PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^2+1, z^4+z^3+z^2+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^2+1, z^4+z^3+z^2+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z), t_1(z), t_2(z), \ldots) = (a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^2+1, z^4+z^3+z^2+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{13}+z^9+z^8+z^7+z^5+z^4+z^3+1,\\
p_1(z)&=z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^6+z^4+z^3,\\
p_2(z)&=z^{20}+z^{16}+z^{10}+z^6+z^4+1,\\
p_3(z)&=z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^6+z^4+z^3,\\
p_4(z)&=z^{16}+z^{14}+z^{13}+z^{12}+z^{10}+z^9+z^7+z^6+z^5+z^4+z^3+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{13}+z^{12}+z^9+z^8+z^7+z^5+z^4+z^3+1,\\
p_1(z)&=z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^6+z^4+z^3,\\
p_2(z)&=z^{20}+z^{16}+z^{10}+z^6+z^4+1,\\
p_3(z)&=z^{18}+z^{15}+z^{13}+z^{12}+z^{11}+z^6+z^4+z^3,\\
p_4(z)&=z^{16}+z^{14}+z^{13}+z^{10}+z^9+z^7+z^6+z^5+z^4+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u]\\
& + x^u M^e[:6u] + x^{2u}M^e[:5u] + x^{4u}M^e[:3u] + x^{5u}M^e[:2u]\\
& + x^{6u}M^e[:u] + x^{2u}M^e[:6u] + x^{3u}M^e[:5u] + x^{5u}M^e[:3u]\\
& + x^{6u}M^e[:2u] + x^{7u}M^e[:u] + x^{5u}M^e[:6u] + x^{6u}M^e[:5u]\\
& + x^{8u}M^e[:3u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{7u}M^e[:5u]\\
& + x^{10u}M^e[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{11u})R^e,
\end{aligned}$$

$$\begin{aligned}
W^e[:12u] = {} & W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{5u} W^e[:u] \\
& + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] \\
& + x^{6u} W^e[:u] + x^{2u} W^e[:6u] + x^{3u} W^e[:5u] + x^{5u} W^e[:3u] \\
& + x^{6u} W^e[:2u] + x^{7u} W^e[:u] + x^{5u} W^e[:6u] + x^{6u} W^e[:5u] \\
& + x^{8u} W^e[:3u] + x^{9u} W^e[:2u] + x^{10u} W^e[:u] + x^{7u} W^e[:5u] \\
& + x^{10u} W^e[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{11u}) R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] = {} & M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{4u} M^o[:2u] + x^{5u} M^o[:u] \\
& + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] \\
& + x^{6u} M^o[:u] + x^{2u} M^o[:6u] + x^{3u} M^o[:5u] + x^{5u} M^o[:3u] \\
& + x^{6u} M^o[:2u] + x^{7u} M^o[:u] + x^{5u} M^o[:6u] + x^{6u} M^o[:5u] \\
& + x^{8u} M^o[:3u] + x^{9u} M^o[:2u] + x^{10u} M^o[:u] + x^{7u} M^o[:5u] \\
& + x^{10u} M^o[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{11u}) R^o, \\
W^o[:12u] = {} & W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{4u} W^o[:2u] + x^{5u} W^o[:u] \\
& + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] \\
& + x^{6u} W^o[:u] + x^{2u} W^o[:6u] + x^{3u} W^o[:5u] + x^{5u} W^o[:3u] \\
& + x^{6u} W^o[:2u] + x^{7u} W^o[:u] + x^{5u} W^o[:6u] + x^{6u} W^o[:5u] \\
& + x^{8u} W^o[:3u] + x^{9u} W^o[:2u] + x^{10u} W^o[:u] + x^{7u} W^o[:5u] \\
& + x^{10u} W^o[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{11u}) R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] = {} & T[:6u] + x^u T[:5u] + x^{3u} T[:3u] + x^{4u} T[:2u] + x^{5u} T[:u] + x^u T[:6u] \\
& + x^{2u} T[:5u] + x^{4u} T[:3u] + x^{5u} T[:2u] + x^{6u} T[:u] + x^{2u} T[:6u] \\
& + x^{3u} T[:5u] + x^{5u} T[:3u] + x^{6u} T[:2u] + x^{7u} T[:u] + x^{5u} T[:6u] \\
& + x^{6u} T[:5u] + x^{8u} T[:3u] + x^{9u} T[:2u] + x^{10u} T[:u] + x^{7u} T[:5u] \\
& + x^{10u} T[:2u] + (x^{6u} + x^{7u} + x^{8u} + x^{11u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
x^6 u &= x^{6u} T[:u] + x^{5u} T[u:2u] + x^{4u} T[2u:3u] + x^{3u} T[3u:4u] \\
&\quad + x^u T[5u:6u] + T[6u:7u], \\
x^7 u &= x^{3u} T[4u:5u] + x^{2u} T[5u:6u] + T[7u:8u], \\
x^8 u &= x^{8u} T[:u] + x^{7u} T[u:2u] + x^{6u} T[2u:3u] + x^{5u} T[3u:4u] \\
&\quad + T[8u:9u], \\
0 &= x^{9u} T[:u] + x^{8u} T[u:2u] + x^{7u} T[2u:3u] + x^{6u} T[3u:4u]
\end{aligned}$$

$$+ x^{5u}T[4u:5u] + T[9u:10u],$$
$$0 = x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u]$$
$$+ x^{5u}T[5u:6u] + T[10u:11u],$$
$$x^1 1u = x^{10u}T[u:2u] + x^{7u}T[4u:5u] + T[11u:12u],$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[1w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$
$$1 = T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$
$$1 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \tag{2.17}$$
$$1 = T[[1w]_2] + T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1717 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.19}$$
$$0 = \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.20}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1000)), \tag{2.21}$$
$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100))$$

$$+\tau(A(s,1001)), \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)), \end{aligned} \tag{2.25}$$

$$1 = \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,111)), \tag{2.26}$$

$$\begin{aligned} 1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.27}$$

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,1001)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$1 = \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{30}), E_{30}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 15$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{4u} M^e[:2u] + x^{5u} M^e[:u] \\ &\quad + x^{6u} R^e, \\ W_{2k} &= W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{4u} W^e[:2u] + x^{5u} W^e[:u] \\ &\quad + x^{6u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{4u} M^o[:2u] + x^{5u} M^o[:u] \\ &\quad + x^{6u} R^o, \\ W_{2k+1} &= W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{4u} W^o[:2u] + x^{5u} W^o[:u] \\ &\quad + x^{6u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^v W^e[:5v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{5v}W^e[:v] \\ &+ x^{6u}R^e\big) \times \big(M^e[:6v] + x^v M^e[:5v] + x^{3v}M^e[:3v] + x^{4v}M^e[:2v] \\ &+ x^{5v}M^e[:v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} W^e[:12v]M^e[:12v] + x^{2v}W^e[:10v]M^e[:10v] + x^{6v}W^e[:6v]M^e[:6v] \\ + x^{8v}W^e[:4v]M^e[:4v] + x^{10v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^7 + x^6,\\
q_1(x) &= x^{17} + x^{16} + x^{14} + x^9 + x^8 + x^7 + x^5 + x^2,\\
q_2(x) &= x^{20} + x^{16} + x^{14} + x^{10} + x^4 + 1,\\
q_3(x) &= x^{17} + x^{16} + x^{14} + x^9 + x^8 + x^7 + x^5 + x^2,\\
q_4(x) &= x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^8 + x^7 + x^6 + x^4.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{92} + x^{90} + x^{84}\\
&\quad + x^{82} + x^{78} + x^{72} + x^{68} + x^{66} + x^{64} + x^{56} + x^{52} + x^{50} + x^{40} + x^{38}\\
&\quad + x^{34} + x^{28} + x^{26} + x^{24},\\
p_3(x) &= x^{108} + x^{104} + x^{96} + x^{92} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{68}\\
&\quad + x^{66} + x^{56} + x^{54} + x^{50} + x^{42} + x^{36} + x^{32} + x^{30} + x^{26} + x^{24} + x^{18},\\
p_6(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{86} + x^{80} + x^{78} + x^{72}\\
&\quad + x^{68} + x^{60} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{28} + x^{26} + x^{20}\\
&\quad + x^{18} + x^{16} + x^{14} + x^{10} + x^6 + x^4 + x^2 + 1,\\
p_9(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{94} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{68}\\
&\quad + x^{60} + x^{58} + x^{48} + x^{46} + x^{40} + x^{36} + x^{34} + x^{30} + x^{20} + x^{18} + x^8\\
&\quad + x^6 + x^4,\\
p_{12}(x) &= x^{114} + x^{112} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{58}\\
&\quad + x^{56} + x^{42} + x^{40} + x^{18} + x^{16} + x^{10} + x^8 + x^2 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} + x^{95} + x^{94} + x^{92} + x^{90} + x^{87}\\
&\quad + x^{86} + x^{85} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{69} + x^{68} + x^{67} + x^{66}
\end{aligned}$$

$$\begin{aligned}
&+ x^{64} + x^{63} + x^{62} + x^{59} + x^{57} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45}\\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{30},\\
p_3(x) &= x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{43} + x^{42} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_6(x) &= x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72}\\
&+ x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{38} + x^{35} + x^{34} + x^{33} + x^{31}\\
&+ x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{19} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75}\\
&+ x^{74} + x^{59} + x^{58} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{82} + x^{81} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62}\\
&+ x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{40}\\
&+ x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14}\\
&+ x^{13} + x^{12} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} + x^{95} + x^{94} + x^{92} + x^{90} + x^{87}\\
&+ x^{86} + x^{85} + x^{81} + x^{80} + x^{76} + x^{75} + x^{73} + x^{69} + x^{68} + x^{67} + x^{66}\\
&+ x^{64} + x^{63} + x^{62} + x^{59} + x^{57} + x^{53} + x^{51} + x^{50} + x^{49} + x^{47} + x^{45}\\
&+ x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{35} + x^{34} + x^{30},\\
p_3(x) &= x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61}\\
&+ x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50}\\
&+ x^{43} + x^{42} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18},\\
p_6(x) &= x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72}\\
&+ x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{38} + x^{35} + x^{34} + x^{33} + x^{31}\\
&+ x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{19} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75}
\end{aligned}$$

$$+ x^{74} + x^{59} + x^{58} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6,$$

$$p_{12}(x) = x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85} + x^{82} + x^{81} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62} + x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{40} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14} + x^{13} + x^{12} + x^8 + x^6 + x^5 + x^2 + x + 1,$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$p_0(x) = x^{102} + x^{100} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{80} + x^{78} + x^{72} + x^{70} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{42} + x^{38} + x^{36},$$

$$p_3(x) = x^{96} + x^{92} + x^{90} + x^{88} + x^{80} + x^{78} + x^{74} + x^{66} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18} + x^{16} + x^{12},$$

$$p_6(x) = x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{38} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + x^8 + x^6 + 1,$$

$$p_9(x) = x^{96} + x^{92} + x^{90} + x^{86} + x^{76} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12} + x^4,$$

$$p_{12}(x) = x^{102} + x^{100} + x^{96} + x^{38} + x^{36} + x^{32} + x^{22} + x^{20} + x^{16} + x^6 + x^4 + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$p_0(x) = x^{102} + x^{100} + x^{94} + x^{92} + x^{76} + x^{72} + x^{70} + x^{64} + x^{62} + x^{56} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{24},$$

