PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^3 + z^2 + z, z^3 + z^2 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^3 + z^2 + z, z^3 + z^2 + z + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^3+z^2+z, z^3+z^2+z+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^6+z^5+z^2+z+1,\\
p_1(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z,\\
p_2(z)&=z^{24}+z^{16}+z^8+z^6+z^4+z^2,\\
p_3(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z,\\
p_4(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^5+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^6+z^5+z^2+z,\\
p_1(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z,\\
p_2(z)&=z^{24}+z^{16}+z^8+z^6+z^4+z^2,\\
p_3(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^5+z^4+z,\\
p_4(z)&=z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^7+z^5+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 32 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{2u} M^e[:6u] \\ & + x^{4u} M^e[:4u] + x^{5u} M^e[:3u] + x^{7u} M^e[:u] + x^{3u} M^e[:6u] \\ & + x^{5u} M^e[:4u] + x^{6u} M^e[:3u] + x^{8u} M^e[:u] + x^{4u} M^e[:6u] \\ & + x^{6u} M^e[:4u] + x^{7u} M^e[:3u] + x^{9u} M^e[:u] + x^{5u} M^e[:6u] \\ & + x^{7u} M^e[:4u] + x^{8u} M^e[:3u] + x^{10u} M^e[:u] + x^{7u} M^e[:5u] \\ & + x^{9u} M^e[:3u] + x^{10u} M^e[:2u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^e,\\
W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{2u}W^e[:6u]\\
&+ x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{7u}W^e[:u] + x^{3u}W^e[:6u]\\
&+ x^{5u}W^e[:4u] + x^{6u}W^e[:3u] + x^{8u}W^e[:u] + x^{4u}W^e[:6u]\\
&+ x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{9u}W^e[:u] + x^{5u}W^e[:6u]\\
&+ x^{7u}W^e[:4u] + x^{8u}W^e[:3u] + x^{10u}W^e[:u] + x^{7u}W^e[:5u]\\
&+ x^{9u}W^e[:3u] + x^{10u}W^e[:2u]\\
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{2u}M^o[:6u]\\
&+ x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{7u}M^o[:u] + x^{3u}M^o[:6u]\\
&+ x^{5u}M^o[:4u] + x^{6u}M^o[:3u] + x^{8u}M^o[:u] + x^{4u}M^o[:6u]\\
&+ x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{9u}M^o[:u] + x^{5u}M^o[:6u]\\
&+ x^{7u}M^o[:4u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u] + x^{7u}M^o[:5u]\\
&+ x^{9u}M^o[:3u] + x^{10u}M^o[:2u]\\
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{2u}W^o[:6u]\\
&+ x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{7u}W^o[:u] + x^{3u}W^o[:6u]\\
&+ x^{5u}W^o[:4u] + x^{6u}W^o[:3u] + x^{8u}W^o[:u] + x^{4u}W^o[:6u]\\
&+ x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{9u}W^o[:u] + x^{5u}W^o[:6u]\\
&+ x^{7u}W^o[:4u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u] + x^{7u}W^o[:5u]\\
&+ x^{9u}W^o[:3u] + x^{10u}W^o[:2u]\\
&+ (x^{6u} + x^{8u} + x^{9u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{5u}T[:u] + x^{2u}T[:6u] + x^{4u}T[:4u]\\
&+ x^{5u}T[:3u] + x^{7u}T[:u] + x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u]\\
&+ x^{8u}T[:u] + x^{4u}T[:6u] + x^{6u}T[:4u] + x^{7u}T[:3u] + x^{9u}T[:u]\\
&+ x^{5u}T[:6u] + x^{7u}T[:4u] + x^{8u}T[:3u] + x^{10u}T[:u] + x^{7u}T[:5u]\\
&+ x^{9u}T[:3u] + x^{10u}T[:2u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
0 &= x^{5u}T[u:2u] + x^{3u}T[3u:4u] + x^{2u}T[4u:5u] + T[6u:7u],\\
0 &= x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + T[7u:8u],
\end{aligned}
$$

$$\begin{aligned}
0 &= x^{7u}T[u:2u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u]\\
&\quad + x^{4u}T[5u:6u] + T[9u:10u],\\
0 &= x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{5u}T[5u:6u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{9u}T[2u:3u] + x^{7u}T[4u:5u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$

$$0 = T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$

$$\begin{aligned}0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1001w]_2],\end{aligned} \tag{2.10}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$

$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$\begin{aligned}0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1001w]_2],\end{aligned} \tag{2.16}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 882 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.19}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.20}$$

$$0 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.21}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,1001)),\end{aligned} \tag{2.22}$$

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.23}$$

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.25}$$

$$0=\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,101))+\tau(A(s,111)), \tag{2.26}$$

$$0=\tau(A(s,1))+\tau(A(s,100))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.27}$$

$$\begin{aligned}0=&\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,11))+\tau(A(s,100))\\ &+\tau(A(s,101))+\tau(A(s,1001)),\end{aligned} \tag{2.28}$$

$$0=\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,101))+\tau(A(s,1010)), \tag{2.29}$$

$$0=\tau(A(s,1))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,1011)), \tag{2.30}$$

for $s=A(i,0^{2k})$. We find that $(A(i,0^{24}),E_{24})=(A(i,0^{20}),E_{20})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2\le k\le 12$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k\ge 2$, and $u=2^{2k-1}$,*

$$M_{2k}=M^e[:6u]+x^{2u}M^e[:4u]+x^{3u}M^e[:3u]+x^{5u}M^e[:u]+x^{6u}R^e,$$
$$W_{2k}=W^e[:6u]+x^{2u}W^e[:4u]+x^{3u}W^e[:3u]+x^{5u}W^e[:u]+x^{6u}R^e.$$

For all integer $k\ge 2$, and $u=2^{2k}$,

$$M_{2k+1}=M^o[:6u]+x^{2u}M^o[:4u]+x^{3u}M^o[:3u]+x^{5u}M^o[:u]+x^{6u}R^o,$$
$$W_{2k+1}=W^o[:6u]+x^{2u}W^o[:4u]+x^{3u}W^o[:3u]+x^{5u}W^o[:u]+x^{6u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n\ge 2$, we claim that

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1}\wedge W_{2k+1},$$
$$M_{2k+1}\wedge W_{2k+1}\Rightarrow M_{2k+2}\wedge W_{2k+2}.$$

We give the proof of

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1},$$

the other ones being similar. Set $u=2^{2k}$ and $v=2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned}W_{2k}M_{2k}=&\big(W^e[:6v]+x^{2v}W^e[:4v]+x^{3v}W^e[:3v]+x^{5v}W^e[:v]+x^{6u}R^e\big)\times\big(\\ &M^e[:6v]+x^{2v}M^e[:4v]+x^{3v}M^e[:3v]+x^{5v}M^e[:v]+x^{6u}R^e\big);\end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned}W^e[:12v]M^e[:12v]+x^{4v}W^e[:8v]M^e[:8v]+x^{6v}W^e[:6v]M^e[:6v]&\\ +x^{10v}W^e[:2v]M^e[:2v].&\end{aligned}$$

