PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^4 + z^3 + z^2 + z, z^2 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^4 + z^3 + z^2 + z, z^2 + z + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 15:46:58 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 32.8 minutes CPU time used.

Theorem 1.1. *When* $(a,b)=(z^4+z^3+z^2+z, z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{16}+z^{15}+z^{13}+z^{12}+z^8+z^6+z^5+z^2+z+1,\\
p_1(z)&=z^{22}+z^{21}+z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^{11}+z^9+z^8+z^7+z,\\
p_2(z)&=z^{24}+z^{20}+z^{18}+z^{12}+z^{10}+z^6+z^4+z^2,\\
p_3(z)&=z^{22}+z^{21}+z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^{11}+z^9+z^8+z^7+z,\\
p_4(z)&=z^{20}+z^{16}+z^{15}+z^{13}+z^{10}+z^8+z^5+z,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{18}+z^{15}+z^{13}+z^8+z^6+z^5+z^2+z,\\
p_1(z)&=z^{22}+z^{21}+z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^{11}+z^9+z^8+z^7+z,\\
p_2(z)&=z^{24}+z^{20}+z^{18}+z^{12}+z^{10}+z^6+z^4+z^2,\\
p_3(z)&=z^{22}+z^{21}+z^{20}+z^{17}+z^{15}+z^{13}+z^{12}+z^{11}+z^9+z^8+z^7+z,\\
p_4(z)&=z^{20}+z^{15}+z^{13}+z^{12}+z^{10}+z^8+z^5+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$
$$M^o = W^e \cdot M^e, \tag{2.4}$$
$$W^e = M^o \cdot W^o, \tag{2.5}$$
$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity
$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$
the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that
$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$
is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that
$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$
is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define
$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad\text{and}\qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*
$$\begin{aligned}M^e[:12u] = {}& M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u]\\ &+ x^{2u}M^e[:6u] + x^{4u}M^e[:4u] + x^{5u}M^e[:3u] + x^{6u}M^e[:2u]\\ &+ x^{7u}M^e[:u] + x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{6u}M^e[:3u]\\ &+ x^{7u}M^e[:2u] + x^{8u}M^e[:u] + x^{5u}M^e[:6u] + x^{7u}M^e[:4u]\\ &+ x^{8u}M^e[:3u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{8u}M^e[:4u]\\ &+ x^{10u}M^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^e,\end{aligned}$$

$$\begin{aligned}W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u]\\&\quad + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{5u}W^e[:3u] + x^{6u}W^e[:2u]\\&\quad + x^{7u}W^e[:u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u]\\&\quad + x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{5u}W^e[:6u] + x^{7u}W^e[:4u]\\&\quad + x^{8u}W^e[:3u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{8u}W^e[:4u]\\&\quad + x^{10u}W^e[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^e.\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u]\\&\quad + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{5u}M^o[:3u] + x^{6u}M^o[:2u]\\&\quad + x^{7u}M^o[:u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u]\\&\quad + x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{5u}M^o[:6u] + x^{7u}M^o[:4u]\\&\quad + x^{8u}M^o[:3u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{8u}M^o[:4u]\\&\quad + x^{10u}M^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^o,\\W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u]\\&\quad + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{5u}W^o[:3u] + x^{6u}W^o[:2u]\\&\quad + x^{7u}W^o[:u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u]\\&\quad + x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{5u}W^o[:6u] + x^{7u}W^o[:4u]\\&\quad + x^{8u}W^o[:3u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{8u}W^o[:4u]\\&\quad + x^{10u}W^o[:2u] + (x^{6u} + x^{8u} + x^{9u} + x^{11u})R^o.\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{5u}T[:u] + x^{2u}T[:6u]\\&\quad + x^{4u}T[:4u] + x^{5u}T[:3u] + x^{6u}T[:2u] + x^{7u}T[:u] + x^{3u}T[:6u]\\&\quad + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{7u}T[:2u] + x^{8u}T[:u] + x^{5u}T[:6u]\\&\quad + x^{7u}T[:4u] + x^{8u}T[:3u] + x^{9u}T[:2u] + x^{10u}T[:u] + x^{8u}T[:4u]\\&\quad + x^{10u}T[:2u].\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}0 &= x^{5u}T[u:2u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u] + x^{2u}T[4u:5u]\\&\quad + T[6u:7u],\\0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u]\\&\quad + x^{2u}T[5u:6u] + T[7u:8u],\\0 &= x^{8u}T[:u] + x^{6u}T[2u:3u] + x^{3u}T[5u:6u] + T[8u:9u],\\0 &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{5u}T[4u:5u] + T[9u:10u],\end{aligned}$$

$$0 = x^{9u}T[u\!:\!2u] + x^{7u}T[3u\!:\!4u] + x^{5u}T[5u\!:\!6u] + T[10u\!:\!11u],$$
$$0 = x^{10u}T[u\!:\!2u] + x^{8u}T[3u\!:\!4u] + T[11u\!:\!12u],$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1472 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.19}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.20}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.21}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1001)), \tag{2.22}$$
$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1010)), \tag{2.23}$$
$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)), \tag{2.25}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))$$

$$+\tau(A(s,101))+\tau(A(s,111)), \tag{2.26}$$

$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,101))+\tau(A(s,1000)), \tag{2.27}$$

$$0=\tau(A(s,\epsilon))+\tau(A(s,10))+\tau(A(s,100))+\tau(A(s,1001)), \tag{2.28}$$

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,101))+\tau(A(s,1010)), \tag{2.29}$$

$$0=\tau(A(s,1))+\tau(A(s,11))+\tau(A(s,1011)), \tag{2.30}$$

for $s=A(i,0^{2k})$. We find that $(A(i,0^{46}),E_{46})=(A(i,0^{18}),E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2\le k\le 23$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k\ge 2$, and $u=2^{2k-1}$,*

$$\begin{aligned}M_{2k}&=M^e[:6u]+x^{2u}M^e[:4u]+x^{3u}M^e[:3u]+x^{4u}M^e[:2u]+x^{5u}M^e[:u]\\&\quad+x^{6u}R^e,\\W_{2k}&=W^e[:6u]+x^{2u}W^e[:4u]+x^{3u}W^e[:3u]+x^{4u}W^e[:2u]+x^{5u}W^e[:u]\\&\quad+x^{6u}R^e.\end{aligned}$$

For all integer $k\ge 2$, and $u=2^{2k}$,

$$\begin{aligned}M_{2k+1}&=M^o[:6u]+x^{2u}M^o[:4u]+x^{3u}M^o[:3u]+x^{4u}M^o[:2u]+x^{5u}M^o[:u]\\&\quad+x^{6u}R^o,\\W_{2k+1}&=W^o[:6u]+x^{2u}W^o[:4u]+x^{3u}W^o[:3u]+x^{4u}W^o[:2u]+x^{5u}W^o[:u]\\&\quad+x^{6u}R^o.\end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n\ge 2$, we claim that

$$\begin{aligned}M_{2k}\wedge W_{2k}&\Rightarrow M_{2k+1}\wedge W_{2k+1},\\M_{2k+1}\wedge W_{2k+1}&\Rightarrow M_{2k+2}\wedge W_{2k+2}.\end{aligned}$$

We give the proof of

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1},$$

the other ones being similar. Set $u=2^{2k}$ and $v=2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned}W_{2k}M_{2k}&=\big(W^e[:6v]+x^{2v}W^e[:4v]+x^{3v}W^e[:3v]+x^{4v}W^e[:2v]+x^{5v}W^e[:v]\\&\quad+x^{6u}R^e\big)\times\big(M^e[:6v]+x^{2v}M^e[:4v]+x^{3v}M^e[:3v]+x^{4v}M^e[:2v]\\&\quad+x^{5v}M^e[:v]+x^{6u}R^e\big);\end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned}&W^e[:12v]M^e[:12v]+x^{4v}W^e[:8v]M^e[:8v]+x^{6v}W^e[:6v]M^e[:6v]\\&\qquad+x^{8v}W^e[:4v]M^e[:4v]+x^{10v}W^e[:2v]M^e[:2v].\end{aligned}$$

For all $n \leq 12$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned}
X &:= x^v, \\
a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\
b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\
c &:= R^e.
\end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned}
\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\
\lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\
\lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\
\lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.
\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{16} + x^{12} + x^{11} + x^9 + x^8 + x^6, \\
q_1(x) &= x^{23} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^7 + x^4 + x^3 + x^2, \\
q_2(x) &= x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^6 + x^4 + 1, \\
q_3(x) &= x^{23} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 + x^7 + x^4 + x^3 + x^2, \\
q_4(x) &= x^{23} + x^{19} + x^{16} + x^{14} + x^{11} + x^9 + x^8 + x^4.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate

$f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{106} + x^{104} + x^{96} + x^{94} + x^{92} + x^{82} \\
&\quad + x^{80} + x^{74} + x^{70} + x^{66} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{36} + x^{34} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{116} + x^{112} + x^{100} + x^{92} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} \\
&\quad + x^{72} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{48} + x^{44} + x^{34} + x^{32} \\
&\quad + x^{30} + x^{28} + x^{22} + x^{18}, \\
p_6(x) &= x^{114} + x^{112} + x^{110} + x^{104} + x^{102} + x^{98} + x^{94} + x^{84} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{72} + x^{68} + x^{64} + x^{62} + x^{56} + x^{54} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} \\
&\quad + x^{32} + x^{28} + x^{26} + x^{10} + x^{8} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{92} + x^{90} + x^{88} + x^{84} \\
&\quad + x^{78} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{42} \\
&\quad + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{20} + x^{18} + x^{12} + x^{4}, \\
p_{12}(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{115} + x^{114} \\
&\quad + x^{113} + x^{112} + x^{110} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} \\
&\quad + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{88} + x^{85} + x^{83} \\
&\quad + x^{80} + x^{79} + x^{78} + x^{73} + x^{71} + x^{70} + x^{69} + x^{65} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{43} \\
&\quad + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} + x^{30}, \\
p_3(x) &= x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{109} + x^{108} \\
&\quad + x^{102} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{62} + x^{60} \\
&\quad + x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44}
\end{aligned}$$

