PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b) = (z^2+1, z^4+z^2+z+1)$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b) = (z^2+1, z^4+z^2+z+1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^2+1,z^4+z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{12}+z^{11}+z^9+z^5+z^3+z^2+z+1,\\
p_1(z)&=z^{18}+z^{13}+z^{12}+z^{10}+z^9+z^8+z^7+z^6+z^2+z,\\
p_2(z)&=z^{20}+z^8+z^6+z^4+z^2+1,\\
p_3(z)&=z^{18}+z^{13}+z^{12}+z^{10}+z^9+z^8+z^7+z^6+z^2+z,\\
p_4(z)&=z^{16}+z^{14}+z^{12}+z^{11}+z^9+z^6+z^5+z^3+z+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^{11}+z^9+z^8+z^5+z^4+z^3+z^2+z+1,\\
p_1(z)&=z^{18}+z^{13}+z^{12}+z^{10}+z^9+z^8+z^7+z^6+z^2+z,\\
p_2(z)&=z^{20}+z^8+z^6+z^4+z^2+1,\\
p_3(z)&=z^{18}+z^{13}+z^{12}+z^{10}+z^9+z^8+z^7+z^6+z^2+z,\\
p_4(z)&=z^{16}+z^{14}+z^{11}+z^9+z^8+z^6+z^5+z^4+z^3+z+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge0}$ and $(Q_n(z))_{n\ge0}$ by

$$\begin{pmatrix}P_n(z)&Q_n(z)\\P_{n-1}(z)&Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z)&1\\1&0\end{pmatrix}\begin{pmatrix}t_{n-1}(z)&1\\1&0\end{pmatrix}\cdots\begin{pmatrix}t_0(z)&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix}t_{2^n-1}(1/x)&1\\1&0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x)&1\\1&0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x)&1\\1&0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix}\bar t_{2^n-1}(1/x)&1\\1&0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x)&1\\1&0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x)&1\\1&0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u] + x^{3u}M^e[:6u] \\ & + x^{6u}M^e[:3u] + x^{7u}M^e[:2u] + x^{8u}M^e[:u] + x^{4u}M^e[:6u] \\ & + x^{7u}M^e[:3u] + x^{8u}M^e[:2u] + x^{9u}M^e[:u] + x^{5u}M^e[:6u] \\ & + x^{8u}M^e[:3u] + x^{9u}M^e[:2u] + x^{10u}M^e[:u] + x^{7u}M^e[:5u] \\ & + (x^{6u} + x^{9u} + x^{10u} + x^{11u})R^e, \end{aligned}$$

$$W^e[:12u] = W^e[:6u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u] + x^{3u}W^e[:6u]$$

$$
\begin{aligned}
&+ x^{6u}W^e[:3u] + x^{7u}W^e[:2u] + x^{8u}W^e[:u] + x^{4u}W^e[:6u]\\
&+ x^{7u}W^e[:3u] + x^{8u}W^e[:2u] + x^{9u}W^e[:u] + x^{5u}W^e[:6u]\\
&+ x^{8u}W^e[:3u] + x^{9u}W^e[:2u] + x^{10u}W^e[:u] + x^{7u}W^e[:5u]\\
&+ (x^{6u} + x^{9u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u] + x^{3u}M^o[:6u]\\
&+ x^{6u}M^o[:3u] + x^{7u}M^o[:2u] + x^{8u}M^o[:u] + x^{4u}M^o[:6u]\\
&+ x^{7u}M^o[:3u] + x^{8u}M^o[:2u] + x^{9u}M^o[:u] + x^{5u}M^o[:6u]\\
&+ x^{8u}M^o[:3u] + x^{9u}M^o[:2u] + x^{10u}M^o[:u] + x^{7u}M^o[:5u]\\
&+ (x^{6u} + x^{9u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u] + x^{3u}W^o[:6u]\\
&+ x^{6u}W^o[:3u] + x^{7u}W^o[:2u] + x^{8u}W^o[:u] + x^{4u}W^o[:6u]\\
&+ x^{7u}W^o[:3u] + x^{8u}W^o[:2u] + x^{9u}W^o[:u] + x^{5u}W^o[:6u]\\
&+ x^{8u}W^o[:3u] + x^{9u}W^o[:2u] + x^{10u}W^o[:u] + x^{7u}W^o[:5u]\\
&+ (x^{6u} + x^{9u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] &= T[:6u] + x^{3u}T[:3u] + x^{4u}T[:2u] + x^{5u}T[:u] + x^{3u}T[:6u] + x^{6u}T[:3u]\\
&+ x^{7u}T[:2u] + x^{8u}T[:u] + x^{4u}T[:6u] + x^{7u}T[:3u] + x^{8u}T[:2u]\\
&+ x^{9u}T[:u] + x^{5u}T[:6u] + x^{8u}T[:3u] + x^{9u}T[:2u] + x^{10u}T[:u]\\
&+ x^{7u}T[:5u] + (x^{6u} + x^{9u} + x^{10u} + x^{11u}).
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
x^6u &= x^{6u}T[:u] + x^{5u}T[u:2u] + x^{4u}T[2u:3u] + x^{3u}T[3u:4u]\\
&\quad + T[6u:7u],\\
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{5u}T[2u:3u] + x^{4u}T[3u:4u]\\
&\quad + x^{3u}T[4u:5u] + T[7u:8u],\\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u]\\
&\quad + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u],\\
x^9u &= x^{5u}T[4u:5u] + x^{4u}T[5u:6u] + T[9u:10u],\\
x^10u &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u]\\
&\quad + x^{5u}T[5u:6u] + T[10u:11u],\\
x^11u &= x^{7u}T[4u:5u] + T[11u:12u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.7}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[111w]_2], \end{aligned} \tag{2.8}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[101w]_2] + T[[1000w]_2], \end{aligned} \tag{2.9}$$