For all $n\le 12$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X:=x^v,$$
$$a_n:=W^e[n\cdot v:(n+1)\cdot v]/X^n,$$

$$\begin{aligned} b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.31) and *rhs* as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t \left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^9 \\ &\quad + x^8 + x^6, \\ q_1(x) &= x^{23} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^9 + x^8 + x^6, \\ q_2(x) &= x^{22} + x^{20} + x^{18} + x^{16} + x^8 + 1, \\ q_3(x) &= x^{23} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^{10} + x^9 + x^8 + x^6, \\ q_4(x) &= x^{23} + x^{19} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^6. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 32 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{124}+x^{122}+x^{114}+x^{112}+x^{110}+x^{108}+x^{106}+x^{96}+x^{94}\\
&\quad+x^{92}+x^{90}+x^{88}+x^{86}+x^{82}+x^{76}+x^{72}+x^{70}+x^{66}+x^{60}+x^{52}\\
&\quad+x^{46}+x^{44}+x^{38}+x^{36},\\
p_3(x) &= x^{122}+x^{112}+x^{108}+x^{106}+x^{104}+x^{102}+x^{96}+x^{94}+x^{92}+x^{90}\\
&\quad+x^{88}+x^{86}+x^{82}+x^{74}+x^{70}+x^{66}+x^{62}+x^{56}+x^{52}+x^{44}+x^{42}\\
&\quad+x^{38}+x^{36}+x^{28}+x^{26}+x^{24},\\
p_6(x) &= x^{120}+x^{112}+x^{110}+x^{108}+x^{102}+x^{98}+x^{94}+x^{90}+x^{88}+x^{80}+x^{78}\\
&\quad+x^{76}+x^{72}+x^{70}+x^{68}+x^{64}+x^{60}+x^{58}+x^{46}+x^{44}+x^{40}+x^{36}\\
&\quad+x^{34}+x^{32}+x^{28}+x^{26}+x^{24}+x^{20}+x^{18}+x^{16}+x^{12}+x^{10}+x^{8}\\
&\quad+x^{4}+x^{2}+1,\\
p_9(x) &= x^{124}+x^{120}+x^{112}+x^{110}+x^{106}+x^{104}+x^{98}+x^{94}+x^{84}+x^{82}\\
&\quad+x^{78}+x^{72}+x^{70}+x^{66}+x^{62}+x^{60}+x^{58}+x^{54}+x^{52}+x^{50}+x^{48}\\
&\quad+x^{44}+x^{42}+x^{30}+x^{26}+x^{24}+x^{14}+x^{12},\\
p_{12}(x) &= x^{124}+x^{122}+x^{120}+x^{116}+x^{114}+x^{112}+x^{92}+x^{90}+x^{88}+x^{84}\\
&\quad+x^{82}+x^{80}+x^{76}+x^{74}+x^{72}+x^{68}+x^{66}+x^{64}+x^{44}+x^{42}+x^{40}\\
&\quad+x^{36}+x^{34}+x^{32}+x^{28}+x^{26}+x^{24}+x^{20}+x^{18}+x^{16}+x^{12}+x^{10}\\
&\quad+x^{8}+x^{4}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{125}+x^{124}+x^{123}+x^{121}+x^{118}+x^{116}+x^{114}+x^{113}+x^{112}\\
&\quad+x^{110}+x^{108}+x^{107}+x^{104}+x^{101}+x^{99}+x^{97}+x^{96}+x^{95}+x^{94}\\
&\quad+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{83}+x^{82}+x^{80}+x^{79}+x^{78}\\
&\quad+x^{76}+x^{74}+x^{73}+x^{71}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{60}\\
&\quad+x^{59}+x^{57}+x^{55}+x^{50}+x^{48}+x^{46}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}\\
&\quad+x^{38}+x^{36},\\
p_3(x) &= x^{124}+x^{122}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}+x^{100}+x^{98}+x^{96}\\
&\quad+x^{92}+x^{90}+x^{88}+x^{76}+x^{74}+x^{72}+x^{62}+x^{54},\\
p_6(x) &= x^{123}+x^{121}+x^{120}+x^{118}+x^{117}+x^{115}+x^{114}+x^{112}+x^{111}+x^{109}\\
&\quad+x^{108}+x^{107}+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}+x^{98}+x^{96}
\end{aligned}$$

$$
\begin{aligned}
&+x^{91}+x^{89}+x^{88}+x^{86}+x^{85}+x^{82}+x^{81}+x^{78}+x^{77}+x^{75}+x^{74}\\
&+x^{72}+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{48}+x^{43}+x^{41}+x^{40}+x^{39}\\
&+x^{38}+x^{36},\\
p_9(x)&=x^{122}+x^{120}+x^{118}+x^{114}+x^{112}+x^{104}+x^{102}+x^{92}+x^{84}+x^{82}\\
&+x^{78}+x^{76}+x^{74}+x^{72}+x^{60}+x^{52}+x^{48}+x^{46}+x^{42}+x^{30}+x^{28}\\
&+x^{24}+x^{22}+x^{18},\\
p_{12}(x)&=x^{121}+x^{119}+x^{118}+x^{116}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}\\
&+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{85}+x^{83}+x^{82}\\
&+x^{80}+x^{73}+x^{71}+x^{70}+x^{68}+x^{65}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}\\
&+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}\\
&+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}+x^{37}+x^{35}+x^{34}+x^{32}\\
&+x^{25}+x^{23}+x^{22}+x^{20}+x^{17}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}\\
&+x^{8}+x^{5}+x^{3}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{126}+x^{125}+x^{124}+x^{123}+x^{121}+x^{118}+x^{116}+x^{114}+x^{113}+x^{112}\\
&+x^{110}+x^{108}+x^{107}+x^{104}+x^{101}+x^{99}+x^{97}+x^{96}+x^{95}+x^{94}\\
&+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{88}+x^{83}+x^{82}+x^{80}+x^{79}+x^{78}\\
&+x^{76}+x^{74}+x^{73}+x^{71}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{63}+x^{60}\\
&+x^{59}+x^{57}+x^{55}+x^{50}+x^{48}+x^{46}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}\\
&+x^{38}+x^{36},\\
p_3(x)&=x^{124}+x^{122}+x^{120}+x^{118}+x^{116}+x^{114}+x^{112}+x^{100}+x^{98}+x^{96}\\
&+x^{92}+x^{90}+x^{88}+x^{76}+x^{74}+x^{72}+x^{62}+x^{54},\\
p_6(x)&=x^{123}+x^{121}+x^{120}+x^{118}+x^{117}+x^{115}+x^{114}+x^{112}+x^{111}+x^{109}\\
&+x^{108}+x^{107}+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}+x^{98}+x^{96}\\
&+x^{91}+x^{89}+x^{88}+x^{86}+x^{85}+x^{82}+x^{81}+x^{78}+x^{77}+x^{75}+x^{74}\\
&+x^{72}+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{48}+x^{43}+x^{41}+x^{40}+x^{39}\\
&+x^{38}+x^{36},\\
p_9(x)&=x^{122}+x^{120}+x^{118}+x^{114}+x^{112}+x^{104}+x^{102}+x^{92}+x^{84}+x^{82}\\
&+x^{78}+x^{76}+x^{74}+x^{72}+x^{60}+x^{52}+x^{48}+x^{46}+x^{42}+x^{30}+x^{28}\\
&+x^{24}+x^{22}+x^{18},\\
p_{12}(x)&=x^{121}+x^{119}+x^{118}+x^{116}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}
\end{aligned}
$$

