

PROOF OF THE CONTINUED FRACTION CONJECTURE FOR

$$(a, b) = (z^4 + z^3 + 1, z^2 + z + 1)$$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b) -Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^4 + z^3 + 1, z^2 + z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$(1.1) \quad \text{CF}(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\ddots}}}}$$

as the limit of the finite continued fractions

$$(1.2) \quad \text{CF}_n(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{\ddots + \frac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)).$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$(1.3) \quad c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \dots$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b) -Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$(1.4) \quad \text{CF}(a, b) := \text{CF}(\mathbf{t}).$$

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Theorem 1.1. *When $(a, b) = (z^4 + z^3 + 1, z^2 + z + 1)$, the two power series $\text{CF}(a, b)$ and $\text{CF}(b, a)$ are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\text{CF}(a, b)$,

$$\begin{aligned} p_0(z) &= z^{18} + z^{15} + z^{13} + z^{11} + z^{10} + z^9 + z^3 + z + 1, \\ p_1(z) &= z^{22} + z^{21} + z^{20} + z^{18} + z^{16} + z^{15} + z^{11} + z^{10} + z^8 + z^5 + z^3 + z, \\ p_2(z) &= z^{24} + z^{20} + z^{18} + z^{16} + z^{12} + z^{10} + z^8 + z^6 + z^4 + z^2 + 1, \\ p_3(z) &= z^{22} + z^{21} + z^{20} + z^{18} + z^{16} + z^{15} + z^{11} + z^{10} + z^8 + z^5 + z^3 + z, \\ p_4(z) &= z^{20} + z^{15} + z^{13} + z^{12} + z^{11} + z^9 + z^8 + z^6 + z^4 + z^3 + z^2 + z + 1, \end{aligned}$$

and for $\text{CF}(b, a)$,

$$\begin{aligned} p_0(z) &= z^{18} + z^{16} + z^{15} + z^{13} + z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^4 + z^3 + z + 1, \\ p_1(z) &= z^{22} + z^{21} + z^{20} + z^{18} + z^{16} + z^{15} + z^{11} + z^{10} + z^8 + z^5 + z^3 + z, \\ p_2(z) &= z^{24} + z^{20} + z^{18} + z^{16} + z^{12} + z^{10} + z^8 + z^6 + z^4 + z^2 + 1, \\ p_3(z) &= z^{22} + z^{21} + z^{20} + z^{18} + z^{16} + z^{15} + z^{11} + z^{10} + z^8 + z^5 + z^3 + z, \\ p_4(z) &= z^{20} + z^{16} + z^{15} + z^{13} + z^{11} + z^9 + z^6 + z^3 + z^2 + z + 1. \end{aligned}$$

2. PROOF

Define the sequences of polynomials $(P_n(z))_{n \geq 0}$ and $(Q_n(z))_{n \geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$(2.1) \quad \text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)).$$

As formula (2.1) is not efficient for calculating $\text{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\ W_n(x) &= x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $\bar{t}$ is the (b, a) -Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n . Hence,

$$(2.2) \quad \text{CF}_{2^{2^n}-1}(a, b) = \frac{\tilde{P}_{2^{2^n}-1}(x)}{\tilde{Q}_{2^{2^n}-1}(x)} = \frac{M_{2^n}(x)_{0,1}}{M_{2^n}(x)_{0,0}}.$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\text{CF}(a, b)$ will be established if it is shown that both $M_{2^n}(x)_{0,1}$ and $M_{2^n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i, j \leq 1$ the four sequences $(M_{2^n}(x)_{i,j})_n$, $(M_{2^{n+1}}(x)_{i,j})_n$, $(W_{2^n}(x)_{i,j})_n$, and $(W_{2^{n+1}}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i, j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions; the polynomials $\phi(T, i, j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2^n}(x))_n$, $(M_{2^{n+1}}(x))_n$, $(W_{2^n}(x))_n$, and $(W_{2^{n+1}}(x))_n$.

Let us explain how the polynomials $\phi(T, i, j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i, j \leq 1$, the coefficients of the polynomial $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x, y) \in \mathbb{F}_2[x, y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o . The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$(2.3) \quad M^e = W^o \cdot M^o,$$

$$(2.4) \quad M^o = W^e \cdot M^e,$$

$$(2.5) \quad W^e = M^o \cdot W^o,$$

$$(2.6) \quad W^o = M^e \cdot W^e.$$

Proof. We give the proof of the identity

$$M_{0,0}^e = W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o,$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W_{0,0}^o M_{0,0}^o$ and $W_{0,1}^o M_{1,0}^o$. We know that

$$P(x, y) = \text{Res}_z (\phi(W^o, 0, 0)(x, z), z^{12} \cdot \phi(M^o, 0, 0)(x, y/z))$$

is an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W_{0,0}^o M_{0,0}^o$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We verify the first point directly. For the second point, we truncate $W_{0,0}^o M_{0,0}^o$ to order 540 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 540, which proves that $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We find the minimal polynomial $\phi_1(x, y)$ of $W_{0,1}^o M_{1,0}^o$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We know that

$$S(x, y) = \text{Res}_z (\phi_0(x, z), \phi_1(x, y + z))$$

is an annihilating of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ , and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. To see the last point, we truncate $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ to order 660 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 660, and therefore $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$.