$$p_3(x) = x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{66} + x^{64} + x^{60} + x^{58} + x^{50} + x^{48} + x^{46} + x^{40} + x^{38} + x^{36} + x^{30} + x^{24} + x^{18},$$

$$p_6(x) = x^{96} + x^{94} + x^{92} + x^{88} + x^{78} + x^{70} + x^{66} + x^{64} + x^{56} + x^{52} + x^{48} + x^{44} + x^{40} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{12} + x^6 + 1,$$

$$p_9(x) = x^{96} + x^{92} + x^{90} + x^{86} + x^{76} + x^{74} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12} + x^4,$$

$$p_{12}(x) = x^{102} + x^{100} + x^{96} + x^{38} + x^{36} + x^{32} + x^{22} + x^{20} + x^{16} + x^6 + x^4 + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} + x^{95} + x^{94} + x^{92} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{72} + x^{71} + x^{67} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{55} + x^{53} + x^{52} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{43} + x^{42} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} \\
&\quad + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72} \\
&\quad + x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{38} + x^{35} + x^{34} + x^{33} + x^{31} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{19} + x^{17} + x^{14} + x^{13} + x^{12}, \\
p_9(x) &= x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{59} + x^{58} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} \\
&\quad + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} \\
&\quad + x^{9} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85} \\
&\quad + x^{82} + x^{81} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62} \\
&\quad + x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{40} \\
&\quad + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14} \\
&\quad + x^{13} + x^{12} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{99} + x^{95} + x^{94} + x^{92} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{72} + x^{71} + x^{67} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{55} + x^{53} + x^{52} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{43} + x^{42} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} \\
&\quad + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72}
\end{aligned}$$

$$\begin{aligned}
&+ x^{69} + x^{68} + x^{67} + x^{65} + x^{62} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{38} + x^{35} + x^{34} + x^{33} + x^{31}\\
&+ x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{19} + x^{17} + x^{14} + x^{13} + x^{12},\\
p_9(x) &= x^{99} + x^{98} + x^{91} + x^{90} + x^{89} + x^{88} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75}\\
&+ x^{74} + x^{59} + x^{58} + x^{55} + x^{54} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{82} + x^{81} + x^{78} + x^{77} + x^{74} + x^{73} + x^{70} + x^{69} + x^{66} + x^{65} + x^{62}\\
&+ x^{61} + x^{58} + x^{57} + x^{54} + x^{53} + x^{50} + x^{49} + x^{46} + x^{45} + x^{44} + x^{40}\\
&+ x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14}\\
&+ x^{13} + x^{12} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88} + x^{82}\\
&+ x^{78} + x^{72} + x^{70} + x^{66} + x^{56} + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{36},\\
p_3(x) &= x^{108} + x^{104} + x^{92} + x^{90} + x^{72} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56}\\
&+ x^{52} + x^{50} + x^{42} + x^{36} + x^{30} + x^{22} + x^{20} + x^{14} + x^{12},\\
p_6(x) &= x^{108} + x^{106} + x^{104} + x^{94} + x^{86} + x^{78} + x^{68} + x^{60} + x^{56} + x^{54} + x^{52}\\
&+ x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{28} + x^{26} + x^{20} + x^{18}\\
&+ x^{12} + x^{8} + x^{6} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{108} + x^{104} + x^{100} + x^{98} + x^{94} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{68}\\
&+ x^{60} + x^{58} + x^{48} + x^{46} + x^{40} + x^{36} + x^{34} + x^{30} + x^{20} + x^{18} + x^{8}\\
&+ x^{6} + x^{4},\\
p_{12}(x) &= x^{114} + x^{112} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{66} + x^{64} + x^{58}\\
&+ x^{56} + x^{42} + x^{40} + x^{18} + x^{16} + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94}\\
&+ x^{93} + x^{92} + x^{90} + x^{84} + x^{83} + x^{82} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72}\\
&+ x^{71} + x^{68} + x^{65} + x^{62} + x^{61} + x^{60} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49}\\
&+ x^{47} + x^{41} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} + x^{26}\\
&+ x^{25} + x^{24},\\
p_3(x) &= x^{96} + x^{94} + x^{93} + x^{89} + x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{55} + x^{52}\\
&+ x^{50} + x^{49} + x^{45} + x^{41} + x^{39} + x^{36} + x^{34} + x^{30} + x^{29} + x^{24} + x^{23}\\
&+ x^{22} + x^{21} + x^{18} + x^{17} + x^{16},\\
p_6(x) &= x^{102} + x^{100} + x^{99} + x^{95} + x^{90} + x^{86} + x^{80} + x^{79} + x^{78} + x^{75} + x^{67}\\
&+ x^{65} + x^{60} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{51} + x^{48} + x^{45} + x^{44}\\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{30} + x^{29} + x^{27} + x^{25}\\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{12} + x^{10}\\
&+ x^{9} + x^{6} + x^{5} + x^{2} + x + 1,\\
p_9(x) &= x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73}\\
&+ x^{72} + x^{64} + x^{62} + x^{61} + x^{58} + x^{57} + x^{56} + x^{48} + x^{46} + x^{45} + x^{42}\\
&+ x^{41} + x^{40} + x^{16} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8},\\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} + x^{72}\\
&+ x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48}\\
&+ x^{44} + x^{42} + x^{41} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25}\\
&+ x^{24} + x^{12} + x^{10} + x^{9} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{100} + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87}\\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{64} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52}\\
&+ x^{50} + x^{49} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34}\\
&+ x^{33} + x^{31} + x^{30},\\
p_3(x) &= x^{81} + x^{80} + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{49}\\
&+ x^{48} + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29}\\
&+ x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18},\\
p_6(x) &= x^{90} + x^{88} + x^{84} + x^{74} + x^{72} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56} + x^{52}\\
&+ x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20}\\
&+ x^{18} + x^{12},\\
p_9(x) &= x^{99} + x^{98} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{86} + x^{83} + x^{82} + x^{81}\\
&+ x^{78} + x^{75} + x^{74} + x^{73} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} + x^{59} + x^{58}\\
&+ x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{19}\\
&+ x^{18} + x^{17} + x^{14} + x^{11} + x^{10} + x^{9} + x^{6},\\
p_{12}(x) &= x^{108} + x^{104} + x^{88} + x^{80} + x^{72} + x^{64} + x^{56} + x^{48} + x^{44} + x^{32} + x^{28}\\
&+ x^{24} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{112} + x^{106} + x^{105} + x^{103} + x^{101} + x^{98} + x^{96} + x^{90} + x^{89} \\
&\quad + x^{88} + x^{85} + x^{83} + x^{81} + x^{80} + x^{79} + x^{76} + x^{72} + x^{70} + x^{69} + x^{67} \\
&\quad + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{54} + x^{52} + x^{50} + x^{46} \\
&\quad + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{36} + x^{31} + x^{30}, \\
p_3(x) &= x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{77} + x^{76} + x^{75} + x^{73} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{66} + x^{55} + x^{54} + x^{53} + x^{51} + x^{48} + x^{47} + x^{46} + x^{44} + x^{41} \\
&\quad + x^{40} + x^{39} + x^{37} + x^{34} + x^{33} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{22} \\
&\quad + x^{21} + x^{20} + x^{18}, \\
p_6(x) &= x^{96} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{56} + x^{52} + x^{50} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{18} + x^{14} + x^{12}, \\
p_9(x) &= x^{105} + x^{104} + x^{102} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{81} + x^{80} \\
&\quad + x^{78} + x^{77} + x^{76} + x^{74} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{58} + x^{49} \\
&\quad + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{25} + x^{24} + x^{22} \\
&\quad + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{114} + x^{112} + x^{102} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84} + x^{70} + x^{68} + x^{54} \\
&\quad + x^{52} + x^{50} + x^{48} + x^{22} + x^{20} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{95} + x^{92} + x^{90} \\
&\quad + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{75} + x^{73} \\
&\quad + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{59} + x^{57} + x^{55} + x^{54} + x^{51} + x^{48} \\
&\quad + x^{45} + x^{44} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36}, \\
p_3(x) &= x^{96} + x^{94} + x^{93} + x^{89} + x^{87} + x^{86} + x^{84} + x^{80} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{72} + x^{71} + x^{70} + x^{65} + x^{62} + x^{60} + x^{55} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{38} + x^{35} + x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} \\
&\quad + x^{20}, \\
p_6(x) &= x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{77} + x^{74} + x^{72} + x^{69} + x^{68} + x^{62} \\
&\quad + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{39} + x^{38} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{15} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1, \\
p_9(x) &= x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{64} + x^{62} + x^{61} + x^{58} + x^{57} + x^{56} + x^{48} + x^{46} + x^{45} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{16} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8},
\end{aligned}$$

$$\begin{aligned}p_{12}(x) &= x^{108}+x^{106}+x^{105}+x^{104}+x^{88}+x^{86}+x^{85}+x^{82}+x^{81}+x^{80}+x^{72}\\&+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}+x^{56}+x^{54}+x^{53}+x^{50}+x^{49}+x^{48}\\&+x^{44}+x^{42}+x^{41}+x^{38}+x^{37}+x^{34}+x^{33}+x^{32}+x^{28}+x^{26}+x^{25}\\&+x^{24}+x^{12}+x^{10}+x^{9}+x^{6}+x^{5}+x^{2}+x+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1].

For $\phi(W^o,0,0)$:

$$\begin{aligned}p_0(x) &= x^{108}+x^{106}+x^{105}+x^{104}+x^{102}+x^{100}+x^{99}+x^{96}+x^{95}+x^{94}\\&+x^{93}+x^{92}+x^{90}+x^{84}+x^{83}+x^{82}+x^{79}+x^{76}+x^{75}+x^{74}+x^{72}\\&+x^{71}+x^{68}+x^{65}+x^{62}+x^{61}+x^{60}+x^{55}+x^{53}+x^{52}+x^{51}+x^{49}\\&+x^{47}+x^{41}+x^{39}+x^{37}+x^{36}+x^{34}+x^{33}+x^{31}+x^{30}+x^{29}+x^{26}\\&+x^{25}+x^{24},\\
p_3(x) &= x^{96}+x^{94}+x^{93}+x^{89}+x^{87}+x^{86}+x^{82}+x^{81}+x^{79}+x^{77}+x^{76}\\&+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{66}+x^{64}+x^{62}+x^{60}+x^{55}+x^{52}\\&+x^{50}+x^{49}+x^{45}+x^{41}+x^{39}+x^{36}+x^{34}+x^{30}+x^{29}+x^{24}+x^{23}\\&+x^{22}+x^{21}+x^{18}+x^{17}+x^{16},\\
p_6(x) &= x^{102}+x^{100}+x^{99}+x^{95}+x^{90}+x^{86}+x^{80}+x^{79}+x^{78}+x^{75}+x^{67}\\&+x^{65}+x^{60}+x^{59}+x^{58}+x^{57}+x^{55}+x^{54}+x^{51}+x^{48}+x^{45}+x^{44}\\&+x^{43}+x^{42}+x^{41}+x^{39}+x^{38}+x^{37}+x^{35}+x^{30}+x^{29}+x^{27}+x^{25}\\&+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{18}+x^{17}+x^{16}+x^{15}+x^{12}+x^{10}\\&+x^{9}+x^{6}+x^{5}+x^{2}+x+1,\\
p_9(x) &= x^{96}+x^{94}+x^{93}+x^{90}+x^{89}+x^{88}+x^{80}+x^{78}+x^{77}+x^{74}+x^{73}\\&+x^{72}+x^{64}+x^{62}+x^{61}+x^{58}+x^{57}+x^{56}+x^{48}+x^{46}+x^{45}+x^{42}\\&+x^{41}+x^{40}+x^{16}+x^{14}+x^{13}+x^{10}+x^{9}+x^{8},\\
p_{12}(x) &= x^{108}+x^{106}+x^{105}+x^{104}+x^{88}+x^{86}+x^{85}+x^{82}+x^{81}+x^{80}+x^{72}\\&+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}+x^{56}+x^{54}+x^{53}+x^{50}+x^{49}+x^{48}\\&+x^{44}+x^{42}+x^{41}+x^{38}+x^{37}+x^{34}+x^{33}+x^{32}+x^{28}+x^{26}+x^{25}\\&+x^{24}+x^{12}+x^{10}+x^{9}+x^{6}+x^{5}+x^{2}+x+1,\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1].