$$
\begin{aligned}
&+x^{40}+x^{37}+x^{36}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}+x^{23}+x^{19}+x^{18},\\
p_6(x)&=x^{123}+x^{121}+x^{120}+x^{118}+x^{115}+x^{113}+x^{112}+x^{110}+x^{109}+x^{107}\\
&+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}\\
&+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}+x^{84}+x^{83}+x^{80}+x^{75}+x^{73}+x^{72}\\
&+x^{71}+x^{70}+x^{69}+x^{68}+x^{66}+x^{63}+x^{60}+x^{51}+x^{49}+x^{48}+x^{46}\\
&+x^{39}+x^{36}+x^{35}+x^{32}+x^{31}+x^{28}+x^{27}+x^{24}+x^{21}+x^{19}+x^{18}\\
&+x^{16}+x^{15}+x^{12},\\
p_9(x)&=x^{122}+x^{120}+x^{119}+x^{117}+x^{116}+x^{115}+x^{114}+x^{106}+x^{103}+x^{102}\\
&+x^{100}+x^{98}+x^{97}+x^{96}+x^{95}+x^{91}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}\\
&+x^{82}+x^{79}+x^{78}+x^{76}+x^{74}+x^{73}+x^{71}+x^{70}+x^{69}+x^{68}+x^{64}\\
&+x^{61}+x^{58}+x^{57}+x^{55}+x^{53}+x^{52}+x^{51}+x^{50}+x^{48}+x^{45}+x^{41}\\
&+x^{40}+x^{36}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{26}+x^{23}+x^{19}+x^{18}\\
&+x^{12}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6},\\
p_{12}(x)&=x^{121}+x^{119}+x^{118}+x^{116}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}\\
&+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}\\
&+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{85}+x^{83}+x^{82}\\
&+x^{80}+x^{25}+x^{23}+x^{22}+x^{20}+x^{17}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}\\
&+x^{10}+x^{8}+x^{5}+x^{3}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{126}+x^{125}+x^{124}+x^{123}+x^{121}+x^{120}+x^{119}+x^{118}+x^{115}+x^{114}\\
&+x^{113}+x^{112}+x^{110}+x^{106}+x^{105}+x^{103}+x^{102}+x^{101}+x^{100}\\
&+x^{99}+x^{98}+x^{96}+x^{95}+x^{93}+x^{92}+x^{90}+x^{89}+x^{88}+x^{85}+x^{83}\\
&+x^{80}+x^{79}+x^{78}+x^{73}+x^{71}+x^{70}+x^{69}+x^{65}+x^{64}+x^{62}+x^{60}\\
&+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}+x^{43}\\
&+x^{40}+x^{39}+x^{37}+x^{36}+x^{35}+x^{34}+x^{30},\\
p_3(x)&=x^{124}+x^{122}+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{112}+x^{109}+x^{108}\\
&+x^{102}+x^{99}+x^{98}+x^{96}+x^{94}+x^{93}+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}\\
&+x^{80}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{72}+x^{69}+x^{68}+x^{62}+x^{60}\\
&+x^{59}+x^{57}+x^{56}+x^{55}+x^{52}+x^{51}+x^{49}+x^{47}+x^{46}+x^{45}+x^{44}\\
&+x^{40}+x^{37}+x^{36}+x^{30}+x^{28}+x^{27}+x^{25}+x^{24}+x^{23}+x^{19}+x^{18},\\
p_6(x)&=x^{123}+x^{121}+x^{120}+x^{118}+x^{115}+x^{113}+x^{112}+x^{110}+x^{109}+x^{107}\\
&+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{98}+x^{97}+x^{95}+x^{94}+x^{92}\\
&+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}+x^{84}+x^{83}+x^{80}+x^{75}+x^{73}+x^{72}\\
&+x^{71}+x^{70}+x^{69}+x^{68}+x^{66}+x^{63}+x^{60}+x^{51}+x^{49}+x^{48}+x^{46}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{21} + x^{19} + x^{18} \\
&+ x^{16} + x^{15} + x^{12}, \\
p_9(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{106} + x^{103} + x^{102} \\
&+ x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
&+ x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} \\
&+ x^{61} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{41} \\
&+ x^{40} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{23} + x^{19} + x^{18} \\
&+ x^{12} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} \\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} \\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82} \\
&+ x^{80} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} \\
&+ x^{10} + x^{8} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{122} + x^{118} + x^{114} + x^{106} + x^{100} + x^{98} + x^{96} + x^{90} + x^{86} \\
&+ x^{82} + x^{78} + x^{76} + x^{72} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{38} + x^{36}, \\
p_3(x) &= x^{126} + x^{124} + x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{96} \\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{60} \\
&+ x^{58} + x^{52} + x^{50} + x^{48} + x^{38} + x^{36} + x^{34} + x^{18} + x^{12}, \\
p_6(x) &= x^{126} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} \\
&+ x^{100} + x^{94} + x^{92} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{66} + x^{64} \\
&+ x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{46} + x^{38} + x^{36} + x^{30} + x^{26} + x^{24} \\
&+ x^{22} + x^{20} + x^{18} + x^{10} + x^{2} + 1, \\
p_9(x) &= x^{124} + x^{120} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{88} \\
&+ x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{64} + x^{58} + x^{56} + x^{54} + x^{46} + x^{42} \\
&+ x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{10} + x^{4}, \\
p_{12}(x) &= x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{92} + x^{90} + x^{88} + x^{84} \\
&+ x^{82} + x^{80} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} \\
&+ x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{126} + x^{124} + x^{118} + x^{112} + x^{108} + x^{104} + x^{94} + x^{92} \\
&+ x^{90} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{62} + x^{60} + x^{58} + x^{54} + x^{42} \\
&+ x^{32} + x^{28} + x^{26} + x^{24},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{108} + x^{102} + x^{100} + x^{96} \\
&\quad + x^{94} + x^{86} + x^{84} + x^{82} + x^{80} + x^{74} + x^{70} + x^{68} + x^{64} + x^{58} + x^{56} \\
&\quad + x^{46} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18}, \\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{106} + x^{104} + x^{100} + x^{98} \\
&\quad + x^{74} + x^{66} + x^{64} + x^{62} + x^{60} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} \\
&\quad + x^{38} + x^{32} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^{8} + x^{2} + 1, \\
p_9(x) &= x^{124} + x^{120} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{88} \\
&\quad + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{64} + x^{58} + x^{56} + x^{54} + x^{46} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{10} + x^{4}, \\
p_{12}(x) &= x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{92} + x^{90} + x^{88} + x^{84} \\
&\quad + x^{82} + x^{80} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} \\
&\quad + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} \\
&\quad + x^{110} + x^{107} + x^{106} + x^{105} + x^{103} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} \\
&\quad + x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{73} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{67} + x^{65} + x^{64} + x^{60} + x^{58} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} \\
&\quad + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{32} + x^{30}, \\
p_3(x) &= x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{109} + x^{108} \\
&\quad + x^{102} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{62} + x^{60} \\
&\quad + x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} \\
&\quad + x^{40} + x^{37} + x^{36} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{19} + x^{18}, \\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{118} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107} \\
&\quad + x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{75} + x^{73} + x^{72} \\
&\quad + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{60} + x^{51} + x^{49} + x^{48} + x^{46} \\
&\quad + x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{21} + x^{19} + x^{18} \\
&\quad + x^{16} + x^{15} + x^{12}, \\
p_9(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{106} + x^{103} + x^{102} \\
&\quad + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} \\
&\quad + x^{61} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{41} \\
&\quad + x^{40} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{23} + x^{19} + x^{18}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6,\\
p_{12}(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}\\
&+ x^{80} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{10} + x^8 + x^5 + x^3 + x^2 + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112}\\
&+ x^{110} + x^{107} + x^{106} + x^{105} + x^{103} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91}\\
&+ x^{89} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{73} + x^{72} + x^{70} + x^{68}\\
&+ x^{67} + x^{65} + x^{64} + x^{60} + x^{58} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44}\\
&+ x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{32} + x^{30},\\
p_3(x) &= x^{124} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{109} + x^{108}\\
&+ x^{102} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{80} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{62} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44}\\
&+ x^{40} + x^{37} + x^{36} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{19} + x^{18},\\
p_6(x) &= x^{123} + x^{121} + x^{120} + x^{118} + x^{115} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107}\\
&+ x^{106} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{92}\\
&+ x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{75} + x^{73} + x^{72}\\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{60} + x^{51} + x^{49} + x^{48} + x^{46}\\
&+ x^{39} + x^{36} + x^{35} + x^{32} + x^{31} + x^{28} + x^{27} + x^{24} + x^{21} + x^{19} + x^{18}\\
&+ x^{16} + x^{15} + x^{12},\\
p_9(x) &= x^{122} + x^{120} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{106} + x^{103} + x^{102}\\
&+ x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{91} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84}\\
&+ x^{82} + x^{79} + x^{78} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64}\\
&+ x^{61} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{41}\\
&+ x^{40} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{23} + x^{19} + x^{18}\\
&+ x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6,\\
p_{12}(x) &= x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{85} + x^{83} + x^{82}\\
&+ x^{80} + x^{25} + x^{23} + x^{22} + x^{20} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{10} + x^8 + x^5 + x^3 + x^2 + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{96} + x^{94} + x^{90} + x^{86} \\
&\quad + x^{78} + x^{74} + x^{70} + x^{68} + x^{64} + x^{60} + x^{52} + x^{46} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{114} + x^{112} + x^{102} + x^{96} + x^{90} + x^{86} + x^{80} + x^{76} + x^{74} + x^{68} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{32} + x^{28} + x^{26} + x^{24} + x^{12}, \\
p_6(x) &= x^{116} + x^{114} + x^{108} + x^{106} + x^{102} + x^{86} + x^{82} + x^{78} + x^{76} + x^{70} + x^{68} \\
&\quad + x^{64} + x^{56} + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{34} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{20} + x^{16} + x^{12} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{92} + x^{90} + x^{88} + x^{84} \\
&\quad + x^{78} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52} + x^{48} + x^{44} + x^{42} \\
&\quad + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{20} + x^{18} + x^{12} + x^{4}, \\
p_{12}(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{106} + x^{104} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} \\
&\quad + x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{32} + x^{31} + x^{30} \\
&\quad + x^{29} + x^{28} + x^{27} + x^{26} + x^{24}, \\
p_3(x) &= x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{87} + x^{84} + x^{83} + x^{81} + x^{77} + x^{75} + x^{74} + x^{71} \\
&\quad + x^{68} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} \\
&\quad + x^{45} + x^{44} + x^{43} + x^{42} + x^{37} + x^{34} + x^{33} + x^{29} + x^{28} + x^{27} + x^{25} \\
&\quad + x^{24} + x^{21} + x^{19} + x^{18} + x^{16}, \\
p_6(x) &= x^{125} + x^{124} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{107} + x^{106} + x^{105} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} \\
&\quad + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{76} + x^{74} + x^{72} \\
&\quad + x^{69} + x^{66} + x^{64} + x^{63} + x^{58} + x^{56} + x^{54} + x^{48} + x^{45} + x^{44} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} \\
&\quad + x^{25} + x^{23} + x^{21} + x^{18} + x^{17} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{125}+x^{123}+x^{122}+x^{120}+x^{113}+x^{111}+x^{110}+x^{108}+x^{97}+x^{95}\\
&\quad+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{88}+x^{29}+x^{27}+x^{26}+x^{24}+x^{17}\\
&\quad+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^{8},\\
p_{12}(x) &= x^{125}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{118}+x^{116}+x^{109}+x^{107}\\
&\quad+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{96}\\
&\quad+x^{85}+x^{83}+x^{82}+x^{80}+x^{29}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}\\
&\quad+x^{20}+x^{5}+x^{3}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{136}+x^{133}+x^{132}+x^{129}+x^{127}+x^{126}+x^{123}+x^{122}+x^{118}+x^{113}\\
&\quad+x^{112}+x^{110}+x^{109}+x^{108}+x^{107}+x^{102}+x^{100}+x^{99}+x^{98}+x^{97}\\
&\quad+x^{96}+x^{94}+x^{91}+x^{83}+x^{81}+x^{80}+x^{77}+x^{76}+x^{74}+x^{73}+x^{71}\\
&\quad+x^{65}+x^{62}+x^{61}+x^{58}+x^{56}+x^{53}+x^{48}+x^{46}+x^{45}+x^{44}+x^{43}\\
&\quad+x^{41}+x^{40}+x^{33}+x^{32}+x^{30},\\
p_3(x) &= x^{129}+x^{127}+x^{125}+x^{123}+x^{122}+x^{121}+x^{120}+x^{117}+x^{116}+x^{113}\\
&\quad+x^{112}+x^{109}+x^{108}+x^{104}+x^{103}+x^{100}+x^{99}+x^{98}+x^{81}+x^{79}\\
&\quad+x^{77}+x^{75}+x^{74}+x^{73}+x^{72}+x^{69}+x^{68}+x^{65}+x^{64}+x^{61}+x^{60}\\
&\quad+x^{57}+x^{56}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{41}+x^{40}+x^{37}\\
&\quad+x^{36}+x^{33}+x^{32}+x^{29}+x^{28}+x^{24}+x^{23}+x^{20}+x^{19}+x^{18},\\
p_6(x) &= x^{130}+x^{126}+x^{116}+x^{114}+x^{112}+x^{108}+x^{106}+x^{102}+x^{100}+x^{96}\\
&\quad+x^{88}+x^{84}+x^{80}+x^{70}+x^{68}+x^{66}+x^{62}+x^{60}+x^{58}+x^{54}+x^{50}\\
&\quad+x^{44}+x^{42}+x^{40}+x^{30}+x^{26}+x^{22}+x^{16}+x^{14}+x^{12},\\
p_9(x) &= x^{131}+x^{129}+x^{124}+x^{123}+x^{122}+x^{120}+x^{117}+x^{116}+x^{115}+x^{114}\\
&\quad+x^{113}+x^{112}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}+x^{105}\\
&\quad+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{98}+x^{96}+x^{95}+x^{94}+x^{93}\\
&\quad+x^{89}+x^{87}+x^{86}+x^{35}+x^{33}+x^{28}+x^{27}+x^{26}+x^{24}+x^{21}+x^{20}\\
&\quad+x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{9}+x^{7}+x^{6},\\
p_{12}(x) &= x^{132}+x^{108}+x^{104}+x^{96}+x^{92}+x^{88}+x^{84}+x^{80}+x^{36}+x^{20}+x^{12}\\
&\quad+x^{8}+x^{4}+1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{125}+x^{124}+x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{116}+x^{110}\\
&\quad+x^{109}+x^{105}+x^{101}+x^{100}+x^{99}+x^{96}+x^{95}+x^{93}+x^{91}+x^{87}\\
&\quad+x^{85}+x^{81}+x^{80}+x^{79}+x^{77}+x^{75}+x^{72}+x^{71}+x^{69}+x^{68}+x^{67}\\
&\quad+x^{66}+x^{64}+x^{62}+x^{61}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{46} + x^{44} + x^{41} + x^{38} + x^{37} + x^{36} + x^{33} + x^{30},\\
p_3(x) &= x^{125} + x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109}\\
&+ x^{108} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94}\\
&+ x^{90} + x^{89} + x^{88} + x^{86} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74}\\
&+ x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{56}\\
&+ x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} + x^{20}\\
&+ x^{19} + x^{18},\\
p_6(x) &= x^{126} + x^{122} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100}\\
&+ x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{64} + x^{58}\\
&+ x^{56} + x^{52} + x^{48} + x^{44} + x^{34} + x^{32} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12},\\
p_9(x) &= x^{127} + x^{125} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{112} + x^{110} + x^{105}\\
&+ x^{104} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} + x^{87} + x^{86} + x^{31}\\
&+ x^{29} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9}\\
&+ x^{7} + x^{6},\\
p_{12}(x) &= x^{128} + x^{116} + x^{112} + x^{104} + x^{96} + x^{88} + x^{84} + x^{80} + x^{32} + x^{20} + x^{8}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113}\\
&+ x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{98} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83}\\
&+ x^{82} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{69} + x^{66} + x^{64} + x^{62} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{53} + x^{51} + x^{50} + x^{45} + x^{44} + x^{41} + x^{39}\\
&+ x^{36},\\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{119} + x^{116} + x^{113} + x^{112} + x^{110} + x^{105} + x^{103}\\
&+ x^{100} + x^{96} + x^{95} + x^{94} + x^{93} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74}\\
&+ x^{72} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52}\\
&+ x^{51} + x^{47} + x^{44} + x^{43} + x^{39} + x^{36} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25}\\
&+ x^{24} + x^{23} + x^{20},\\
p_6(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{115} + x^{113} + x^{112} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{91}\\
&+ x^{90} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{74} + x^{64} + x^{61} + x^{60} + x^{59}\\
&+ x^{54} + x^{51} + x^{47} + x^{43} + x^{40} + x^{39} + x^{37} + x^{33} + x^{30} + x^{28} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{21} + x^{18} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{2} + 1,\\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{113} + x^{111} + x^{110} + x^{108} + x^{97} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{29} + x^{27} + x^{26} + x^{24} + x^{17}\\
&+ x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8},\\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{109} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96}\\
&+ x^{85} + x^{83} + x^{82} + x^{80} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22}\\
&+ x^{20} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{115} + x^{113} + x^{112} + x^{111}\\
&+ x^{106} + x^{104} + x^{100} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89}\\
&+ x^{84} + x^{82} + x^{81} + x^{78} + x^{76} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{57} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49}\\
&+ x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{39} + x^{35} + x^{32} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{27} + x^{26} + x^{24},\\
p_3(x) &= x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98}\\
&+ x^{94} + x^{93} + x^{92} + x^{87} + x^{84} + x^{83} + x^{81} + x^{77} + x^{75} + x^{74} + x^{71}\\
&+ x^{68} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47}\\
&+ x^{45} + x^{44} + x^{43} + x^{42} + x^{37} + x^{34} + x^{33} + x^{29} + x^{28} + x^{27} + x^{25}\\
&+ x^{24} + x^{21} + x^{19} + x^{18} + x^{16},\\
p_6(x) &= x^{125} + x^{124} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109}\\
&+ x^{107} + x^{106} + x^{105} + x^{102} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93}\\
&+ x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{76} + x^{74} + x^{72}\\
&+ x^{69} + x^{66} + x^{64} + x^{63} + x^{58} + x^{56} + x^{54} + x^{48} + x^{45} + x^{44} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26}\\
&+ x^{25} + x^{23} + x^{21} + x^{18} + x^{17} + x^{5} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{113} + x^{111} + x^{110} + x^{108} + x^{97} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{29} + x^{27} + x^{26} + x^{24} + x^{17}\\
&+ x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8},\\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{109} + x^{107}\\
&+ x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96}\\
&+ x^{85} + x^{83} + x^{82} + x^{80} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22}\\
&+ x^{20} + x^{5} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{116} + x^{110} \\
&\quad + x^{109} + x^{105} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{91} + x^{87} \\
&\quad + x^{85} + x^{81} + x^{80} + x^{79} + x^{77} + x^{75} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{64} + x^{62} + x^{61} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{46} + x^{44} + x^{41} + x^{38} + x^{37} + x^{36} + x^{33} + x^{30}, \\
p_3(x) &= x^{125} + x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{111} + x^{109} \\
&\quad + x^{108} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} \\
&\quad + x^{90} + x^{89} + x^{88} + x^{86} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} \\
&\quad + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} + x^{20} \\
&\quad + x^{19} + x^{18}, \\
p_6(x) &= x^{126} + x^{122} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} \\
&\quad + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{64} + x^{58} \\
&\quad + x^{56} + x^{52} + x^{48} + x^{44} + x^{34} + x^{32} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{127} + x^{125} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{112} + x^{110} + x^{105} \\
&\quad + x^{104} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{89} + x^{87} + x^{86} + x^{31} \\
&\quad + x^{29} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9} \\
&\quad + x^{7} + x^{6}, \\
p_{12}(x) &= x^{128} + x^{116} + x^{112} + x^{104} + x^{96} + x^{88} + x^{84} + x^{80} + x^{32} + x^{20} + x^{8} \\
&\quad + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{136} + x^{133} + x^{132} + x^{129} + x^{127} + x^{126} + x^{123} + x^{122} + x^{118} + x^{113} \\
&\quad + x^{112} + x^{110} + x^{109} + x^{108} + x^{107} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} \\
&\quad + x^{96} + x^{94} + x^{91} + x^{83} + x^{81} + x^{80} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{65} + x^{62} + x^{61} + x^{58} + x^{56} + x^{53} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{129} + x^{127} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{116} + x^{113} \\
&\quad + x^{112} + x^{109} + x^{108} + x^{104} + x^{103} + x^{100} + x^{99} + x^{98} + x^{81} + x^{79} \\
&\quad + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{69} + x^{68} + x^{65} + x^{64} + x^{61} + x^{60} \\
&\quad + x^{57} + x^{56} + x^{53} + x^{52} + x^{49} + x^{48} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} \\
&\quad + x^{36} + x^{33} + x^{32} + x^{29} + x^{28} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18},
\end{aligned}
$$