$$+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}$$
$$+ x^{80} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59}$$
$$+ x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48}$$
$$+ x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32}$$
$$+ x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10}$$
$$+ x^{8} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{96} + x^{94} + x^{92} + x^{90} + x^{84} + x^{82} + x^{74} + x^{70} + x^{66} + x^{60} + x^{58} \\
&\quad + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{36}, \\
p_3(x) &= x^{120} + x^{118} + x^{112} + x^{106} + x^{104} + x^{98} + x^{92} + x^{84} + x^{82} + x^{78} + x^{76} \\
&\quad + x^{72} + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{24}, \\
p_6(x) &= x^{120} + x^{114} + x^{112} + x^{108} + x^{106} + x^{98} + x^{94} + x^{90} + x^{88} + x^{86} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{72} + x^{68} + x^{64} + x^{60} + x^{54} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{118} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{94} + x^{84} + x^{82} \\
&\quad + x^{78} + x^{76} + x^{62} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{42} + x^{30} + x^{28} \\
&\quad + x^{26} + x^{22} + x^{18} + x^{14} + x^{12}, \\
p_{12}(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} \\
&\quad + x^{60} + x^{58} + x^{56} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} \\
&\quad + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{116} + x^{114} + x^{112} + x^{106} + x^{100} + x^{94} + x^{92} + x^{86} \\
&\quad + x^{84} + x^{60} + x^{56} + x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{36}, \\
p_3(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{92} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{62} + x^{54} + x^{50} \\
&\quad + x^{48} + x^{46} + x^{44} + x^{38} + x^{36} + x^{30} + x^{28} + x^{26} + x^{24}, \\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{86} \\
&\quad + x^{80} + x^{78} + x^{76} + x^{72} + x^{68} + x^{64} + x^{56} + x^{54} + x^{44} + x^{40} + x^{38}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{36} + x^{34} + x^{32} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} \\
&+ x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{118} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{94} + x^{84} + x^{82} \\
&+ x^{78} + x^{76} + x^{62} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{42} + x^{30} + x^{28} \\
&+ x^{26} + x^{22} + x^{18} + x^{14} + x^{12}, \\
p_{12}(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{94} + x^{92} \\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} \\
&+ x^{60} + x^{58} + x^{56} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} \\
&+ x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{119} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} \\
&+ x^{104} + x^{103} + x^{99} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} \\
&+ x^{81} + x^{77} + x^{70} + x^{69} + x^{62} + x^{59} + x^{57} + x^{55} + x^{50} + x^{48} + x^{46} \\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96} \\
&+ x^{92} + x^{90} + x^{88} + x^{76} + x^{74} + x^{72} + x^{62} + x^{54}, \\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{111} + x^{109} \\
&+ x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} \\
&+ x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{74} \\
&+ x^{72} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} \\
&+ x^{38} + x^{36}, \\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{114} + x^{112} + x^{104} + x^{102} + x^{92} + x^{84} + x^{82} \\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{60} + x^{52} + x^{48} + x^{46} + x^{42} + x^{30} + x^{28} \\
&+ x^{24} + x^{22} + x^{18}, \\
p_{12}(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} \\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} \\
&+ x^{80} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\
&+ x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
&+ x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32} \\
&+ x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} \\
&+ x^{8} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{119} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109}\\
&\quad + x^{104} + x^{103} + x^{99} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82}\\
&\quad + x^{81} + x^{77} + x^{70} + x^{69} + x^{62} + x^{59} + x^{57} + x^{55} + x^{50} + x^{48} + x^{46}\\
&\quad + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96}\\
&\quad + x^{92} + x^{90} + x^{88} + x^{76} + x^{74} + x^{72} + x^{62} + x^{54},\\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114} + x^{112} + x^{111} + x^{109}\\
&\quad + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96}\\
&\quad + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{74}\\
&\quad + x^{72} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39}\\
&\quad + x^{38} + x^{36},\\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{114} + x^{112} + x^{104} + x^{102} + x^{92} + x^{84} + x^{82}\\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{60} + x^{52} + x^{48} + x^{46} + x^{42} + x^{30} + x^{28}\\
&\quad + x^{24} + x^{22} + x^{18},\\
p_{12}(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&\quad + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&\quad + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}\\
&\quad + x^{80} + x^{73} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59}\\
&\quad + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48}\\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32}\\
&\quad + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10}\\
&\quad + x^{8} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{122} + x^{120} + x^{116} + x^{114} + x^{110} + x^{106} + x^{102} + x^{96} + x^{94}\\
&\quad + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{76} + x^{72} + x^{70} + x^{66} + x^{62} + x^{60}\\
&\quad + x^{58} + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{36},\\
p_3(x) &= x^{120} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{84}\\
&\quad + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48}\\
&\quad + x^{46} + x^{42} + x^{28} + x^{26} + x^{24},\\
p_6(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98}\\
&\quad + x^{96} + x^{92} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66}\\
&\quad + x^{64} + x^{58} + x^{56} + x^{54} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{20}
\end{aligned}$$

$$\begin{aligned}
&\quad + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{124} + x^{120} + x^{112} + x^{110} + x^{106} + x^{104} + x^{98} + x^{94} + x^{84} + x^{82}\\
&\quad + x^{78} + x^{72} + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48}\\
&\quad + x^{44} + x^{42} + x^{30} + x^{26} + x^{24} + x^{14} + x^{12},\\
p_{12}(x) &= x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{92} + x^{90} + x^{88} + x^{84}\\
&\quad + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{44} + x^{42} + x^{40}\\
&\quad + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10}\\
&\quad + x^{8} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{111}\\
&\quad + x^{110} + x^{108} + x^{107} + x^{104} + x^{103} + x^{101} + x^{96} + x^{94} + x^{93} + x^{92}\\
&\quad + x^{89} + x^{88} + x^{86} + x^{81} + x^{77} + x^{76} + x^{72} + x^{69} + x^{66} + x^{62} + x^{61}\\
&\quad + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38}\\
&\quad + x^{36},\\
p_3(x) &= x^{125} + x^{124} + x^{120} + x^{119} + x^{118} + x^{114} + x^{112} + x^{111} + x^{108} + x^{105}\\
&\quad + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{86}\\
&\quad + x^{83} + x^{80} + x^{79} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&\quad + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54}\\
&\quad + x^{52} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_6(x) &= x^{125} + x^{124} + x^{115} + x^{108} + x^{107} + x^{104} + x^{101} + x^{93} + x^{92} + x^{91}\\
&\quad + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{70} + x^{67} + x^{66}\\
&\quad + x^{64} + x^{59} + x^{54} + x^{53} + x^{52} + x^{48} + x^{46} + x^{45} + x^{43} + x^{40} + x^{37}\\
&\quad + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{26} + x^{19} + x^{16} + x^{13} + x^{11} + x^{10}\\
&\quad + x^{8} + x^{5} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{109} + x^{107} + x^{106} + x^{104} + x^{77} + x^{75}\\
&\quad + x^{74} + x^{72} + x^{61} + x^{59} + x^{58} + x^{56} + x^{29} + x^{27} + x^{26} + x^{24},\\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111}\\
&\quad + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&\quad + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}\\
&\quad + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63}\\
&\quad + x^{62} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48}\\
&\quad + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32}\\
&\quad + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14}
\end{aligned}$$