For $\phi(W^o,0,1)$:

$$\begin{aligned}p_0(x) &= x^{114}+x^{112}+x^{106}+x^{105}+x^{103}+x^{101}+x^{98}+x^{96}+x^{90}+x^{89}\\&+x^{88}+x^{85}+x^{83}+x^{81}+x^{80}+x^{79}+x^{76}+x^{72}+x^{70}+x^{69}+x^{67}\\&+x^{65}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{57}+x^{54}+x^{52}+x^{50}+x^{46}\\&+x^{45}+x^{44}+x^{43}+x^{41}+x^{39}+x^{38}+x^{36}+x^{31}+x^{30},\\
p_3(x) &= x^{87}+x^{86}+x^{85}+x^{83}+x^{80}+x^{77}+x^{76}+x^{75}+x^{73}+x^{70}+x^{69}\\&+x^{68}+x^{66}+x^{55}+x^{54}+x^{53}+x^{51}+x^{48}+x^{47}+x^{46}+x^{44}+x^{41}\end{aligned}$$

$$
\begin{aligned}
&+ x^{40} + x^{39} + x^{37} + x^{34} + x^{33} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{22} \\
&+ x^{21} + x^{20} + x^{18}, \\
p_6(x) &= x^{96} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{56} + x^{52} + x^{50} + x^{44} + x^{42} \\
&+ x^{40} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{18} + x^{14} + x^{12}, \\
p_9(x) &= x^{105} + x^{104} + x^{102} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{81} + x^{80} \\
&+ x^{78} + x^{77} + x^{76} + x^{74} + x^{65} + x^{64} + x^{62} + x^{61} + x^{60} + x^{58} + x^{49} \\
&+ x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{25} + x^{24} + x^{22} \\
&+ x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{114} + x^{112} + x^{102} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84} + x^{70} + x^{68} + x^{54} \\
&+ x^{52} + x^{50} + x^{48} + x^{22} + x^{20} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{104} + x^{100} + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} \\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&+ x^{68} + x^{64} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} \\
&+ x^{50} + x^{49} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37} + x^{34} \\
&+ x^{33} + x^{31} + x^{30}, \\
p_3(x) &= x^{81} + x^{80} + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{49} \\
&+ x^{48} + x^{46} + x^{43} + x^{42} + x^{41} + x^{39} + x^{37} + x^{35} + x^{33} + x^{31} + x^{29} \\
&+ x^{27} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{88} + x^{84} + x^{74} + x^{72} + x^{70} + x^{66} + x^{60} + x^{58} + x^{56} + x^{52} \\
&+ x^{50} + x^{48} + x^{44} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} \\
&+ x^{18} + x^{12}, \\
p_9(x) &= x^{99} + x^{98} + x^{97} + x^{94} + x^{91} + x^{90} + x^{89} + x^{86} + x^{83} + x^{82} + x^{81} \\
&+ x^{78} + x^{75} + x^{74} + x^{73} + x^{70} + x^{67} + x^{66} + x^{65} + x^{62} + x^{59} + x^{58} \\
&+ x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{46} + x^{43} + x^{42} + x^{41} + x^{38} + x^{19} \\
&+ x^{18} + x^{17} + x^{14} + x^{11} + x^{10} + x^{9} + x^{6}, \\
p_{12}(x) &= x^{108} + x^{104} + x^{88} + x^{80} + x^{72} + x^{64} + x^{56} + x^{48} + x^{44} + x^{32} + x^{28} \\
&+ x^{24} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{95} + x^{92} + x^{90} \\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{78} + x^{75} + x^{73} \\
&+ x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{59} + x^{57} + x^{55} + x^{54} + x^{51} + x^{48} \\
&+ x^{45} + x^{44} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{96} + x^{94} + x^{93} + x^{89} + x^{87} + x^{86} + x^{84} + x^{80} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{75} + x^{72} + x^{71} + x^{70} + x^{65} + x^{62} + x^{60} + x^{55} + x^{48} + x^{46} + x^{45} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{38} + x^{35} + x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} \\
&\quad + x^{20}, \\
p_6(x) &= x^{102} + x^{100} + x^{99} + x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{77} + x^{74} + x^{72} + x^{69} + x^{68} + x^{62} \\
&\quad + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{43} + x^{42} + x^{41} \\
&\quad + x^{40} + x^{39} + x^{38} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23} + x^{22} + x^{20} \\
&\quad + x^{19} + x^{17} + x^{16} + x^{15} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1, \\
p_9(x) &= x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} \\
&\quad + x^{72} + x^{64} + x^{62} + x^{61} + x^{58} + x^{57} + x^{56} + x^{48} + x^{46} + x^{45} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{16} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8}, \\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} + x^{72} \\
&\quad + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{56} + x^{54} + x^{53} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{41} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} \\
&\quad + x^{24} + x^{12} + x^{10} + x^{9} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	430	[508, 489]	860	[1211, 674]	1290	[1553, 1554]
1	[3, 4]	431	[708, 590]	861	[549, 1248]	1291	[1555, 1556]
2	[5, 6]	432	[42, 709]	862	[1140, 1249]	1292	[1557, 1558]
3	[7, 8]	433	[141, 710]	863	[1250, 1224]	1293	[869, 1559]
4	[9, 10]	434	[711, 712]	864	[202, 674]	1294	[821, 1494]
5	[11, 12]	435	[713, 714]	865	[1251, 494]	1295	[1533, 356]
6	[13, 14]	436	[504, 715]	866	[211, 1252]	1296	[831, 1037]
7	[15, 16]	437	[716, 717]	867	[175, 616]	1297	[1560, 1561]
8	[17, 18]	438	[234, 125]	868	[1141, 453]	1298	[91, 1562]
9	[19, 20]	439	[169, 504]	869	[162, 1253]	1299	[282, 1563]
10	[21, 22]	440	[718, 719]	870	[651, 700]	1300	[1564, 1565]
11	[23, 24]	441	[669, 225]	871	[610, 522]	1301	[1566, 946]
12	[25, 26]	442	[720, 667]	872	[1233, 1235]	1302	[842, 1567]
13	[27, 28]	443	[721, 722]	873	[1254, 1255]	1303	[30, 244]
14	[29, 30]	444	[723, 724]	874	[1256, 693]	1304	[1568, 98]
15	[31, 32]	445	[725, 726]	875	[1257, 1258]	1305	[1569, 274]
16	[33, 34]	446	[312, 238]	876	[1259, 1239]	1306	[1570, 1571]
17	[35, 9]	447	[727, 728]	877	[168, 681]	1307	[384, 305]

18	[36, 37]	448	[729, 730]	878	[157, 175]	1308	[1572, 1573]
19	[38, 39]	449	[79, 731]	879	[689, 141]	1309	[765, 1574]
20	[40, 41]	450	[732, 733]	880	[1260, 1261]	1310	[1486, 838]
21	[42, 43]	451	[734, 735]	881	[1227, 1262]	1311	[1108, 1575]
22	[44, 45]	452	[252, 736]	882	[1180, 450]	1312	[1576, 1577]
23	[46, 47]	453	[737, 738]	883	[1263, 1240]	1313	[1578, 428]
24	[48, 49]	454	[739, 740]	884	[577, 1257]	1314	[71, 1126]
25	[50, 51]	455	[741, 742]	885	[1264, 1265]	1315	[1579, 1580]
26	[52, 53]	456	[280, 743]	886	[1211, 1266]	1316	[806, 1520]
27	[54, 55]	457	[744, 745]	887	[572, 60]	1317	[1581, 1111]
28	[56, 57]	458	[101, 746]	888	[1267, 170]	1318	[1485, 807]
29	[58, 59]	459	[747, 748]	889	[1268, 533]	1319	[1582, 1064]
30	[60, 61]	460	[749, 750]	890	[1269, 619]	1320	[1558, 283]
31	[62, 63]	461	[751, 752]	891	[631, 1270]	1321	[905, 1109]
32	[64, 65]	462	[753, 754]	892	[218, 580]	1322	[1519, 1048]
33	[66, 67]	463	[755, 756]	893	[1271, 1272]	1323	[784, 1583]
34	[68, 69]	464	[757, 758]	894	[1273, 486]	1324	[1584, 426]
35	[70, 71]	465	[759, 760]	895	[681, 629]	1325	[1583, 1585]
36	[72, 73]	466	[761, 762]	896	[1234, 37]	1326	[1586, 397]
37	[74, 75]	467	[746, 763]	897	[690, 1274]	1327	[733, 1445]
38	[76, 77]	468	[764, 765]	898	[1275, 1242]	1328	[1587, 1457]
39	[78, 79]	469	[766, 767]	899	[578, 1276]	1329	[1588, 1589]
40	[80, 81]	470	[768, 769]	900	[719, 1277]	1330	[306, 1479]
41	[82, 83]	471	[770, 771]	901	[625, 1278]	1331	[1549, 375]
42	[84, 85]	472	[772, 744]	902	[581, 623]	1332	[1590, 1591]
43	[86, 87]	473	[250, 418]	903	[8, 1173]	1333	[1592, 1548]
44	[88, 89]	474	[429, 773]	904	[1271, 228]	1334	[914, 1130]
45	[90, 91]	475	[774, 775]	905	[1279, 1280]	1335	[1429, 999]
46	[92, 93]	476	[257, 776]	906	[1142, 173]	1336	[989, 925]
47	[94, 95]	477	[777, 774]	907	[546, 1217]	1337	[979, 1593]
48	[96, 97]	478	[778, 779]	908	[47, 1189]	1338	[1594, 1092]
49	[98, 99]	479	[775, 780]	909	[664, 152]	1339	[1473, 415]
50	[100, 101]	480	[776, 781]	910	[192, 210]	1340	[1470, 1595]
51	[102, 103]	481	[779, 782]	911	[1269, 1281]	1341	[795, 23]
52	[104, 105]	482	[783, 784]	912	[527, 1194]	1342	[1042, 1596]
53	[106, 107]	483	[785, 786]	913	[1282, 1283]	1343	[1597, 1049]
54	[108, 109]	484	[787, 788]	914	[1179, 235]	1344	[1598, 1599]
55	[110, 111]	485	[789, 790]	915	[617, 157]	1345	[907, 1572]
56	[112, 113]	486	[791, 792]	916	[1284, 640]	1346	[1600, 354]
57	[114, 115]	487	[793, 794]	917	[1187, 1285]	1347	[1079, 256]
58	[116, 117]	488	[408, 795]	918	[1186, 1285]	1348	[1601, 1602]
59	[118, 119]	489	[796, 106]	919	[208, 495]	1349	[1044, 1603]
60	[120, 121]	490	[797, 798]	920	[1154, 44]	1350	[1604, 1605]
61	[122, 123]	491	[799, 295]	921	[1260, 196]	1351	[1606, 265]