Finally, the first 16 terms of $M_{0,0}^e$ and $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] &= M^e[:6u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{3u} M^e[:6u] + x^{6u} M^e[:3u] \\ &\quad + x^{8u} M^e[:u] + x^{5u} M^e[:6u] + x^{8u} M^e[:3u] + x^{10u} M^e[:u] \\ &\quad + x^{6u} M^e[:6u] + x^{7u} M^e[:5u] + x^{8u} M^e[:4u] \\ &\quad + (x^{6u} + x^{9u} + x^{11u}) R^e, \\ W^e[:12u] &= W^e[:6u] + x^{3u} W^e[:3u] + x^{5u} W^e[:u] + x^{3u} W^e[:6u] + x^{6u} W^e[:3u] \\ &\quad + x^{8u} W^e[:u] + x^{5u} W^e[:6u] + x^{8u} W^e[:3u] + x^{10u} W^e[:u] \end{aligned}$$

$$\begin{aligned}
& + x^{6u}W^e[:6u] + x^{7u}W^e[:5u] + x^{8u}W^e[:4u] \\
& + (x^{6u} + x^{9u} + x^{11u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u] \\
& + x^{8u}M^o[:u] + x^{5u}M^o[:6u] + x^{8u}M^o[:3u] + x^{10u}M^o[:u] \\
& + x^{6u}M^o[:6u] + x^{7u}M^o[:5u] + x^{8u}M^o[:4u] \\
& + (x^{6u} + x^{9u} + x^{11u})R^o, \\
W^o[:12u] &= W^o[:6u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u] \\
& + x^{8u}W^o[:u] + x^{5u}W^o[:6u] + x^{8u}W^o[:3u] + x^{10u}W^o[:u] \\
& + x^{6u}W^o[:6u] + x^{7u}W^o[:5u] + x^{8u}W^o[:4u] \\
& + (x^{6u} + x^{9u} + x^{11u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k 's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M_{0,0}^o$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] &= T[:6u] + x^{3u}T[:3u] + x^{5u}T[:u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{8u}T[:u] \\
& + x^{5u}T[:6u] + x^{8u}T[:3u] + x^{10u}T[:u] + x^{6u}T[:6u] + x^{7u}T[:5u] \\
& + x^{8u}T[:4u] + (x^{6u} + x^{9u} + x^{11u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
x^{6u} &= x^{5u}T[u:2u] + x^{3u}T[3u:4u] + T[6u:7u], \\
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + T[7u:8u], \\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] \\
& + T[8u:9u], \\
x^{9u} &= x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] \\
& + T[10u:11u], \\
x^{11u} &= x^{8u}T[3u:4u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$(2.7) \quad 0 = T[[1w]_2] + T[[11w]_2] + T[[110w]_2],$$

$$(2.8) \quad 0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2],$$

$$(2.9) \quad 0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2],$$

$$(2.10) \quad 0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1001w]_2],$$

$$(2.11) \quad 0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2],$$

$$(2.12) \quad 0 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2],$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$(2.13) \quad 1 = T[[1w]_2] + T[[11w]_2] + T[[110w]_2],$$

$$(2.14) \quad 0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2],$$

$$(2.15) \quad 0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2],$$

$$(2.16) \quad 1 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1001w]_2],$$

$$(2.17) \quad 0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2],$$

$$(2.18) \quad 1 = T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2],$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 175 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s , and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$(2.19) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 110)),$$

$$(2.20) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 111)),$$

$$(2.21) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) \\ + \tau(A(s, 1000)),$$

$$(2.22) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1001)),$$

$$(2.23) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ + \tau(A(s, 1010)),$$

$$(2.24) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1011)),$$

for all $s \in E_{2k}$, and

$$(2.25) \quad 1 = \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 110)),$$

$$(2.26) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 111)),$$

$$(2.27) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 101)) \\ + \tau(A(s, 1000)),$$

$$(2.28) \quad 1 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1001)),$$

$$(2.29) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) \\ + \tau(A(s, 1010)),$$

$$(2.30) \quad 1 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1011)),$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{14}), E_{14}) = (A(i, 0^{10}), E_{10})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 7$, which turns out to be true. $\square$

In the following lemma, we express M_{2k} , M_{2k+1} , W_{2k} , and W_{2k+1} in terms of M^e , M^o , W^e , and W^o .