$$0 = T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] \\ & + T[[1010w]_2], \end{aligned} \tag{2.11}$$

$$0 = T[[100w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[110w]_2], \tag{2.13}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[111w]_2], \end{aligned} \tag{2.14}$$

$$\begin{aligned} 0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ & + T[[101w]_2] + T[[1000w]_2], \end{aligned} \tag{2.15}$$

$$1 = T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$\begin{aligned} 1 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] \\ & + T[[1010w]_2], \end{aligned} \tag{2.17}$$

$$1 = T[[100w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1513 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)), \tag{2.19}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.20}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.21}$$

$$0 = \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.22}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,1010)), \end{aligned} \tag{2.23}$$

$$0 = \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,110)), \tag{2.25}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,111)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.27}$$

$$1 = \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.28}$$

$$\begin{aligned} 1 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$1 = \tau(A(s,100)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{22}), E_{22}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 11$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u}M^e[:3u] + x^{4u}M^e[:2u] + x^{5u}M^e[:u] + x^{6u}R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u}W^e[:3u] + x^{4u}W^e[:2u] + x^{5u}W^e[:u] + x^{6u}R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^{3u}M^o[:3u] + x^{4u}M^o[:2u] + x^{5u}M^o[:u] + x^{6u}R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u}W^o[:3u] + x^{4u}W^o[:2u] + x^{5u}W^o[:u] + x^{6u}R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = {} & \big(W^e[:6v] + x^{3v}W^e[:3v] + x^{4v}W^e[:2v] + x^{5v}W^e[:v] + x^{6u}R^e\big) \times \big(\\ & M^e[:6v] + x^{3v}M^e[:3v] + x^{4v}M^e[:2v] + x^{5v}M^e[:v] + x^{6u}R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} & W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v] + x^{8v}W^e[:4v]M^e[:4v] \\ & \qquad + x^{10v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \leq 12$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{11} + x^9 + x^8 + x^6, \\ q_1(x) &= x^{19} + x^{18} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^2, \\ q_2(x) &= x^{20} + x^{18} + x^{16} + x^{14} + x^{12} + 1, \\ q_3(x) &= x^{19} + x^{18} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^2, \\ q_4(x) &= x^{20} + x^{19} + x^{17} + x^{15} + x^{14} + x^{11} + x^9 + x^8 + x^6 + x^4. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate

$f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{108} + x^{106} + x^{100} + x^{96} + x^{94} + x^{90} + x^{88} + x^{84} + x^{82} + x^{76} \\
&\quad + x^{68} + x^{66} + x^{58} + x^{54} + x^{50} + x^{44} + x^{42} + x^{38} + x^{36} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{94} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{58} + x^{56} + x^{48} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{26} + x^{22}, \\
p_6(x) &= x^{112} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{74} \\
&\quad + x^{68} + x^{64} + x^{56} + x^{52} + x^{44} + x^{40} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18} \\
&\quad + x^{16} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{112} + x^{110} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{74} \\
&\quad + x^{72} + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{46} + x^{44} + x^{38} + x^{36} + x^{32} \\
&\quad + x^{28} + x^{24} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{42} \\
&\quad + x^{40} + x^{26} + x^{24} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{63} + x^{58} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{43} + x^{41} + x^{39} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{30}, \\
p_3(x) &= x^{99} + x^{96} + x^{89} + x^{86} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} \\
&\quad + x^{70} + x^{63} + x^{61} + x^{60} + x^{58} + x^{43} + x^{40} + x^{33} + x^{30} + x^{23} + x^{21} \\
&\quad + x^{20} + x^{18}, \\
p_6(x) &= x^{102} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{84} + x^{83} + x^{82} \\
&\quad + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{63} \\
&\quad + x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{41} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25}\\
&+ x^{23} + x^{20} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_9(x) &= x^{105} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{83}\\
&+ x^{80} + x^{77} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{14} + x^{11} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{91}\\
&+ x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70}\\
&+ x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22}\\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{89} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} + x^{77}\\
&+ x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{63} + x^{58}\\
&+ x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{43} + x^{41} + x^{39} + x^{38}\\
&+ x^{36} + x^{34} + x^{32} + x^{30},\\
p_3(x) &= x^{99} + x^{96} + x^{89} + x^{86} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73}\\
&+ x^{70} + x^{63} + x^{61} + x^{60} + x^{58} + x^{43} + x^{40} + x^{33} + x^{30} + x^{23} + x^{21}\\
&+ x^{20} + x^{18},\\
p_6(x) &= x^{102} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{84} + x^{83} + x^{82}\\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{63}\\
&+ x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{41} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25}\\
&+ x^{23} + x^{20} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_9(x) &= x^{105} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{83}\\
&+ x^{80} + x^{77} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{14} + x^{11} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{91}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} \\
&+ x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43} \\
&+ x^{42} + x^{39} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} \\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{86} + x^{84} + x^{74} + x^{70} + x^{64} + x^{62} \\
&\quad + x^{58} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{100} + x^{96} + x^{94} + x^{86} + x^{84} + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{54} \\
&\quad + x^{52} + x^{50} + x^{48} + x^{38} + x^{36} + x^{34} + x^{30} + x^{26} + x^{22} + x^{20} + x^{18} \\
&\quad + x^{14} + x^{12}, \\
p_6(x) &= x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{66} + x^{64} + x^{62} + x^{52} + x^{48} + x^{46} + x^{42} + x^{34} + x^{30} + x^{28} + x^{24} \\
&\quad + x^{22} + x^{14} + x^{10} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{100} + x^{96} + x^{94} + x^{92} + x^{82} + x^{80} + x^{76} + x^{72} + x^{66} + x^{58} + x^{54} \\
&\quad + x^{50} + x^{44} + x^{42} + x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{12} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{54} + x^{50} + x^{48} + x^{38} + x^{34} + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{72} + x^{68} \\
&\quad + x^{66} + x^{62} + x^{60} + x^{52} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{28} + x^{26} \\
&\quad + x^{24}, \\
p_3(x) &= x^{100} + x^{92} + x^{86} + x^{84} + x^{80} + x^{78} + x^{68} + x^{62} + x^{56} + x^{52} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{38} + x^{32} + x^{30} + x^{28} + x^{20}, \\
p_6(x) &= x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{84} + x^{70} + x^{68} + x^{64} + x^{60} + x^{58} \\
&\quad + x^{56} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{18} \\
&\quad + x^{14} + x^{12} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{100} + x^{96} + x^{94} + x^{92} + x^{82} + x^{80} + x^{76} + x^{72} + x^{66} + x^{58} + x^{54} \\
&\quad + x^{50} + x^{44} + x^{42} + x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{12} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{54} + x^{50} + x^{48} + x^{38} + x^{34} + x^{32} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{98} + x^{97} \\
&\quad + x^{95} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{76} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{56} + x^{53} + x^{52}\\
&+ x^{51} + x^{45} + x^{44} + x^{42} + x^{35} + x^{34} + x^{30},\\
p_3(x) &= x^{99} + x^{96} + x^{89} + x^{86} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73}\\
&+ x^{70} + x^{63} + x^{61} + x^{60} + x^{58} + x^{43} + x^{40} + x^{33} + x^{30} + x^{23} + x^{21}\\
&+ x^{20} + x^{18},\\
p_6(x) &= x^{102} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{84} + x^{83} + x^{82}\\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{63}\\
&+ x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{41} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25}\\
&+ x^{23} + x^{20} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_9(x) &= x^{105} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{83}\\
&+ x^{80} + x^{77} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{60}\\
&+ x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17}\\
&+ x^{14} + x^{11} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) &= x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{91}\\
&+ x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70}\\
&+ x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43}\\
&+ x^{42} + x^{39} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22}\\
&+ x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{99} + x^{98} + x^{97}\\
&+ x^{95} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78}\\
&+ x^{76} + x^{72} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{56} + x^{53} + x^{52}\\
&+ x^{51} + x^{45} + x^{44} + x^{42} + x^{35} + x^{34} + x^{30},\\
p_3(x) &= x^{99} + x^{96} + x^{89} + x^{86} + x^{83} + x^{80} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73}\\
&+ x^{70} + x^{63} + x^{61} + x^{60} + x^{58} + x^{43} + x^{40} + x^{33} + x^{30} + x^{23} + x^{21}\\
&+ x^{20} + x^{18},\\
p_6(x) &= x^{102} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{84} + x^{83} + x^{82}\\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{67} + x^{66} + x^{63}\\
&+ x^{61} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{45}\\
&+ x^{41} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25}\\
&+ x^{23} + x^{20} + x^{16} + x^{15} + x^{14} + x^{12},
\end{aligned}
$$