62	[124, 125]	492	[800, 801]	922	[1286, 686]	1352	[323, 1607]
63	[126, 127]	493	[802, 803]	923	[1172, 609]	1353	[1608, 936]
64	[128, 129]	494	[804, 805]	924	[516, 1257]	1354	[1052, 299]
65	[130, 131]	495	[390, 806]	925	[1231, 449]	1355	[1609, 933]
66	[132, 133]	496	[807, 808]	926	[32, 1287]	1356	[302, 1610]
67	[134, 135]	497	[809, 387]	927	[1288, 234]	1357	[1573, 1449]
68	[136, 137]	498	[810, 811]	928	[1249, 1289]	1358	[919, 1068]
69	[138, 139]	499	[812, 813]	929	[1290, 1291]	1359	[1498, 1454]
70	[140, 141]	500	[371, 242]	930	[226, 659]	1360	[1509, 1066]
71	[142, 143]	501	[814, 815]	931	[692, 1292]	1361	[75, 1611]
72	[144, 145]	502	[816, 817]	932	[451, 197]	1362	[1516, 1612]
73	[33, 146]	503	[818, 819]	933	[1293, 1294]	1363	[373, 1541]
74	[147, 9]	504	[820, 821]	934	[1295, 590]	1364	[1613, 799]
75	[148, 149]	505	[83, 822]	935	[1296, 603]	1365	[388, 1057]
76	[150, 151]	506	[823, 824]	936	[1273, 1297]	1366	[1054, 935]
77	[152, 153]	507	[825, 826]	937	[1153, 1253]	1367	[1614, 1615]
78	[154, 57]	508	[99, 827]	938	[571, 213]	1368	[1616, 1617]
79	[155, 156]	509	[828, 829]	939	[720, 1298]	1369	[24, 1023]
80	[157, 158]	510	[830, 831]	940	[590, 217]	1370	[1063, 1100]
81	[159, 160]	511	[123, 293]	941	[580, 1299]	1371	[1618, 1619]
82	[161, 60]	512	[832, 833]	942	[1300, 1301]	1372	[981, 1398]
83	[162, 163]	513	[834, 835]	943	[522, 1254]	1373	[998, 68]
84	[164, 165]	514	[836, 837]	944	[468, 627]	1374	[1620, 1621]
85	[166, 167]	515	[838, 839]	945	[32, 1215]	1375	[849, 1525]
86	[168, 169]	516	[726, 840]	946	[227, 1272]	1376	[1440, 1083]
87	[170, 171]	517	[841, 842]	947	[553, 1302]	1377	[1047, 1622]
88	[172, 173]	518	[843, 844]	948	[1303, 1304]	1378	[978, 1517]
89	[129, 174]	519	[441, 845]	949	[1303, 1305]	1379	[1623, 1076]
90	[175, 176]	520	[846, 847]	950	[170, 1144]	1380	[330, 258]
91	[177, 178]	521	[848, 849]	951	[533, 160]	1381	[1624, 407]
92	[179, 180]	522	[850, 851]	952	[1306, 219]	1382	[1127, 16]
93	[181, 182]	523	[852, 853]	953	[36, 624]	1383	[286, 1404]
94	[35, 183]	524	[854, 855]	954	[1307, 1139]	1384	[1458, 1625]
95	[184, 185]	525	[856, 857]	955	[1266, 713]	1385	[328, 972]
96	[186, 187]	526	[824, 858]	956	[1252, 1226]	1386	[949, 1578]
97	[188, 189]	527	[859, 860]	957	[568, 1269]	1387	[1626, 1415]
98	[190, 191]	528	[861, 862]	958	[1308, 1297]	1388	[815, 1627]
99	[192, 193]	529	[290, 334]	959	[1309, 454]	1389	[1628, 645]
100	[14, 194]	530	[863, 864]	960	[141, 650]	1390	[139, 497]
101	[144, 41]	531	[865, 336]	961	[1160, 1310]	1391	[1290, 1349]
102	[195, 196]	532	[866, 867]	962	[1258, 554]	1392	[1145, 169]
103	[197, 198]	533	[868, 869]	963	[1311, 1312]	1393	[456, 535]
104	[199, 200]	534	[771, 870]	964	[517, 1258]	1394	[1154, 1331]
105	[201, 202]	535	[871, 872]	965	[694, 1313]	1395	[1208, 648]

106	[203, 204]	536	[873, 874]	966	[496, 1314]	1396	[1380, 578]
107	[205, 206]	537	[875, 876]	967	[166, 1315]	1397	[1275, 215]
108	[207, 208]	538	[877, 878]	968	[1316, 211]	1398	[595, 1629]
109	[209, 210]	539	[879, 880]	969	[1317, 154]	1399	[1630, 1317]
110	[211, 203]	540	[881, 72]	970	[717, 1318]	1400	[1385, 1631]
111	[212, 213]	541	[882, 424]	971	[1221, 499]	1401	[128, 1152]
112	[214, 215]	542	[883, 884]	972	[583, 1174]	1402	[700, 1388]
113	[216, 217]	543	[885, 886]	973	[1237, 1319]	1403	[1228, 1357]
114	[148, 218]	544	[887, 29]	974	[1320, 1321]	1404	[523, 1255]
115	[219, 220]	545	[888, 889]	975	[480, 1187]	1405	[1632, 1633]
116	[221, 222]	546	[890, 891]	976	[644, 643]	1406	[161, 573]
117	[223, 224]	547	[892, 893]	977	[613, 1293]	1407	[604, 589]
118	[225, 226]	548	[894, 260]	978	[509, 1322]	1408	[41, 689]
119	[227, 228]	549	[895, 90]	979	[1219, 1323]	1409	[1634, 1635]
120	[229, 230]	550	[896, 897]	980	[452, 595]	1410	[135, 1210]
121	[231, 232]	551	[750, 898]	981	[574, 1324]	1411	[127, 1163]
122	[233, 211]	552	[878, 895]	982	[1325, 1326]	1412	[677, 50]
123	[234, 235]	553	[840, 899]	983	[201, 1211]	1413	[1159, 1233]
124	[236, 237]	554	[748, 900]	984	[127, 1327]	1414	[1166, 218]
125	[238, 239]	555	[901, 902]	985	[1328, 1249]	1415	[1279, 1636]
126	[240, 241]	556	[903, 904]	986	[1215, 1161]	1416	[1637, 1638]
127	[242, 243]	557	[893, 247]	987	[1329, 1330]	1417	[164, 1289]
128	[244, 86]	558	[773, 905]	988	[559, 1229]	1418	[498, 1222]
129	[245, 246]	559	[906, 907]	989	[1286, 32]	1419	[1639, 1238]
130	[247, 248]	560	[908, 382]	990	[722, 1331]	1420	[564, 1640]
131	[249, 250]	561	[909, 910]	991	[1332, 1333]	1421	[713, 1347]
132	[251, 252]	562	[911, 912]	992	[1218, 1262]	1422	[549, 137]
133	[253, 82]	563	[913, 914]	993	[560, 721]	1423	[1641, 50]
134	[254, 255]	564	[822, 915]	994	[1334, 561]	1424	[1276, 1227]
135	[256, 257]	565	[916, 108]	995	[1306, 697]	1425	[1388, 684]
136	[258, 259]	566	[917, 918]	996	[1335, 599]	1426	[1207, 544]
137	[260, 261]	567	[109, 919]	997	[135, 201]	1427	[486, 1350]
138	[262, 21]	568	[920, 921]	998	[171, 1197]	1428	[125, 628]
139	[263, 264]	569	[858, 922]	999	[1336, 459]	1429	[453, 1629]
140	[265, 266]	570	[923, 924]	1000	[1282, 482]	1430	[689, 649]
141	[267, 268]	571	[925, 926]	1001	[501, 645]	1431	[1358, 673]
142	[269, 270]	572	[927, 122]	1002	[530, 1337]	1432	[1347, 134]
143	[271, 272]	573	[928, 929]	1003	[459, 571]	1433	[649, 710]
144	[273, 110]	574	[930, 931]	1004	[1250, 1338]	1434	[576, 1191]
145	[274, 267]	575	[932, 313]	1005	[1266, 675]	1435	[1258, 1302]
146	[275, 276]	576	[933, 934]	1006	[512, 688]	1436	[1632, 661]
147	[277, 278]	577	[935, 818]	1007	[216, 507]	1437	[181, 1337]
148	[279, 280]	578	[936, 937]	1008	[1311, 224]	1438	[686, 1287]
149	[281, 282]	579	[97, 938]	1009	[1257, 553]	1439	[613, 1372]