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u} M^e[:3u] + x^{5u} M^e[:u] + x^{6u} R^e,$$

$$W_{2k} = W^e[:6u] + x^{3u}W^e[:3u] + x^{5u}W^e[:u] + x^{6u}R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u}M^o[:3u] + x^{5u}M^o[:u] + x^{6u}R^o,$$

$$W_{2k+1} = W^o[:6u] + x^{3u}W^o[:3u] + x^{5u}W^o[:u] + x^{6u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k} , W_{2k} , M_{2k+1} , and W_{2k+1} . For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} (2.31) \quad W_{2k}M_{2k} &= (W^e[:6v] + x^{3v}W^e[:3v] + x^{5v}W^e[:v] + x^{6v}R^e) \times (M^e[:6v] \\ &\quad + x^{3v}M^e[:3v] + x^{5v}M^e[:v] + x^{6v}R^e); \end{aligned}$$

Call this expression *lhs*. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v} , to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v] + x^{10v}W^e[:2v]M^e[:2v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.31) and *rhs* as polynomials in $\mathbb{F}_2[a_1, \dots, a_{12}, b_1, \dots, b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e , because in the expressions concerned, the products of W^e -terms and M^e -terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\text{CF}(a, b)$; for $\text{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n \rightarrow \infty} M_{2n,j,k} &= M_{j,k}^e, \\ \lim_{n \rightarrow \infty} M_{2n+1,j,k} &= M_{j,k}^o, \\ \lim_{n \rightarrow \infty} W_{2n,j,k} &= W_{j,k}^e, \\ \lim_{n \rightarrow \infty} W_{2n+1,j,k} &= W_{j,k}^o. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\text{CF}(a, b) = \frac{M_{0,1}^e(x)}{M_{0,0}^e(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M_{0,1}^e$ and $M_{0,0}^e$. Therefore

$$P(x, y) = \text{Res}_t (\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y))$$

is an annihilating polynomial of $f(x) = M_{0,1}^e/M_{0,0}^e$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{24} + x^{23} + x^{21} + x^{15} + x^{14} + x^{13} + x^{11} + x^9 + x^6, \\ q_1(x) &= x^{23} + x^{21} + x^{19} + x^{16} + x^{14} + x^{13} + x^9 + x^8 + x^6 + x^4 + x^3 + x^2, \\ q_2(x) &= x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^8 + x^6 + x^4 + 1, \\ q_3(x) &= x^{23} + x^{21} + x^{19} + x^{16} + x^{14} + x^{13} + x^9 + x^8 + x^6 + x^4 + x^3 + x^2, \\ q_4(x) &= x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^9 \\ &\quad + x^4. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\text{CF}(a, b) = f(1/z)$. $\square$

3. ANNEXE

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned} p_0(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{110} + x^{106} + x^{102} + x^{92} + x^{88} + x^{86} \\ &\quad + x^{84} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} + x^{62} + x^{60} + x^{50} + x^{48} \\ &\quad + x^{44} + x^{42} + x^{38} + x^{36} + x^{26} + x^{24}, \\ p_3(x) &= x^{118} + x^{116} + x^{112} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} \\ &\quad + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{68} + x^{62} \\ &\quad + x^{60} + x^{52} + x^{48} + x^{42} + x^{38} + x^{36} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18}, \\ p_6(x) &= x^{118} + x^{114} + x^{112} + x^{104} + x^{102} + x^{100} + x^{98} + x^{92} + x^{86} + x^{82} \\ &\quad + x^{68} + x^{60} + x^{58} + x^{54} + x^{44} + x^{40} + x^{26} + x^{20} + x^{18} + x^{10} + x^8 \end{aligned}$$

$$\begin{aligned}
& + x^4 + x^2 + 1, \\
p_9(x) &= x^{118} + x^{116} + x^{114} + x^{106} + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} \\
& + x^{78} + x^{76} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{52} + x^{48} + x^{42} + x^{40} \\
& + x^{38} + x^{28} + x^{26} + x^{24} + x^{22} + x^{14} + x^8 + x^4, \\
p_{12}(x) &= x^{120} + x^{114} + x^{106} + x^{98} + x^{96} + x^{56} + x^{50} + x^{42} + x^{34} + x^{32} + x^{24} \\
& + x^{18} + x^{10} + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{112} \\
& + x^{111} + x^{109} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{96} + x^{91} + x^{90} \\
& + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} \\
& + x^{74} + x^{70} + x^{69} + x^{68} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
& + x^{55} + x^{53} + x^{49} + x^{48} + x^{47} + x^{45} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{30}, \\
p_3(x) &= x^{117} + x^{116} + x^{109} + x^{108} + x^{103} + x^{102} + x^{93} + x^{92} + x^{91} + x^{90} \\
& + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{67} + x^{66} + x^{59} \\
& + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{45} + x^{44} + x^{39} + x^{38} + x^{37} + x^{36} \\
& + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
& + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} \\
& + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{74} \\
& + x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{45} \\
& + x^{43} + x^{42} + x^{37} + x^{36} + x^{34} + x^{28} + x^{27} + x^{21} + x^{19} + x^{18} + x^{16} \\
& + x^{15} + x^{14} + x^{12}, \\
p_9(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{107} + x^{106} + x^{95} + x^{94} \\
& + x^{91} + x^{90} + x^{87} + x^{86} + x^{75} + x^{74} + x^{67} + x^{66} + x^{63} + x^{62} + x^{55} \\
& + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} \\
& + x^{11} + x^{10} + x^7 + x^6, \\
p_{12}(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} \\
& + x^{114} + x^{113} + x^{111} + x^{108} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} \\
& + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\
& + x^{47} + x^{44} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} \\
& + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^6 \\
& + x^5 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1]$.