$$+ x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} \\
&\quad + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{58} + x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{36}, \\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{119} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} \\
&\quad + x^{109} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} \\
&\quad + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{63} \\
&\quad + x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54}, \\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{102} + x^{96} \\
&\quad + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{58} + x^{52} + x^{50} \\
&\quad + x^{44} + x^{42} + x^{36}, \\
p_9(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} \\
&\quad + x^{116} + x^{114} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} \\
&\quad + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74} \\
&\quad + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{31} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_{12}(x) &= x^{128} + x^{112} + x^{100} + x^{96} + x^{84} + x^{64} + x^{52} + x^{48} + x^{36} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{119} + x^{118} + x^{116} + x^{115} + x^{112} + x^{109} + x^{100} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} + x^{78} + x^{66} + x^{63} \\
&\quad + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{45} + x^{36}, \\
p_3(x) &= x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} \\
&\quad + x^{111} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{97} + x^{95} + x^{94} + x^{92} \\
&\quad + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{65} + x^{63} + x^{62} \\
&\quad + x^{60} + x^{59} + x^{57} + x^{56} + x^{54},
\end{aligned}$$

$$
\begin{aligned}
p_6(x) &= x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{102} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{88} + x^{82} + x^{78} + x^{76} + x^{72} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{36}, \\
p_9(x) &= x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{98} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} \\
&\quad + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{52} + x^{50} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_{12}(x) &= x^{132} + x^{120} + x^{108} + x^{104} + x^{96} + x^{92} + x^{80} + x^{72} + x^{60} + x^{56} + x^{48} \\
&\quad + x^{44} + x^{32} + x^{24} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} \\
&\quad + x^{112} + x^{106} + x^{104} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{87} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} \\
&\quad + x^{50} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{119} + x^{118} + x^{114} + x^{112} + x^{111} + x^{109} + x^{108} \\
&\quad + x^{106} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} \\
&\quad + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{50} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_6(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{111} + x^{109} + x^{106} \\
&\quad + x^{105} + x^{102} + x^{101} + x^{97} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} \\
&\quad + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{55} + x^{54} + x^{53} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{27} + x^{24} \\
&\quad + x^{19} + x^{16} + x^{13} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1, \\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{109} + x^{107} + x^{106} + x^{104} + x^{77} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{61} + x^{59} + x^{58} + x^{56} + x^{29} + x^{27} + x^{26} + x^{24}, \\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111}
\end{aligned}
$$

$$+ x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}$$
$$+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}$$
$$+ x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63}$$
$$+ x^{62} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48}$$
$$+ x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32}$$
$$+ x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14}$$
$$+ x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{115} + x^{111} \\
& + x^{110} + x^{108} + x^{107} + x^{104} + x^{103} + x^{101} + x^{96} + x^{94} + x^{93} + x^{92} \\
& + x^{89} + x^{88} + x^{86} + x^{81} + x^{77} + x^{76} + x^{72} + x^{69} + x^{66} + x^{62} + x^{61} \\
& + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} \\
& + x^{36}, \\
p_3(x) = {} & x^{125} + x^{124} + x^{120} + x^{119} + x^{118} + x^{114} + x^{112} + x^{111} + x^{108} + x^{105} \\
& + x^{102} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{91} + x^{90} + x^{89} + x^{86} \\
& + x^{83} + x^{80} + x^{79} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} \\
& + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} \\
& + x^{52} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36}, \\
p_6(x) = {} & x^{125} + x^{124} + x^{115} + x^{108} + x^{107} + x^{104} + x^{101} + x^{93} + x^{92} + x^{91} \\
& + x^{90} + x^{88} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{77} + x^{70} + x^{67} + x^{66} \\
& + x^{64} + x^{59} + x^{54} + x^{53} + x^{52} + x^{48} + x^{46} + x^{45} + x^{43} + x^{40} + x^{37} \\
& + x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{26} + x^{19} + x^{16} + x^{13} + x^{11} + x^{10} \\
& + x^{8} + x^{5} + x^{3} + x^{2} + 1, \\
p_9(x) = {} & x^{125} + x^{123} + x^{122} + x^{120} + x^{109} + x^{107} + x^{106} + x^{104} + x^{77} + x^{75} \\
& + x^{74} + x^{72} + x^{61} + x^{59} + x^{58} + x^{56} + x^{29} + x^{27} + x^{26} + x^{24}, \\
p_{12}(x) = {} & x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} \\
& + x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} \\
& + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} \\
& + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63} \\
& + x^{62} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
& + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32} \\
& + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} \\
& + x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{119} + x^{118} + x^{116} + x^{115} + x^{112} + x^{109} + x^{100} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{83} + x^{81} + x^{80} + x^{78} + x^{66} + x^{63} \\
&\quad + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{45} + x^{36}, \\
p_3(x) &= x^{129} + x^{127} + x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{112} \\
&\quad + x^{111} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{97} + x^{95} + x^{94} + x^{92} \\
&\quad + x^{91} + x^{88} + x^{87} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{65} + x^{63} + x^{62} \\
&\quad + x^{60} + x^{59} + x^{57} + x^{56} + x^{54}, \\
p_6(x) &= x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{102} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{88} + x^{82} + x^{78} + x^{76} + x^{72} + x^{62} + x^{56} + x^{54} + x^{50} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{36}, \\
p_9(x) &= x^{131} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} \\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{98} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} \\
&\quad + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{52} + x^{50} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_{12}(x) &= x^{132} + x^{120} + x^{108} + x^{104} + x^{96} + x^{92} + x^{80} + x^{72} + x^{60} + x^{56} + x^{48} \\
&\quad + x^{44} + x^{32} + x^{24} + x^{20} + x^{12} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{129} + x^{128} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} \\
&\quad + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{103} + x^{101} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{61} + x^{58} + x^{56} + x^{54} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{36}, \\
p_3(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{119} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} \\
&\quad + x^{109} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{63}\\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54},\\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{102} + x^{96}\\
&+ x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{58} + x^{52} + x^{50}\\
&+ x^{44} + x^{42} + x^{36},\\
p_9(x) &= x^{127} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117}\\
&+ x^{116} + x^{114} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104}\\
&+ x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{79} + x^{77} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{61} + x^{60} + x^{59}\\
&+ x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{31} + x^{29} + x^{28}\\
&+ x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_{12}(x) &= x^{128} + x^{112} + x^{100} + x^{96} + x^{84} + x^{64} + x^{52} + x^{48} + x^{36} + x^4 + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{121} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{112} + x^{106} + x^{104} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{91} + x^{90} + x^{87}\\
&+ x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51}\\
&+ x^{50} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{119} + x^{118} + x^{114} + x^{112} + x^{111} + x^{109} + x^{108}\\
&+ x^{106} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&+ x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{74} + x^{73}\\
&+ x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{50} + x^{48} + x^{43} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36},\\
p_6(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{119} + x^{116} + x^{115} + x^{111} + x^{109} + x^{106}\\
&+ x^{105} + x^{102} + x^{101} + x^{97} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{84} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68}\\
&+ x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{55} + x^{54} + x^{53} + x^{48} + x^{46} + x^{45}\\
&+ x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{30} + x^{27} + x^{24}\\
&+ x^{19} + x^{16} + x^{13} + x^{11} + x^{10} + x^8 + x^5 + x^3 + x^2 + 1,\\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{109} + x^{107} + x^{106} + x^{104} + x^{77} + x^{75}\\
&+ x^{74} + x^{72} + x^{61} + x^{59} + x^{58} + x^{56} + x^{29} + x^{27} + x^{26} + x^{24},\\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111}\\
&+ x^{110} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}
\end{aligned}
$$

