} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{33} + x^{31} + x^{28} + x^{24}, \\
p_3(x) &= x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} \\
&\quad + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{45} + x^{44} + x^{42} + x^{41} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} \\
&\quad + x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^{9}, \\
p_6(x) &= x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{85} \\
&\quad + x^{83} + x^{82} + x^{80} + x^{77} + x^{74} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{29} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} \\
&\quad + x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) &= x^{104} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{79} + x^{77} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} \\
&\quad + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{46} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{26} + x^{25} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5} \\
&\quad + x^{4} + x^{3}, \\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{71} + x^{68} + x^{63} \\
&\quad + x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8} \\
&\quad + x^{3} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{104} + x^{101} + x^{98} + x^{97} + x^{95} + x^{90} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{85} + x^{80} + x^{79} + x^{78} + x^{71} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{33} + x^{31} + x^{28} + x^{24}, \\
p_3(x) &= x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} \\
&\quad + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{51} + x^{49} + x^{48} \\
&+ x^{45} + x^{44} + x^{42} + x^{41} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} \\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^{9}, \\
p_6(x) &= x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{85} \\
&+ x^{83} + x^{82} + x^{80} + x^{77} + x^{74} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} \\
&+ x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{29} + x^{27} + x^{26} + x^{25} \\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} \\
&+ x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) &= x^{104} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{84} + x^{83} \\
&+ x^{82} + x^{81} + x^{79} + x^{77} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} \\
&+ x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{46} + x^{45} \\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{29} + x^{28} \\
&+ x^{26} + x^{25} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5} \\
&+ x^{4} + x^{3}, \\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{71} + x^{68} + x^{63} \\
&+ x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8} \\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{106} + x^{88} + x^{84} + x^{82} + x^{78} + x^{74} + x^{68} \\
&+ x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{48} + x^{42} + x^{36}, \\
p_3(x) &= x^{114} + x^{108} + x^{98} + x^{94} + x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{70} + x^{66} \\
&+ x^{64} + x^{54} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{28} \\
&+ x^{26} + x^{22} + x^{20} + x^{18} + x^{14} + x^{6}, \\
p_6(x) &= x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{96} + x^{94} + x^{90} + x^{88} + x^{82} + x^{78} \\
&+ x^{74} + x^{66} + x^{62} + x^{58} + x^{56} + x^{52} + x^{44} + x^{42} + x^{40} + x^{36} + x^{30} \\
&+ x^{28} + x^{26} + x^{14} + x^{12} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{90} + x^{84} \\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{60} + x^{56} + x^{52} + x^{50} \\
&+ x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{32} + x^{16} + x^{12} + x^{2}, \\
p_{12}(x) &= x^{112} + x^{104} + x^{88} + x^{72} + x^{56} + x^{48} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
p_0(x) = x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100}
$$

$$
\begin{aligned}
&+ x^{96} + x^{94} + x^{92} + x^{90} + x^{84} + x^{82} + x^{80} + x^{78} + x^{70} + x^{64} + x^{62} \\
&+ x^{58} + x^{54} + x^{52} + x^{48} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{28} + x^{26} \\
&+ x^{24} + x^{16} + x^{12}, \\
p_3(x) &= x^{114} + x^{110} + x^{108} + x^{104} + x^{102} + x^{100} + x^{92} + x^{90} + x^{86} + x^{82} \\
&+ x^{78} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} + x^{56} + x^{52} + x^{50} + x^{44} + x^{42} \\
&+ x^{38} + x^{28} + x^{18} + x^{14} + x^{8}, \\
p_6(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{92} + x^{88} + x^{86} \\
&+ x^{80} + x^{74} + x^{68} + x^{60} + x^{58} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{34} \\
&+ x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{16} + x^{14} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{90} + x^{84} \\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{60} + x^{56} + x^{52} + x^{50} \\
&+ x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{32} + x^{16} + x^{12} + x^{2}, \\
p_{12}(x) &= x^{112} + x^{104} + x^{88} + x^{72} + x^{56} + x^{48} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].
For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} \\
&+ x^{90} + x^{88} + x^{87} + x^{84} + x^{81} + x^{75} + x^{73} + x^{72} + x^{68} + x^{66} + x^{65} \\
&+ x^{63} + x^{58} + x^{57} + x^{54} + x^{50} + x^{48} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} \\
&+ x^{39} + x^{34} + x^{30} + x^{28} + x^{27} + x^{24} + x^{23} + x^{21} + x^{18}, \\
p_3(x) &= x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87} \\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} \\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{51} + x^{49} + x^{48} \\
&+ x^{45} + x^{44} + x^{42} + x^{41} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} \\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^{9}, \\
p_6(x) &= x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{85} \\
&+ x^{83} + x^{82} + x^{80} + x^{77} + x^{74} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} \\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} \\
&+ x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{29} + x^{27} + x^{26} + x^{25} \\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} \\
&+ x^{11} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) &= x^{104} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{84} + x^{83} \\
&+ x^{82} + x^{81} + x^{79} + x^{77} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} \\
&+ x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{46} + x^{45} \\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{29} + x^{28}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{26} + x^{25} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5}\\
&+ x^{4} + x^{3},\\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{71} + x^{68} + x^{63}\\
&+ x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94}\\
&+ x^{90} + x^{88} + x^{87} + x^{84} + x^{81} + x^{75} + x^{73} + x^{72} + x^{68} + x^{66} + x^{65}\\
&+ x^{63} + x^{58} + x^{57} + x^{54} + x^{50} + x^{48} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{34} + x^{30} + x^{28} + x^{27} + x^{24} + x^{23} + x^{21} + x^{18},\\
p_3(x) &= x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{95} + x^{94} + x^{91} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76}\\
&+ x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{55} + x^{51} + x^{49} + x^{48}\\
&+ x^{45} + x^{44} + x^{42} + x^{41} + x^{34} + x^{32} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23}\\
&+ x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12} + x^{10} + x^{9},\\
p_6(x) &= x^{105} + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{88} + x^{85}\\
&+ x^{83} + x^{82} + x^{80} + x^{77} + x^{74} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{46} + x^{44} + x^{43} + x^{40} + x^{39} + x^{36} + x^{29} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12}\\
&+ x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{104} + x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{91} + x^{90} + x^{87} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{79} + x^{77} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64}\\
&+ x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{29} + x^{28}\\
&+ x^{26} + x^{25} + x^{20} + x^{18} + x^{15} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5}\\
&+ x^{4} + x^{3},\\
p_{12}(x) &= x^{103} + x^{100} + x^{95} + x^{92} + x^{91} + x^{88} + x^{83} + x^{80} + x^{71} + x^{68} + x^{63}\\
&+ x^{60} + x^{59} + x^{56} + x^{51} + x^{48} + x^{23} + x^{20} + x^{15} + x^{12} + x^{11} + x^{8}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
p_0(x) = x^{96} + x^{94} + x^{92} + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64}
$$