150	[283, 284]	580	[939, 940]	1010	[701, 663]	1440	[1642, 1643]
151	[285, 286]	581	[941, 942]	1011	[575, 607]	1441	[603, 613]
152	[287, 288]	582	[943, 944]	1012	[1339, 466]	1442	[607, 1644]
153	[289, 290]	583	[16, 945]	1013	[491, 668]	1443	[665, 1645]
154	[291, 292]	584	[946, 947]	1014	[1340, 695]	1444	[1646, 1647]
155	[293, 294]	585	[948, 949]	1015	[1341, 1245]	1445	[494, 1648]
156	[295, 296]	586	[950, 951]	1016	[1342, 1343]	1446	[1649, 502]
157	[297, 298]	587	[952, 953]	1017	[445, 1167]	1447	[1370, 1650]
158	[299, 300]	588	[954, 955]	1018	[571, 682]	1448	[1136, 690]
159	[301, 302]	589	[956, 957]	1019	[1344, 1322]	1449	[652, 1189]
160	[303, 304]	590	[958, 959]	1020	[1161, 39]	1450	[1651, 655]
161	[305, 306]	591	[880, 960]	1021	[1345, 1346]	1451	[1652, 1647]
162	[307, 308]	592	[817, 961]	1022	[1347, 600]	1452	[1186, 477]
163	[309, 310]	593	[962, 963]	1023	[47, 1206]	1453	[1183, 1184]
164	[311, 312]	594	[964, 965]	1024	[1213, 1318]	1454	[1187, 477]
165	[313, 100]	595	[966, 967]	1025	[1348, 1349]	1455	[145, 140]
166	[314, 315]	596	[968, 969]	1026	[1297, 1350]	1456	[697, 1653]
167	[316, 317]	597	[970, 971]	1027	[1351, 54]	1457	[195, 1261]
168	[318, 319]	598	[972, 973]	1028	[1295, 216]	1458	[1654, 1655]
169	[320, 321]	599	[974, 868]	1029	[683, 690]	1459	[1656, 565]
170	[322, 323]	600	[975, 976]	1030	[1173, 1352]	1460	[134, 566]
171	[73, 324]	601	[977, 978]	1031	[1244, 1353]	1461	[629, 715]
172	[325, 326]	602	[826, 979]	1032	[1354, 587]	1462	[1657, 1269]
173	[327, 328]	603	[980, 981]	1033	[1185, 1186]	1463	[696, 219]
174	[329, 330]	604	[982, 983]	1034	[476, 642]	1464	[698, 447]
175	[331, 332]	605	[984, 985]	1035	[481, 642]	1465	[183, 1658]
176	[298, 333]	606	[986, 987]	1036	[1285, 478]	1466	[147, 183]
177	[334, 335]	607	[988, 989]	1037	[1143, 177]	1467	[1659, 234]
178	[336, 337]	608	[990, 991]	1038	[1240, 1355]	1468	[1351, 1143]
179	[272, 338]	609	[992, 993]	1039	[569, 1281]	1469	[447, 621]
180	[339, 340]	610	[994, 852]	1040	[1204, 1148]	1470	[1201, 1151]
181	[341, 342]	611	[995, 787]	1041	[536, 151]	1471	[1660, 1182]
182	[343, 344]	612	[996, 997]	1042	[453, 596]	1472	[1388, 198]
183	[345, 279]	613	[798, 998]	1043	[158, 176]	1473	[40, 145]
184	[346, 347]	614	[999, 1000]	1044	[160, 1356]	1474	[1138, 1661]
185	[348, 349]	615	[1001, 78]	1045	[545, 633]	1475	[1225, 1359]
186	[350, 351]	616	[884, 4]	1046	[606, 576]	1476	[1252, 204]
187	[352, 353]	617	[993, 1002]	1047	[1241, 522]	1477	[1662, 655]
188	[354, 355]	618	[1003, 825]	1048	[1195, 154]	1478	[180, 1276]
189	[356, 357]	619	[991, 430]	1049	[1308, 486]	1479	[459, 212]
190	[358, 359]	620	[1004, 1005]	1050	[130, 1324]	1480	[717, 1663]
191	[360, 361]	621	[1006, 1007]	1051	[1227, 1246]	1481	[657, 1315]
192	[362, 363]	622	[427, 1008]	1052	[1230, 544]	1482	[1268, 159]
193	[364, 365]	623	[1009, 1010]	1053	[1133, 1357]	1483	[1285, 1183]

194	[366, 367]	624	[1011, 1012]	1054	[1331, 45]	1484	[1215, 38]
195	[368, 369]	625	[898, 1013]	1055	[1358, 1247]	1485	[12, 143]
196	[370, 371]	626	[1014, 901]	1056	[1359, 1225]	1486	[12, 1369]
197	[372, 373]	627	[1015, 1016]	1057	[588, 605]	1487	[515, 1277]
198	[111, 374]	628	[237, 1017]	1058	[721, 1154]	1488	[618, 654]
199	[375, 376]	629	[319, 1018]	1059	[697, 220]	1489	[1297, 487]
200	[377, 378]	630	[1019, 1020]	1060	[658, 1168]	1490	[1283, 483]
201	[255, 379]	631	[1021, 1022]	1061	[223, 1312]	1491	[1642, 563]
202	[380, 381]	632	[1023, 734]	1062	[542, 658]	1492	[1264, 1346]
203	[382, 383]	633	[1024, 1025]	1063	[1259, 585]	1493	[674, 713]
204	[65, 384]	634	[889, 950]	1064	[557, 1360]	1494	[615, 1664]
205	[385, 386]	635	[910, 1026]	1065	[1170, 1361]	1495	[1379, 229]
206	[387, 388]	636	[1027, 797]	1066	[1362, 1304]	1496	[1649, 186]
207	[389, 390]	637	[1028, 843]	1067	[1363, 1364]	1497	[1665, 1371]
208	[391, 392]	638	[1029, 755]	1068	[1365, 1195]	1498	[1184, 476]
209	[393, 394]	639	[760, 1030]	1069	[1366, 656]	1499	[678, 51]
210	[395, 396]	640	[1031, 948]	1070	[1367, 1368]	1500	[231, 1270]
211	[397, 398]	641	[1032, 329]	1071	[711, 133]	1501	[1634, 1326]
212	[399, 400]	642	[1033, 1034]	1072	[566, 1210]	1502	[675, 714]
213	[401, 402]	643	[643, 643]	1073	[615, 676]	1503	[483, 1323]
214	[403, 404]	644	[1035, 1036]	1074	[1164, 496]	1504	[1657, 569]
215	[405, 406]	645	[1037, 366]	1075	[1206, 1190]	1505	[670, 1274]
216	[407, 408]	646	[18, 1038]	1076	[142, 1369]	1506	[660, 1633]
217	[409, 410]	647	[1039, 1040]	1077	[1267, 1198]	1507	[218, 518]
218	[411, 237]	648	[359, 438]	1078	[1370, 1371]	1508	[1340, 1313]
219	[412, 413]	649	[1041, 1042]	1079	[478, 475]	1509	[1666, 1636]
220	[414, 412]	650	[1043, 1044]	1080	[191, 1196]	1510	[1331, 1135]
221	[415, 416]	651	[1045, 372]	1081	[13, 559]	1511	[55, 178]
222	[351, 417]	652	[1046, 1047]	1082	[1329, 1333]	1512	[1662, 1366]
223	[418, 419]	653	[1048, 1001]	1083	[1372, 1373]	1513	[1365, 1317]
224	[420, 421]	654	[1049, 1050]	1084	[625, 1361]	1514	[639, 1667]
225	[422, 423]	655	[1051, 1052]	1085	[1300, 1374]	1515	[1652, 1668]
226	[424, 425]	656	[1053, 1054]	1086	[611, 523]	1516	[152, 1645]
227	[426, 427]	657	[1055, 1056]	1087	[1375, 1360]	1517	[1293, 1373]
228	[428, 429]	658	[1057, 1058]	1088	[1251, 208]	1518	[637, 472]
229	[430, 431]	659	[423, 1059]	1089	[609, 1376]	1519	[1669, 1283]
230	[432, 433]	660	[1060, 1061]	1090	[668, 1372]	1520	[654, 615]
231	[434, 435]	661	[1062, 1063]	1091	[1189, 1377]	1521	[1206, 1377]
232	[436, 437]	662	[1064, 1065]	1092	[1188, 152]	1522	[8, 583]
233	[438, 439]	663	[1066, 1067]	1093	[460, 1378]	1523	[651, 451]
234	[440, 441]	664	[1068, 370]	1094	[1147, 1205]	1524	[1240, 591]
235	[442, 443]	665	[1069, 1070]	1095	[645, 1327]	1525	[518, 1299]
236	[444, 445]	666	[1071, 1072]	1096	[1379, 460]	1526	[1639, 1319]
237	[446, 447]	667	[1073, 1074]	1097	[508, 602]	1527	[608, 445]

238	[448, 449]	668	[951, 1075]	1098	[1380, 180]	1528	[666, 1298]
239	[450, 451]	669	[934, 291]	1099	[445, 1376]	1529	[41, 140]
240	[452, 453]	670	[1020, 994]	1100	[1254, 703]	1530	[1366, 1220]
241	[58, 454]	671	[1076, 263]	1101	[555, 1200]	1531	[1348, 1291]
242	[455, 203]	672	[827, 1077]	1102	[185, 1173]	1532	[653, 188]
243	[456, 457]	673	[1078, 958]	1103	[1341, 1353]	1533	[1317, 56]
244	[458, 459]	674	[379, 1079]	1104	[700, 197]	1534	[217, 508]
245	[460, 230]	675	[381, 975]	1105	[708, 216]	1535	[1288, 1179]
246	[52, 461]	676	[355, 1080]	1106	[718, 515]	1536	[1670, 1631]
247	[462, 463]	677	[1081, 1082]	1107	[455, 1252]	1537	[644, 644]
248	[464, 465]	678	[1083, 882]	1108	[583, 1352]	1538	[136, 1248]
249	[466, 467]	679	[1084, 1085]	1109	[706, 1343]	1539	[601, 460]
250	[468, 469]	680	[1086, 1087]	1110	[1243, 1381]	1540	[1637, 1364]
251	[129, 470]	681	[1088, 1041]	1111	[1339, 1149]	1541	[1630, 1195]
252	[471, 472]	682	[924, 1089]	1112	[183, 10]	1542	[1656, 1640]
253	[473, 474]	683	[1090, 432]	1113	[1152, 174]	1543	[1210, 202]
254	[475, 476]	684	[1091, 956]	1114	[1382, 652]	1544	[503, 592]
255	[477, 478]	685	[87, 785]	1115	[1383, 1384]	1545	[562, 1643]
256	[479, 480]	686	[1092, 76]	1116	[1385, 1386]	1546	[540, 1355]
257	[475, 481]	687	[1093, 1094]	1117	[570, 212]	1547	[1671, 1672]
258	[482, 483]	688	[1095, 1096]	1118	[233, 455]	1548	[1198, 1144]
259	[56, 484]	689	[81, 1043]	1119	[458, 570]	1549	[1309, 59]
260	[172, 485]	690	[1097, 1098]	1120	[1383, 1387]	1550	[472, 534]
261	[486, 487]	691	[762, 1099]	1121	[573, 61]	1551	[654, 189]
262	[217, 488]	692	[1100, 1101]	1122	[451, 1388]	1552	[534, 457]
263	[488, 489]	693	[1102, 764]	1123	[1344, 510]	1553	[1335, 200]
264	[490, 491]	694	[790, 325]	1124	[1334, 721]	1554	[675, 1347]
265	[49, 136]	695	[1067, 1103]	1125	[532, 14]	1555	[1219, 635]
266	[154, 484]	696	[1104, 414]	1126	[1372, 1294]	1556	[506, 1203]
267	[492, 493]	697	[433, 1105]	1127	[1296, 491]	1557	[699, 621]
268	[494, 495]	698	[839, 753]	1128	[1345, 1265]	1558	[626, 469]
269	[496, 497]	699	[1106, 871]	1129	[1359, 1359]	1559	[208, 1648]
270	[498, 499]	700	[1107, 1091]	1130	[1283, 1219]	1560	[139, 1314]
271	[500, 501]	701	[365, 1108]	1131	[235, 550]	1561	[1673, 1674]
272	[502, 503]	702	[1109, 1110]	1132	[1389, 294]	1562	[205, 1203]
273	[186, 503]	703	[1111, 1112]	1133	[1390, 1391]	1563	[515, 1156]
274	[207, 494]	704	[1113, 1114]	1134	[1392, 96]	1564	[1170, 1278]
275	[504, 505]	705	[1115, 732]	1135	[1393, 436]	1565	[520, 465]
276	[506, 206]	706	[1080, 927]	1136	[1394, 1395]	1566	[517, 553]
277	[507, 508]	707	[1116, 887]	1137	[1396, 1397]	1567	[1233, 1310]
278	[509, 510]	708	[1117, 409]	1138	[1398, 1399]	1568	[1675, 1384]
279	[511, 450]	709	[1118, 318]	1139	[1400, 1401]	1569	[1676, 1677]
280	[512, 513]	710	[294, 1095]	1140	[1402, 1403]	1570	[1287, 1161]
281	[514, 515]	711	[268, 1119]	1141	[1114, 271]	1571	[647, 1209]