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{112} \\
&\quad + x^{111} + x^{109} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{96} + x^{91} + x^{90} \\
&\quad + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{70} + x^{69} + x^{68} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{49} + x^{48} + x^{47} + x^{45} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} \\
&\quad + x^{34} + x^{30}, \\
p_3(x) &= x^{117} + x^{116} + x^{109} + x^{108} + x^{103} + x^{102} + x^{93} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{67} + x^{66} + x^{59} \\
&\quad + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{45} + x^{44} + x^{39} + x^{38} + x^{37} + x^{36} \\
&\quad + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{120} + x^{119} + x^{118} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} \\
&\quad + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{74} \\
&\quad + x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{45} \\
&\quad + x^{43} + x^{42} + x^{37} + x^{36} + x^{34} + x^{28} + x^{27} + x^{21} + x^{19} + x^{18} + x^{16} \\
&\quad + x^{15} + x^{14} + x^{12}, \\
p_9(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{107} + x^{106} + x^{95} + x^{94} \\
&\quad + x^{91} + x^{90} + x^{87} + x^{86} + x^{75} + x^{74} + x^{67} + x^{66} + x^{63} + x^{62} + x^{55} \\
&\quad + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} \\
&\quad + x^{11} + x^{10} + x^7 + x^6, \\
p_{12}(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{111} + x^{108} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} \\
&\quad + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{44} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^6 \\
&\quad + x^5 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1]$.

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{108} + x^{104} + x^{96} \\
&\quad + x^{92} + x^{90} + x^{84} + x^{82} + x^{70} + x^{64} + x^{58} + x^{48} + x^{46} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{36}, \\
p_3(x) &= x^{130} + x^{126} + x^{122} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{92} \\
&\quad + x^{90} + x^{82} + x^{76} + x^{72} + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{52} + x^{46} \\
&\quad + x^{44} + x^{36} + x^{28} + x^{18} + x^{12},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{130} + x^{112} + x^{110} + x^{104} + x^{102} + x^{96} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} \\
&\quad + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{44} + x^{40} + x^{36} \\
&\quad + x^{26} + x^{24} + x^{18} + x^{16} + x^{12} + x^{10} + x^2 + 1, \\
p_9(x) &= x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{110} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{70} + x^{66} + x^{64} \\
&\quad + x^{60} + x^{54} + x^{48} + x^{46} + x^{40} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{16} + x^{12} + x^{10} + x^4, \\
p_{12}(x) &= x^{132} + x^{130} + x^{128} + x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{44} + x^{42} + x^{40} + x^{12} + x^{10} + x^8 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{126} + x^{124} + x^{120} + x^{116} + x^{108} + x^{100} + x^{98} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{86} + x^{78} + x^{76} + x^{68} + x^{66} + x^{58} + x^{54} + x^{52} + x^{50} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{130} + x^{124} + x^{118} + x^{114} + x^{108} + x^{104} + x^{98} + x^{94} + x^{92} + x^{90} \\
&\quad + x^{86} + x^{76} + x^{74} + x^{72} + x^{62} + x^{60} + x^{58} + x^{56} + x^{46} + x^{44} + x^{38} \\
&\quad + x^{34} + x^{28} + x^{26} + x^{22} + x^{18}, \\
p_6(x) &= x^{130} + x^{128} + x^{122} + x^{120} + x^{110} + x^{108} + x^{106} + x^{94} + x^{92} + x^{90} \\
&\quad + x^{76} + x^{68} + x^{66} + x^{62} + x^{58} + x^{50} + x^{48} + x^{44} + x^{36} + x^{34} + x^{20} \\
&\quad + x^{18} + x^{14} + x^8 + x^2 + 1, \\
p_9(x) &= x^{130} + x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{110} + x^{106} + x^{104} + x^{100} \\
&\quad + x^{98} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{70} + x^{66} + x^{64} \\
&\quad + x^{60} + x^{54} + x^{48} + x^{46} + x^{40} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{16} + x^{12} + x^{10} + x^4, \\
p_{12}(x) &= x^{132} + x^{130} + x^{128} + x^{108} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{44} + x^{42} + x^{40} + x^{12} + x^{10} + x^8 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{113} + x^{112} + x^{110} \\
&\quad + x^{109} + x^{106} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{89} + x^{85} \\
&\quad + x^{84} + x^{82} + x^{81} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{63} \\
&\quad + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30}, \\
p_3(x) &= x^{117} + x^{116} + x^{109} + x^{108} + x^{103} + x^{102} + x^{93} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{67} + x^{66} + x^{59}
\end{aligned}$$

$$\begin{aligned}
& + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{45} + x^{44} + x^{39} + x^{38} + x^{37} + x^{36} \\
& + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) = & x^{120} + x^{119} + x^{118} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
& + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} \\
& + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{74} \\
& + x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{45} \\
& + x^{43} + x^{42} + x^{37} + x^{36} + x^{34} + x^{28} + x^{27} + x^{21} + x^{19} + x^{18} + x^{16} \\
& + x^{15} + x^{14} + x^{12}, \\
p_9(x) = & x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{107} + x^{106} + x^{95} + x^{94} \\
& + x^{91} + x^{90} + x^{87} + x^{86} + x^{75} + x^{74} + x^{67} + x^{66} + x^{63} + x^{62} + x^{55} \\
& + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} \\
& + x^{11} + x^{10} + x^7 + x^6, \\
p_{12}(x) = & x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} \\
& + x^{114} + x^{113} + x^{111} + x^{108} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} \\
& + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\
& + x^{47} + x^{44} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} \\
& + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^6 \\
& + x^5 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1]$.