$$+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}$$
$$+ x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{63}$$
$$+ x^{62} + x^{60} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48}$$
$$+ x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{35} + x^{34} + x^{32}$$
$$+ x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14}$$
$$+ x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	177	[276, 277]	354	[199, 479]	531	[638, 6]	708	[769, 459]
1	[3, 4]	178	[278, 279]	355	[480, 481]	532	[639, 66]	709	[493, 398]
2	[5, 6]	179	[280, 111]	356	[482, 201]	533	[384, 640]	710	[770, 647]
3	[7, 8]	180	[281, 282]	357	[483, 191]	534	[641, 390]	711	[626, 666]
4	[9, 10]	181	[53, 134]	358	[453, 212]	535	[642, 214]	712	[771, 88]
5	[11, 12]	182	[159, 170]	359	[484, 73]	536	[643, 415]	713	[57, 66]
6	[13, 14]	183	[104, 283]	360	[485, 486]	537	[644, 223]	714	[99, 392]
7	[15, 16]	184	[284, 34]	361	[487, 207]	538	[424, 645]	715	[772, 199]
8	[17, 18]	185	[285, 286]	362	[91, 488]	539	[646, 647]	716	[401, 416]
9	[19, 20]	186	[287, 288]	363	[489, 459]	540	[476, 262]	717	[773, 463]
10	[21, 22]	187	[289, 290]	364	[490, 491]	541	[648, 437]	718	[774, 182]
11	[23, 24]	188	[291, 111]	365	[258, 439]	542	[387, 649]	719	[775, 776]
12	[25, 26]	189	[292, 293]	366	[492, 205]	543	[650, 651]	720	[777, 255]
13	[27, 28]	190	[294, 295]	367	[493, 82]	544	[4, 430]	721	[76, 96]
14	[29, 30]	191	[296, 297]	368	[494, 495]	545	[652, 233]	722	[778, 214]
15	[31, 32]	192	[298, 291]	369	[441, 224]	546	[414, 653]	723	[412, 433]
16	[33, 34]	193	[36, 143]	370	[16, 208]	547	[654, 385]	724	[469, 263]
17	[35, 36]	194	[299, 300]	371	[259, 29]	548	[212, 251]	725	[779, 412]
18	[37, 14]	195	[195, 195]	372	[496, 403]	549	[426, 402]	726	[780, 439]
19	[38, 39]	196	[301, 302]	373	[497, 498]	550	[177, 189]	727	[634, 214]
20	[40, 41]	197	[303, 130]	374	[499, 500]	551	[655, 429]	728	[432, 781]
21	[42, 43]	198	[304, 305]	375	[501, 385]	552	[656, 657]	729	[669, 481]
22	[44, 45]	199	[306, 117]	376	[434, 242]	553	[658, 223]	730	[782, 90]
23	[46, 45]	200	[307, 173]	377	[502, 73]	554	[637, 659]	731	[783, 255]
24	[47, 48]	201	[49, 34]	378	[503, 463]	555	[660, 229]	732	[180, 479]
25	[38, 49]	202	[308, 36]	379	[385, 475]	556	[661, 621]	733	[784, 675]
26	[50, 51]	203	[309, 120]	380	[504, 94]	557	[662, 621]	734	[386, 385]
27	[52, 53]	204	[310, 128]	381	[271, 202]	558	[663, 84]	735	[785, 211]
28	[39, 34]	205	[311, 41]	382	[222, 505]	559	[664, 197]	736	[786, 425]
29	[54, 55]	206	[312, 313]	383	[506, 455]	560	[665, 666]	737	[787, 216]

30	[56, 14]	207	[314, 139]	384	[123, 135]	561	[197, 207]	738	[788, 776]
31	[57, 58]	208	[32, 281]	385	[507, 375]	562	[17, 442]	739	[789, 92]
32	[59, 60]	209	[134, 277]	386	[278, 7]	563	[667, 69]	740	[790, 457]
33	[61, 20]	210	[310, 315]	387	[307, 175]	564	[668, 23]	741	[504, 232]
34	[62, 63]	211	[316, 51]	388	[508, 509]	565	[669, 670]	742	[791, 28]
35	[64, 65]	212	[317, 318]	389	[330, 309]	566	[461, 242]	743	[792, 74]
36	[66, 67]	213	[319, 295]	390	[510, 375]	567	[671, 250]	744	[446, 793]
37	[68, 69]	214	[320, 321]	391	[340, 511]	568	[672, 217]	745	[794, 495]
38	[70, 62]	215	[322, 323]	392	[512, 141]	569	[673, 236]	746	[420, 78]
39	[71, 72]	216	[324, 54]	393	[513, 514]	570	[674, 675]	747	[795, 430]
40	[73, 18]	217	[277, 41]	394	[515, 145]	571	[404, 93]	748	[796, 450]
41	[74, 75]	218	[325, 326]	395	[165, 281]	572	[676, 473]	749	[797, 500]
42	[76, 77]	219	[327, 154]	396	[516, 49]	573	[468, 63]	750	[798, 253]
43	[78, 79]	220	[328, 10]	397	[517, 158]	574	[677, 21]	751	[465, 799]
44	[80, 81]	221	[329, 132]	398	[345, 336]	575	[226, 444]	752	[800, 423]
45	[82, 80]	222	[330, 331]	399	[518, 519]	576	[185, 678]	753	[409, 202]
46	[83, 84]	223	[332, 111]	400	[520, 154]	577	[679, 459]	754	[801, 63]
47	[85, 86]	224	[333, 282]	401	[521, 522]	578	[680, 77]	755	[802, 211]
48	[87, 88]	225	[279, 162]	402	[523, 321]	579	[95, 77]	756	[803, 430]
49	[89, 90]	226	[42, 334]	403	[524, 154]	580	[464, 220]	757	[85, 649]
50	[91, 92]	227	[335, 336]	404	[333, 164]	581	[681, 221]	758	[804, 248]
51	[93, 94]	228	[337, 338]	405	[525, 526]	582	[640, 75]	759	[805, 776]
52	[95, 96]	229	[339, 281]	406	[527, 528]	583	[631, 640]	760	[471, 806]
53	[97, 98]	230	[340, 302]	407	[296, 529]	584	[617, 6]	761	[660, 246]
54	[99, 100]	231	[341, 41]	408	[321, 297]	585	[234, 493]	762	[798, 689]
55	[63, 101]	232	[342, 343]	409	[530, 531]	586	[682, 491]	763	[807, 79]
56	[102, 103]	233	[344, 122]	410	[116, 532]	587	[683, 237]	764	[808, 651]
57	[104, 105]	234	[345, 346]	411	[533, 534]	588	[684, 444]	765	[257, 475]
58	[106, 107]	235	[175, 166]	412	[535, 523]	589	[685, 394]	766	[790, 6]
59	[108, 109]	236	[347, 45]	413	[536, 537]	590	[683, 24]	767	[195, 723]
60	[110, 111]	237	[348, 145]	414	[538, 536]	591	[686, 687]	768	[574, 743]
61	[112, 113]	238	[45, 144]	415	[539, 143]	592	[688, 250]	769	[748, 346]
62	[114, 115]	239	[144, 44]	416	[540, 522]	593	[252, 689]	770	[809, 810]
63	[116, 117]	240	[349, 350]	417	[354, 541]	594	[690, 65]	771	[811, 291]
64	[8, 118]	241	[351, 352]	418	[542, 283]	595	[616, 260]	772	[812, 305]
65	[119, 120]	242	[353, 166]	419	[543, 282]	596	[691, 486]	773	[813, 814]
66	[121, 122]	243	[354, 113]	420	[308, 142]	597	[692, 186]	774	[599, 815]
67	[105, 123]	244	[331, 120]	421	[313, 526]	598	[693, 481]	775	[115, 318]
68	[124, 32]	245	[355, 356]	422	[110, 544]	599	[633, 465]	776	[282, 158]
69	[125, 117]	246	[357, 293]	423	[545, 120]	600	[694, 412]	777	[527, 737]
70	[126, 116]	247	[358, 359]	424	[546, 159]	601	[695, 626]	778	[816, 817]
71	[127, 128]	248	[360, 297]	425	[547, 548]	602	[269, 488]	779	[818, 819]
72	[129, 130]	249	[131, 323]	426	[549, 321]	603	[497, 221]	780	[820, 507]
73	[131, 132]	250	[361, 162]	427	[112, 541]	604	[396, 696]	781	[534, 582]