$$
\begin{aligned}
p_9(x) = {} & x^{105} + x^{103} + x^{102} + x^{100} + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{88} + x^{83} \\
& + x^{80} + x^{77} + x^{74} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{63} + x^{60} \\
& + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{44} + x^{43} \\
& + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} \\
& + x^{14} + x^{11} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) = {} & x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{100} + x^{96} + x^{95} + x^{94} + x^{91} \\
& + x^{90} + x^{87} + x^{86} + x^{83} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} \\
& + x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43} \\
& + x^{42} + x^{39} + x^{38} + x^{36} + x^{32} + x^{31} + x^{30} + x^{27} + x^{26} + x^{23} + x^{22} \\
& + x^{19} + x^{18} + x^{15} + x^{14} + x^{12} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{114} + x^{110} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} \\
& + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{48} \\
& + x^{42} + x^{38} + x^{36}, \\
p_3(x) = {} & x^{112} + x^{110} + x^{102} + x^{98} + x^{94} + x^{92} + x^{88} + x^{80} + x^{78} + x^{76} + x^{74} \\
& + x^{68} + x^{66} + x^{64} + x^{54} + x^{52} + x^{50} + x^{36} + x^{14} + x^{12}, \\
p_6(x) = {} & x^{112} + x^{110} + x^{104} + x^{102} + x^{98} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} \\
& + x^{74} + x^{72} + x^{70} + x^{68} + x^{64} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{42} \\
& + x^{38} + x^{36} + x^{34} + x^{30} + x^{26} + x^{24} + x^{22} + x^{18} + x^{16} + x^{14} + x^{6} \\
& + x^{4} + x^{2} + 1, \\
p_9(x) = {} & x^{112} + x^{110} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{82} + x^{74} \\
& + x^{72} + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{46} + x^{44} + x^{38} + x^{36} + x^{32} \\
& + x^{28} + x^{24} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6} + x^{4}, \\
p_{12}(x) = {} & x^{114} + x^{112} + x^{106} + x^{104} + x^{90} + x^{88} + x^{82} + x^{80} + x^{74} + x^{72} + x^{42} \\
& + x^{40} + x^{26} + x^{24} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{86} \\
& + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{76} + x^{73} + x^{72} + x^{70} + x^{65} + x^{64} \\
& + x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{42} \\
& + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24}, \\
p_3(x) = {} & x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} \\
& + x^{86} + x^{83} + x^{82} + x^{80} + x^{79} + x^{76} + x^{73} + x^{71} + x^{67} + x^{66} + x^{61}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{59}+x^{58}+x^{57}+x^{54}+x^{50}+x^{49}+x^{47}+x^{42}+x^{41}+x^{39}+x^{35}\\
&+x^{31}+x^{29}+x^{28}+x^{27}+x^{22}+x^{20}+x^{19}+x^{18}+x^{16},\\
p_6(x)&=x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}+x^{97}\\
&+x^{94}+x^{93}+x^{92}+x^{89}+x^{88}+x^{85}+x^{79}+x^{78}+x^{76}+x^{75}+x^{74}\\
&+x^{69}+x^{62}+x^{58}+x^{57}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{46}+x^{45}\\
&+x^{41}+x^{34}+x^{33}+x^{32}+x^{29}+x^{27}+x^{26}+x^{21}+x^{19}+x^{16}+x^{15}\\
&+x^{11}+x^{10}+x^{8}+x^{4}+x^{3}+x^{2}+1,\\
p_9(x)&=x^{104}+x^{103}+x^{102}+x^{100}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}+x^{84}+x^{76}\\
&+x^{75}+x^{74}+x^{71}+x^{70}+x^{68}+x^{60}+x^{59}+x^{58}+x^{56}+x^{40}+x^{39}\\
&+x^{38}+x^{36}+x^{28}+x^{27}+x^{26}+x^{23}+x^{22}+x^{20}+x^{12}+x^{11}+x^{10}\\
&+x^{8},\\
p_{12}(x)&=x^{108}+x^{107}+x^{106}+x^{104}+x^{96}+x^{95}+x^{94}+x^{92}+x^{84}+x^{83}+x^{82}\\
&+x^{79}+x^{78}+x^{76}+x^{68}+x^{67}+x^{66}+x^{63}+x^{62}+x^{59}+x^{58}+x^{56}\\
&+x^{52}+x^{51}+x^{50}+x^{48}+x^{44}+x^{43}+x^{42}+x^{40}+x^{32}+x^{31}+x^{30}\\
&+x^{28}+x^{20}+x^{19}+x^{18}+x^{15}+x^{14}+x^{11}+x^{10}+x^{8}+x^{4}+x^{3}+x^{2}\\
&+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0].