282	[516, 517]	712	[1120, 894]	1142	[1404, 1405]	1572	[1678, 502]
283	[518, 519]	713	[781, 778]	1143	[1406, 906]	1573	[1316, 455]
284	[520, 521]	714	[1036, 1033]	1144	[1407, 1408]	1574	[1679, 1301]
285	[522, 523]	715	[321, 1121]	1145	[969, 866]	1575	[464, 521]
286	[524, 525]	716	[1122, 1123]	1146	[1082, 988]	1576	[1149, 221]
287	[206, 526]	717	[103, 1124]	1147	[1409, 1410]	1577	[1375, 558]
288	[527, 528]	718	[1125, 102]	1148	[1411, 1412]	1578	[1654, 705]
289	[529, 54]	719	[1126, 1127]	1149	[1413, 1414]	1579	[189, 1664]
290	[502, 187]	720	[1128, 1129]	1150	[944, 873]	1580	[1328, 164]
291	[530, 182]	721	[378, 88]	1151	[1415, 1416]	1581	[1263, 540]
292	[531, 532]	722	[1130, 1131]	1152	[1417, 1418]	1582	[37, 1170]
293	[533, 155]	723	[1132, 649]	1153	[1065, 1419]	1583	[665, 153]
294	[534, 535]	724	[1133, 1134]	1154	[439, 1393]	1584	[542, 226]
295	[536, 537]	725	[44, 1135]	1155	[413, 399]	1585	[1382, 47]
296	[60, 538]	726	[179, 578]	1156	[837, 1051]	1586	[1676, 474]
297	[539, 540]	727	[1136, 670]	1157	[1420, 1421]	1587	[1287, 38]
298	[541, 542]	728	[1137, 185]	1158	[1422, 1423]	1588	[229, 1378]
299	[543, 146]	729	[1138, 1139]	1159	[1424, 1019]	1589	[1193, 528]
300	[544, 13]	730	[714, 600]	1160	[1425, 1426]	1590	[1249, 165]
301	[545, 546]	731	[1140, 164]	1161	[398, 1427]	1591	[1673, 1368]
302	[492, 547]	732	[1141, 595]	1162	[1428, 1411]	1592	[586, 501]
303	[548, 549]	733	[1142, 485]	1163	[241, 1429]	1593	[716, 1213]
304	[125, 550]	734	[1143, 55]	1164	[1430, 809]	1594	[1646, 1668]
305	[551, 170]	735	[1144, 646]	1165	[1431, 1432]	1595	[621, 1236]
306	[552, 49]	736	[1145, 681]	1166	[431, 939]	1596	[1678, 186]
307	[553, 554]	737	[50, 1146]	1167	[724, 1433]	1597	[1680, 1681]
308	[555, 556]	738	[1147, 1148]	1168	[953, 1434]	1598	[1203, 630]
309	[447, 462]	739	[1149, 467]	1169	[1435, 1436]	1599	[620, 461]
310	[557, 558]	740	[1150, 1151]	1170	[1437, 1438]	1600	[1682, 1683]
311	[559, 194]	741	[641, 1152]	1171	[740, 962]	1601	[1223, 1338]
312	[560, 561]	742	[1153, 163]	1172	[1439, 1440]	1602	[714, 134]
313	[562, 563]	743	[561, 1154]	1173	[347, 1441]	1603	[138, 496]
314	[564, 565]	744	[1155, 466]	1174	[349, 1442]	1604	[500, 126]
315	[566, 201]	745	[719, 1156]	1175	[1443, 888]	1605	[638, 456]
316	[55, 567]	746	[1157, 1158]	1176	[1444, 1088]	1606	[587, 175]
317	[568, 569]	747	[1159, 1160]	1177	[1445, 1446]	1607	[219, 1653]
318	[570, 571]	748	[446, 699]	1178	[1447, 240]	1608	[1320, 1672]
319	[159, 155]	749	[1161, 1162]	1179	[1448, 896]	1609	[1332, 1330]
320	[572, 573]	750	[685, 32]	1180	[1449, 1107]	1610	[546, 634]
321	[574, 131]	751	[645, 1163]	1181	[367, 1450]	1611	[9, 1658]
322	[575, 576]	752	[1164, 139]	1182	[1451, 64]	1612	[471, 638]
323	[577, 517]	753	[1165, 1158]	1183	[1452, 380]	1613	[1682, 1681]
324	[578, 579]	754	[531, 13]	1184	[1453, 917]	1614	[1307, 1661]
325	[580, 519]	755	[1166, 149]	1185	[1454, 777]	1615	[202, 1266]

326	[581, 582]	756	[609, 1167]	1186	[976, 254]	1616	[155, 1356]
327	[185, 583]	757	[226, 1168]	1187	[1034, 1452]	1617	[1641, 678]
328	[584, 585]	758	[1169, 556]	1188	[1446, 289]	1618	[1666, 1280]
329	[586, 126]	759	[624, 1170]	1189	[1397, 1455]	1619	[704, 1655]
330	[587, 158]	760	[1171, 525]	1190	[1012, 1456]	1620	[611, 1254]
331	[588, 589]	761	[1172, 445]	1191	[1457, 992]	1621	[1150, 1202]
332	[590, 507]	762	[1155, 1149]	1192	[857, 1458]	1622	[576, 1644]
333	[540, 591]	763	[180, 579]	1193	[1459, 1460]	1623	[722, 44]
334	[54, 177]	764	[662, 702]	1194	[1461, 1462]	1624	[1665, 1650]
335	[187, 592]	765	[1173, 1174]	1195	[1463, 114]	1625	[1362, 1305]
336	[593, 594]	766	[1175, 1176]	1196	[1000, 1003]	1626	[1363, 1638]
337	[595, 596]	767	[1177, 1178]	1197	[1464, 1465]	1627	[573, 538]
338	[501, 127]	768	[552, 548]	1198	[1466, 1464]	1628	[1684, 733]
339	[190, 597]	769	[1179, 125]	1199	[1467, 1468]	1629	[736, 1550]
340	[598, 547]	770	[1180, 651]	1200	[1469, 1470]	1630	[1105, 112]
341	[199, 599]	771	[1165, 1181]	1201	[1087, 757]	1631	[1685, 1031]
342	[600, 566]	772	[514, 719]	1202	[1471, 766]	1632	[1595, 1686]
343	[160, 156]	773	[1177, 1182]	1203	[1472, 1473]	1633	[1687, 854]
344	[601, 229]	774	[477, 1183]	1204	[1474, 1475]	1634	[1627, 1032]
345	[488, 602]	775	[1184, 481]	1205	[1476, 1477]	1635	[1688, 1045]
346	[490, 603]	776	[479, 1185]	1206	[93, 1478]	1636	[1689, 909]
347	[604, 605]	777	[478, 1184]	1207	[1479, 820]	1637	[1026, 1690]
348	[606, 607]	778	[642, 1185]	1208	[1480, 1481]	1638	[1691, 783]
349	[541, 225]	779	[480, 1186]	1209	[1482, 80]	1639	[1030, 1692]
350	[608, 609]	780	[1183, 475]	1210	[780, 1453]	1640	[1693, 339]
351	[610, 611]	781	[481, 479]	1211	[918, 1483]	1641	[1694, 1545]
352	[612, 513]	782	[1185, 1187]	1212	[1484, 1485]	1642	[1695, 1696]
353	[491, 613]	783	[1188, 665]	1213	[960, 1396]	1643	[1468, 1606]
354	[614, 191]	784	[679, 594]	1214	[6, 1486]	1644	[987, 749]
355	[150, 537]	785	[169, 629]	1215	[63, 1428]	1645	[1697, 908]
356	[188, 615]	786	[1189, 1190]	1216	[1414, 1487]	1646	[965, 1435]
357	[158, 616]	787	[184, 8]	1217	[1488, 1117]	1647	[1698, 856]
358	[617, 587]	788	[607, 1191]	1218	[937, 1489]	1648	[1699, 273]
359	[618, 188]	789	[699, 462]	1219	[1490, 796]	1649	[264, 352]
360	[569, 619]	790	[1171, 1192]	1220	[1491, 345]	1650	[767, 1015]
361	[620, 53]	791	[678, 1146]	1221	[1492, 1493]	1651	[1700, 1491]
362	[621, 463]	792	[1193, 1194]	1222	[1494, 1495]	1652	[1010, 116]
363	[622, 623]	793	[48, 548]	1223	[296, 1496]	1653	[1123, 311]
364	[624, 625]	794	[1195, 56]	1224	[1497, 814]	1654	[1559, 1701]
365	[626, 627]	795	[597, 1196]	1225	[782, 1498]	1655	[1702, 741]
366	[235, 628]	796	[1144, 1197]	1226	[742, 1113]	1656	[1489, 389]
367	[629, 505]	797	[1198, 171]	1227	[1050, 1443]	1657	[1703, 724]
368	[206, 630]	798	[548, 136]	1228	[1499, 1500]	1658	[278, 1097]
369	[631, 232]	799	[1199, 633]	1229	[28, 316]	1659	[1426, 1006]