74	[133, 134]	251	[167, 51]	428	[550, 521]	605	[697, 66]	782	[744, 116]
75	[135, 130]	252	[362, 363]	429	[551, 523]	606	[698, 626]	783	[382, 7]
76	[136, 137]	253	[364, 365]	430	[536, 375]	607	[699, 651]	784	[106, 135]
77	[138, 139]	254	[366, 367]	431	[552, 54]	608	[260, 87]	785	[604, 344]
78	[140, 141]	255	[368, 132]	432	[553, 548]	609	[700, 225]	786	[547, 821]
79	[142, 143]	256	[369, 53]	433	[554, 297]	610	[70, 468]	787	[822, 758]
80	[144, 145]	257	[162, 370]	434	[555, 40]	611	[701, 244]	788	[301, 511]
81	[146, 146]	258	[7, 162]	435	[556, 318]	612	[664, 206]	789	[823, 814]
82	[147, 146]	259	[371, 56]	436	[535, 549]	613	[702, 431]	790	[766, 287]
83	[147, 147]	260	[372, 170]	437	[557, 195]	614	[413, 623]	791	[824, 825]
84	[148, 144]	261	[373, 37]	438	[518, 558]	615	[379, 569]	792	[826, 827]
85	[149, 150]	262	[374, 375]	439	[370, 365]	616	[703, 288]	793	[828, 115]
86	[151, 122]	263	[376, 167]	440	[559, 290]	617	[280, 544]	794	[762, 329]
87	[152, 153]	264	[377, 134]	441	[174, 173]	618	[704, 47]	795	[108, 552]
88	[55, 154]	265	[322, 132]	442	[560, 561]	619	[362, 509]	796	[829, 710]
89	[155, 156]	266	[378, 370]	443	[562, 563]	620	[705, 359]	797	[830, 561]
90	[157, 158]	267	[379, 326]	444	[564, 146]	621	[706, 321]	798	[811, 332]
91	[159, 160]	268	[380, 152]	445	[283, 123]	622	[707, 48]	799	[359, 548]
92	[161, 162]	269	[54, 327]	446	[565, 138]	623	[708, 153]	800	[831, 725]
93	[163, 123]	270	[355, 139]	447	[508, 363]	624	[709, 710]	801	[590, 832]
94	[164, 158]	271	[381, 309]	448	[566, 288]	625	[46, 711]	802	[833, 526]
95	[124, 165]	272	[369, 377]	449	[567, 55]	626	[588, 144]	803	[716, 752]
96	[166, 167]	273	[382, 279]	450	[568, 117]	627	[712, 305]	804	[834, 835]
97	[168, 109]	274	[383, 170]	451	[325, 569]	628	[507, 537]	805	[741, 138]
98	[169, 170]	275	[127, 315]	452	[570, 365]	629	[713, 14]	806	[836, 48]
99	[171, 172]	276	[384, 74]	453	[571, 167]	630	[714, 47]	807	[837, 159]
100	[109, 54]	277	[385, 258]	454	[572, 573]	631	[344, 715]	808	[838, 352]
101	[173, 166]	278	[386, 257]	455	[172, 288]	632	[716, 717]	809	[681, 498]
102	[174, 175]	279	[387, 86]	456	[574, 137]	633	[718, 719]	810	[839, 236]
103	[176, 139]	280	[388, 193]	457	[575, 561]	634	[295, 598]	811	[692, 678]
104	[15, 177]	281	[58, 67]	458	[576, 2]	635	[720, 522]	812	[840, 616]
105	[178, 179]	282	[60, 100]	459	[577, 146]	636	[721, 722]	813	[697, 58]
106	[180, 181]	283	[389, 390]	460	[381, 331]	637	[723, 195]	814	[841, 437]
107	[182, 30]	284	[391, 251]	461	[284, 578]	638	[724, 725]	815	[842, 629]
108	[59, 99]	285	[60, 392]	462	[372, 160]	639	[726, 313]	816	[480, 670]
109	[183, 184]	286	[393, 394]	463	[579, 54]	640	[727, 595]	817	[843, 242]
110	[185, 186]	287	[233, 395]	464	[580, 329]	641	[728, 531]	818	[61, 198]
111	[187, 188]	288	[6, 186]	465	[581, 582]	642	[723, 723]	819	[844, 647]
112	[16, 189]	289	[396, 397]	466	[529, 523]	643	[614, 519]	820	[780, 501]
113	[190, 191]	290	[398, 80]	467	[583, 156]	644	[729, 152]	821	[845, 20]
114	[192, 193]	291	[399, 400]	468	[584, 51]	645	[730, 10]	822	[768, 177]
115	[194, 195]	292	[401, 402]	469	[585, 586]	646	[306, 532]	823	[791, 209]
116	[196, 197]	293	[403, 221]	470	[587, 153]	647	[731, 281]	824	[846, 806]
117	[198, 28]	294	[404, 405]	471	[588, 348]	648	[732, 521]	825	[847, 675]