For $\phi(M^o,0,1)$:

$$
\begin{aligned}
p_0(x)&=x^{108}+x^{105}+x^{104}+x^{101}+x^{98}+x^{97}+x^{95}+x^{92}+x^{91}+x^{90}+x^{87}\\
&+x^{85}+x^{84}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{68}+x^{67}+x^{64}+x^{60}\\
&+x^{58}+x^{57}+x^{56}+x^{54}+x^{52}+x^{51}+x^{50}+x^{48}+x^{46}+x^{43}+x^{42}\\
&+x^{40}+x^{38}+x^{36}+x^{35}+x^{33}+x^{30},\\
p_3(x)&=x^{99}+x^{98}+x^{97}+x^{96}+x^{94}+x^{93}+x^{91}+x^{90}+x^{89}+x^{87}+x^{85}\\
&+x^{83}+x^{81}+x^{79}+x^{77}+x^{75}+x^{73}+x^{71}+x^{69}+x^{66}+x^{64}+x^{63}\\
&+x^{62}+x^{58}+x^{43}+x^{42}+x^{41}+x^{40}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}\\
&+x^{31}+x^{29}+x^{26}+x^{24}+x^{23}+x^{22}+x^{18},\\
p_6(x)&=x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{90}+x^{84}+x^{80}+x^{78}+x^{76}+x^{70}\\
&+x^{62}+x^{58}+x^{56}+x^{54}+x^{48}+x^{44}+x^{38}+x^{36}+x^{28}+x^{26}+x^{24}\\
&+x^{22}+x^{20}+x^{16}+x^{12},\\
p_9(x)&=x^{105}+x^{104}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{94}+x^{92}+x^{91}+x^{90}\\
&+x^{89}+x^{88}+x^{86}+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{78}+x^{76}+x^{75}\\
&+x^{74}+x^{73}+x^{72}+x^{70}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}+x^{60}\\
&+x^{59}+x^{58}+x^{54}+x^{41}+x^{40}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{30}\\
&+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{22}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}\\
&+x^{14}+x^{12}+x^{11}+x^{10}+x^{6},
\end{aligned}
$$

$$\begin{aligned}p_{12}(x) &= x^{108}+x^{104}+x^{96}+x^{92}+x^{84}+x^{76}+x^{68}+x^{56}+x^{52}+x^{48}+x^{44}\\&\quad+x^{40}+x^{32}+x^{28}+x^{20}+x^{8}+x^{4}+1,\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}p_0(x) &= x^{114}+x^{112}+x^{111}+x^{110}+x^{109}+x^{107}+x^{106}+x^{105}+x^{104}+x^{103}\\&\quad+x^{100}+x^{99}+x^{98}+x^{97}+x^{95}+x^{91}+x^{88}+x^{85}+x^{83}+x^{82}+x^{80}\\&\quad+x^{79}+x^{77}+x^{75}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{66}+x^{63}+x^{62}\\&\quad+x^{60}+x^{59}+x^{56}+x^{55}+x^{54}+x^{53}+x^{51}+x^{49}+x^{48}+x^{47}+x^{42}\\&\quad+x^{38}+x^{37}+x^{36}+x^{34}+x^{33}+x^{30},\\
p_3(x) &= x^{105}+x^{104}+x^{101}+x^{99}+x^{98}+x^{96}+x^{94}+x^{93}+x^{92}+x^{89}+x^{87}\\&\quad+x^{86}+x^{85}+x^{82}+x^{81}+x^{79}+x^{78}+x^{77}+x^{74}+x^{72}+x^{71}+x^{70}\\&\quad+x^{67}+x^{65}+x^{64}+x^{63}+x^{60}+x^{58}+x^{49}+x^{48}+x^{45}+x^{43}+x^{42}\\&\quad+x^{40}+x^{38}+x^{37}+x^{36}+x^{32}+x^{31}+x^{30}+x^{27}+x^{25}+x^{24}+x^{23}\\&\quad+x^{20}+x^{18},\\
p_6(x) &= x^{108}+x^{98}+x^{94}+x^{90}+x^{84}+x^{82}+x^{76}+x^{74}+x^{72}+x^{66}+x^{64}\\&\quad+x^{62}+x^{60}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{46}+x^{44}+x^{42}+x^{40}\\&\quad+x^{36}+x^{32}+x^{26}+x^{20}+x^{14}+x^{12},\\
p_9(x) &= x^{111}+x^{110}+x^{109}+x^{108}+x^{106}+x^{105}+x^{101}+x^{99}+x^{98}+x^{96}\\&\quad+x^{94}+x^{91}+x^{90}+x^{89}+x^{88}+x^{86}+x^{85}+x^{83}+x^{82}+x^{80}+x^{78}\\&\quad+x^{75}+x^{74}+x^{73}+x^{72}+x^{70}+x^{69}+x^{67}+x^{66}+x^{64}+x^{63}+x^{61}\\&\quad+x^{60}+x^{59}+x^{56}+x^{54}+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{37}\\&\quad+x^{35}+x^{34}+x^{32}+x^{30}+x^{27}+x^{26}+x^{25}+x^{24}+x^{22}+x^{21}+x^{19}\\&\quad+x^{18}+x^{16}+x^{15}+x^{13}+x^{12}+x^{11}+x^{8}+x^{6},\\
p_{12}(x) &= x^{114}+x^{112}+x^{110}+x^{108}+x^{94}+x^{92}+x^{90}+x^{88}+x^{82}+x^{80}+x^{78}\\&\quad+x^{76}+x^{74}+x^{72}+x^{58}+x^{56}+x^{46}+x^{44}+x^{30}+x^{28}+x^{26}+x^{24}\\&\quad+x^{10}+x^{8}+x^{2}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}p_0(x) &= x^{108}+x^{107}+x^{105}+x^{103}+x^{100}+x^{94}+x^{93}+x^{92}+x^{89}+x^{87}+x^{86}\\&\quad+x^{85}+x^{81}+x^{80}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{67}+x^{66}\\&\quad+x^{64}+x^{63}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}+x^{51}+x^{49}+x^{48}+x^{45}\\&\quad+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{36},\\
p_3(x) &= x^{104}+x^{103}+x^{101}+x^{99}+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}\\&\quad+x^{82}+x^{81}+x^{76}+x^{75}+x^{72}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}+x^{62}\\&\quad+x^{61}+x^{57}+x^{56}+x^{54}+x^{50}+x^{49}+x^{47}+x^{44}+x^{43}+x^{42}+x^{39}\end{aligned}$$