$$\begin{aligned}
p_6(x) &= x^{130} + x^{126} + x^{116} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{96} \\
&\quad + x^{88} + x^{84} + x^{80} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{50} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{30} + x^{26} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{131} + x^{129} + x^{124} + x^{123} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} \\
&\quad + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{89} + x^{87} + x^{86} + x^{35} + x^{33} + x^{28} + x^{27} + x^{26} + x^{24} + x^{21} + x^{20} \\
&\quad + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{9} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{132} + x^{108} + x^{104} + x^{96} + x^{92} + x^{88} + x^{84} + x^{80} + x^{36} + x^{20} + x^{12} \\
&\quad + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{113} \\
&\quad + x^{112} + x^{109} + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{98} + x^{93} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} \\
&\quad + x^{82} + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{69} + x^{66} + x^{64} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{57} + x^{53} + x^{51} + x^{50} + x^{45} + x^{44} + x^{41} + x^{39} \\
&\quad + x^{36}, \\
p_3(x) &= x^{125} + x^{124} + x^{123} + x^{119} + x^{116} + x^{113} + x^{112} + x^{110} + x^{105} + x^{103} \\
&\quad + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} \\
&\quad + x^{72} + x^{69} + x^{66} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{47} + x^{44} + x^{43} + x^{39} + x^{36} + x^{33} + x^{28} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{24} + x^{23} + x^{20}, \\
p_6(x) &= x^{125} + x^{124} + x^{123} + x^{120} + x^{119} + x^{118} + x^{115} + x^{113} + x^{112} + x^{105} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{95} + x^{93} + x^{91} \\
&\quad + x^{90} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} + x^{74} + x^{64} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{54} + x^{51} + x^{47} + x^{43} + x^{40} + x^{39} + x^{37} + x^{33} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{25} + x^{24} + x^{23} + x^{21} + x^{18} + x^{14} + x^{13} + x^{11} + x^{10} + x^{8} + x^{5} + x^{3} \\
&\quad + x^{2} + 1, \\
p_9(x) &= x^{125} + x^{123} + x^{122} + x^{120} + x^{113} + x^{111} + x^{110} + x^{108} + x^{97} + x^{95} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{88} + x^{29} + x^{27} + x^{26} + x^{24} + x^{17} \\
&\quad + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{8}, \\
p_{12}(x) &= x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{109} + x^{107} \\
&\quad + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{96} \\
&\quad + x^{85} + x^{83} + x^{82} + x^{80} + x^{29} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22}
\end{aligned}$$