370	[632, 159]	800	[467, 222]	1230	[794, 27]	1660	[89, 1704]
371	[472, 456]	801	[1169, 1200]	1231	[1501, 1502]	1661	[1705, 980]
372	[209, 193]	802	[37, 625]	1232	[1503, 1504]	1662	[1706, 1609]
373	[633, 634]	803	[1201, 1202]	1233	[1433, 346]	1663	[1571, 881]
374	[483, 635]	804	[1203, 526]	1234	[922, 1437]	1664	[340, 251]
375	[191, 636]	805	[1204, 1205]	1235	[386, 1472]	1665	[119, 930]
376	[637, 638]	806	[652, 1206]	1236	[728, 1505]	1666	[1610, 1707]
377	[639, 640]	807	[38, 1162]	1237	[855, 1506]	1667	[1110, 1618]
378	[641, 129]	808	[1207, 531]	1238	[1507, 1508]	1668	[1103, 1626]
379	[476, 479]	809	[1208, 1209]	1239	[1016, 1509]	1669	[1605, 63]
380	[642, 480]	810	[132, 712]	1240	[1510, 770]	1670	[904, 1564]
381	[643, 644]	811	[1210, 1211]	1241	[808, 331]	1671	[310, 846]
382	[126, 645]	812	[177, 567]	1242	[1511, 1512]	1672	[1708, 1708]
383	[171, 646]	813	[1212, 1213]	1243	[1121, 1513]	1673	[1709, 1710]
384	[597, 636]	814	[1199, 546]	1244	[337, 358]	1674	[374, 1711]
385	[532, 559]	815	[593, 680]	1245	[1514, 1406]	1675	[1621, 1060]
386	[647, 648]	816	[1214, 1160]	1246	[938, 405]	1676	[1577, 307]
387	[649, 650]	817	[1137, 8]	1247	[1515, 320]	1677	[1625, 1685]
388	[511, 651]	818	[1157, 1181]	1248	[959, 1516]	1678	[1098, 1528]
389	[46, 652]	819	[686, 1215]	1249	[1517, 92]	1679	[947, 1712]
390	[653, 654]	820	[49, 549]	1250	[77, 1467]	1680	[396, 941]
391	[655, 656]	821	[503, 1216]	1251	[1518, 1519]	1681	[1619, 1540]
392	[657, 167]	822	[633, 1217]	1252	[1520, 1521]	1682	[1508, 362]
393	[658, 659]	823	[579, 1218]	1253	[1522, 1014]	1683	[1416, 1698]
394	[660, 661]	824	[687, 709]	1254	[1002, 1523]	1684	[1713, 494]
395	[149, 580]	825	[482, 1219]	1255	[851, 1524]	1685	[1342, 707]
396	[662, 663]	826	[187, 1216]	1256	[1525, 1526]	1686	[1679, 1374]
397	[664, 665]	827	[1152, 470]	1257	[1527, 401]	1687	[466, 221]
398	[598, 493]	828	[655, 1220]	1258	[1528, 1529]	1688	[1675, 1387]
399	[666, 667]	829	[1221, 1222]	1259	[876, 800]	1689	[1284, 1667]
400	[603, 668]	830	[529, 1143]	1260	[1530, 1531]	1690	[551, 1198]
401	[669, 542]	831	[638, 534]	1261	[1532, 1533]	1691	[1670, 1386]
402	[670, 671]	832	[1223, 1224]	1262	[1521, 861]	1692	[622, 582]
403	[672, 673]	833	[1225, 1225]	1263	[1534, 1053]	1693	[1671, 1321]
404	[674, 675]	834	[203, 1226]	1264	[261, 1535]	1694	[1276, 1218]
405	[189, 676]	835	[1212, 717]	1265	[1536, 865]	1695	[1213, 1663]
406	[677, 678]	836	[579, 1227]	1266	[1537, 1035]	1696	[1367, 1674]
407	[614, 597]	837	[1228, 1134]	1267	[20, 1407]	1697	[1669, 482]
408	[679, 680]	838	[14, 1229]	1268	[769, 303]	1698	[1175, 1381]
409	[681, 504]	839	[1230, 531]	1269	[22, 1028]	1699	[632, 533]
410	[212, 682]	840	[1231, 1232]	1270	[1538, 1539]	1700	[507, 488]
411	[683, 670]	841	[1214, 1233]	1271	[421, 393]	1701	[1354, 157]
412	[197, 684]	842	[1234, 624]	1272	[1540, 1471]	1702	[473, 1677]
413	[685, 686]	843	[448, 1232]	1273	[1541, 793]	1703	[668, 1293]

414	[687, 43]	844	[544, 532]	1274	[1395, 1510]	1704	[1659, 1179]
415	[612, 688]	845	[1160, 1235]	1275	[1542, 1543]	1705	[1680, 1683]
416	[145, 689]	846	[462, 1236]	1276	[338, 1544]	1706	[140, 649]
417	[690, 671]	847	[1237, 1238]	1277	[1545, 19]	1707	[1336, 570]
418	[467, 691]	848	[149, 518]	1278	[95, 1546]	1708	[1660, 1178]
419	[692, 693]	849	[584, 1239]	1279	[1427, 1518]	1709	[1325, 1635]
420	[225, 658]	850	[539, 1240]	1280	[1547, 301]	1710	[600, 135]
421	[694, 695]	851	[1241, 611]	1281	[921, 725]	1711	[1651, 1366]
422	[696, 697]	852	[214, 1242]	1282	[1548, 1490]	1712	[1256, 1292]
423	[698, 699]	853	[561, 722]	1283	[1401, 1549]	1713	[1714, 1699]
424	[543, 34]	854	[524, 1192]	1284	[803, 1084]	1714	[1715, 186]
425	[450, 700]	855	[221, 691]	1285	[1483, 1537]	1715	[1716, 818]
426	[701, 702]	856	[1243, 1176]	1286	[1550, 1551]	1716	[31, 686]
427	[523, 703]	857	[1244, 1245]	1287	[1551, 1552]		
428	[704, 705]	858	[1218, 1246]	1288	[1124, 442]		
429	[706, 707]	859	[672, 1247]	1289	[1403, 1425]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	287	1	574	0	861	1	1148	0	1435	0
1	0	288	0	575	1	862	0	1149	1	1436	0
2	0	289	0	576	1	863	1	1150	0	1437	0
3	0	290	0	577	1	864	0	1151	0	1438	0
4	0	291	1	578	0	865	1	1152	0	1439	0
5	0	292	1	579	0	866	1	1153	0	1440	0
6	0	293	1	580	1	867	1	1154	0	1441	0
7	0	294	1	581	1	868	1	1155	1	1442	0
8	1	295	1	582	0	869	1	1156	0	1443	0
9	0	296	0	583	1	870	0	1157	0	1444	0
10	0	297	0	584	1	871	0	1158	1	1445	0
11	0	298	0	585	0	872	1	1159	1	1446	0
12	1	299	0	586	1	873	1	1160	0	1447	0
13	0	300	0	587	1	874	0	1161	1	1448	0
14	1	301	0	588	1	875	1	1162	0	1449	0
15	0	302	0	589	1	876	0	1163	1	1450	0
16	1	303	0	590	1	877	0	1164	0	1451	1
17	1	304	0	591	0	878	0	1165	1	1452	1
18	1	305	1	592	1	879	0	1166	0	1453	1
19	0	306	0	593	0	880	0	1167	1	1454	0
20	0	307	1	594	0	881	1	1168	1	1455	0
21	0	308	1	595	0	882	0	1169	0	1456	0
22	1	309	0	596	0	883	1	1170	0	1457	1
23	0	310	0	597	1	884	1	1171	0	1458	1
24	1	311	0	598	1	885	0	1172	0	1459	1

25	1	312	1	599	1	886	1	1173	0	1460	0
26	0	313	1	600	1	887	0	1174	0	1461	0
27	0	314	0	601	0	888	0	1175	0	1462	1
28	0	315	0	602	1	889	0	1176	0	1463	0
29	1	316	1	603	0	890	1	1177	0	1464	0
30	0	317	0	604	0	891	1	1178	0	1465	1
31	0	318	0	605	0	892	1	1179	0	1466	0
32	1	319	0	606	0	893	0	1180	0	1467	1
33	1	320	0	607	0	894	0	1181	0	1468	1
34	0	321	0	608	1	895	1	1182	1	1469	0
35	1	322	1	609	0	896	0	1183	1	1470	1
36	1	323	1	610	0	897	0	1184	1	1471	1
37	0	324	0	611	1	898	1	1185	0	1472	0
38	0	325	1	612	1	899	0	1186	1	1473	0
39	1	326	1	613	0	900	0	1187	0	1474	0
40	0	327	0	614	0	901	1	1188	0	1475	0
41	1	328	1	615	1	902	1	1189	1	1476	1
42	0	329	1	616	1	903	1	1190	0	1477	1
43	0	330	1	617	1	904	0	1191	1	1478	1
44	1	331	1	618	1	905	0	1192	0	1479	1
45	1	332	1	619	1	906	0	1193	1	1480	1
46	0	333	1	620	1	907	1	1194	0	1481	1
47	1	334	0	621	0	908	1	1195	0	1482	0
48	1	335	1	622	0	909	1	1196	0	1483	1
49	1	336	0	623	1	910	0	1197	0	1484	0
50	1	337	0	624	1	911	1	1198	0	1485	1
51	1	338	1	625	1	912	0	1199	1	1486	1
52	0	339	1	626	1	913	0	1200	0	1487	1
53	0	340	1	627	1	914	0	1201	1	1488	1
54	0	341	0	628	1	915	1	1202	1	1489	1
55	1	342	1	629	0	916	1	1203	0	1490	1
56	0	343	0	630	0	917	0	1204	0	1491	0
57	1	344	0	631	1	918	1	1205	1	1492	0
58	1	345	1	632	1	919	1	1206	0	1493	0
59	0	346	0	633	0	920	0	1207	1	1494	1
60	0	347	0	634	0	921	0	1208	1	1495	1
61	1	348	0	635	0	922	0	1209	0	1496	0
62	0	349	0	636	1	923	0	1210	1	1497	1
63	0	350	1	637	1	924	0	1211	1	1498	1
64	1	351	0	638	1	925	1	1212	0	1499	0
65	1	352	1	639	0	926	1	1213	0	1500	0
66	1	353	1	640	0	927	0	1214	0	1501	1
67	0	354	0	641	0	928	1	1215	0	1502	0
68	0	355	0	642	0	929	1	1216	0	1503	0

69	1	356	0	643	0	930	0	1217	1	1504	1
70	1	357	0	644	1	931	1	1218	0	1505	1
71	0	358	1	645	1	932	1	1219	1	1506	1
72	1	359	1	646	1	933	1	1220	0	1507	1
73	1	360	0	647	0	934	1	1221	0	1508	1
74	0	361	1	648	1	935	1	1222	1	1509	1
75	1	362	0	649	1	936	0	1223	0	1510	0
76	0	363	0	650	0	937	0	1224	1	1511	1
77	1	364	1	651	0	938	1	1225	0	1512	1
78	1	365	1	652	0	939	1	1226	0	1513	1
79	1	366	1	653	0	940	1	1227	1	1514	0
80	0	367	0	654	1	941	1	1228	0	1515	1
81	0	368	1	655	1	942	0	1229	0	1516	1
82	1	369	1	656	1	943	0	1230	0	1517	1
83	1	370	1	657	1	944	0	1231	1	1518	1
84	0	371	0	658	1	945	1	1232	0	1519	1
85	0	372	1	659	0	946	1	1233	1	1520	1
86	0	373	0	660	1	947	1	1234	0	1521	0
87	1	374	0	661	1	948	0	1235	0	1522	1
88	1	375	0	662	0	949	0	1236	1	1523	0
89	1	376	1	663	1	950	1	1237	1	1524	0
90	1	377	0	664	1	951	1	1238	1	1525	0
91	0	378	0	665	0	952	1	1239	1	1526	0
92	0	379	0	666	0	953	1	1240	0	1527	1
93	0	380	0	667	1	954	1	1241	1	1528	0
94	1	381	0	668	1	955	1	1242	1	1529	1
95	0	382	0	669	1	956	1	1243	1	1530	0
96	1	383	1	670	1	957	0	1244	0	1531	0
97	0	384	1	671	0	958	1	1245	0	1532	0
98	1	385	1	672	0	959	0	1246	1	1533	1
99	0	386	0	673	0	960	0	1247	1	1534	1
100	1	387	1	674	0	961	0	1248	0	1535	0
101	1	388	1	675	0	962	0	1249	1	1536	0
102	1	389	0	676	0	963	1	1250	1	1537	1
103	1	390	0	677	0	964	0	1251	1	1538	0
104	0	391	1	678	0	965	0	1252	1	1539	0
105	0	392	1	679	1	966	0	1253	1	1540	1
106	0	393	1	680	1	967	0	1254	1	1541	0
107	1	394	1	681	1	968	0	1255	1	1542	1
108	0	395	0	682	0	969	1	1256	0	1543	1
109	1	396	0	683	1	970	1	1257	1	1544	0
110	1	397	1	684	1	971	0	1258	0	1545	1
111	0	398	1	685	1	972	1	1259	0	1546	1
112	0	399	0	686	0	973	1	1260	0	1547	0