$$
\begin{aligned}
&+ x^{38} + x^{35} + x^{34} + x^{33} + x^{29} + x^{27} + x^{24} + x^{23} + x^{20},\\
p_6(x) &= x^{106} + x^{105} + x^{102} + x^{100} + x^{99} + x^{97} + x^{92} + x^{91} + x^{89} + x^{85} + x^{84}\\
&+ x^{83} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{51} + x^{49} + x^{47}\\
&+ x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{35} + x^{33} + x^{29} + x^{27} + x^{25} + x^{24}\\
&+ x^{22} + x^{21} + x^{20} + x^{18} + x^{16} + x^{14} + x^{4} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{104} + x^{103} + x^{102} + x^{100} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{84} + x^{76}\\
&+ x^{75} + x^{74} + x^{71} + x^{70} + x^{68} + x^{60} + x^{59} + x^{58} + x^{56} + x^{40} + x^{39}\\
&+ x^{38} + x^{36} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{20} + x^{12} + x^{11} + x^{10}\\
&+ x^{8},\\
p_{12}(x) &= x^{108} + x^{107} + x^{106} + x^{104} + x^{96} + x^{95} + x^{94} + x^{92} + x^{84} + x^{83} + x^{82}\\
&+ x^{79} + x^{78} + x^{76} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{56}\\
&+ x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43} + x^{42} + x^{40} + x^{32} + x^{31} + x^{30}\\
&+ x^{28} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{8} + x^{4} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{107} + x^{105} + x^{104} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{86}\\
&+ x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{76} + x^{73} + x^{72} + x^{70} + x^{65} + x^{64}\\
&+ x^{63} + x^{60} + x^{57} + x^{55} + x^{54} + x^{50} + x^{49} + x^{47} + x^{45} + x^{44} + x^{42}\\
&+ x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24},\\
p_3(x) &= x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89}\\
&+ x^{86} + x^{83} + x^{82} + x^{80} + x^{79} + x^{76} + x^{73} + x^{71} + x^{67} + x^{66} + x^{61}\\
&+ x^{59} + x^{58} + x^{57} + x^{54} + x^{50} + x^{49} + x^{47} + x^{42} + x^{41} + x^{39} + x^{35}\\
&+ x^{31} + x^{29} + x^{28} + x^{27} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16},\\
p_6(x) &= x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97}\\
&+ x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{85} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74}\\
&+ x^{69} + x^{62} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{46} + x^{45}\\
&+ x^{41} + x^{34} + x^{33} + x^{32} + x^{29} + x^{27} + x^{26} + x^{21} + x^{19} + x^{16} + x^{15}\\
&+ x^{11} + x^{10} + x^{8} + x^{4} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{104} + x^{103} + x^{102} + x^{100} + x^{92} + x^{91} + x^{90} + x^{87} + x^{86} + x^{84} + x^{76}\\
&+ x^{75} + x^{74} + x^{71} + x^{70} + x^{68} + x^{60} + x^{59} + x^{58} + x^{56} + x^{40} + x^{39}\\
&+ x^{38} + x^{36} + x^{28} + x^{27} + x^{26} + x^{23} + x^{22} + x^{20} + x^{12} + x^{11} + x^{10}\\
&+ x^{8},\\
p_{12}(x) &= x^{108} + x^{107} + x^{106} + x^{104} + x^{96} + x^{95} + x^{94} + x^{92} + x^{84} + x^{83} + x^{82}
\end{aligned}
$$