$$+ x^{20} + x^5 + x^3 + x^2 + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	369	[546, 144]	738	[162, 456]	1107	[904, 838]
1	[3, 4]	370	[547, 167]	739	[194, 478]	1108	[1254, 134]
2	[5, 6]	371	[548, 413]	740	[146, 525]	1109	[1225, 412]
3	[7, 8]	372	[549, 550]	741	[476, 144]	1110	[193, 477]
4	[9, 10]	373	[551, 552]	742	[946, 904]	1111	[1255, 121]
5	[11, 12]	374	[553, 554]	743	[151, 173]	1112	[1256, 491]
6	[13, 14]	375	[555, 556]	744	[947, 815]	1113	[1257, 1199]
7	[15, 16]	376	[557, 558]	745	[529, 840]	1114	[913, 160]
8	[17, 18]	377	[559, 560]	746	[171, 473]	1115	[1258, 382]
9	[19, 20]	378	[561, 562]	747	[432, 426]	1116	[831, 902]
10	[21, 22]	379	[563, 564]	748	[948, 937]	1117	[1259, 435]
11	[23, 24]	380	[565, 15]	749	[821, 388]	1118	[1260, 31]
12	[25, 26]	381	[566, 567]	750	[949, 815]	1119	[182, 133]
13	[27, 28]	382	[568, 228]	751	[194, 855]	1120	[1261, 35]
14	[29, 30]	383	[569, 570]	752	[540, 191]	1121	[37, 53]
15	[31, 32]	384	[571, 268]	753	[7, 59]	1122	[1262, 455]
16	[33, 34]	385	[80, 572]	754	[950, 881]	1123	[1263, 950]
17	[35, 36]	386	[573, 243]	755	[506, 128]	1124	[846, 538]
18	[37, 38]	387	[574, 575]	756	[944, 824]	1125	[929, 432]
19	[39, 40]	388	[550, 576]	757	[548, 412]	1126	[1264, 512]
20	[41, 42]	389	[577, 578]	758	[951, 461]	1127	[1185, 40]
21	[41, 43]	390	[579, 277]	759	[408, 424]	1128	[378, 901]
22	[44, 45]	391	[580, 581]	760	[952, 848]	1129	[1265, 171]
23	[46, 47]	392	[582, 19]	761	[953, 525]	1130	[1266, 1267]
24	[48, 49]	393	[583, 584]	762	[954, 431]	1131	[195, 520]
25	[50, 51]	394	[302, 585]	763	[955, 146]	1132	[1268, 524]
26	[52, 53]	395	[62, 586]	764	[956, 393]	1133	[164, 146]
27	[54, 55]	396	[329, 587]	765	[534, 519]	1134	[1255, 382]
28	[56, 57]	397	[588, 589]	766	[874, 443]	1135	[12, 58]
29	[58, 59]	398	[369, 222]	767	[957, 53]	1136	[848, 526]
30	[12, 60]	399	[590, 591]	768	[163, 502]	1137	[431, 175]
31	[61, 62]	400	[592, 593]	769	[958, 869]	1138	[1269, 444]
32	[30, 63]	401	[594, 595]	770	[509, 192]	1139	[870, 922]
33	[64, 65]	402	[596, 597]	771	[829, 477]	1140	[962, 1270]
34	[66, 67]	403	[598, 599]	772	[959, 436]	1141	[893, 456]
35	[68, 69]	404	[600, 601]	773	[477, 855]	1142	[938, 815]
36	[70, 71]	405	[258, 602]	774	[960, 503]	1143	[1271, 450]

37	[72, 73]	406	[603, 604]	775	[405, 490]	1144	[1272, 908]
38	[74, 75]	407	[605, 230]	776	[961, 190]	1145	[1273, 546]
39	[76, 77]	408	[556, 606]	777	[955, 165]	1146	[407, 881]
40	[65, 78]	409	[607, 608]	778	[962, 460]	1147	[1274, 205]
41	[79, 80]	410	[609, 610]	779	[482, 408]	1148	[957, 38]
42	[81, 82]	411	[315, 611]	780	[855, 458]	1149	[1275, 916]
43	[83, 29]	412	[612, 613]	781	[963, 401]	1150	[1276, 12]
44	[84, 85]	413	[614, 615]	782	[964, 924]	1151	[203, 206]
45	[86, 87]	414	[616, 617]	783	[153, 425]	1152	[508, 209]
46	[88, 89]	415	[618, 619]	784	[965, 966]	1153	[1277, 1278]
47	[90, 91]	416	[620, 621]	785	[176, 156]	1154	[387, 437]
48	[92, 93]	417	[622, 113]	786	[967, 968]	1155	[1279, 1219]
49	[94, 95]	418	[560, 623]	787	[823, 188]	1156	[854, 34]
50	[96, 97]	419	[624, 625]	788	[969, 461]	1157	[1280, 1281]
51	[98, 99]	420	[626, 627]	789	[970, 881]	1158	[1282, 1283]
52	[100, 101]	421	[628, 551]	790	[971, 136]	1159	[903, 544]
53	[102, 103]	422	[365, 629]	791	[416, 202]	1160	[1284, 1285]
54	[104, 105]	423	[630, 631]	792	[514, 135]	1161	[959, 496]
55	[106, 107]	424	[632, 633]	793	[545, 58]	1162	[1286, 1287]
56	[108, 109]	425	[327, 634]	794	[972, 55]	1163	[1288, 1283]
57	[110, 111]	426	[635, 636]	795	[461, 470]	1164	[160, 913]
58	[24, 112]	427	[637, 638]	796	[973, 526]	1165	[453, 816]
59	[113, 114]	428	[639, 211]	797	[391, 470]	1166	[815, 49]
60	[115, 116]	429	[242, 355]	798	[974, 809]	1167	[1289, 155]
61	[117, 118]	430	[640, 306]	799	[128, 516]	1168	[1224, 1290]
62	[119, 120]	431	[641, 642]	800	[975, 434]	1169	[521, 1223]
63	[121, 122]	432	[643, 612]	801	[455, 837]	1170	[1243, 395]
64	[123, 124]	433	[644, 590]	802	[44, 375]	1171	[1291, 404]
65	[59, 125]	434	[95, 631]	803	[976, 449]	1172	[1211, 478]
66	[126, 127]	435	[645, 646]	804	[977, 37]	1173	[1292, 37]
67	[128, 129]	436	[647, 115]	805	[978, 400]	1174	[51, 443]
68	[130, 131]	437	[648, 325]	806	[969, 391]	1175	[1293, 440]
69	[132, 133]	438	[649, 650]	807	[441, 36]	1176	[490, 398]
70	[134, 135]	439	[651, 652]	808	[979, 324]	1177	[1294, 1278]
71	[136, 127]	440	[653, 285]	809	[980, 981]	1178	[1295, 1204]
72	[137, 138]	441	[654, 655]	810	[982, 983]	1179	[1240, 12]
73	[139, 140]	442	[656, 657]	811	[984, 574]	1180	[880, 120]
74	[141, 142]	443	[658, 659]	812	[985, 598]	1181	[499, 208]
75	[122, 143]	444	[660, 648]	813	[986, 987]	1182	[871, 477]
76	[144, 145]	445	[584, 295]	814	[988, 986]	1183	[1296, 316]
77	[146, 147]	446	[661, 662]	815	[989, 990]	1184	[1297, 1298]
78	[124, 148]	447	[663, 664]	816	[991, 992]	1185	[599, 1099]
79	[149, 150]	448	[665, 666]	817	[993, 994]	1186	[1299, 1300]
80	[151, 152]	449	[667, 4]	818	[675, 995]	1187	[1301, 1302]