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = & x^{126} + x^{125} + x^{124} + x^{122} + x^{121} + x^{119} + x^{116} + x^{113} + x^{112} + x^{110} \\
& + x^{109} + x^{106} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{89} + x^{85} \\
& + x^{84} + x^{82} + x^{81} + x^{79} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{63} \\
& + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{45} \\
& + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30}, \\
p_3(x) = & x^{117} + x^{116} + x^{109} + x^{108} + x^{103} + x^{102} + x^{93} + x^{92} + x^{91} + x^{90} \\
& + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{67} + x^{66} + x^{59} \\
& + x^{58} + x^{55} + x^{54} + x^{51} + x^{50} + x^{45} + x^{44} + x^{39} + x^{38} + x^{37} + x^{36} \\
& + x^{27} + x^{26} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) = & x^{120} + x^{119} + x^{118} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} \\
& + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{93} \\
& + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{74} \\
& + x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{59} + x^{56} + x^{55} + x^{54} + x^{51} + x^{45} \\
& + x^{43} + x^{42} + x^{37} + x^{36} + x^{34} + x^{28} + x^{27} + x^{21} + x^{19} + x^{18} + x^{16} \\
& + x^{15} + x^{14} + x^{12},
\end{aligned}$$

$$\begin{aligned}
p_9(x) &= x^{123} + x^{122} + x^{119} + x^{118} + x^{115} + x^{114} + x^{107} + x^{106} + x^{95} + x^{94} \\
&\quad + x^{91} + x^{90} + x^{87} + x^{86} + x^{75} + x^{74} + x^{67} + x^{66} + x^{63} + x^{62} + x^{55} \\
&\quad + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{39} + x^{38} + x^{35} + x^{34} \\
&\quad + x^{11} + x^{10} + x^7 + x^6, \\
p_{12}(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{111} + x^{108} + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} \\
&\quad + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{47} + x^{44} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{12} + x^6 \\
&\quad + x^5 + x^4 + x^3 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1]$.

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{88} + x^{80} + x^{76} + x^{74} + x^{70} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{46} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{104} + x^{90} + x^{84} + x^{82} + x^{80} \\
&\quad + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{58} + x^{56} + x^{54} + x^{32} + x^{28} + x^{26} \\
&\quad + x^{22} + x^{16} + x^{12}, \\
p_6(x) &= x^{118} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{100} + x^{96} + x^{92} + x^{86} \\
&\quad + x^{84} + x^{80} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} + x^{50} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{36} + x^{34} + x^{32} + x^{26} + x^{14} + x^4 + x^2 + 1, \\
p_9(x) &= x^{118} + x^{116} + x^{114} + x^{106} + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{82} \\
&\quad + x^{78} + x^{76} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{52} + x^{48} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{28} + x^{26} + x^{24} + x^{22} + x^{14} + x^8 + x^4, \\
p_{12}(x) &= x^{120} + x^{114} + x^{106} + x^{98} + x^{96} + x^{56} + x^{50} + x^{42} + x^{34} + x^{32} + x^{24} \\
&\quad + x^{18} + x^{10} + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{122} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} \\
&\quad + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{92} + x^{91} + x^{89} + x^{87} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} \\
&\quad + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} \\
&\quad + x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24}, \\
p_3(x) &= x^{122} + x^{121} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{104}
\end{aligned}$$

$$\begin{aligned}
& + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{94} + x^{93} + x^{88} + x^{86} \\
& + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} \\
& + x^{63} + x^{60} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{48} + x^{44} + x^{43} + x^{40} \\
& + x^{37} + x^{36} + x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} \\
& + x^{20} + x^{19} + x^{16}, \\
p_6(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} \\
& + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} \\
& + x^{89} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{74} + x^{73} + x^{64} + x^{61} \\
& + x^{57} + x^{56} + x^{55} + x^{53} + x^{46} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18} \\
& + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^3 + 1, \\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{111} \\
& + x^{110} + x^{109} + x^{107} + x^{104} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
& + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{40} + x^{26} + x^{25} + x^{24} \\
& + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^8, \\
p_{12}(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} \\
& + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{46} \\
& + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{24} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^3 \\
& + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1]$.