113	0	400	0	687	1	974	1	1261	0	1548	0
114	1	401	1	688	1	975	1	1262	0	1549	0
115	1	402	1	689	0	976	1	1263	1	1550	0
116	1	403	0	690	0	977	0	1264	0	1551	1
117	0	404	0	691	1	978	1	1265	0	1552	1
118	0	405	0	692	1	979	1	1266	1	1553	1
119	1	406	0	693	0	980	0	1267	0	1554	0
120	0	407	0	694	0	981	0	1268	0	1555	1
121	0	408	1	695	0	982	0	1269	1	1556	0
122	1	409	1	696	0	983	0	1270	0	1557	1
123	1	410	0	697	0	984	0	1271	0	1558	1
124	0	411	1	698	0	985	1	1272	1	1559	1
125	0	412	1	699	1	986	0	1273	0	1560	1
126	0	413	1	700	0	987	0	1274	1	1561	0
127	0	414	1	701	1	988	0	1275	1	1562	1
128	1	415	1	702	0	989	0	1276	1	1563	1
129	1	416	0	703	0	990	1	1277	1	1564	0
130	1	417	0	704	0	991	1	1278	0	1565	1
131	0	418	0	705	1	992	0	1279	0	1566	0
132	1	419	1	706	0	993	1	1280	0	1567	1
133	0	420	0	707	1	994	0	1281	0	1568	0
134	0	421	0	708	0	995	1	1282	0	1569	1
135	1	422	0	709	1	996	1	1283	1	1570	1
136	0	423	0	710	1	997	1	1284	1	1571	0
137	1	424	0	711	0	998	1	1285	1	1572	1
138	1	425	1	712	1	999	0	1286	0	1573	0
139	1	426	1	713	1	1000	0	1287	1	1574	1
140	1	427	0	714	1	1001	1	1288	0	1575	0
141	0	428	0	715	0	1002	1	1289	0	1576	1
142	0	429	0	716	1	1003	1	1290	1	1577	1
143	0	430	0	717	1	1004	1	1291	1	1578	1
144	1	431	0	718	1	1005	1	1292	1	1579	0
145	0	432	0	719	0	1006	0	1293	1	1580	1
146	1	433	0	720	1	1007	0	1294	0	1581	1
147	0	434	0	721	0	1008	1	1295	1	1582	0
148	1	435	1	722	1	1009	1	1296	1	1583	0
149	0	436	1	723	0	1010	1	1297	1	1584	1
150	0	437	1	724	1	1011	1	1298	0	1585	1
151	0	438	1	725	1	1012	0	1299	0	1586	1
152	1	439	0	726	0	1013	1	1300	0	1587	1
153	0	440	1	727	0	1014	1	1301	0	1588	0
154	1	441	1	728	1	1015	1	1302	0	1589	1
155	1	442	1	729	0	1016	1	1303	0	1590	1
156	1	443	0	730	1	1017	1	1304	0	1591	0

157	0	444	0	731	0	1018	1	1305	1	1592	1
158	0	445	1	732	1	1019	0	1306	1	1593	1
159	0	446	1	733	0	1020	1	1307	1	1594	0
160	0	447	0	734	1	1021	1	1308	1	1595	0
161	1	448	0	735	0	1022	0	1309	0	1596	1
162	1	449	1	736	1	1023	1	1310	1	1597	0
163	0	450	1	737	1	1024	0	1311	1	1598	0
164	0	451	1	738	1	1025	0	1312	1	1599	1
165	1	452	0	739	1	1026	1	1313	1	1600	1
166	0	453	1	740	0	1027	1	1314	0	1601	0
167	1	454	1	741	0	1028	1	1315	0	1602	1
168	0	455	0	742	0	1029	1	1316	0	1603	1
169	0	456	0	743	1	1030	0	1317	1	1604	0
170	1	457	1	744	1	1031	0	1318	1	1605	1
171	1	458	1	745	0	1032	0	1319	0	1606	1
172	1	459	1	746	0	1033	0	1320	1	1607	1
173	0	460	1	747	1	1034	0	1321	0	1608	1
174	1	461	1	748	1	1035	1	1322	1	1609	1
175	1	462	1	749	1	1036	1	1323	1	1610	1
176	0	463	0	750	1	1037	1	1324	1	1611	0
177	0	464	0	751	1	1038	0	1325	0	1612	0
178	0	465	1	752	0	1039	0	1326	1	1613	1
179	0	466	0	753	1	1040	0	1327	0	1614	1
180	1	467	0	754	1	1041	1	1328	1	1615	0
181	0	468	0	755	0	1042	1	1329	0	1616	1
182	0	469	0	756	0	1043	0	1330	0	1617	1
183	1	470	0	757	0	1044	0	1331	0	1618	1
184	0	471	0	758	0	1045	0	1332	1	1619	0
185	0	472	0	759	1	1046	0	1333	1	1620	1
186	1	473	0	760	0	1047	1	1334	0	1621	0
187	1	474	0	761	0	1048	0	1335	1	1622	1
188	0	475	0	762	1	1049	1	1336	0	1623	1
189	0	476	0	763	1	1050	1	1337	1	1624	1
190	1	477	0	764	0	1051	1	1338	0	1625	1
191	0	478	0	765	0	1052	0	1339	0	1626	0
192	0	479	1	766	0	1053	1	1340	1	1627	1
193	1	480	1	767	0	1054	0	1341	1	1628	0
194	1	481	1	768	0	1055	1	1342	1	1629	1
195	1	482	0	769	0	1056	1	1343	0	1630	0
196	1	483	0	770	0	1057	1	1344	0	1631	1
197	1	484	0	771	1	1058	0	1345	1	1632	0
198	0	485	1	772	0	1059	0	1346	1	1633	0
199	0	486	0	773	0	1060	1	1347	0	1634	1
200	0	487	1	774	0	1061	0	1348	0	1635	0

201	0	488	1	775	1	1062	1	1349	0	1636	1
202	0	489	0	776	1	1063	0	1350	0	1637	1
203	0	490	0	777	0	1064	0	1351	1	1638	0
204	1	491	1	778	0	1065	0	1352	1	1639	0
205	1	492	0	779	1	1066	1	1353	1	1640	0
206	1	493	0	780	1	1067	0	1354	0	1641	1
207	0	494	0	781	1	1068	1	1355	1	1642	0
208	1	495	0	782	0	1069	0	1356	0	1643	1
209	1	496	0	783	0	1070	1	1357	0	1644	0
210	0	497	1	784	1	1071	0	1358	1	1645	1
211	1	498	1	785	0	1072	0	1359	1	1646	0
212	0	499	0	786	1	1073	1	1360	1	1647	0
213	1	500	0	787	0	1074	0	1361	1	1648	1
214	0	501	1	788	0	1075	0	1362	1	1649	0
215	0	502	0	789	1	1076	0	1363	0	1650	0
216	0	503	0	790	0	1077	0	1364	1	1651	0
217	1	504	1	791	0	1078	0	1365	1	1652	1
218	1	505	1	792	1	1079	0	1366	0	1653	0
219	1	506	0	793	1	1080	0	1367	1	1654	1
220	1	507	0	794	0	1081	0	1368	1	1655	0
221	1	508	0	795	1	1082	0	1369	1	1656	1
222	0	509	1	796	0	1083	0	1370	0	1657	1
223	0	510	0	797	0	1084	1	1371	1	1658	1
224	0	511	1	798	0	1085	0	1372	0	1659	1
225	0	512	0	799	1	1086	1	1373	1	1660	1
226	0	513	0	800	0	1087	1	1374	1	1661	0
227	1	514	0	801	0	1088	1	1375	1	1662	1
228	0	515	1	802	0	1089	0	1376	0	1663	0
229	0	516	0	803	1	1090	1	1377	1	1664	1
230	0	517	0	804	0	1091	1	1378	1	1665	1
231	0	518	0	805	0	1092	0	1379	1	1666	1
232	1	519	1	806	0	1093	1	1380	1	1667	1
233	1	520	1	807	0	1094	1	1381	1	1668	1
234	1	521	0	808	1	1095	1	1382	1	1669	1
235	1	522	0	809	1	1096	1	1383	1	1670	0
236	0	523	0	810	1	1097	0	1384	1	1671	0
237	1	524	1	811	1	1098	1	1385	1	1672	1
238	0	525	1	812	0	1099	1	1386	0	1673	0
239	1	526	1	813	0	1100	1	1387	0	1674	0
240	0	527	0	814	1	1101	1	1388	0	1675	0
241	1	528	1	815	0	1102	0	1389	0	1676	1
242	0	529	0	816	0	1103	1	1390	1	1677	1
243	0	530	1	817	1	1104	0	1391	1	1678	1
244	1	531	1	818	0	1105	0	1392	1	1679	1

245	1	532	1	819	0	1106	1	1393	0	1680	0
246	0	533	1	820	1	1107	0	1394	0	1681	0
247	1	534	1	821	0	1108	1	1395	1	1682	1
248	0	535	0	822	0	1109	0	1396	1	1683	1
249	0	536	1	823	0	1110	1	1397	1	1684	0
250	0	537	1	824	1	1111	0	1398	0	1685	1
251	1	538	0	825	0	1112	1	1399	0	1686	1
252	0	539	0	826	1	1113	0	1400	1	1687	0
253	0	540	1	827	0	1114	1	1401	1	1688	0
254	0	541	0	828	1	1115	1	1402	0	1689	1
255	0	542	1	829	0	1116	1	1403	0	1690	1
256	1	543	0	830	0	1117	0	1404	0	1691	0
257	0	544	0	831	1	1118	1	1405	0	1692	0
258	0	545	0	832	0	1119	1	1406	1	1693	0
259	0	546	1	833	0	1120	1	1407	0	1694	1
260	1	547	1	834	0	1121	1	1408	1	1695	0
261	0	548	0	835	0	1122	1	1409	1	1696	1
262	1	549	1	836	0	1123	0	1410	1	1697	1
263	1	550	0	837	0	1124	0	1411	0	1698	0
264	0	551	1	838	1	1125	1	1412	0	1699	1
265	1	552	0	839	0	1126	0	1413	1	1700	0
266	1	553	1	840	1	1127	1	1414	0	1701	0
267	0	554	1	841	0	1128	1	1415	0	1702	0
268	0	555	1	842	0	1129	1	1416	1	1703	1
269	0	556	1	843	0	1130	1	1417	0	1704	1
270	1	557	0	844	0	1131	1	1418	1	1705	0
271	0	558	0	845	0	1132	0	1419	0	1706	1
272	0	559	0	846	1	1133	1	1420	0	1707	0
273	1	560	1	847	1	1134	1	1421	1	1708	1
274	0	561	1	848	0	1135	0	1422	1	1709	0
275	1	562	1	849	1	1136	0	1423	1	1710	1
276	0	563	0	850	0	1137	1	1424	1	1711	0
277	0	564	0	851	1	1138	0	1425	0	1712	0
278	1	565	1	852	0	1139	1	1426	1	1713	0
279	1	566	0	853	1	1140	0	1427	0	1714	0
280	0	567	1	854	1	1141	1	1428	0	1715	0
281	0	568	0	855	1	1142	0	1429	1	1716	0
282	0	569	0	856	1	1143	1	1430	0		
283	0	570	0	857	0	1144	0	1431	1		
284	1	571	1	858	0	1145	1	1432	0		
285	0	572	0	859	0	1146	0	1433	1		
286	1	573	1	860	1	1147	1	1434	1		

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