118	[199, 181]	295	[406, 274]	472	[589, 145]	649	[733, 107]	826	[654, 257]
119	[200, 201]	296	[407, 248]	473	[145, 45]	650	[276, 40]	827	[848, 626]
120	[202, 203]	297	[402, 408]	474	[590, 150]	651	[734, 134]	828	[849, 400]
121	[204, 205]	298	[192, 187]	475	[367, 573]	652	[31, 165]	829	[676, 850]
122	[206, 207]	299	[409, 265]	476	[591, 592]	653	[735, 290]	830	[677, 236]
123	[177, 208]	300	[405, 94]	477	[593, 594]	654	[736, 737]	831	[796, 419]
124	[27, 209]	301	[410, 405]	478	[373, 56]	655	[551, 549]	832	[851, 93]
125	[210, 211]	302	[411, 412]	479	[305, 595]	656	[52, 377]	833	[686, 486]
126	[19, 198]	303	[413, 65]	480	[155, 596]	657	[738, 122]	834	[196, 206]
127	[103, 212]	304	[414, 415]	481	[107, 130]	658	[728, 739]	835	[852, 491]
128	[213, 214]	305	[416, 408]	482	[597, 115]	659	[297, 523]	836	[853, 207]
129	[215, 216]	306	[417, 397]	483	[299, 598]	660	[376, 316]	837	[840, 268]
130	[208, 217]	307	[418, 419]	484	[599, 592]	661	[581, 719]	838	[632, 99]
131	[218, 219]	308	[420, 216]	485	[523, 554]	662	[360, 529]	839	[854, 855]
132	[220, 221]	309	[272, 270]	486	[600, 195]	663	[740, 345]	840	[837, 372]
133	[222, 223]	310	[66, 421]	487	[542, 105]	664	[741, 314]	841	[856, 857]
134	[224, 101]	311	[422, 423]	488	[601, 573]	665	[742, 350]	842	[858, 722]
135	[179, 225]	312	[424, 219]	489	[602, 345]	666	[711, 144]	843	[859, 304]
136	[226, 227]	313	[425, 426]	490	[176, 356]	667	[136, 743]	844	[860, 342]
137	[228, 229]	314	[427, 26]	491	[603, 123]	668	[148, 348]	845	[861, 595]
138	[230, 231]	315	[428, 429]	492	[604, 121]	669	[744, 306]	846	[742, 862]
139	[232, 233]	316	[430, 431]	493	[605, 350]	670	[745, 313]	847	[863, 827]
140	[234, 84]	317	[432, 433]	494	[285, 606]	671	[35, 142]	848	[864, 865]
141	[20, 28]	318	[193, 188]	495	[607, 143]	672	[583, 596]	849	[820, 536]
142	[103, 235]	319	[434, 435]	496	[608, 606]	673	[746, 514]	850	[866, 43]
143	[216, 79]	320	[436, 437]	497	[168, 552]	674	[121, 715]	851	[867, 810]
144	[84, 82]	321	[429, 426]	498	[609, 36]	675	[747, 166]	852	[868, 869]
145	[236, 22]	322	[438, 439]	499	[608, 286]	676	[748, 336]	853	[829, 764]
146	[237, 238]	323	[440, 182]	500	[118, 365]	677	[707, 749]	854	[438, 501]
147	[23, 237]	324	[441, 63]	501	[610, 367]	678	[12, 152]	855	[870, 463]
148	[239, 236]	325	[73, 442]	502	[371, 37]	679	[146, 147]	856	[204, 627]
149	[24, 240]	326	[443, 444]	503	[611, 327]	680	[750, 343]	857	[871, 74]
150	[241, 242]	327	[184, 445]	504	[612, 344]	681	[751, 752]	858	[801, 224]
151	[243, 75]	328	[446, 251]	505	[613, 172]	682	[753, 338]	859	[248, 799]
152	[24, 238]	329	[447, 448]	506	[614, 558]	683	[754, 43]	860	[485, 687]
153	[94, 233]	330	[273, 449]	507	[615, 616]	684	[347, 711]	861	[786, 429]
154	[244, 88]	331	[418, 450]	508	[617, 457]	685	[348, 44]	862	[872, 219]
155	[245, 246]	332	[451, 452]	509	[618, 23]	686	[540, 755]	863	[658, 505]
156	[247, 248]	333	[453, 235]	510	[619, 88]	687	[756, 300]	864	[615, 268]
157	[249, 250]	334	[454, 455]	511	[620, 621]	688	[757, 758]	865	[873, 250]
158	[235, 251]	335	[456, 233]	512	[622, 81]	689	[759, 367]	866	[874, 657]
159	[252, 253]	336	[457, 186]	513	[64, 623]	690	[760, 366]	867	[774, 29]
160	[254, 255]	337	[406, 449]	514	[624, 184]	691	[761, 534]	868	[195, 194]
161	[256, 199]	338	[458, 459]	515	[625, 626]	692	[762, 368]	869	[875, 452]

162	[257, 258]	339	[460, 260]	516	[492, 627]	693	[343, 293]	870	[876, 730]
163	[259, 182]	340	[461, 435]	517	[628, 629]	694	[549, 554]	871	[877, 878]
164	[260, 244]	341	[462, 463]	518	[215, 78]	695	[754, 334]	872	[879, 141]
165	[261, 262]	342	[464, 403]	519	[630, 472]	696	[763, 343]	873	[880, 559]
166	[263, 264]	343	[465, 466]	520	[631, 74]	697	[709, 764]	874	[824, 586]
167	[265, 203]	344	[467, 468]	521	[632, 60]	698	[765, 563]	875	[546, 372]
168	[261, 266]	345	[469, 270]	522	[189, 217]	699	[33, 578]	876	[239, 21]
169	[267, 79]	346	[470, 193]	523	[248, 466]	700	[766, 172]	877	[655, 425]
170	[268, 260]	347	[471, 472]	524	[245, 229]	701	[611, 55]	878	[881, 448]
171	[269, 92]	348	[81, 473]	525	[633, 248]	702	[8, 370]	879	[846, 472]
172	[270, 264]	349	[474, 103]	526	[439, 385]	703	[410, 93]	880	[81, 850]
173	[271, 265]	350	[475, 439]	527	[634, 635]	704	[701, 87]	881	[298, 332]
174	[272, 263]	351	[476, 266]	528	[636, 65]	705	[58, 421]		
175	[273, 274]	352	[477, 472]	529	[191, 637]	706	[767, 191]		
176	[275, 94]	353	[478, 188]	530	[194, 194]	707	[768, 16]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	127	1	254	1	381	0	508	1	635	0	762	1
1	0	128	1	255	0	382	1	509	0	636	0	763	0
2	0	129	1	256	0	383	0	510	0	637	1	764	1
3	0	130	0	257	0	384	0	511	1	638	1	765	0
4	0	131	0	258	0	385	1	512	1	639	0	766	0
5	0	132	1	259	0	386	0	513	1	640	0	767	0
6	1	133	1	260	1	387	1	514	1	641	0	768	1
7	0	134	0	261	1	388	1	515	0	642	1	769	0
8	1	135	1	262	1	389	1	516	1	643	0	770	0
9	0	136	0	263	1	390	0	517	1	644	1	771	1
10	0	137	0	264	0	391	0	518	1	645	1	772	0
11	0	138	0	265	1	392	1	519	1	646	0	773	1
12	0	139	0	266	0	393	1	520	1	647	1	774	1
13	1	140	1	267	1	394	0	521	1	648	1	775	1
14	1	141	0	268	0	395	1	522	1	649	1	776	0
15	0	142	1	269	1	396	1	523	1	650	0	777	0
16	1	143	0	270	0	397	1	524	0	651	0	778	0
17	1	144	0	271	0	398	1	525	1	652	0	779	1
18	1	145	1	272	0	399	1	526	1	653	1	780	0
19	0	146	1	273	1	400	1	527	0	654	1	781	0
20	0	147	0	274	0	401	1	528	0	655	0	782	1
21	0	148	0	275	1	402	1	529	0	656	1	783	1
22	0	149	0	276	0	403	0	530	1	657	1	784	1
23	0	150	0	277	1	404	0	531	1	658	0	785	1
24	0	151	0	278	0	405	1	532	0	659	1	786	1
25	0	152	0	279	1	406	0	533	0	660	1	787	1