$$+ x^{79} + x^{78} + x^{76} + x^{68} + x^{67} + x^{66} + x^{63} + x^{62} + x^{59} + x^{58} + x^{56} + x^{52} + x^{51} + x^{50} + x^{48} + x^{44} + x^{43} + x^{42} + x^{40} + x^{32} + x^{31} + x^{30} + x^{28} + x^{20} + x^{19} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{8} + x^{4} + x^{3} + x^{2} + 1,$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} \\
& + x^{100} + x^{99} + x^{98} + x^{97} + x^{95} + x^{91} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} \\
& + x^{79} + x^{77} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} \\
& + x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{49} + x^{48} + x^{47} + x^{42} \\
& + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{30}, \\
p_3(x) = {} & x^{105} + x^{104} + x^{101} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{92} + x^{89} + x^{87} \\
& + x^{86} + x^{85} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} \\
& + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{58} + x^{49} + x^{48} + x^{45} + x^{43} + x^{42} \\
& + x^{40} + x^{38} + x^{37} + x^{36} + x^{32} + x^{31} + x^{30} + x^{27} + x^{25} + x^{24} + x^{23} \\
& + x^{20} + x^{18}, \\
p_6(x) = {} & x^{108} + x^{98} + x^{94} + x^{90} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} \\
& + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
& + x^{36} + x^{32} + x^{26} + x^{20} + x^{14} + x^{12}, \\
p_9(x) = {} & x^{111} + x^{110} + x^{109} + x^{108} + x^{106} + x^{105} + x^{101} + x^{99} + x^{98} + x^{96} \\
& + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} \\
& + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} \\
& + x^{60} + x^{59} + x^{56} + x^{54} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{37} \\
& + x^{35} + x^{34} + x^{32} + x^{30} + x^{27} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} \\
& + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{11} + x^{8} + x^{6}, \\
p_{12}(x) = {} & x^{114} + x^{112} + x^{110} + x^{108} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{78} \\
& + x^{76} + x^{74} + x^{72} + x^{58} + x^{56} + x^{46} + x^{44} + x^{30} + x^{28} + x^{26} + x^{24} \\
& + x^{10} + x^{8} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{108} + x^{105} + x^{104} + x^{101} + x^{98} + x^{97} + x^{95} + x^{92} + x^{91} + x^{90} + x^{87} \\
& + x^{85} + x^{84} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{68} + x^{67} + x^{64} + x^{60} \\
& + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{43} + x^{42} \\
& + x^{40} + x^{38} + x^{36} + x^{35} + x^{33} + x^{30}, \\
p_3(x) = {} & x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{85}
\end{aligned}$$

$$
\begin{aligned}
&+x^{83}+x^{81}+x^{79}+x^{77}+x^{75}+x^{73}+x^{71}+x^{69}+x^{66}+x^{64}+x^{63}\\
&+x^{62}+x^{58}+x^{43}+x^{42}+x^{41}+x^{40}+x^{38}+x^{37}+x^{35}+x^{34}+x^{33}\\
&+x^{31}+x^{29}+x^{26}+x^{24}+x^{23}+x^{22}+x^{18},\\
p_6(x)=\,&x^{102}+x^{100}+x^{98}+x^{96}+x^{94}+x^{90}+x^{84}+x^{80}+x^{78}+x^{76}+x^{70}\\
&+x^{62}+x^{58}+x^{56}+x^{54}+x^{48}+x^{44}+x^{38}+x^{36}+x^{28}+x^{26}+x^{24}\\
&+x^{22}+x^{20}+x^{16}+x^{12},\\
p_9(x)=\,&x^{105}+x^{104}+x^{100}+x^{99}+x^{98}+x^{97}+x^{96}+x^{94}+x^{92}+x^{91}+x^{90}\\
&+x^{89}+x^{88}+x^{86}+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{78}+x^{76}+x^{75}\\
&+x^{74}+x^{73}+x^{72}+x^{70}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}+x^{60}\\
&+x^{59}+x^{58}+x^{54}+x^{41}+x^{40}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{30}\\
&+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{22}+x^{20}+x^{19}+x^{18}+x^{17}+x^{16}\\
&+x^{14}+x^{12}+x^{11}+x^{10}+x^{6},\\
p_{12}(x)=\,&x^{108}+x^{104}+x^{96}+x^{92}+x^{84}+x^{76}+x^{68}+x^{56}+x^{52}+x^{48}+x^{44}\\
&+x^{40}+x^{32}+x^{28}+x^{20}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)=\,&x^{108}+x^{107}+x^{105}+x^{103}+x^{100}+x^{94}+x^{93}+x^{92}+x^{89}+x^{87}+x^{86}\\
&+x^{85}+x^{81}+x^{80}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{67}+x^{66}\\
&+x^{64}+x^{63}+x^{58}+x^{57}+x^{55}+x^{54}+x^{52}+x^{51}+x^{49}+x^{48}+x^{45}\\
&+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{36},\\
p_3(x)=\,&x^{104}+x^{103}+x^{101}+x^{99}+x^{95}+x^{93}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}\\
&+x^{82}+x^{81}+x^{76}+x^{75}+x^{72}+x^{68}+x^{67}+x^{66}+x^{64}+x^{63}+x^{62}\\
&+x^{61}+x^{57}+x^{56}+x^{54}+x^{50}+x^{49}+x^{47}+x^{44}+x^{43}+x^{42}+x^{39}\\
&+x^{38}+x^{35}+x^{34}+x^{33}+x^{29}+x^{27}+x^{24}+x^{23}+x^{20},\\
p_6(x)=\,&x^{106}+x^{105}+x^{102}+x^{100}+x^{99}+x^{97}+x^{92}+x^{91}+x^{89}+x^{85}+x^{84}\\
&+x^{83}+x^{82}+x^{80}+x^{79}+x^{77}+x^{76}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}\\
&+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{51}+x^{49}+x^{47}\\
&+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{35}+x^{33}+x^{29}+x^{27}+x^{25}+x^{24}\\
&+x^{22}+x^{21}+x^{20}+x^{18}+x^{16}+x^{14}+x^{4}+x^{3}+x^{2}+1,\\
p_9(x)=\,&x^{104}+x^{103}+x^{102}+x^{100}+x^{92}+x^{91}+x^{90}+x^{87}+x^{86}+x^{84}+x^{76}\\
&+x^{75}+x^{74}+x^{71}+x^{70}+x^{68}+x^{60}+x^{59}+x^{58}+x^{56}+x^{40}+x^{39}\\
&+x^{38}+x^{36}+x^{28}+x^{27}+x^{26}+x^{23}+x^{22}+x^{20}+x^{12}+x^{11}+x^{10}\\
&+x^{8},\\
p_{12}(x)=\,&x^{108}+x^{107}+x^{106}+x^{104}+x^{96}+x^{95}+x^{94}+x^{92}+x^{84}+x^{83}+x^{82}\\
&+x^{79}+x^{78}+x^{76}+x^{68}+x^{67}+x^{66}+x^{63}+x^{62}+x^{59}+x^{58}+x^{56}
\end{aligned}
$$