$$
\begin{aligned}
&+x^{60}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{40}+x^{38}\\
&+x^{36}+x^{32}+x^{30}+x^{28}+x^{24},\\
p_3(x) &= x^{88}+x^{86}+x^{82}+x^{78}+x^{74}+x^{70}+x^{66}+x^{56}+x^{50}+x^{48}+x^{46}\\
&+x^{38}+x^{32}+x^{30}+x^{22}+x^{18}+x^{10}+x^{6},\\
p_6(x) &= x^{92}+x^{90}+x^{86}+x^{84}+x^{74}+x^{66}+x^{64}+x^{62}+x^{58}+x^{56}+x^{54}\\
&+x^{46}+x^{44}+x^{36}+x^{32}+x^{22}+x^{20}+x^{18}+x^{16}+x^{8}+x^{6}+x^{4}+x^{2}\\
&+1,\\
p_9(x) &= x^{94}+x^{92}+x^{90}+x^{82}+x^{80}+x^{78}+x^{76}+x^{74}+x^{72}+x^{66}+x^{64}\\
&+x^{62}+x^{58}+x^{54}+x^{50}+x^{48}+x^{46}+x^{42}+x^{36}+x^{26}+x^{22}+x^{18}\\
&+x^{16}+x^{12}+x^{8}+x^{2},\\
p_{12}(x) &= x^{94}+x^{88}+x^{86}+x^{80}+x^{62}+x^{56}+x^{54}+x^{48}+x^{14}+x^{8}+x^{6}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108}+x^{107}+x^{106}+x^{104}+x^{102}+x^{98}+x^{97}+x^{94}+x^{91}+x^{90}+x^{89}\\
&+x^{86}+x^{85}+x^{81}+x^{80}+x^{79}+x^{76}+x^{75}+x^{71}+x^{69}+x^{67}+x^{65}\\
&+x^{62}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{46}+x^{43}\\
&+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}\\
&+x^{23}+x^{21}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12},\\
p_3(x) &= x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{99}+x^{98}+x^{92}\\
&+x^{91}+x^{90}+x^{86}+x^{85}+x^{84}+x^{83}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}\\
&+x^{68}+x^{66}+x^{64}+x^{63}+x^{62}+x^{61}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}\\
&+x^{52}+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{38}+x^{34}+x^{31}+x^{27}\\
&+x^{26}+x^{25}+x^{22}+x^{21}+x^{17}+x^{13}+x^{11}+x^{10}+x^{8},\\
p_6(x) &= x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{97}+x^{94}+x^{91}+x^{89}+x^{88}+x^{87}\\
&+x^{85}+x^{83}+x^{81}+x^{80}+x^{77}+x^{76}+x^{73}+x^{71}+x^{69}+x^{68}+x^{67}\\
&+x^{62}+x^{60}+x^{57}+x^{54}+x^{53}+x^{48}+x^{46}+x^{44}+x^{42}+x^{41}+x^{39}\\
&+x^{38}+x^{35}+x^{34}+x^{32}+x^{30}+x^{26}+x^{25}+x^{24}+x^{23}+x^{21}+x^{18}\\
&+x^{17}+x^{15}+x^{13}+x^{12}+x^{11}+x^{9}+x^{8}+x^{7}+x^{6}+x^{4}+x^{3}+1,\\
p_9(x) &= x^{107}+x^{104}+x^{99}+x^{96}+x^{91}+x^{88}+x^{87}+x^{84}+x^{75}+x^{72}+x^{67}\\
&+x^{64}+x^{59}+x^{56}+x^{55}+x^{52}+x^{27}+x^{24}+x^{19}+x^{16}+x^{11}+x^{8}\\
&+x^{7}+x^{4},\\
p_{12}(x) &= x^{107}+x^{104}+x^{103}+x^{100}+x^{99}+x^{96}+x^{95}+x^{92}+x^{87}+x^{84}+x^{83}\\
&+x^{80}+x^{75}+x^{72}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}+x^{60}+x^{55}+x^{52}\\
&+x^{51}+x^{48}+x^{27}+x^{24}+x^{23}+x^{20}+x^{19}+x^{16}+x^{15}+x^{12}+x^{7}\\
&+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{117} + x^{114} + x^{112} + x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{92} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{68} + x^{65} + x^{64} + x^{61} + x^{60} + x^{55} + x^{54} + x^{49} \\
&\quad + x^{48} + x^{45} + x^{44} + x^{38} + x^{37} + x^{33} + x^{31} + x^{29} + x^{27} + x^{26} + x^{24}, \\
p_3(x) &= x^{113} + x^{110} + x^{109} + x^{108} + x^{104} + x^{103} + x^{102} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{92} + x^{91} + x^{90} + x^{89} + x^{65} + x^{62} + x^{61} + x^{60} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{42} + x^{38} + x^{37} + x^{36} + x^{33} + x^{32} \\
&\quad + x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{16} + x^{12} \\
&\quad + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88} \\
&\quad + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{62} + x^{58} + x^{54} + x^{46} \\
&\quad + x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{28} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} \\
&\quad + x^{8} + x^{6}, \\
p_9(x) &= x^{115} + x^{112} + x^{111} + x^{110} + x^{109} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} \\
&\quad + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{84} + x^{80} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58} \\
&\quad + x^{52} + x^{51} + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{22} + x^{20} + x^{19} \\
&\quad + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{4} + x^{3}, \\
p_{12}(x) &= x^{116} + x^{112} + x^{108} + x^{100} + x^{76} + x^{68} + x^{52} + x^{48} + x^{36} + x^{32} + x^{28} \\
&\quad + x^{20} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{100} + x^{99} + x^{94} + x^{93} + x^{90} + x^{87} + x^{86} + x^{78} + x^{77} + x^{72} + x^{67} \\
&\quad + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27} + x^{26} \\
&\quad + x^{23} + x^{18}, \\
p_3(x) &= x^{99} + x^{97} + x^{96} + x^{90} + x^{89} + x^{51} + x^{49} + x^{48} + x^{42} + x^{41} + x^{27} \\
&\quad + x^{25} + x^{24} + x^{19} + x^{18} + x^{16} + x^{10} + x^{9}, \\
p_6(x) &= x^{100} + x^{98} + x^{94} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{64} + x^{60} \\
&\quad + x^{54} + x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{24} + x^{20} + x^{14} + x^{10} + x^{6}, \\
p_9(x) &= x^{101} + x^{99} + x^{98} + x^{95} + x^{93} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{69} + x^{67} + x^{66} + x^{63} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{21} + x^{19} + x^{18} + x^{15} + x^{13} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5}
\end{aligned}$$

$$
\begin{aligned}
&+x^{4}+x^{3},\\
p_{12}(x) &= x^{102}+x^{100}+x^{94}+x^{92}+x^{82}+x^{80}+x^{70}+x^{68}+x^{62}+x^{60}+x^{50}\\
&\quad+x^{48}+x^{22}+x^{20}+x^{14}+x^{12}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(M^o,1,1)$:

$$
\begin{aligned}
p_0(x) &= x^{108}+x^{103}+x^{99}+x^{96}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{87}\\
&\quad+x^{86}+x^{85}+x^{84}+x^{83}+x^{77}+x^{76}+x^{74}+x^{73}+x^{72}+x^{71}+x^{67}\\
&\quad+x^{65}+x^{64}+x^{62}+x^{61}+x^{56}+x^{53}+x^{49}+x^{47}+x^{45}+x^{43}+x^{36}\\
&\quad+x^{35}+x^{34}+x^{32}+x^{30},\\
p_3(x) &= x^{107}+x^{106}+x^{104}+x^{101}+x^{100}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}\\
&\quad+x^{88}+x^{86}+x^{85}+x^{84}+x^{83}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{68}\\
&\quad+x^{66}+x^{64}+x^{63}+x^{62}+x^{61}+x^{58}+x^{56}+x^{54}+x^{53}+x^{51}+x^{49}\\
&\quad+x^{48}+x^{43}+x^{41}+x^{40}+x^{38}+x^{34}+x^{33}+x^{30}+x^{28}+x^{26}+x^{25}\\
&\quad+x^{24}+x^{22}+x^{18}+x^{15}+x^{14}+x^{12},\\
p_6(x) &= x^{107}+x^{106}+x^{104}+x^{102}+x^{101}+x^{100}+x^{98}+x^{91}+x^{89}+x^{88}\\
&\quad+x^{87}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{74}+x^{73}+x^{66}+x^{65}+x^{64}\\
&\quad+x^{60}+x^{53}+x^{49}+x^{48}+x^{44}+x^{42}+x^{41}+x^{38}+x^{36}+x^{34}+x^{31}\\
&\quad+x^{30}+x^{29}+x^{28}+x^{27}+x^{23}+x^{22}+x^{21}+x^{18}+x^{15}+x^{14}+x^{12}\\
&\quad+x^{10}+x^{3}+1,\\
p_9(x) &= x^{107}+x^{104}+x^{99}+x^{96}+x^{91}+x^{88}+x^{87}+x^{84}+x^{75}+x^{72}+x^{67}\\
&\quad+x^{64}+x^{59}+x^{56}+x^{55}+x^{52}+x^{27}+x^{24}+x^{19}+x^{16}+x^{11}+x^{8}\\
&\quad+x^{7}+x^{4},\\
p_{12}(x) &= x^{107}+x^{104}+x^{103}+x^{100}+x^{99}+x^{96}+x^{95}+x^{92}+x^{87}+x^{84}+x^{83}\\
&\quad+x^{80}+x^{75}+x^{72}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}+x^{60}+x^{55}+x^{52}\\
&\quad+x^{51}+x^{48}+x^{27}+x^{24}+x^{23}+x^{20}+x^{19}+x^{16}+x^{15}+x^{12}+x^{7}\\
&\quad+x^{4}+x^{3}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1].