81	[153, 144]	450	[668, 669]	819	[720, 996]	1188	[699, 565]
82	[154, 146]	451	[670, 671]	820	[997, 615]	1189	[752, 557]
83	[155, 156]	452	[672, 348]	821	[103, 998]	1190	[787, 1303]
84	[157, 158]	453	[673, 649]	822	[999, 245]	1191	[1304, 30]
85	[159, 160]	454	[674, 675]	823	[1000, 1001]	1192	[1305, 1038]
86	[161, 162]	455	[676, 677]	824	[319, 1002]	1193	[1306, 1307]
87	[31, 163]	456	[290, 607]	825	[1003, 594]	1194	[1308, 999]
88	[164, 165]	457	[678, 679]	826	[1004, 756]	1195	[1093, 1307]
89	[122, 127]	458	[680, 681]	827	[1005, 696]	1196	[1309, 1310]
90	[166, 167]	459	[682, 247]	828	[691, 988]	1197	[1302, 563]
91	[168, 169]	460	[683, 684]	829	[1006, 754]	1198	[1311, 365]
92	[170, 171]	461	[685, 265]	830	[1007, 1008]	1199	[1312, 97]
93	[172, 173]	462	[686, 687]	831	[617, 1009]	1200	[1313, 1310]
94	[174, 175]	463	[677, 688]	832	[1010, 1010]	1201	[1314, 1124]
95	[176, 177]	464	[689, 690]	833	[790, 1011]	1202	[1315, 595]
96	[155, 177]	465	[570, 691]	834	[1012, 1013]	1203	[1316, 1016]
97	[178, 179]	466	[692, 693]	835	[1014, 710]	1204	[1317, 781]
98	[180, 181]	467	[694, 695]	836	[1015, 1005]	1205	[1318, 1319]
99	[182, 183]	468	[696, 689]	837	[233, 277]	1206	[558, 1320]
100	[184, 185]	469	[697, 297]	838	[211, 1016]	1207	[1321, 1018]
101	[186, 183]	470	[698, 699]	839	[1017, 741]	1208	[1322, 1323]
102	[187, 135]	471	[700, 337]	840	[772, 1018]	1209	[1324, 755]
103	[177, 188]	472	[701, 702]	841	[1019, 1020]	1210	[631, 1325]
104	[189, 190]	473	[703, 704]	842	[1021, 360]	1211	[1326, 577]
105	[191, 192]	474	[705, 706]	843	[646, 692]	1212	[1327, 243]
106	[193, 194]	475	[707, 635]	844	[331, 81]	1213	[20, 1170]
107	[195, 196]	476	[708, 21]	845	[97, 1022]	1214	[1328, 1033]
108	[197, 198]	477	[709, 680]	846	[1023, 1024]	1215	[601, 1329]
109	[199, 200]	478	[710, 711]	847	[1025, 1026]	1216	[1330, 75]
110	[57, 201]	479	[712, 713]	848	[1027, 1028]	1217	[1331, 1094]
111	[198, 202]	480	[714, 210]	849	[18, 1029]	1218	[1011, 1332]
112	[47, 191]	481	[715, 556]	850	[1030, 704]	1219	[1333, 563]
113	[203, 204]	482	[716, 632]	851	[602, 70]	1220	[1057, 1064]
114	[205, 206]	483	[717, 718]	852	[1031, 1032]	1221	[1334, 740]
115	[207, 208]	484	[719, 720]	853	[1033, 614]	1222	[738, 1335]
116	[209, 140]	485	[721, 703]	854	[1034, 1035]	1223	[1336, 1337]
117	[210, 211]	486	[604, 722]	855	[244, 83]	1224	[77, 1338]
118	[212, 213]	487	[723, 724]	856	[1036, 264]	1225	[633, 303]
119	[214, 215]	488	[725, 678]	857	[1037, 777]	1226	[1084, 1339]
120	[216, 105]	489	[726, 727]	858	[761, 1038]	1227	[727, 107]
121	[217, 218]	490	[367, 728]	859	[1039, 94]	1228	[1340, 789]
122	[219, 220]	491	[729, 730]	860	[1040, 1041]	1229	[597, 275]
123	[221, 222]	492	[731, 732]	861	[687, 351]	1230	[1161, 717]
124	[223, 224]	493	[733, 734]	862	[711, 364]	1231	[1341, 78]

125	[16, 225]	494	[735, 736]	863	[1042, 313]	1232	[1342, 1343]
126	[226, 227]	495	[737, 738]	864	[311, 796]	1233	[1094, 1337]
127	[228, 229]	496	[615, 739]	865	[1043, 739]	1234	[1344, 291]
128	[230, 231]	497	[740, 247]	866	[1044, 1045]	1235	[1345, 1346]
129	[232, 233]	498	[741, 360]	867	[1046, 1047]	1236	[1347, 1348]
130	[234, 235]	499	[742, 743]	868	[1048, 229]	1237	[1349, 1350]
131	[236, 219]	500	[744, 555]	869	[1049, 1050]	1238	[1351, 1068]
132	[237, 238]	501	[586, 79]	870	[1051, 1052]	1239	[1352, 325]
133	[239, 240]	502	[745, 746]	871	[1053, 1054]	1240	[1353, 1354]
134	[241, 242]	503	[294, 747]	872	[1055, 1056]	1241	[317, 733]
135	[243, 244]	504	[748, 749]	873	[63, 1057]	1242	[1355, 795]
136	[245, 246]	505	[750, 275]	874	[1058, 1059]	1243	[797, 1128]
137	[247, 248]	506	[751, 752]	875	[1060, 644]	1244	[1356, 1340]
138	[249, 250]	507	[753, 754]	876	[1061, 1062]	1245	[1103, 1072]
139	[251, 252]	508	[755, 736]	877	[1063, 678]	1246	[1357, 1358]
140	[253, 254]	509	[756, 757]	878	[1064, 1065]	1247	[1359, 1360]
141	[255, 256]	510	[758, 652]	879	[636, 766]	1248	[1002, 640]
142	[257, 258]	511	[759, 662]	880	[1066, 1067]	1249	[22, 1361]
143	[259, 260]	512	[760, 761]	881	[1068, 232]	1250	[1362, 792]
144	[261, 262]	513	[762, 763]	882	[1069, 1051]	1251	[1363, 801]
145	[263, 264]	514	[764, 765]	883	[587, 1070]	1252	[1364, 1365]
146	[265, 266]	515	[766, 767]	884	[1071, 1072]	1253	[73, 584]
147	[267, 268]	516	[718, 768]	885	[1073, 1074]	1254	[1366, 747]
148	[222, 269]	517	[769, 770]	886	[1075, 753]	1255	[775, 250]
149	[270, 271]	518	[771, 772]	887	[638, 1076]	1256	[1367, 1368]
150	[272, 226]	519	[621, 357]	888	[1077, 1078]	1257	[1369, 1370]
151	[273, 274]	520	[305, 773]	889	[1079, 102]	1258	[1343, 219]
152	[275, 19]	521	[774, 775]	890	[1080, 1081]	1259	[1371, 647]
153	[276, 263]	522	[776, 666]	891	[613, 1082]	1260	[1372, 293]
154	[277, 267]	523	[575, 777]	892	[1083, 1084]	1261	[1065, 1073]
155	[278, 279]	524	[778, 235]	893	[292, 588]	1262	[1373, 233]
156	[280, 281]	525	[296, 555]	894	[1085, 361]	1263	[1374, 87]
157	[282, 71]	526	[300, 779]	895	[1086, 258]	1264	[610, 239]
158	[283, 284]	527	[780, 280]	896	[724, 772]	1265	[1156, 703]
159	[285, 286]	528	[657, 781]	897	[1087, 1088]	1266	[218, 1375]
160	[287, 288]	529	[782, 783]	898	[1070, 1089]	1267	[1376, 717]
161	[289, 290]	530	[784, 785]	899	[1090, 599]	1268	[799, 759]
162	[291, 292]	531	[786, 787]	900	[554, 1091]	1269	[747, 253]
163	[293, 294]	532	[788, 658]	901	[1092, 1093]	1270	[1377, 1378]
164	[295, 296]	533	[789, 790]	902	[562, 1094]	1271	[1368, 1092]
165	[297, 298]	534	[224, 791]	903	[1095, 1096]	1272	[262, 1302]
166	[299, 300]	535	[256, 108]	904	[1097, 1098]	1273	[1348, 1127]
167	[301, 302]	536	[792, 793]	905	[1099, 1100]	1274	[1098, 6]
168	[303, 304]	537	[794, 349]	906	[1101, 1070]	1275	[1100, 560]

169	[116, 305]	538	[652, 795]	907	[1102, 1103]	1276	[1379, 115]
170	[306, 307]	539	[796, 797]	908	[1104, 732]	1277	[1380, 1002]
171	[308, 309]	540	[798, 799]	909	[235, 573]	1278	[1381, 1338]
172	[310, 311]	541	[800, 801]	910	[783, 658]	1279	[1382, 1383]
173	[312, 313]	542	[606, 802]	911	[1105, 226]	1280	[1009, 1348]
174	[314, 315]	543	[803, 96]	912	[347, 287]	1281	[1384, 640]
175	[316, 317]	544	[802, 617]	913	[913, 913]	1282	[1385, 103]
176	[318, 319]	545	[804, 17]	914	[785, 598]	1283	[1386, 796]
177	[320, 321]	546	[805, 774]	915	[1106, 1107]	1284	[1176, 780]
178	[322, 323]	547	[806, 82]	916	[1108, 1109]	1285	[1387, 1361]
179	[324, 94]	548	[767, 807]	917	[1110, 305]	1286	[341, 1050]
180	[325, 289]	549	[808, 809]	918	[107, 751]	1287	[1388, 1125]
181	[326, 327]	550	[165, 147]	919	[1111, 551]	1288	[1389, 1390]
182	[328, 329]	551	[810, 143]	920	[1112, 1102]	1289	[1378, 1139]
183	[4, 330]	552	[59, 169]	921	[1113, 1114]	1290	[1391, 1383]
184	[284, 331]	553	[811, 812]	922	[1035, 792]	1291	[1026, 1392]
185	[332, 280]	554	[813, 814]	923	[260, 1033]	1292	[1067, 284]
186	[333, 334]	555	[478, 458]	924	[1115, 1116]	1293	[1393, 225]
187	[335, 336]	556	[815, 816]	925	[1117, 259]	1294	[1394, 1395]
188	[337, 338]	557	[440, 520]	926	[309, 102]	1295	[1396, 982]
189	[339, 22]	558	[817, 163]	927	[1118, 86]	1296	[1264, 186]
190	[340, 3]	559	[818, 176]	928	[1119, 272]	1297	[1397, 1398]
191	[89, 341]	560	[819, 145]	929	[1120, 1121]	1298	[457, 497]
192	[91, 342]	561	[820, 447]	930	[1122, 800]	1299	[1399, 1400]
193	[343, 244]	562	[458, 42]	931	[1123, 88]	1300	[973, 14]
194	[344, 345]	563	[821, 437]	932	[1124, 1125]	1301	[1213, 908]
195	[346, 347]	564	[490, 456]	933	[1126, 1127]	1302	[161, 490]
196	[348, 316]	565	[455, 401]	934	[1128, 725]	1303	[953, 147]
197	[349, 350]	566	[822, 121]	935	[1129, 745]	1304	[1258, 121]
198	[351, 352]	567	[823, 824]	936	[1130, 1131]	1305	[1401, 906]
199	[353, 354]	568	[825, 400]	937	[1132, 1133]	1306	[1272, 1214]
200	[355, 353]	569	[826, 2]	938	[1134, 269]	1307	[1402, 438]
201	[356, 110]	570	[827, 828]	939	[6, 1135]	1308	[1403, 31]
202	[357, 356]	571	[829, 194]	940	[1136, 1137]	1309	[1208, 1252]
203	[358, 359]	572	[530, 173]	941	[1138, 1139]	1310	[1234, 208]
204	[360, 361]	573	[830, 177]	942	[1140, 659]	1311	[1404, 1270]
205	[362, 363]	574	[831, 541]	943	[768, 320]	1312	[1276, 417]
206	[364, 365]	575	[832, 525]	944	[704, 1141]	1313	[921, 436]
207	[366, 367]	576	[809, 179]	945	[1142, 613]	1314	[1405, 1406]
208	[368, 369]	577	[171, 120]	946	[359, 619]	1315	[409, 512]
209	[370, 371]	578	[472, 120]	947	[1143, 991]	1316	[961, 857]
210	[372, 373]	579	[421, 444]	948	[1144, 260]	1317	[947, 453]
211	[374, 375]	580	[833, 480]	949	[1145, 1146]	1318	[1407, 47]
212	[376, 377]	581	[487, 443]	950	[1147, 1148]	1319	[890, 163]