26	0	153	1	280	1	407	0	534	0	661	0	788	1
27	1	154	1	281	1	408	0	535	1	662	1	789	0
28	1	155	0	282	0	409	1	536	0	663	0	790	0
29	1	156	0	283	1	410	1	537	1	664	0	791	0
30	0	157	0	284	0	411	0	538	1	665	0	792	1
31	0	158	1	285	0	412	1	539	0	666	1	793	0
32	0	159	0	286	1	413	0	540	0	667	0	794	1
33	1	160	1	287	1	414	1	541	1	668	0	795	0
34	0	161	0	288	1	415	0	542	1	669	1	796	0
35	1	162	0	289	1	416	0	543	0	670	1	797	1
36	0	163	0	290	1	417	0	544	0	671	1	798	1
37	1	164	1	291	1	418	1	545	0	672	1	799	1
38	0	165	1	292	1	419	0	546	1	673	0	800	0
39	1	166	1	293	0	420	0	547	1	674	0	801	1
40	0	167	1	294	0	421	1	548	0	675	0	802	0
41	1	168	1	295	0	422	0	549	0	676	0	803	1
42	0	169	1	296	0	423	0	550	0	677	1	804	1
43	1	170	0	297	1	424	1	551	0	678	0	805	0
44	0	171	1	298	0	425	1	552	1	679	1	806	0
45	0	172	0	299	1	426	0	553	0	680	1	807	0
46	0	173	0	300	1	427	1	554	1	681	0	808	1
47	0	174	0	301	1	428	0	555	1	682	1	809	0
48	0	175	1	302	0	429	0	556	0	683	1	810	1
49	0	176	1	303	0	430	0	557	1	684	1	811	1
50	0	177	0	304	1	431	1	558	0	685	1	812	0
51	0	178	0	305	0	432	0	559	0	686	0	813	1
52	1	179	1	306	0	433	1	560	0	687	1	814	0
53	1	180	1	307	1	434	1	561	0	688	0	815	1
54	1	181	1	308	0	435	0	562	1	689	0	816	0
55	1	182	0	309	0	436	1	563	0	690	1	817	1
56	0	183	0	310	0	437	1	564	0	691	1	818	1
57	0	184	0	311	0	438	1	565	1	692	1	819	1
58	1	185	0	312	1	439	1	566	0	693	0	820	0
59	0	186	1	313	1	440	0	567	1	694	0	821	1
60	0	187	1	314	1	441	0	568	1	695	1	822	1
61	1	188	1	315	0	442	0	569	0	696	0	823	0
62	0	189	1	316	0	443	1	570	0	697	1	824	0
63	1	190	0	317	0	444	0	571	0	698	0	825	0
64	1	191	0	318	0	445	1	572	0	699	1	826	1
65	1	192	0	319	1	446	1	573	1	700	0	827	1
66	0	193	0	320	1	447	1	574	1	701	0	828	0
67	0	194	1	321	0	448	0	575	0	702	1	829	0
68	1	195	0	322	1	449	1	576	0	703	1	830	1
69	0	196	1	323	0	450	1	577	1	704	0	831	0

70	0	197	0	324	0	451	0	578	1	705	1	832	1
71	1	198	1	325	0	452	0	579	1	706	0	833	0
72	1	199	0	326	1	453	0	580	0	707	1	834	1
73	0	200	1	327	0	454	0	581	0	708	0	835	0
74	1	201	0	328	1	455	0	582	0	709	1	836	0
75	1	202	0	329	1	456	1	583	1	710	0	837	0
76	0	203	0	330	1	457	0	584	1	711	1	838	1
77	0	204	0	331	1	458	0	585	1	712	1	839	1
78	1	205	0	332	0	459	1	586	1	713	0	840	0
79	1	206	1	333	0	460	0	587	1	714	1	841	0
80	0	207	1	334	0	461	0	588	1	715	0	842	1
81	1	208	0	335	1	462	1	589	1	716	1	843	1
82	0	209	0	336	0	463	1	590	1	717	1	844	1
83	0	210	0	337	0	464	0	591	0	718	1	845	1
84	0	211	0	338	0	465	0	592	0	719	1	846	0
85	0	212	0	339	0	466	0	593	0	720	0	847	0
86	0	213	1	340	0	467	1	594	1	721	0	848	1
87	0	214	1	341	1	468	1	595	1	722	0	849	0
88	1	215	1	342	0	469	1	596	1	723	1	850	0
89	0	216	0	343	0	470	1	597	1	724	1	851	1
90	0	217	1	344	1	471	1	598	0	725	1	852	0
91	0	218	0	345	1	472	1	599	1	726	0	853	0
92	0	219	0	346	1	473	1	600	0	727	0	854	1
93	0	220	1	347	1	474	1	601	1	728	0	855	0
94	1	221	1	348	1	475	1	602	1	729	1	856	0
95	1	222	1	349	1	476	0	603	1	730	1	857	0
96	1	223	0	350	1	477	0	604	1	731	1	858	1
97	1	224	0	351	0	478	1	605	1	732	1	859	1
98	1	225	1	352	0	479	0	606	0	733	1	860	1
99	1	226	0	353	1	480	0	607	1	734	0	861	1
100	0	227	1	354	0	481	0	608	1	735	1	862	0
101	0	228	0	355	0	482	1	609	0	736	1	863	0
102	0	229	0	356	1	483	1	610	0	737	1	864	1
103	1	230	0	357	1	484	1	611	0	738	1	865	1
104	0	231	1	358	0	485	1	612	0	739	0	866	0
105	0	232	0	359	1	486	0	613	1	740	0	867	1
106	1	233	1	360	1	487	1	614	0	741	0	868	0
107	0	234	1	361	1	488	1	615	1	742	0	869	1
108	0	235	1	362	0	489	1	616	1	743	1	870	0
109	0	236	1	363	1	490	1	617	1	744	1	871	0
110	0	237	1	364	1	491	1	618	0	745	1	872	0
111	1	238	0	365	0	492	1	619	0	746	0	873	1
112	1	239	0	366	1	493	1	620	1	747	0	874	0
113	0	240	1	367	1	494	0	621	0	748	0	875	1

114	0	241	0	368	0	495	1	622	1	749	1	876	0
115	1	242	1	369	0	496	1	623	0	750	1	877	0
116	1	243	0	370	1	497	1	624	1	751	0	878	0
117	1	244	1	371	0	498	0	625	0	752	0	879	0
118	0	245	0	372	1	499	1	626	1	753	1	880	1
119	1	246	1	373	1	500	0	627	1	754	1	881	0
120	0	247	0	374	1	501	0	628	1	755	0		
121	0	248	1	375	0	502	0	629	0	756	1		
122	1	249	0	376	1	503	0	630	1	757	0		
123	0	250	1	377	0	504	0	631	1	758	1		
124	1	251	1	378	0	505	1	632	1	759	0		
125	0	252	0	379	1	506	0	633	1	760	1		
126	0	253	1	380	0	507	1	634	0	761	1		

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