For $\phi(W^o,0,0)$:

$$
\begin{aligned}
p_0(x) &= x^{108}+x^{107}+x^{106}+x^{104}+x^{102}+x^{98}+x^{97}+x^{94}+x^{91}+x^{90}+x^{89}\\
&\quad+x^{86}+x^{85}+x^{81}+x^{80}+x^{79}+x^{76}+x^{75}+x^{71}+x^{69}+x^{67}+x^{65}\\
&\quad+x^{62}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{46}+x^{43}\\
&\quad+x^{39}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}\\
&\quad+x^{23}+x^{21}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}+x^{12},\\
p_3(x) &= x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{99}+x^{98}+x^{92}\\
&\quad+x^{91}+x^{90}+x^{86}+x^{85}+x^{84}+x^{83}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{68} + x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53}\\
&+ x^{52} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{34} + x^{31} + x^{27}\\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{17} + x^{13} + x^{11} + x^{10} + x^{8},\\
p_6(x) &= x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{97} + x^{94} + x^{91} + x^{89} + x^{88} + x^{87}\\
&+ x^{85} + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{62} + x^{60} + x^{57} + x^{54} + x^{53} + x^{48} + x^{46} + x^{44} + x^{42} + x^{41} + x^{39}\\
&+ x^{38} + x^{35} + x^{34} + x^{32} + x^{30} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{18}\\
&+ x^{17} + x^{15} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + 1,\\
p_9(x) &= x^{107} + x^{104} + x^{99} + x^{96} + x^{91} + x^{88} + x^{87} + x^{84} + x^{75} + x^{72} + x^{67}\\
&+ x^{64} + x^{59} + x^{56} + x^{55} + x^{52} + x^{27} + x^{24} + x^{19} + x^{16} + x^{11} + x^{8}\\
&+ x^{7} + x^{4},\\
p_{12}(x) &= x^{107} + x^{104} + x^{103} + x^{100} + x^{99} + x^{96} + x^{95} + x^{92} + x^{87} + x^{84} + x^{83}\\
&+ x^{80} + x^{75} + x^{72} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{55} + x^{52}\\
&+ x^{51} + x^{48} + x^{27} + x^{24} + x^{23} + x^{20} + x^{19} + x^{16} + x^{15} + x^{12} + x^{7}\\
&+ x^{4} + x^{3} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{100} + x^{99} + x^{94} + x^{93} + x^{90} + x^{87} + x^{86} + x^{78} + x^{77} + x^{72} + x^{67}\\
&+ x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{48} + x^{46} + x^{45}\\
&+ x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27} + x^{26}\\
&+ x^{23} + x^{18},\\
p_3(x) &= x^{99} + x^{97} + x^{96} + x^{90} + x^{89} + x^{51} + x^{49} + x^{48} + x^{42} + x^{41} + x^{27}\\
&+ x^{25} + x^{24} + x^{19} + x^{18} + x^{16} + x^{10} + x^{9},\\
p_6(x) &= x^{100} + x^{98} + x^{94} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{64} + x^{60}\\
&+ x^{54} + x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{24} + x^{20} + x^{14} + x^{10} + x^{6},\\
p_9(x) &= x^{101} + x^{99} + x^{98} + x^{95} + x^{93} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&+ x^{83} + x^{69} + x^{67} + x^{66} + x^{63} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{21} + x^{19} + x^{18} + x^{15} + x^{13} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5}\\
&+ x^{4} + x^{3},\\
p_{12}(x) &= x^{102} + x^{100} + x^{94} + x^{92} + x^{82} + x^{80} + x^{70} + x^{68} + x^{62} + x^{60} + x^{50}\\
&+ x^{48} + x^{22} + x^{20} + x^{14} + x^{12} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{117} + x^{114} + x^{112} + x^{104} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95}\\
&+ x^{92} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{74} + x^{72} + x^{70} + x^{68} + x^{65} + x^{64} + x^{61} + x^{60} + x^{55} + x^{54} + x^{49}\\
&+ x^{48} + x^{45} + x^{44} + x^{38} + x^{37} + x^{33} + x^{31} + x^{29} + x^{27} + x^{26} + x^{24},\\
p_3(x) &= x^{113} + x^{110} + x^{109} + x^{108} + x^{104} + x^{103} + x^{102} + x^{98} + x^{97} + x^{96}\\
&+ x^{92} + x^{91} + x^{90} + x^{89} + x^{65} + x^{62} + x^{61} + x^{60} + x^{56} + x^{55} + x^{54}\\
&+ x^{50} + x^{49} + x^{48} + x^{44} + x^{43} + x^{42} + x^{38} + x^{37} + x^{36} + x^{33} + x^{32}\\
&+ x^{31} + x^{29} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{16} + x^{12}\\
&+ x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88}\\
&+ x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{62} + x^{58} + x^{54} + x^{46}\\
&+ x^{44} + x^{40} + x^{38} + x^{34} + x^{32} + x^{28} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10}\\
&+ x^{8} + x^{6},\\
p_9(x) &= x^{115} + x^{112} + x^{111} + x^{110} + x^{109} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{84} + x^{80} + x^{79} + x^{78} + x^{77}\\
&+ x^{73} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{58}\\
&+ x^{52} + x^{51} + x^{35} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{22} + x^{20} + x^{19}\\
&+ x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{4} + x^{3},\\
p_{12}(x) &= x^{116} + x^{112} + x^{108} + x^{100} + x^{76} + x^{68} + x^{52} + x^{48} + x^{36} + x^{32} + x^{28}\\
&+ x^{20} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{103} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67}\\
&+ x^{65} + x^{64} + x^{62} + x^{61} + x^{56} + x^{53} + x^{49} + x^{47} + x^{45} + x^{43} + x^{36}\\
&+ x^{35} + x^{34} + x^{32} + x^{30},\\
p_3(x) &= x^{107} + x^{106} + x^{104} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93}\\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{68}\\
&+ x^{66} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49}\\
&+ x^{48} + x^{43} + x^{41} + x^{40} + x^{38} + x^{34} + x^{33} + x^{30} + x^{28} + x^{26} + x^{25}\\
&+ x^{24} + x^{22} + x^{18} + x^{15} + x^{14} + x^{12},\\
p_6(x) &= x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{91} + x^{89} + x^{88}\\
&+ x^{87} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{74} + x^{73} + x^{66} + x^{65} + x^{64}\\
&+ x^{60} + x^{53} + x^{49} + x^{48} + x^{44} + x^{42} + x^{41} + x^{38} + x^{36} + x^{34} + x^{31}\\
&+ x^{30} + x^{29} + x^{28} + x^{27} + x^{23} + x^{22} + x^{21} + x^{18} + x^{15} + x^{14} + x^{12}\\
&+ x^{10} + x^{3} + 1,\\
p_9(x) &= x^{107} + x^{104} + x^{99} + x^{96} + x^{91} + x^{88} + x^{87} + x^{84} + x^{75} + x^{72} + x^{67}
\end{aligned}
$$