213	[378, 379]	582	[834, 190]	951	[1149, 698]	1320	[1221, 418]
214	[380, 381]	583	[835, 477]	952	[1150, 29]	1321	[1408, 1252]
215	[382, 122]	584	[836, 837]	953	[266, 1151]	1322	[1409, 1281]
216	[383, 45]	585	[394, 56]	954	[1152, 773]	1323	[860, 196]
217	[384, 44]	586	[118, 838]	955	[666, 1119]	1324	[1401, 1290]
218	[385, 386]	587	[141, 538]	956	[1153, 1154]	1325	[177, 824]
219	[387, 388]	588	[839, 838]	957	[1155, 1156]	1326	[1410, 472]
220	[389, 42]	589	[119, 473]	958	[1157, 731]	1327	[11, 417]
221	[390, 391]	590	[172, 152]	959	[1158, 694]	1328	[1217, 1252]
222	[60, 8]	591	[543, 840]	960	[681, 1159]	1329	[58, 8]
223	[392, 393]	592	[841, 842]	961	[1160, 1161]	1330	[1250, 1247]
224	[394, 202]	593	[843, 468]	962	[1162, 234]	1331	[1411, 1223]
225	[32, 395]	594	[844, 458]	963	[1163, 1164]	1332	[482, 881]
226	[133, 396]	595	[817, 32]	964	[781, 1165]	1333	[1412, 209]
227	[397, 398]	596	[845, 181]	965	[619, 1166]	1334	[1413, 419]
228	[399, 42]	597	[33, 39]	966	[1167, 66]	1335	[163, 395]
229	[400, 401]	598	[846, 142]	967	[1168, 576]	1336	[1251, 1209]
230	[402, 403]	599	[156, 824]	968	[1169, 1170]	1337	[875, 49]
231	[402, 36]	600	[847, 848]	969	[1171, 1172]	1338	[1206, 411]
232	[404, 34]	601	[849, 129]	970	[1173, 1174]	1339	[133, 858]
233	[405, 162]	602	[382, 136]	971	[1175, 1176]	1340	[537, 142]
234	[406, 135]	603	[850, 505]	972	[1177, 1000]	1341	[523, 1414]
235	[407, 408]	604	[199, 199]	973	[1178, 620]	1342	[1415, 1214]
236	[409, 186]	605	[851, 402]	974	[1179, 607]	1343	[844, 389]
237	[410, 124]	606	[512, 183]	975	[572, 1180]	1344	[1416, 474]
238	[411, 412]	607	[470, 165]	976	[307, 1181]	1345	[1417, 1287]
239	[136, 143]	608	[156, 188]	977	[1089, 74]	1346	[462, 179]
240	[411, 413]	609	[852, 482]	978	[1182, 768]	1347	[1222, 522]
241	[414, 415]	610	[853, 411]	979	[1183, 174]	1348	[533, 458]
242	[416, 56]	611	[480, 204]	980	[1184, 415]	1349	[1418, 867]
243	[408, 128]	612	[854, 39]	981	[861, 57]	1350	[867, 38]
244	[417, 58]	613	[816, 433]	982	[1185, 922]	1351	[1419, 404]
245	[418, 192]	614	[139, 373]	983	[501, 434]	1352	[1268, 1414]
246	[395, 386]	615	[855, 389]	984	[1186, 1187]	1353	[1420, 854]
247	[419, 142]	616	[856, 857]	985	[865, 12]	1354	[935, 32]
248	[420, 60]	617	[183, 858]	986	[160, 160]	1355	[1289, 176]
249	[421, 209]	618	[859, 815]	987	[468, 814]	1356	[1421, 1400]
250	[422, 396]	619	[860, 520]	988	[828, 468]	1357	[1422, 1406]
251	[423, 419]	620	[201, 202]	989	[1188, 118]	1358	[13, 526]
252	[424, 129]	621	[861, 519]	990	[149, 838]	1359	[1411, 522]
253	[425, 10]	622	[862, 205]	991	[420, 58]	1360	[898, 433]
254	[426, 412]	623	[424, 516]	992	[450, 901]	1361	[48, 816]
255	[427, 428]	624	[863, 485]	993	[1189, 876]	1362	[1215, 1290]
256	[429, 199]	625	[197, 201]	994	[839, 150]	1363	[1415, 908]

257	[430, 8]	626	[864, 402]	995	[183, 396]	1364	[1423, 1414]
258	[431, 432]	627	[442, 488]	996	[960, 386]	1365	[853, 426]
259	[433, 434]	628	[60, 59]	997	[835, 194]	1366	[1202, 461]
260	[435, 436]	629	[515, 129]	998	[849, 516]	1367	[1424, 817]
261	[373, 437]	630	[865, 417]	999	[909, 395]	1368	[1402, 433]
262	[438, 434]	631	[175, 411]	1000	[1190, 396]	1369	[1425, 1193]
263	[438, 439]	632	[866, 206]	1001	[975, 439]	1370	[126, 143]
264	[440, 196]	633	[51, 488]	1002	[383, 375]	1371	[1426, 207]
265	[441, 403]	634	[162, 398]	1003	[1191, 817]	1372	[1427, 476]
266	[442, 443]	635	[205, 204]	1004	[1192, 1193]	1373	[859, 453]
267	[444, 373]	636	[867, 53]	1005	[814, 464]	1374	[1412, 444]
268	[49, 438]	637	[868, 869]	1006	[891, 121]	1375	[816, 438]
269	[391, 154]	638	[870, 40]	1007	[1194, 447]	1376	[1408, 1209]
270	[445, 446]	639	[871, 194]	1008	[395, 503]	1377	[1428, 1223]
271	[121, 136]	640	[872, 478]	1009	[399, 43]	1378	[878, 162]
272	[447, 436]	641	[873, 442]	1010	[52, 38]	1379	[1269, 209]
273	[448, 449]	642	[874, 488]	1011	[124, 179]	1380	[907, 1214]
274	[450, 379]	643	[875, 816]	1012	[1195, 1196]	1381	[905, 1290]
275	[451, 45]	644	[876, 544]	1013	[963, 837]	1382	[187, 507]
276	[452, 440]	645	[877, 842]	1014	[1197, 930]	1383	[475, 175]
277	[453, 49]	646	[465, 814]	1015	[1198, 1199]	1384	[189, 857]
278	[454, 455]	647	[878, 490]	1016	[373, 388]	1385	[1429, 460]
279	[397, 456]	648	[879, 125]	1017	[1200, 381]	1386	[951, 391]
280	[457, 192]	649	[880, 473]	1018	[140, 437]	1387	[856, 190]
281	[458, 43]	650	[876, 840]	1019	[1201, 968]	1388	[1246, 1232]
282	[459, 460]	651	[495, 425]	1020	[810, 127]	1389	[1430, 1236]
283	[390, 461]	652	[881, 424]	1021	[1202, 391]	1390	[819, 10]
284	[462, 148]	653	[882, 158]	1022	[417, 60]	1391	[1429, 1270]
285	[463, 464]	654	[883, 884]	1023	[1203, 1204]	1392	[379, 902]
286	[465, 466]	655	[47, 418]	1024	[202, 519]	1393	[949, 453]
287	[467, 468]	656	[885, 131]	1025	[1205, 174]	1394	[1431, 1285]
288	[466, 464]	657	[479, 34]	1026	[1206, 426]	1395	[836, 401]
289	[469, 470]	658	[406, 507]	1027	[1207, 428]	1396	[1432, 1398]
290	[471, 156]	659	[425, 145]	1028	[535, 198]	1397	[1433, 1313]
291	[472, 473]	660	[886, 134]	1029	[866, 204]	1398	[1434, 1100]
292	[474, 173]	661	[527, 176]	1030	[1208, 1209]	1399	[679, 1133]
293	[475, 432]	662	[887, 541]	1031	[1210, 916]	1400	[1435, 289]
294	[476, 425]	663	[888, 889]	1032	[872, 855]	1401	[713, 1436]
295	[477, 478]	664	[467, 813]	1033	[1211, 855]	1402	[1332, 651]
296	[479, 39]	665	[374, 45]	1034	[1212, 514]	1403	[1437, 779]
297	[480, 206]	666	[890, 32]	1035	[832, 147]	1404	[1438, 342]
298	[35, 403]	667	[891, 382]	1036	[1213, 1214]	1405	[371, 628]
299	[481, 482]	668	[892, 383]	1037	[1215, 906]	1406	[1439, 789]
300	[483, 128]	669	[893, 398]	1038	[848, 14]	1407	[1329, 1116]