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{133} + x^{132} + x^{131} + x^{130} + x^{127} + x^{126} + x^{122} + x^{120} + x^{116} \\
& + x^{114} + x^{112} + x^{107} + x^{105} + x^{104} + x^{100} + x^{97} + x^{93} + x^{85} + x^{80} \\
& + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{62} \\
& + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{47} + x^{44} \\
& + x^{43} + x^{42} + x^{36} + x^{33} + x^{30}, \\
p_3(x) &= x^{127} + x^{123} + x^{121} + x^{120} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{104} \\
& + x^{101} + x^{100} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} \\
& + x^{80} + x^{79} + x^{78} + x^{77} + x^{73} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} \\
& + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{45} + x^{42} \\
& + x^{41} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{22} + x^{20} \\
& + x^{19} + x^{18}, \\
p_6(x) &= x^{130} + x^{122} + x^{116} + x^{114} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
& + x^{92} + x^{86} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{48}
\end{aligned}$$

$$\begin{aligned}
& + x^{44} + x^{40} + x^{38} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} + x^{12}, \\
p_9(x) = & x^{133} + x^{129} + x^{127} + x^{126} + x^{125} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} \\
& + x^{110} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{69} + x^{65} + x^{63} + x^{62} \\
& + x^{61} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} \\
& + x^{38} + x^{37} + x^{33} + x^{31} + x^{30} + x^{29} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15} \\
& + x^{14} + x^{11} + x^{10} + x^9 + x^7 + x^6, \\
p_{12}(x) = & x^{136} + x^{124} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{72} + x^{60} + x^{48} \\
& + x^{44} + x^{36} + x^{32} + x^{28} + x^{16} + x^{12} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1]$.

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = & x^{128} + x^{125} + x^{123} + x^{121} + x^{120} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} \\
& + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{95} + x^{94} + x^{92} \\
& + x^{91} + x^{88} + x^{87} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{71} + x^{70} \\
& + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{58} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} \\
& + x^{47} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32} + x^{30}, \\
p_3(x) = & x^{119} + x^{113} + x^{112} + x^{109} + x^{108} + x^{104} + x^{101} + x^{100} + x^{98} + x^{96} \\
& + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} + x^{80} + x^{78} + x^{77} + x^{75} \\
& + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{55} + x^{54} \\
& + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} \\
& + x^{34} + x^{33} + x^{30} + x^{28} + x^{27} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) = & x^{122} + x^{118} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{84} + x^{82} \\
& + x^{80} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} + x^{48} + x^{46} \\
& + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) = & x^{125} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{107} + x^{106} + x^{105} \\
& + x^{103} + x^{102} + x^{61} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{43} + x^{42} \\
& + x^{41} + x^{39} + x^{38} + x^{29} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{11} \\
& + x^{10} + x^9 + x^7 + x^6, \\
p_{12}(x) = & x^{128} + x^{124} + x^{120} + x^{112} + x^{108} + x^{100} + x^{96} + x^{64} + x^{60} + x^{56} \\
& + x^{48} + x^{44} + x^{36} + x^{28} + x^{24} + x^{16} + x^{12} + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0]$.

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = & x^{126} + x^{125} + x^{121} + x^{118} + x^{116} + x^{115} + x^{113} + x^{110} + x^{109} + x^{107} \\
& + x^{106} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85} \\
& + x^{84} + x^{80} + x^{79} + x^{78} + x^{75} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} \\
& + x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{52} + x^{51} + x^{48} + x^{44} + x^{43} + x^{42}
\end{aligned}$$

$$\begin{aligned}
& + x^{41} + x^{39} + x^{38} + x^{36}, \\
p_3(x) &= x^{122} + x^{121} + x^{118} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{106} + x^{102} \\
& + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{86} \\
& + x^{80} + x^{79} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{63} + x^{59} \\
& + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{45} + x^{43} + x^{42} \\
& + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} \\
& + x^{22} + x^{20}, \\
p_6(x) &= x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{105} \\
& + x^{104} + x^{103} + x^{100} + x^{95} + x^{93} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} \\
& + x^{78} + x^{75} + x^{73} + x^{72} + x^{68} + x^{67} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} \\
& + x^{56} + x^{55} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{37} \\
& + x^{36} + x^{33} + x^{26} + x^{21} + x^{18} + x^{14} + x^6 + x^5 + x^4 + x^3 + 1, \\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{111} \\
& + x^{110} + x^{109} + x^{107} + x^{104} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
& + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{40} + x^{26} + x^{25} + x^{24} \\
& + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^8, \\
p_{12}(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} \\
& + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{46} \\
& + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{24} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^3 \\
& + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1]$.

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{125} + x^{122} + x^{118} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} \\
& + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{92} + x^{91} + x^{89} + x^{87} \\
& + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{73} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} \\
& + x^{65} + x^{64} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} \\
& + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} \\
& + x^{32} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24}, \\
p_3(x) &= x^{122} + x^{121} + x^{117} + x^{114} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{104} \\
& + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{94} + x^{93} + x^{88} + x^{86} \\
& + x^{85} + x^{83} + x^{82} + x^{80} + x^{79} + x^{75} + x^{72} + x^{71} + x^{69} + x^{66} + x^{64} \\
& + x^{63} + x^{60} + x^{58} + x^{57} + x^{55} + x^{53} + x^{52} + x^{48} + x^{44} + x^{43} + x^{40} \\
& + x^{37} + x^{36} + x^{33} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{22} + x^{21} \\
& + x^{20} + x^{19} + x^{16},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{124} + x^{123} + x^{121} + x^{119} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{102} + x^{96} + x^{93} + x^{92} + x^{91} + x^{90} \\
&\quad + x^{89} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{74} + x^{73} + x^{64} + x^{61} \\
&\quad + x^{57} + x^{56} + x^{55} + x^{53} + x^{46} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{35} \\
&\quad + x^{34} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{18} \\
&\quad + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^3 + 1, \\
p_9(x) &= x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{111} \\
&\quad + x^{110} + x^{109} + x^{107} + x^{104} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{40} + x^{26} + x^{25} + x^{24} \\
&\quad + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^8, \\
p_{12}(x) &= x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{46} \\
&\quad + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{27} + x^{24} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^3 \\
&\quad + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1]$.