$$+x^{64}+x^{59}+x^{56}+x^{55}+x^{52}+x^{27}+x^{24}+x^{19}+x^{16}+x^{11}+x^{8}+x^{7}+x^{4},$$

$$p_{12}(x)=x^{107}+x^{104}+x^{103}+x^{100}+x^{99}+x^{96}+x^{95}+x^{92}+x^{87}+x^{84}+x^{83}+x^{80}+x^{75}+x^{72}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}+x^{60}+x^{55}+x^{52}+x^{51}+x^{48}+x^{27}+x^{24}+x^{23}+x^{20}+x^{19}+x^{16}+x^{15}+x^{12}+x^{7}+x^{4}+x^{3}+1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	283	[472, 44]	566	[791, 171]	849	[986, 987]
1	[3, 4]	284	[169, 415]	567	[792, 42]	850	[988, 989]
2	[5, 6]	285	[197, 183]	568	[793, 398]	851	[990, 298]
3	[7, 8]	286	[473, 139]	569	[167, 506]	852	[956, 991]
4	[9, 10]	287	[158, 201]	570	[794, 428]	853	[992, 993]
5	[11, 12]	288	[215, 474]	571	[795, 164]	854	[994, 995]
6	[13, 14]	289	[216, 474]	572	[533, 431]	855	[996, 997]
7	[15, 16]	290	[475, 476]	573	[154, 169]	856	[613, 998]
8	[17, 18]	291	[477, 397]	574	[796, 797]	857	[999, 1000]
9	[19, 20]	292	[210, 53]	575	[398, 535]	858	[1001, 646]
10	[21, 22]	293	[478, 178]	576	[798, 443]	859	[624, 120]
11	[23, 24]	294	[479, 400]	577	[213, 55]	860	[1002, 1003]
12	[25, 26]	295	[480, 144]	578	[799, 174]	861	[903, 1004]
13	[27, 28]	296	[481, 482]	579	[800, 801]	862	[240, 1005]
14	[29, 30]	297	[483, 432]	580	[408, 202]	863	[1006, 1007]
15	[31, 32]	298	[413, 484]	581	[126, 194]	864	[1008, 1009]
16	[33, 34]	299	[485, 134]	582	[802, 540]	865	[608, 572]
17	[35, 36]	300	[486, 34]	583	[803, 191]	866	[1010, 1011]
18	[37, 38]	301	[411, 487]	584	[156, 544]	867	[1012, 1013]
19	[39, 40]	302	[488, 411]	585	[404, 183]	868	[1014, 1015]
20	[41, 42]	303	[453, 191]	586	[804, 544]	869	[30, 245]
21	[43, 44]	304	[195, 133]	587	[182, 198]	870	[266, 308]
22	[43, 45]	305	[127, 489]	588	[423, 417]	871	[1016, 658]
23	[46, 47]	306	[490, 129]	589	[805, 40]	872	[1017, 372]
24	[48, 49]	307	[466, 491]	590	[526, 37]	873	[1018, 1019]
25	[50, 51]	308	[134, 14]	591	[430, 534]	874	[1020, 24]
26	[52, 53]	309	[447, 35]	592	[185, 507]	875	[1003, 1021]
27	[54, 55]	310	[177, 57]	593	[215, 435]	876	[933, 1022]
28	[56, 57]	311	[147, 492]	594	[438, 216]	877	[1023, 1004]
29	[58, 59]	312	[154, 144]	595	[806, 518]	878	[1024, 113]
30	[60, 61]	313	[493, 494]	596	[500, 447]	879	[1025, 592]

31	[62, 63]	314	[153, 49]	597	[169, 145]	880	[993, 579]
32	[64, 65]	315	[495, 177]	598	[46, 511]	881	[1026, 1027]
33	[66, 67]	316	[196, 459]	599	[791, 202]	882	[997, 974]
34	[68, 69]	317	[457, 457]	600	[446, 501]	883	[20, 644]
35	[70, 71]	318	[138, 160]	601	[204, 801]	884	[945, 1028]
36	[72, 73]	319	[161, 395]	602	[509, 123]	885	[1029, 1030]
37	[74, 75]	320	[496, 497]	603	[807, 428]	886	[1031, 527]
38	[76, 77]	321	[52, 211]	604	[53, 808]	887	[1032, 449]
39	[78, 79]	322	[49, 144]	605	[399, 797]	888	[454, 1033]
40	[80, 81]	323	[498, 174]	606	[809, 178]	889	[851, 412]
41	[82, 83]	324	[424, 499]	607	[529, 533]	890	[1034, 1033]
42	[84, 85]	325	[500, 501]	608	[48, 154]	891	[489, 542]
43	[86, 87]	326	[420, 415]	609	[403, 419]	892	[532, 179]
44	[88, 79]	327	[177, 502]	610	[810, 400]	893	[419, 183]
45	[89, 90]	328	[473, 160]	611	[212, 135]	894	[427, 863]
46	[91, 92]	329	[201, 140]	612	[779, 530]	895	[441, 528]
47	[93, 93]	330	[475, 503]	613	[512, 212]	896	[1035, 125]
48	[94, 95]	331	[208, 504]	614	[780, 36]	897	[406, 801]
49	[96, 97]	332	[505, 506]	615	[811, 413]	898	[786, 491]
50	[98, 99]	333	[478, 507]	616	[188, 129]	899	[828, 448]
51	[100, 101]	334	[508, 125]	617	[812, 450]	900	[1036, 131]
52	[102, 103]	335	[203, 194]	618	[813, 463]	901	[489, 133]
53	[104, 105]	336	[509, 407]	619	[501, 780]	902	[1037, 195]
54	[106, 107]	337	[510, 177]	620	[437, 438]	903	[465, 1038]
55	[108, 109]	338	[142, 140]	621	[814, 474]	904	[1039, 440]
56	[110, 111]	339	[458, 200]	622	[546, 435]	905	[506, 497]
57	[112, 113]	340	[137, 511]	623	[815, 215]	906	[868, 841]
58	[114, 115]	341	[512, 432]	624	[435, 214]	907	[862, 178]
59	[116, 117]	342	[513, 32]	625	[816, 215]	908	[871, 545]
60	[118, 119]	343	[190, 514]	626	[815, 216]	909	[510, 545]
61	[120, 118]	344	[515, 174]	627	[479, 797]	910	[840, 537]
62	[121, 43]	345	[516, 206]	628	[817, 150]	911	[1040, 144]
63	[122, 123]	346	[517, 518]	629	[818, 167]	912	[845, 14]
64	[124, 125]	347	[519, 127]	630	[413, 463]	913	[811, 38]
65	[126, 127]	348	[520, 465]	631	[819, 413]	914	[1041, 880]
66	[128, 129]	349	[130, 450]	632	[213, 136]	915	[883, 858]
67	[130, 131]	350	[13, 521]	633	[820, 511]	916	[1042, 43]
68	[132, 133]	351	[522, 134]	634	[806, 40]	917	[857, 45]
69	[42, 134]	352	[414, 145]	635	[426, 522]	918	[860, 168]
70	[135, 136]	353	[523, 460]	636	[505, 168]	919	[837, 476]
71	[137, 47]	354	[524, 181]	637	[821, 537]	920	[496, 875]
72	[138, 139]	355	[418, 457]	638	[822, 137]	921	[850, 509]
73	[140, 61]	356	[457, 418]	639	[143, 417]	922	[800, 205]
74	[141, 142]	357	[404, 198]	640	[517, 40]	923	[835, 854]