301	[484, 485]	670	[894, 532]	1039	[943, 176]	1408	[1440, 665]
302	[486, 200]	671	[828, 813]	1040	[1216, 812]	1409	[1008, 1441]
303	[487, 488]	672	[895, 499]	1041	[464, 468]	1410	[112, 216]
304	[489, 403]	673	[896, 876]	1042	[1217, 1209]	1411	[1001, 1343]
305	[208, 490]	674	[897, 499]	1043	[511, 186]	1412	[763, 1442]
306	[132, 183]	675	[898, 438]	1044	[1218, 1219]	1413	[1443, 257]
307	[491, 196]	676	[899, 889]	1045	[512, 133]	1414	[1444, 248]
308	[492, 377]	677	[900, 466]	1046	[1220, 933]	1415	[669, 1445]
309	[471, 177]	678	[901, 902]	1047	[1221, 191]	1416	[1320, 312]
310	[493, 494]	679	[879, 169]	1048	[1222, 1223]	1417	[334, 1446]
311	[495, 144]	680	[903, 840]	1049	[1224, 906]	1418	[992, 1136]
312	[435, 496]	681	[904, 150]	1050	[430, 59]	1419	[1447, 66]
313	[418, 497]	682	[905, 906]	1051	[887, 902]	1420	[1050, 567]
314	[498, 480]	683	[907, 908]	1052	[1225, 413]	1421	[238, 1447]
315	[489, 36]	684	[909, 502]	1053	[1226, 476]	1422	[330, 1365]
316	[499, 405]	685	[910, 442]	1054	[469, 154]	1423	[552, 3]
317	[186, 133]	686	[911, 518]	1055	[1227, 474]	1424	[1448, 1449]
318	[500, 383]	687	[200, 200]	1056	[118, 150]	1425	[1450, 1382]
319	[501, 439]	688	[900, 814]	1057	[167, 14]	1426	[1451, 368]
320	[140, 388]	689	[464, 813]	1058	[1228, 941]	1427	[1452, 327]
321	[502, 503]	690	[813, 466]	1059	[916, 191]	1428	[281, 1360]
322	[504, 505]	691	[912, 160]	1060	[1229, 172]	1429	[1453, 1454]
323	[57, 198]	692	[159, 913]	1061	[1230, 494]	1430	[1455, 587]
324	[506, 424]	693	[814, 828]	1062	[971, 122]	1431	[983, 684]
325	[134, 507]	694	[466, 828]	1063	[1231, 1232]	1432	[1456, 338]
326	[508, 444]	695	[827, 464]	1064	[950, 408]	1433	[1457, 1196]
327	[509, 497]	696	[843, 813]	1065	[920, 379]	1434	[1458, 1414]
328	[510, 141]	697	[914, 35]	1066	[1233, 446]	1435	[459, 1270]
329	[8, 169]	698	[915, 208]	1067	[546, 425]	1436	[154, 165]
330	[8, 125]	699	[916, 418]	1068	[1234, 405]	1437	[818, 155]
331	[461, 154]	700	[917, 491]	1069	[1235, 1236]	1438	[1458, 524]
332	[511, 512]	701	[918, 919]	1070	[1237, 154]	1439	[1244, 1247]
333	[513, 514]	702	[920, 901]	1071	[1, 186]	1440	[1459, 1267]
334	[515, 516]	703	[921, 496]	1072	[1190, 858]	1441	[32, 502]
335	[517, 518]	704	[34, 922]	1073	[809, 148]	1442	[901, 541]
336	[519, 201]	705	[923, 924]	1074	[174, 432]	1443	[1460, 431]
337	[389, 43]	706	[925, 136]	1075	[1238, 950]	1444	[1231, 1247]
338	[491, 520]	707	[926, 867]	1076	[385, 503]	1445	[49, 433]
339	[521, 522]	708	[927, 44]	1077	[1239, 1187]	1446	[175, 426]
340	[523, 524]	709	[928, 904]	1078	[9, 145]	1447	[881, 128]
341	[165, 525]	710	[929, 175]	1079	[1240, 417]	1448	[1461, 930]
342	[167, 526]	711	[930, 379]	1080	[1241, 172]	1449	[1237, 470]
343	[527, 155]	712	[931, 141]	1081	[474, 152]	1450	[1462, 1463]
344	[528, 529]	713	[426, 413]	1082	[39, 922]	1451	[1464, 546]

345	[530, 152]	714	[932, 933]	1083	[1242, 207]	1452	[208, 162]
346	[531, 532]	715	[1, 512]	1084	[1243, 502]	1453	[1465, 1463]
347	[468, 466]	716	[934, 51]	1085	[1244, 1232]	1454	[178, 148]
348	[533, 389]	717	[935, 163]	1086	[822, 382]	1455	[1423, 524]
349	[198, 56]	718	[372, 140]	1087	[1245, 450]	1456	[1428, 522]
350	[429, 200]	719	[936, 937]	1088	[970, 408]	1457	[279, 1452]
351	[534, 57]	720	[422, 858]	1089	[925, 122]	1458	[1109, 1466]
352	[200, 199]	721	[938, 453]	1090	[1246, 1247]	1459	[220, 1378]
353	[56, 519]	722	[486, 199]	1091	[463, 828]	1460	[1467, 643]
354	[201, 56]	723	[939, 884]	1092	[191, 497]	1461	[225, 1468]
355	[535, 201]	724	[444, 140]	1093	[433, 439]	1462	[1303, 755]
356	[519, 198]	725	[419, 538]	1094	[447, 496]	1463	[1469, 793]
357	[202, 57]	726	[940, 941]	1095	[1248, 966]	1464	[268, 261]
358	[536, 185]	727	[209, 373]	1096	[930, 901]	1465	[252, 1470]
359	[404, 39]	728	[529, 544]	1097	[1249, 919]	1466	[432, 411]
360	[537, 538]	729	[942, 2]	1098	[830, 156]	1467	[1471, 854]
361	[144, 10]	730	[912, 913]	1099	[168, 125]	1468	[34, 40]
362	[539, 138]	731	[451, 375]	1100	[483, 424]	1469	[1404, 460]
363	[540, 418]	732	[915, 405]	1101	[1250, 1232]	1470	[470, 146]
364	[514, 507]	733	[400, 837]	1102	[502, 386]	1471	[82, 76]
365	[379, 541]	734	[207, 405]	1103	[478, 389]		
366	[542, 529]	735	[943, 155]	1104	[834, 857]		
367	[543, 544]	736	[944, 188]	1105	[1251, 1252]		
368	[545, 60]	737	[945, 435]	1106	[1253, 472]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	246	0	492	1	738	0	984	0	1230	0
1	0	247	1	493	1	739	1	985	1	1231	0
2	0	248	1	494	0	740	0	986	1	1232	0
3	0	249	0	495	1	741	0	987	0	1233	1
4	1	250	0	496	0	742	1	988	0	1234	0
5	0	251	1	497	0	743	0	989	1	1235	0
6	1	252	0	498	0	744	0	990	0	1236	0
7	0	253	0	499	1	745	1	991	1	1237	1
8	0	254	0	500	0	746	1	992	1	1238	0
9	1	255	0	501	0	747	0	993	1	1239	1
10	0	256	0	502	1	748	1	994	1	1240	1
11	0	257	1	503	0	749	0	995	1	1241	1
12	1	258	1	504	1	750	1	996	1	1242	1
13	1	259	0	505	1	751	1	997	0	1243	1
14	1	260	0	506	1	752	1	998	1	1244	1
15	0	261	1	507	0	753	0	999	0	1245	1
16	0	262	1	508	1	754	1	1000	1	1246	0

17	0	263	1	509	0	755	1	1001	1	1247	1
18	1	264	0	510	0	756	0	1002	0	1248	0
19	1	265	0	511	1	757	1	1003	0	1249	1
20	0	266	1	512	0	758	0	1004	0	1250	1
21	0	267	0	513	1	759	1	1005	0	1251	0
22	1	268	0	514	1	760	0	1006	1	1252	1
23	0	269	1	515	0	761	1	1007	1	1253	1
24	1	270	0	516	0	762	1	1008	0	1254	1
25	1	271	1	517	1	763	1	1009	1	1255	1
26	0	272	1	518	1	764	1	1010	0	1256	1
27	1	273	0	519	0	765	1	1011	0	1257	1
28	1	274	1	520	0	766	0	1012	0	1258	0
29	1	275	1	521	1	767	1	1013	1	1259	0
30	1	276	0	522	1	768	0	1014	0	1260	0
31	0	277	0	523	0	769	1	1015	0	1261	1
32	1	278	1	524	1	770	0	1016	1	1262	0
33	0	279	1	525	1	771	1	1017	1	1263	1
34	0	280	1	526	0	772	0	1018	0	1264	0
35	0	281	0	527	0	773	0	1019	1	1265	0
36	0	282	1	528	1	774	1	1020	1	1266	0
37	1	283	0	529	1	775	1	1021	1	1267	1
38	0	284	0	530	1	776	1	1022	0	1268	1
39	1	285	1	531	0	777	1	1023	1	1269	0
40	1	286	0	532	1	778	1	1024	0	1270	0
41	0	287	1	533	1	779	0	1025	1	1271	0
42	0	288	1	534	1	780	0	1026	1	1272	1
43	1	289	0	535	0	781	1	1027	1	1273	1
44	1	290	0	536	1	782	1	1028	0	1274	1
45	0	291	0	537	0	783	0	1029	0	1275	0
46	0	292	0	538	0	784	1	1030	0	1276	0
47	0	293	0	539	0	785	0	1031	1	1277	1
48	1	294	0	540	1	786	0	1032	1	1278	1
49	0	295	0	541	1	787	1	1033	0	1279	1
50	1	296	1	542	0	788	1	1034	0	1280	1
51	0	297	1	543	1	789	1	1035	0	1281	1
52	0	298	0	544	1	790	1	1036	0	1282	0
53	1	299	0	545	0	791	0	1037	1	1283	0
54	1	300	0	546	1	792	1	1038	1	1284	1
55	0	301	0	547	1	793	0	1039	0	1285	0
56	1	302	1	548	1	794	0	1040	1	1286	1
57	1	303	1	549	0	795	0	1041	1	1287	0
58	1	304	1	550	1	796	0	1042	1	1288	1
59	1	305	0	551	1	797	1	1043	1	1289	1
60	0	306	1	552	1	798	1	1044	0	1290	0