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{128} + x^{125} + x^{123} + x^{121} + x^{120} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{99} + x^{95} + x^{94} + x^{92} \\
&\quad + x^{91} + x^{88} + x^{87} + x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{71} + x^{70} \\
&\quad + x^{67} + x^{66} + x^{63} + x^{61} + x^{59} + x^{58} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{42} + x^{40} + x^{37} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{119} + x^{113} + x^{112} + x^{109} + x^{108} + x^{104} + x^{101} + x^{100} + x^{98} + x^{96} \\
&\quad + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} + x^{80} + x^{78} + x^{77} + x^{75} \\
&\quad + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} + x^{62} + x^{60} + x^{59} + x^{55} + x^{54} \\
&\quad + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} \\
&\quad + x^{34} + x^{33} + x^{30} + x^{28} + x^{27} + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{122} + x^{118} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{84} + x^{82} \\
&\quad + x^{80} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} + x^{48} + x^{46} \\
&\quad + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{125} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{107} + x^{106} + x^{105} \\
&\quad + x^{103} + x^{102} + x^{61} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{29} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{11} \\
&\quad + x^{10} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{128} + x^{124} + x^{120} + x^{112} + x^{108} + x^{100} + x^{96} + x^{64} + x^{60} + x^{56}
\end{aligned}$$

$$+ x^{48} + x^{44} + x^{36} + x^{28} + x^{24} + x^{16} + x^{12} + x^4 + 1,$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0]$.

For $\phi(W^o, 1, 0)$:

$$\begin{aligned} p_0(x) = & x^{136} + x^{133} + x^{132} + x^{131} + x^{130} + x^{127} + x^{126} + x^{122} + x^{120} + x^{116} \\ & + x^{114} + x^{112} + x^{107} + x^{105} + x^{104} + x^{100} + x^{97} + x^{93} + x^{85} + x^{80} \\ & + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{62} \\ & + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{47} + x^{44} \\ & + x^{43} + x^{42} + x^{36} + x^{33} + x^{30}, \end{aligned}$$

$$\begin{aligned} p_3(x) = & x^{127} + x^{123} + x^{121} + x^{120} + x^{113} + x^{111} + x^{109} + x^{107} + x^{106} + x^{104} \\ & + x^{101} + x^{100} + x^{98} + x^{96} + x^{94} + x^{93} + x^{91} + x^{88} + x^{86} + x^{85} + x^{83} \\ & + x^{80} + x^{79} + x^{78} + x^{77} + x^{73} + x^{70} + x^{69} + x^{67} + x^{65} + x^{64} + x^{62} \\ & + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{45} + x^{42} \\ & + x^{41} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{22} + x^{20} \\ & + x^{19} + x^{18}, \end{aligned}$$

$$\begin{aligned} p_6(x) = & x^{130} + x^{122} + x^{116} + x^{114} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\ & + x^{92} + x^{86} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{48} \\ & + x^{44} + x^{40} + x^{38} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} + x^{12}, \end{aligned}$$

$$\begin{aligned} p_9(x) = & x^{133} + x^{129} + x^{127} + x^{126} + x^{125} + x^{119} + x^{118} + x^{115} + x^{114} + x^{111} \\ & + x^{110} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{69} + x^{65} + x^{63} + x^{62} \\ & + x^{61} + x^{55} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} \\ & + x^{38} + x^{37} + x^{33} + x^{31} + x^{30} + x^{29} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15} \\ & + x^{14} + x^{11} + x^{10} + x^9 + x^7 + x^6, \end{aligned}$$

$$\begin{aligned} p_{12}(x) = & x^{136} + x^{124} + x^{112} + x^{108} + x^{104} + x^{100} + x^{96} + x^{72} + x^{60} + x^{48} \\ & + x^{44} + x^{36} + x^{32} + x^{28} + x^{16} + x^{12} + x^8 + x^4 + 1, \end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1]$.

For $\phi(W^o, 1, 1)$:

$$\begin{aligned} p_0(x) = & x^{126} + x^{125} + x^{121} + x^{118} + x^{116} + x^{115} + x^{113} + x^{110} + x^{109} + x^{107} \\ & + x^{106} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{88} + x^{85} \\ & + x^{84} + x^{80} + x^{79} + x^{78} + x^{75} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} \\ & + x^{63} + x^{62} + x^{61} + x^{58} + x^{57} + x^{52} + x^{51} + x^{48} + x^{44} + x^{43} + x^{42} \\ & + x^{41} + x^{39} + x^{38} + x^{36}, \end{aligned}$$

$$\begin{aligned} p_3(x) = & x^{122} + x^{121} + x^{118} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{106} + x^{102} \\ & + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{86} \\ & + x^{80} + x^{79} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{63} + x^{59} \\ & + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{45} + x^{43} + x^{42} \end{aligned}$$