75	[143, 59]	358	[525, 425]	641	[823, 173]	924	[849, 491]
76	[144, 145]	359	[526, 173]	642	[824, 206]	925	[804, 157]
77	[146, 47]	360	[472, 45]	643	[825, 44]	926	[820, 47]
78	[147, 148]	361	[527, 206]	644	[38, 484]	927	[1043, 778]
79	[149, 150]	362	[207, 395]	645	[42, 539]	928	[809, 507]
80	[151, 152]	363	[477, 528]	646	[826, 514]	929	[796, 400]
81	[153, 154]	364	[482, 166]	647	[189, 827]	930	[1044, 778]
82	[155, 137]	365	[185, 178]	648	[828, 411]	931	[817, 535]
83	[156, 157]	366	[529, 530]	649	[803, 514]	932	[858, 131]
84	[140, 158]	367	[531, 185]	650	[829, 35]	933	[854, 10]
85	[159, 160]	368	[155, 146]	651	[32, 830]	934	[1045, 201]
86	[161, 162]	369	[532, 459]	652	[436, 434]	935	[491, 882]
87	[163, 164]	370	[394, 162]	653	[547, 436]	936	[1046, 158]
88	[165, 166]	371	[431, 504]	654	[217, 438]	937	[1047, 200]
89	[167, 168]	372	[165, 396]	655	[439, 831]	938	[802, 854]
90	[49, 169]	373	[533, 534]	656	[429, 504]	939	[1036, 450]
91	[170, 171]	374	[149, 535]	657	[781, 543]	940	[54, 136]
92	[172, 173]	375	[536, 537]	658	[527, 175]	941	[885, 460]
93	[174, 175]	376	[538, 133]	659	[832, 421]	942	[814, 435]
94	[176, 177]	377	[485, 539]	660	[507, 179]	943	[816, 216]
95	[178, 179]	378	[453, 514]	661	[416, 502]	944	[461, 522]
96	[142, 60]	379	[540, 10]	662	[524, 543]	945	[1048, 36]
97	[180, 181]	380	[541, 411]	663	[60, 158]	946	[805, 518]
98	[182, 183]	381	[195, 542]	664	[833, 509]	947	[1049, 194]
99	[184, 185]	382	[501, 35]	665	[516, 175]	948	[1032, 487]
100	[186, 51]	383	[456, 418]	666	[834, 425]	949	[1031, 174]
101	[187, 168]	384	[180, 543]	667	[823, 37]	950	[822, 146]
102	[188, 189]	385	[535, 544]	668	[835, 540]	951	[838, 863]
103	[190, 191]	386	[545, 57]	669	[412, 34]	952	[166, 397]
104	[192, 193]	387	[434, 436]	670	[132, 542]	953	[879, 440]
105	[194, 195]	388	[474, 438]	671	[173, 413]	954	[864, 875]
106	[196, 179]	389	[474, 214]	672	[836, 43]	955	[1050, 189]
107	[197, 198]	390	[435, 438]	673	[471, 407]	956	[33, 8]
108	[199, 200]	391	[546, 474]	674	[837, 503]	957	[833, 471]
109	[201, 60]	392	[547, 434]	675	[163, 808]	958	[1051, 407]
110	[170, 202]	393	[548, 273]	676	[838, 428]	959	[1052, 418]
111	[203, 127]	394	[549, 293]	677	[534, 504]	960	[843, 210]
112	[204, 205]	395	[550, 551]	678	[839, 162]	961	[483, 212]
113	[174, 206]	396	[552, 553]	679	[211, 808]	962	[1053, 800]
114	[207, 162]	397	[303, 554]	680	[840, 841]	963	[1054, 212]
115	[208, 209]	398	[555, 556]	681	[842, 150]	964	[860, 506]
116	[210, 211]	399	[557, 558]	682	[843, 152]	965	[821, 841]
117	[212, 213]	400	[559, 560]	683	[844, 398]	966	[515, 527]
118	[214, 215]	401	[561, 562]	684	[845, 521]	967	[1051, 123]

119	[214, 216]	402	[563, 564]	685	[846, 539]	968	[1055, 478]
120	[217, 214]	403	[565, 360]	686	[469, 61]	969	[1056, 211]
121	[218, 88]	404	[566, 567]	687	[465, 830]	970	[810, 797]
122	[219, 220]	405	[568, 569]	688	[790, 139]	971	[1057, 148]
123	[26, 221]	406	[570, 571]	689	[847, 447]	972	[1058, 148]
124	[222, 223]	407	[572, 573]	690	[490, 189]	973	[842, 535]
125	[224, 225]	408	[574, 575]	691	[7, 34]	974	[129, 783]
126	[226, 227]	409	[576, 577]	692	[176, 545]	975	[1034, 455]
127	[228, 229]	410	[578, 579]	693	[848, 543]	976	[785, 849]
128	[230, 231]	411	[580, 581]	694	[410, 448]	977	[789, 60]
129	[232, 233]	412	[582, 583]	695	[849, 467]	978	[1059, 417]
130	[234, 235]	413	[584, 585]	696	[406, 205]	979	[881, 137]
131	[6, 236]	414	[586, 587]	697	[121, 825]	980	[150, 544]
132	[237, 238]	415	[588, 338]	698	[146, 511]	981	[523, 422]
133	[239, 240]	416	[589, 590]	699	[850, 471]	982	[1052, 457]
134	[241, 242]	417	[591, 592]	700	[825, 45]	983	[878, 419]
135	[243, 244]	418	[593, 594]	701	[35, 193]	984	[1039, 831]
136	[245, 246]	419	[595, 596]	702	[522, 539]	985	[53, 164]
137	[247, 248]	420	[597, 598]	703	[851, 33]	986	[520, 32]
138	[249, 250]	421	[599, 600]	704	[787, 852]	987	[826, 191]
139	[251, 252]	422	[601, 602]	705	[792, 522]	988	[1055, 185]
140	[253, 254]	423	[603, 604]	706	[539, 14]	989	[866, 396]
141	[255, 256]	424	[605, 606]	707	[853, 854]	990	[1060, 501]
142	[257, 258]	425	[607, 608]	708	[855, 467]	991	[189, 783]
143	[259, 260]	426	[609, 241]	709	[856, 406]	992	[405, 800]
144	[261, 262]	427	[610, 611]	710	[857, 44]	993	[824, 175]
145	[263, 264]	428	[612, 613]	711	[780, 193]	994	[124, 874]
146	[265, 266]	429	[614, 615]	712	[846, 134]	995	[409, 173]
147	[267, 268]	430	[616, 617]	713	[858, 450]	996	[1050, 129]
148	[269, 270]	431	[618, 619]	714	[793, 149]	997	[786, 467]
149	[271, 272]	432	[246, 108]	715	[493, 482]	998	[1056, 53]
150	[273, 274]	433	[620, 621]	716	[402, 478]	999	[867, 476]
151	[275, 276]	434	[434, 434]	717	[798, 421]	1000	[396, 528]
152	[277, 278]	435	[622, 389]	718	[859, 140]	1001	[128, 189]
153	[279, 280]	436	[623, 624]	719	[818, 860]	1002	[853, 540]
154	[281, 263]	437	[258, 625]	720	[855, 491]	1003	[486, 8]
155	[282, 283]	438	[626, 255]	721	[448, 487]	1004	[884, 788]
156	[284, 285]	439	[627, 628]	722	[488, 448]	1005	[1059, 59]
157	[286, 287]	440	[629, 218]	723	[813, 484]	1006	[481, 494]
158	[288, 289]	441	[630, 631]	724	[127, 195]	1007	[844, 149]
159	[290, 291]	442	[632, 633]	725	[443, 460]	1008	[538, 542]
160	[292, 293]	443	[634, 635]	726	[861, 148]	1009	[819, 38]
161	[294, 295]	444	[636, 611]	727	[862, 507]	1010	[513, 465]
162	[296, 297]	445	[637, 573]	728	[794, 863]	1011	[812, 131]