61	0	307	1	553	0	799	1	1045	0	1291	1
62	0	308	1	554	1	800	1	1046	0	1292	1
63	1	309	0	555	1	801	0	1047	0	1293	1
64	0	310	1	556	1	802	1	1048	0	1294	0
65	1	311	1	557	0	803	1	1049	0	1295	0
66	0	312	0	558	1	804	0	1050	1	1296	0
67	1	313	0	559	1	805	1	1051	0	1297	1
68	0	314	0	560	0	806	1	1052	0	1298	1
69	1	315	1	561	0	807	0	1053	1	1299	0
70	0	316	1	562	0	808	0	1054	0	1300	0
71	0	317	1	563	0	809	1	1055	1	1301	0
72	1	318	0	564	1	810	1	1056	0	1302	0
73	1	319	0	565	0	811	0	1057	0	1303	1
74	0	320	0	566	0	812	1	1058	0	1304	0
75	1	321	1	567	1	813	1	1059	1	1305	0
76	1	322	1	568	0	814	0	1060	0	1306	1
77	0	323	1	569	0	815	1	1061	0	1307	0
78	0	324	1	570	0	816	1	1062	1	1308	1
79	0	325	0	571	1	817	1	1063	0	1309	0
80	0	326	1	572	1	818	1	1064	1	1310	0
81	0	327	0	573	1	819	0	1065	1	1311	0
82	0	328	0	574	1	820	0	1066	1	1312	0
83	1	329	0	575	0	821	0	1067	1	1313	1
84	1	330	0	576	1	822	0	1068	0	1314	1
85	1	331	0	577	1	823	1	1069	0	1315	1
86	0	332	1	578	0	824	0	1070	1	1316	1
87	0	333	1	579	0	825	0	1071	0	1317	0
88	0	334	0	580	1	826	0	1072	1	1318	1
89	1	335	1	581	1	827	0	1073	1	1319	1
90	0	336	0	582	0	828	0	1074	0	1320	0
91	1	337	1	583	0	829	1	1075	0	1321	1
92	1	338	1	584	0	830	1	1076	0	1322	0
93	1	339	1	585	1	831	1	1077	1	1323	1
94	0	340	0	586	0	832	0	1078	1	1324	0
95	0	341	1	587	0	833	1	1079	1	1325	0
96	1	342	0	588	1	834	0	1080	1	1326	0
97	1	343	0	589	0	835	0	1081	0	1327	0
98	0	344	1	590	1	836	0	1082	1	1328	1
99	0	345	1	591	1	837	1	1083	1	1329	1
100	0	346	0	592	1	838	0	1084	1	1330	1
101	1	347	0	593	0	839	1	1085	1	1331	1
102	1	348	1	594	0	840	0	1086	0	1332	0
103	0	349	1	595	1	841	1	1087	1	1333	1
104	1	350	0	596	1	842	1	1088	1	1334	0

105	1	351	1	597	0	843	0	1089	0	1335	0
106	0	352	0	598	1	844	0	1090	0	1336	0
107	0	353	1	599	1	845	1	1091	1	1337	0
108	1	354	0	600	1	846	1	1092	1	1338	1
109	1	355	0	601	1	847	1	1093	0	1339	0
110	1	356	0	602	0	848	1	1094	1	1340	0
111	1	357	0	603	0	849	1	1095	0	1341	0
112	0	358	1	604	1	850	0	1096	0	1342	0
113	1	359	1	605	0	851	0	1097	1	1343	0
114	0	360	0	606	0	852	1	1098	1	1344	0
115	0	361	1	607	1	853	0	1099	1	1345	0
116	1	362	0	608	1	854	0	1100	0	1346	0
117	0	363	1	609	1	855	0	1101	1	1347	0
118	0	364	1	610	0	856	0	1102	1	1348	1
119	0	365	0	611	1	857	1	1103	1	1349	1
120	0	366	0	612	0	858	1	1104	0	1350	0
121	1	367	1	613	1	859	0	1105	0	1351	0
122	1	368	0	614	1	860	1	1106	1	1352	1
123	0	369	1	615	0	861	0	1107	1	1353	1
124	0	370	1	616	0	862	0	1108	1	1354	0
125	0	371	1	617	1	863	1	1109	0	1355	1
126	0	372	0	618	0	864	1	1110	0	1356	1
127	1	373	1	619	1	865	1	1111	1	1357	0
128	1	374	0	620	0	866	0	1112	1	1358	1
129	1	375	1	621	0	867	0	1113	1	1359	1
130	0	376	0	622	0	868	0	1114	0	1360	1
131	1	377	1	623	0	869	0	1115	0	1361	1
132	1	378	0	624	1	870	0	1116	1	1362	1
133	0	379	0	625	1	871	1	1117	0	1363	0
134	0	380	0	626	1	872	1	1118	0	1364	1
135	1	381	0	627	1	873	1	1119	0	1365	0
136	0	382	0	628	0	874	0	1120	1	1366	1
137	1	383	0	629	0	875	0	1121	1	1367	1
138	0	384	1	630	1	876	0	1122	0	1368	0
139	1	385	0	631	1	877	0	1123	1	1369	1
140	0	386	1	632	0	878	1	1124	1	1370	0
141	0	387	1	633	0	879	0	1125	1	1371	0
142	1	388	1	634	0	880	1	1126	0	1372	0
143	0	389	1	635	0	881	0	1127	1	1373	0
144	1	390	0	636	0	882	0	1128	0	1374	1
145	1	391	1	637	0	883	0	1129	0	1375	1
146	0	392	0	638	0	884	0	1130	0	1376	1
147	0	393	0	639	1	885	1	1131	0	1377	0
148	0	394	1	640	1	886	0	1132	1	1378	1

149	0	395	0	641	1	887	0	1133	0	1379	0
150	1	396	0	642	0	888	1	1134	1	1380	1
151	0	397	1	643	0	889	1	1135	1	1381	1
152	1	398	1	644	0	890	1	1136	1	1382	1
153	0	399	1	645	0	891	1	1137	1	1383	0
154	0	400	1	646	0	892	1	1138	0	1384	1
155	1	401	0	647	1	893	0	1139	0	1385	0
156	1	402	1	648	0	894	1	1140	1	1386	0
157	1	403	1	649	1	895	0	1141	0	1387	0
158	0	404	1	650	0	896	0	1142	1	1388	0
159	1	405	1	651	1	897	1	1143	0	1389	1
160	1	406	0	652	0	898	1	1144	1	1390	0
161	0	407	0	653	0	899	0	1145	1	1391	0
162	0	408	1	654	0	900	1	1146	0	1392	0
163	0	409	1	655	0	901	1	1147	1	1393	1
164	0	410	1	656	1	902	0	1148	1	1394	0
165	1	411	1	657	1	903	0	1149	0	1395	0
166	0	412	0	658	0	904	1	1150	0	1396	0
167	0	413	1	659	0	905	1	1151	1	1397	1
168	1	414	0	660	0	906	1	1152	1	1398	0
169	1	415	0	661	0	907	1	1153	1	1399	0
170	1	416	0	662	0	908	0	1154	1	1400	1
171	1	417	0	663	1	909	0	1155	1	1401	0
172	1	418	0	664	1	910	0	1156	0	1402	0
173	0	419	1	665	0	911	0	1157	1	1403	1
174	0	420	1	666	1	912	0	1158	0	1404	0
175	1	421	0	667	1	913	0	1159	0	1405	1
176	0	422	0	668	1	914	0	1160	1	1406	1
177	0	423	1	669	0	915	1	1161	0	1407	1
178	1	424	0	670	1	916	1	1162	1	1408	1
179	1	425	0	671	0	917	0	1163	1	1409	0
180	0	426	0	672	0	918	0	1164	1	1410	0
181	1	427	0	673	0	919	1	1165	0	1411	1
182	0	428	1	674	1	920	1	1166	1	1412	1
183	1	429	0	675	1	921	1	1167	1	1413	0
184	0	430	1	676	0	922	0	1168	0	1414	0
185	1	431	1	677	1	923	0	1169	1	1415	0
186	1	432	0	678	1	924	0	1170	1	1416	0
187	1	433	0	679	0	925	0	1171	1	1417	0
188	1	434	0	680	0	926	0	1172	0	1418	1
189	1	435	0	681	1	927	0	1173	1	1419	0
190	0	436	1	682	1	928	0	1174	0	1420	1
191	1	437	0	683	1	929	1	1175	1	1421	1
192	1	438	1	684	0	930	0	1176	1	1422	0

193	0	439	1	685	0	931	1	1177	0	1423	1
194	1	440	0	686	0	932	1	1178	0	1424	1
195	0	441	0	687	0	933	0	1179	1	1425	1
196	1	442	1	688	1	934	0	1180	1	1426	0
197	1	443	0	689	1	935	0	1181	1	1427	0
198	1	444	0	690	1	936	0	1182	1	1428	0
199	1	445	0	691	0	937	1	1183	0	1429	0
200	0	446	0	692	1	938	1	1184	1	1430	1
201	0	447	1	693	0	939	1	1185	1	1431	0
202	0	448	0	694	1	940	1	1186	0	1432	0
203	1	449	1	695	0	941	0	1187	0	1433	1
204	0	450	1	696	0	942	1	1188	1	1434	0
205	0	451	1	697	0	943	0	1189	1	1435	1
206	1	452	0	698	1	944	0	1190	1	1436	0
207	0	453	0	699	1	945	1	1191	0	1437	1
208	0	454	1	700	0	946	1	1192	0	1438	0
209	1	455	0	701	0	947	0	1193	1	1439	1
210	0	456	0	702	1	948	1	1194	1	1440	1
211	0	457	1	703	1	949	1	1195	0	1441	1
212	0	458	0	704	0	950	1	1196	0	1442	1
213	0	459	1	705	0	951	0	1197	0	1443	0
214	0	460	1	706	0	952	0	1198	0	1444	0
215	0	461	0	707	0	953	1	1199	0	1445	0
216	0	462	0	708	0	954	1	1200	1	1446	1
217	1	463	1	709	0	955	1	1201	1	1447	0
218	0	464	1	710	1	956	1	1202	1	1448	1
219	1	465	0	711	0	957	1	1203	1	1449	1
220	1	466	1	712	1	958	1	1204	0	1450	1
221	0	467	1	713	0	959	0	1205	1	1451	0
222	0	468	0	714	1	960	1	1206	1	1452	0
223	0	469	0	715	0	961	1	1207	1	1453	0
224	1	470	1	716	0	962	1	1208	0	1454	1
225	1	471	0	717	0	963	1	1209	0	1455	1
226	0	472	0	718	0	964	1	1210	1	1456	0
227	1	473	1	719	0	965	1	1211	0	1457	1
228	1	474	0	720	0	966	1	1212	0	1458	0
229	1	475	0	721	1	967	0	1213	0	1459	1
230	1	476	0	722	1	968	1	1214	1	1460	0
231	1	477	0	723	1	969	1	1215	1	1461	1
232	1	478	1	724	0	970	1	1216	1	1462	1
233	1	479	1	725	1	971	1	1217	1	1463	0
234	0	480	1	726	1	972	0	1218	0	1464	0
235	0	481	0	727	1	973	0	1219	1	1465	0
236	1	482	0	728	1	974	1	1220	0	1466	0

237	1	483	0	729	1	975	1	1221	0	1467	0
238	1	484	0	730	0	976	1	1222	0	1468	0
239	0	485	1	731	1	977	0	1223	0	1469	0
240	1	486	1	732	1	978	1	1224	0	1470	1
241	0	487	1	733	1	979	0	1225	0	1471	0
242	0	488	1	734	0	980	1	1226	1		
243	1	489	1	735	0	981	0	1227	1		
244	0	490	1	736	0	982	1	1228	0		
245	0	491	1	737	1	983	0	1229	0		

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