$$\begin{aligned}
& + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} \\
& + x^{22} + x^{20}, \\
p_6(x) = & x^{124} + x^{123} + x^{122} + x^{119} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{105} \\
& + x^{104} + x^{103} + x^{100} + x^{95} + x^{93} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{81} \\
& + x^{78} + x^{75} + x^{73} + x^{72} + x^{68} + x^{67} + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} \\
& + x^{56} + x^{55} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{37} \\
& + x^{36} + x^{33} + x^{26} + x^{21} + x^{18} + x^{14} + x^6 + x^5 + x^4 + x^3 + 1, \\
p_9(x) = & x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{114} + x^{113} + x^{111} \\
& + x^{110} + x^{109} + x^{107} + x^{104} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
& + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{45} + x^{43} + x^{40} + x^{26} + x^{25} + x^{24} \\
& + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{11} + x^8, \\
p_{12}(x) = & x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{110} + x^{109} + x^{108} + x^{107} + x^{104} \\
& + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{62} + x^{61} + x^{60} + x^{59} + x^{56} + x^{46} \\
& + x^{45} + x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{24} + x^{14} + x^{13} + x^{12} + x^{11} + x^8 + x^6 + x^5 + x^4 + x^3 \\
& + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1]$.

Below is the transition function and output function of an 2-automaton that generates $T = M_{0,0}^0$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	36	[67, 68]	72	[108, 109]	108	[63, 72]	144	[169, 170]
1	[3, 4]	37	[69, 25]	73	[86, 110]	109	[26, 135]	145	[171, 172]
2	[5, 6]	38	[70, 27]	74	[111, 112]	110	[74, 69]	146	[173, 23]
3	[7, 8]	39	[23, 71]	75	[113, 114]	111	[73, 76]	147	[23, 141]
4	[9, 10]	40	[60, 72]	76	[115, 116]	112	[74, 74]	148	[26, 81]
5	[11, 12]	41	[69, 73]	77	[117, 118]	113	[143, 23]	149	[69, 24]
6	[13, 14]	42	[21, 74]	78	[119, 120]	114	[80, 135]	150	[141, 82]
7	[15, 16]	43	[69, 74]	79	[118, 121]	115	[73, 59]	151	[138, 131]
8	[17, 18]	44	[75, 23]	80	[122, 123]	116	[69, 76]	152	[73, 24]
9	[19, 20]	45	[63, 61]	81	[124, 125]	117	[144, 140]	153	[62, 79]
10	[21, 21]	46	[76, 59]	82	[126, 127]	118	[69, 21]	154	[139, 72]
11	[22, 23]	47	[26, 72]	83	[128, 126]	119	[17, 137]	155	[139, 61]
12	[24, 25]	48	[74, 76]	84	[41, 129]	120	[83, 142]	156	[65, 71]
13	[26, 27]	49	[69, 59]	85	[130, 23]	121	[28, 134]	157	[144, 72]
14	[28, 29]	50	[21, 69]	86	[74, 59]	122	[142, 77]	158	[74, 25]
15	[30, 31]	51	[66, 77]	87	[131, 132]	123	[133, 145]	159	[24, 59]
16	[32, 33]	52	[78, 79]	88	[133, 134]	124	[70, 68]	160	[60, 81]
17	[34, 35]	53	[80, 81]	89	[80, 68]	125	[26, 61]	161	[131, 137]
18	[36, 37]	54	[26, 64]	90	[60, 135]	126	[26, 68]	162	[78, 145]

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	26	1	52	0	78	0	104	1	130	0
1	0	27	1	53	1	79	0	105	1	131	0
2	0	28	1	54	1	80	1	106	0	132	1
3	0	29	0	55	1	81	1	107	1	133	1
4	1	30	0	56	0	82	1	108	0	134	1
5	0	31	0	57	0	83	0	109	1	135	0
6	1	32	0	58	0	84	0	110	1	136	0
7	0	33	0	59	0	85	0	111	1	137	0
8	0	34	0	60	0	86	1	112	1	138	1
9	1	35	0	61	1	87	0	113	0	139	0
10	0	36	1	62	0	88	1	114	1	140	1
11	0	37	0	63	0	89	1	115	1	141	1
12	1	38	1	64	0	90	0	116	0	142	1
13	1	39	1	65	0	91	0	117	0	143	0
14	1	40	0	66	1	92	1	118	0	144	0
15	0	41	0	67	1	93	0	119	0	145	1
16	0	42	0	68	0	94	0	120	0	146	0
17	0	43	0	69	0	95	0	121	1	147	1
18	1	44	0	70	1	96	0	122	1	148	1
19	1	45	0	71	0	97	0	123	1	149	0
20	0	46	1	72	0	98	1	124	1	150	1
21	0	47	1	73	1	99	0	125	1	151	1
22	0	48	1	74	1	100	1	126	1	152	1
23	1	49	0	75	0	101	0	127	1	153	0

24	1	50	0	76	1	102	0	128	0	154	0		
25	0	51	1	77	0	103	0	129	0	155	0		

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