163	[298, 299]	446	[638, 70]	729	[864, 497]	1012	[1044, 877]
164	[300, 301]	447	[73, 639]	730	[865, 471]	1013	[480, 169]
165	[302, 303]	448	[640, 641]	731	[471, 123]	1014	[1061, 830]
166	[304, 305]	449	[642, 643]	732	[468, 419]	1015	[450, 784]
167	[306, 307]	450	[644, 645]	733	[507, 459]	1016	[836, 825]
168	[308, 309]	451	[646, 647]	734	[799, 527]	1017	[1062, 154]
169	[95, 310]	452	[635, 350]	735	[38, 463]	1018	[807, 863]
170	[311, 312]	453	[648, 649]	736	[173, 38]	1019	[506, 875]
171	[313, 314]	454	[276, 650]	737	[866, 166]	1020	[870, 167]
172	[315, 316]	455	[16, 651]	738	[867, 503]	1021	[540, 880]
173	[317, 318]	456	[652, 653]	739	[431, 209]	1022	[131, 451]
174	[319, 320]	457	[654, 593]	740	[868, 537]	1023	[452, 190]
175	[321, 322]	458	[655, 656]	741	[869, 213]	1024	[865, 509]
176	[323, 112]	459	[657, 555]	742	[870, 860]	1025	[536, 841]
177	[324, 325]	460	[336, 658]	743	[531, 478]	1026	[1042, 825]
178	[326, 327]	461	[659, 106]	744	[871, 177]	1027	[1063, 205]
179	[328, 329]	462	[660, 661]	745	[872, 800]	1028	[1064, 35]
180	[330, 331]	463	[662, 663]	746	[873, 200]	1029	[525, 499]
181	[332, 333]	464	[664, 665]	747	[859, 60]	1030	[1049, 127]
182	[334, 335]	465	[666, 667]	748	[508, 874]	1031	[1065, 321]
183	[113, 336]	466	[668, 669]	749	[172, 37]	1032	[1066, 1067]
184	[337, 326]	467	[670, 671]	750	[839, 395]	1033	[67, 1068]
185	[338, 339]	468	[672, 673]	751	[168, 875]	1034	[1069, 1070]
186	[340, 341]	469	[390, 257]	752	[876, 877]	1035	[1071, 221]
187	[342, 343]	470	[551, 26]	753	[878, 404]	1036	[1072, 898]
188	[344, 345]	471	[674, 675]	754	[879, 831]	1037	[1073, 1074]
189	[346, 347]	472	[676, 677]	755	[795, 808]	1038	[696, 922]
190	[348, 349]	473	[678, 679]	756	[854, 880]	1039	[1075, 117]
191	[350, 351]	474	[388, 652]	757	[847, 501]	1040	[1076, 588]
192	[352, 353]	475	[680, 681]	758	[464, 32]	1041	[1077, 689]
193	[354, 355]	476	[682, 683]	759	[881, 146]	1042	[1078, 1079]
194	[356, 239]	477	[684, 685]	760	[541, 448]	1043	[1068, 560]
195	[227, 357]	478	[686, 657]	761	[134, 521]	1044	[1080, 1081]
196	[316, 23]	479	[687, 270]	762	[194, 489]	1045	[943, 253]
197	[358, 359]	480	[355, 688]	763	[442, 55]	1046	[942, 620]
198	[360, 361]	481	[689, 552]	764	[545, 502]	1047	[1082, 1083]
199	[362, 363]	482	[690, 691]	765	[467, 882]	1048	[980, 1084]
200	[364, 365]	483	[692, 385]	766	[883, 812]	1049	[1085, 586]
201	[119, 288]	484	[318, 356]	767	[884, 852]	1050	[1086, 897]
202	[366, 367]	485	[264, 693]	768	[885, 422]	1051	[1087, 1088]
203	[368, 369]	486	[694, 695]	769	[834, 499]	1052	[625, 654]
204	[370, 371]	487	[22, 696]	770	[519, 194]	1053	[367, 1089]
205	[372, 312]	488	[697, 642]	771	[872, 406]	1054	[1090, 27]
206	[373, 374]	489	[369, 698]	772	[873, 444]	1055	[1091, 940]

207	[375, 252]	490	[699, 700]	773	[482, 396]	1056	[1092, 932]
208	[376, 377]	491	[701, 702]	774	[432, 213]	1057	[1093, 890]
209	[378, 379]	492	[703, 704]	775	[445, 874]	1058	[1094, 1095]
210	[380, 300]	493	[705, 706]	776	[886, 336]	1059	[1096, 1097]
211	[381, 382]	494	[707, 708]	777	[887, 888]	1060	[1098, 925]
212	[383, 384]	495	[709, 710]	778	[889, 890]	1061	[1099, 990]
213	[385, 386]	496	[711, 712]	779	[567, 891]	1062	[1100, 261]
214	[387, 388]	497	[713, 267]	780	[892, 893]	1063	[1101, 1102]
215	[389, 390]	498	[714, 373]	781	[894, 895]	1064	[287, 1103]
216	[391, 387]	499	[715, 716]	782	[896, 757]	1065	[1062, 49]
217	[392, 391]	500	[717, 284]	783	[897, 700]	1066	[1104, 499]
218	[393, 149]	501	[718, 72]	784	[898, 687]	1067	[1060, 447]
219	[394, 395]	502	[602, 673]	785	[596, 684]	1068	[129, 827]
220	[396, 397]	503	[719, 341]	786	[899, 900]	1069	[539, 521]
221	[398, 150]	504	[720, 721]	787	[901, 902]	1070	[1105, 195]
222	[399, 400]	505	[722, 713]	788	[349, 903]	1071	[876, 778]
223	[401, 213]	506	[723, 724]	789	[621, 392]	1072	[31, 465]
224	[151, 210]	507	[564, 725]	790	[904, 905]	1073	[1046, 61]
225	[402, 185]	508	[726, 727]	791	[906, 907]	1074	[848, 181]
226	[403, 404]	509	[728, 729]	792	[908, 632]	1075	[777, 877]
227	[135, 55]	510	[730, 602]	793	[909, 763]	1076	[1045, 142]
228	[141, 201]	511	[283, 731]	794	[910, 911]	1077	[1106, 202]
229	[159, 139]	512	[732, 733]	795	[912, 913]	1078	[1054, 432]
230	[405, 406]	513	[734, 643]	796	[914, 704]	1079	[530, 534]
231	[122, 407]	514	[735, 736]	797	[915, 888]	1080	[1061, 1038]
232	[408, 171]	515	[341, 737]	798	[916, 917]	1081	[491, 1107]
233	[409, 37]	516	[738, 739]	799	[743, 918]	1082	[1108, 503]
234	[410, 411]	517	[740, 741]	800	[919, 920]	1083	[211, 164]
235	[412, 8]	518	[742, 743]	801	[569, 575]	1084	[56, 502]
236	[37, 413]	519	[744, 597]	802	[921, 922]	1085	[832, 443]
237	[414, 415]	520	[745, 696]	803	[923, 924]	1086	[1109, 825]
238	[416, 57]	521	[746, 317]	804	[925, 926]	1087	[1110, 831]
239	[58, 417]	522	[747, 29]	805	[927, 928]	1088	[534, 209]
240	[418, 418]	523	[748, 749]	806	[929, 90]	1089	[530, 431]
241	[144, 415]	524	[750, 751]	807	[930, 931]	1090	[1111, 137]
242	[419, 198]	525	[752, 322]	808	[932, 933]	1091	[1112, 404]
243	[420, 145]	526	[753, 584]	809	[934, 328]	1092	[1113, 854]
244	[421, 422]	527	[754, 755]	810	[721, 935]	1093	[1041, 10]
245	[423, 59]	528	[708, 756]	811	[936, 937]	1094	[782, 827]
246	[61, 201]	529	[757, 618]	812	[938, 939]	1095	[467, 1107]
247	[424, 425]	530	[758, 3]	813	[940, 941]	1096	[1108, 476]
248	[426, 42]	531	[759, 660]	814	[594, 942]	1097	[168, 497]
249	[427, 428]	532	[577, 98]	815	[256, 943]	1098	[495, 545]
250	[429, 209]	533	[760, 720]	816	[289, 622]	1099	[39, 518]

251	[430, 431]	534	[761, 762]	817	[663, 286]	1100	[1114, 443]
252	[432, 135]	535	[763, 764]	818	[944, 945]	1101	[1110, 440]
253	[433, 434]	536	[756, 765]	819	[109, 662]	1102	[166, 528]
254	[216, 435]	537	[766, 767]	820	[946, 947]	1103	[1047, 444]
255	[436, 436]	538	[280, 768]	821	[948, 767]	1104	[1115, 1116]
256	[437, 214]	539	[83, 340]	822	[949, 93]	1105	[329, 772]
257	[433, 436]	540	[769, 770]	823	[950, 76]	1106	[1117, 1118]
258	[438, 215]	541	[771, 22]	824	[951, 952]	1107	[987, 1119]
259	[439, 440]	542	[772, 84]	825	[953, 954]	1108	[1120, 774]
260	[441, 397]	543	[773, 774]	826	[955, 956]	1109	[24, 89]
261	[442, 136]	544	[556, 96]	827	[231, 897]	1110	[1121, 1122]
262	[443, 422]	545	[775, 705]	828	[957, 958]	1111	[1123, 731]
263	[199, 444]	546	[653, 626]	829	[959, 354]	1112	[1124, 1125]
264	[158, 142]	547	[254, 623]	830	[63, 21]	1113	[1126, 21]
265	[445, 125]	548	[776, 421]	831	[960, 961]	1114	[1127, 601]
266	[446, 447]	549	[777, 778]	832	[962, 361]	1115	[861, 492]
267	[448, 449]	550	[779, 533]	833	[963, 964]	1116	[869, 135]
268	[450, 451]	551	[393, 398]	834	[965, 374]	1117	[1043, 877]
269	[452, 453]	552	[192, 36]	835	[966, 967]	1118	[401, 135]
270	[454, 455]	553	[447, 780]	836	[968, 969]	1119	[32, 1038]
271	[456, 457]	554	[411, 449]	837	[970, 333]	1120	[1057, 492]
272	[458, 444]	555	[61, 142]	838	[971, 365]	1121	[1058, 492]
273	[178, 459]	556	[781, 181]	839	[972, 973]	1122	[1040, 169]
274	[421, 460]	557	[782, 783]	840	[974, 975]	1123	[1053, 406]
275	[461, 42]	558	[131, 784]	841	[976, 975]	1124	[856, 800]
276	[462, 463]	559	[785, 786]	842	[977, 978]	1125	[1063, 801]
277	[464, 465]	560	[787, 788]	843	[600, 381]	1126	[1109, 43]
278	[466, 467]	561	[789, 140]	844	[979, 980]	1127	[470, 509]
279	[468, 404]	562	[790, 160]	845	[733, 981]		
280	[150, 157]	563	[776, 443]	846	[982, 746]		
281	[469, 158]	564	[535, 157]	847	[983, 892]		
282	[470, 471]	565	[498, 527]	848	[984, 985]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	189	1	378	1	567	1	756	0	945	0
1	0	190	0	379	1	568	0	757	0	946	0
2	1	191	0	380	0	569	1	758	1	947	1
3	0	192	1	381	0	570	0	759	1	948	1
4	1	193	0	382	0	571	1	760	0	949	0
5	1	194	0	383	1	572	0	761	1	950	0
6	0	195	0	384	0	573	0	762	0	951	0
7	0	196	0	385	1	574	0	763	1	952	0
8	0	197	0	386	1	575	0	764	1	953	0

9	1	198	0	387	0	576	1	765	1	954	0
10	1	199	0	388	1	577	1	766	0	955	0
11	1	200	1	389	1	578	1	767	0	956	0
12	1	201	0	390	0	579	1	768	0	957	1
13	0	202	0	391	0	580	0	769	1	958	0
14	0	203	0	392	0	581	0	770	1	959	0
15	0	204	1	393	0	582	1	771	0	960	0
16	0	205	0	394	1	583	1	772	1	961	0
17	0	206	0	395	1	584	0	773	1	962	1
18	1	207	0	396	1	585	1	774	0	963	1
19	1	208	0	397	1	586	1	775	1	964	0
20	0	209	1	398	0	587	1	776	0	965	1
21	1	210	0	399	0	588	1	777	1	966	1
22	1	211	0	400	1	589	0	778	1	967	0
23	1	212	1	401	1	590	0	779	1	968	1
24	0	213	1	402	0	591	1	780	1	969	0
25	1	214	0	403	0	592	1	781	1	970	1
26	1	215	1	404	1	593	1	782	0	971	0
27	0	216	0	405	0	594	1	783	0	972	0
28	1	217	0	406	0	595	0	784	1	973	1
29	0	218	0	407	0	596	1	785	1	974	0
30	0	219	1	408	0	597	0	786	1	975	0
31	0	220	1	409	1	598	1	787	1	976	1
32	0	221	0	410	1	599	1	788	1	977	1
33	0	222	0	411	0	600	0	789	1	978	0
34	1	223	1	412	1	601	1	790	1	979	1
35	0	224	1	413	0	602	0	791	1	980	0
36	1	225	0	414	1	603	1	792	1	981	1
37	1	226	0	415	1	604	1	793	0	982	0
38	1	227	0	416	0	605	0	794	0	983	0
39	1	228	1	417	1	606	0	795	1	984	1
40	1	229	0	418	1	607	0	796	0	985	1
41	0	230	0	419	0	608	0	797	0	986	0
42	1	231	1	420	0	609	0	798	1	987	0
43	1	232	0	421	1	610	1	799	1	988	1
44	0	233	1	422	1	611	1	800	1	989	1
45	1	234	1	423	1	612	1	801	1	990	1
46	1	235	1	424	0	613	1	802	1	991	1
47	1	236	1	425	0	614	1	803	1	992	0
48	0	237	1	426	0	615	1	804	1	993	0
49	1	238	0	427	1	616	1	805	0	994	0
50	1	239	0	428	1	617	1	806	0	995	1
51	0	240	1	429	1	618	0	807	1	996	0
52	1	241	1	430	1	619	0	808	0	997	1

53	1	242	0	431	0	620	1	809	0	998	0
54	0	243	0	432	0	621	1	810	1	999	1
55	0	244	1	433	1	622	0	811	1	1000	1
56	1	245	1	434	0	623	1	812	1	1001	0
57	1	246	0	435	0	624	0	813	0	1002	0
58	0	247	0	436	1	625	0	814	1	1003	1
59	0	248	0	437	1	626	1	815	1	1004	0
60	0	249	1	438	1	627	1	816	0	1005	0
61	0	250	1	439	1	628	0	817	0	1006	0
62	0	251	1	440	1	629	1	818	1	1007	1
63	1	252	0	441	0	630	0	819	0	1008	0
64	0	253	1	442	1	631	0	820	0	1009	0
65	0	254	0	443	0	632	1	821	1	1010	1
66	0	255	1	444	0	633	0	822	0	1011	1
67	1	256	1	445	1	634	0	823	0	1012	1
68	1	257	1	446	0	635	0	824	0	1013	1
69	1	258	1	447	1	636	0	825	0	1014	1
70	0	259	1	448	1	637	1	826	0	1015	1
71	0	260	0	449	0	638	0	827	1	1016	1
72	1	261	1	450	1	639	1	828	1	1017	0
73	1	262	0	451	0	640	1	829	0	1018	1
74	1	263	0	452	0	641	0	830	1	1019	0
75	1	264	1	453	1	642	0	831	0	1020	0
76	1	265	1	454	1	643	0	832	1	1021	1
77	1	266	0	455	0	644	1	833	1	1022	0
78	1	267	1	456	1	645	1	834	1	1023	0
79	1	268	1	457	0	646	0	835	1	1024	0
80	1	269	0	458	1	647	1	836	1	1025	0
81	1	270	1	459	1	648	1	837	1	1026	1
82	0	271	1	460	0	649	1	838	0	1027	0
83	0	272	1	461	1	650	0	839	0	1028	1
84	1	273	0	462	1	651	0	840	0	1029	0
85	0	274	1	463	0	652	1	841	1	1030	1
86	1	275	1	464	1	653	0	842	1	1031	0
87	0	276	1	465	1	654	0	843	0	1032	1
88	0	277	1	466	1	655	1	844	1	1033	1
89	1	278	1	467	1	656	1	845	1	1034	0
90	1	279	1	468	1	657	1	846	0	1035	0
91	1	280	0	469	0	658	0	847	0	1036	0
92	1	281	0	470	0	659	1	848	1	1037	1
93	1	282	0	471	1	660	1	849	0	1038	0
94	0	283	0	472	0	661	0	850	1	1039	1
95	0	284	0	473	0	662	0	851	1	1040	0
96	1	285	0	474	1	663	0	852	0	1041	0

97	0	286	0	475	0	664	1	853	0	1042	1
98	1	287	1	476	0	665	1	854	0	1043	0
99	0	288	1	477	1	666	1	855	0	1044	1
100	0	289	0	478	0	667	0	856	1	1045	0
101	1	290	0	479	1	668	1	857	1	1046	1
102	1	291	1	480	1	669	1	858	0	1047	0
103	0	292	0	481	0	670	1	859	0	1048	0
104	1	293	0	482	1	671	0	860	0	1049	1
105	0	294	1	483	0	672	1	861	1	1050	0
106	0	295	1	484	1	673	1	862	1	1051	0
107	0	296	0	485	1	674	1	863	0	1052	0
108	0	297	0	486	1	675	0	864	0	1053	1
109	0	298	0	487	1	676	0	865	0	1054	1
110	1	299	1	488	0	677	1	866	1	1055	1
111	0	300	1	489	1	678	0	867	1	1056	0
112	1	301	0	490	1	679	0	868	1	1057	0
113	1	302	0	491	0	680	0	869	0	1058	0
114	0	303	1	492	1	681	1	870	0	1059	0
115	0	304	0	493	1	682	0	871	1	1060	1
116	0	305	1	494	0	683	1	872	0	1061	1
117	1	306	1	495	1	684	1	873	1	1062	0
118	0	307	1	496	1	685	0	874	0	1063	0
119	0	308	1	497	0	686	0	875	1	1064	1
120	0	309	1	498	0	687	1	876	0	1065	0
121	0	310	0	499	1	688	1	877	0	1066	1
122	1	311	1	500	1	689	0	878	0	1067	1
123	1	312	0	501	0	690	1	879	0	1068	0
124	0	313	1	502	0	691	0	880	0	1069	0
125	1	314	1	503	1	692	0	881	1	1070	0
126	0	315	1	504	0	693	1	882	1	1071	0
127	1	316	0	505	0	694	1	883	0	1072	0
128	0	317	0	506	0	695	0	884	0	1073	1
129	0	318	1	507	1	696	0	885	0	1074	1
130	1	319	1	508	1	697	0	886	0	1075	1
131	0	320	1	509	0	698	1	887	1	1076	0
132	1	321	1	510	0	699	1	888	1	1077	0
133	0	322	1	511	0	700	0	889	1	1078	1
134	1	323	0	512	1	701	0	890	0	1079	1
135	0	324	0	513	1	702	0	891	1	1080	1
136	1	325	1	514	1	703	1	892	1	1081	0
137	0	326	0	515	1	704	1	893	0	1082	0
138	1	327	0	516	1	705	1	894	1	1083	0
139	1	328	0	517	1	706	0	895	0	1084	1
140	1	329	0	518	0	707	0	896	0	1085	1

141	1	330	0	519	1	708	0	897	0	1086	0
142	1	331	0	520	0	709	1	898	1	1087	0
143	1	332	0	521	1	710	1	899	1	1088	1
144	1	333	0	522	0	711	1	900	0	1089	1
145	0	334	1	523	1	712	0	901	1	1090	1
146	1	335	0	524	0	713	0	902	1	1091	1
147	1	336	0	525	0	714	0	903	1	1092	0
148	0	337	0	526	0	715	1	904	1	1093	0
149	1	338	1	527	0	716	0	905	0	1094	0
150	0	339	1	528	0	717	1	906	1	1095	1
151	1	340	0	529	0	718	0	907	1	1096	0
152	1	341	1	530	1	719	1	908	1	1097	1
153	1	342	1	531	1	720	0	909	0	1098	1
154	0	343	0	532	1	721	1	910	0	1099	1
155	0	344	1	533	0	722	0	911	0	1100	0
156	0	345	1	534	1	723	0	912	1	1101	0
157	0	346	1	535	1	724	1	913	1	1102	0
158	1	347	1	536	0	725	0	914	0	1103	0
159	0	348	0	537	0	726	1	915	0	1104	1
160	0	349	1	538	0	727	1	916	1	1105	0
161	1	350	0	539	0	728	0	917	1	1106	0
162	0	351	0	540	1	729	0	918	0	1107	0
163	0	352	1	541	0	730	0	919	1	1108	0
164	1	353	1	542	1	731	1	920	1	1109	0
165	0	354	0	543	1	732	1	921	1	1110	0
166	0	355	1	544	1	733	1	922	1	1111	1
167	1	356	0	545	1	734	1	923	1	1112	1
168	1	357	1	546	0	735	1	924	0	1113	0
169	0	358	0	547	0	736	0	925	1	1114	0
170	1	359	0	548	0	737	1	926	0	1115	1
171	1	360	0	549	1	738	1	927	0	1116	0
172	1	361	0	550	1	739	0	928	0	1117	0
173	0	362	0	551	0	740	1	929	0	1118	1
174	1	363	1	552	1	741	0	930	1	1119	0
175	1	364	1	553	1	742	0	931	0	1120	0
176	0	365	1	554	0	743	1	932	0	1121	0
177	0	366	0	555	0	744	1	933	0	1122	0
178	0	367	1	556	1	745	0	934	0	1123	1
179	0	368	0	557	0	746	1	935	0	1124	1
180	0	369	1	558	0	747	0	936	1	1125	0
181	0	370	1	559	1	748	1	937	0	1126	0
182	1	371	0	560	1	749	1	938	1	1127	0
183	1	372	0	561	1	750	0	939	0		
184	0	373	0	562	1	751	1	940	0		

185	1	374	1	563	0	752	0	941	0		
186	0	375	0	564	1	753	0	942	1		
187	1	376	0	565	0	754	0	943	0		
188	1	377	1	566	1	755	1	944	1		

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