$$+x^{52}+x^{51}+x^{50}+x^{48}+x^{44}+x^{43}+x^{42}+x^{40}+x^{32}+x^{31}+x^{30}$$
$$+x^{28}+x^{20}+x^{19}+x^{18}+x^{15}+x^{14}+x^{11}+x^{10}+x^{8}+x^{4}+x^{3}+x^{2}$$
$$+1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	379	[632, 35]	758	[1102, 1103]	1137	[1350, 1282]
1	[3, 4]	380	[633, 634]	759	[437, 1100]	1138	[254, 310]
2	[5, 6]	381	[549, 188]	760	[1104, 127]	1139	[1021, 63]
3	[7, 8]	382	[522, 635]	761	[1105, 679]	1140	[1351, 1285]
4	[9, 10]	383	[636, 637]	762	[1106, 1107]	1141	[873, 783]
5	[11, 12]	384	[638, 639]	763	[638, 457]	1142	[1352, 1353]
6	[13, 14]	385	[640, 598]	764	[1108, 1063]	1143	[1316, 1303]
7	[15, 16]	386	[641, 642]	765	[534, 1109]	1144	[1354, 352]
8	[17, 18]	387	[643, 41]	766	[455, 1110]	1145	[1355, 16]
9	[19, 20]	388	[644, 645]	767	[1111, 1112]	1146	[387, 842]
10	[21, 22]	389	[479, 646]	768	[1113, 1114]	1147	[1356, 979]
11	[23, 24]	390	[647, 512]	769	[1115, 1116]	1148	[1357, 1358]
12	[25, 26]	391	[648, 649]	770	[1117, 61]	1149	[1359, 1360]
13	[27, 28]	392	[650, 561]	771	[1118, 1114]	1150	[826, 1361]
14	[29, 30]	393	[518, 651]	772	[1088, 1060]	1151	[1362, 724]
15	[31, 32]	394	[652, 653]	773	[485, 635]	1152	[1363, 1298]
16	[33, 34]	395	[654, 541]	774	[608, 553]	1153	[1364, 836]
17	[35, 36]	396	[221, 43]	775	[1056, 193]	1154	[1339, 1365]
18	[37, 38]	397	[655, 527]	776	[224, 696]	1155	[940, 921]
19	[39, 40]	398	[656, 227]	777	[525, 1119]	1156	[1366, 1357]
20	[41, 42]	399	[657, 658]	778	[1074, 1120]	1157	[79, 1367]
21	[43, 44]	400	[659, 513]	779	[1085, 1066]	1158	[971, 1368]
22	[45, 46]	401	[231, 660]	780	[586, 1067]	1159	[1369, 1346]
23	[47, 48]	402	[533, 463]	781	[578, 1121]	1160	[915, 955]
24	[49, 50]	403	[661, 598]	782	[562, 582]	1161	[1370, 86]
25	[51, 51]	404	[662, 663]	783	[1122, 1094]	1162	[1037, 1371]
26	[52, 53]	405	[664, 124]	784	[462, 1123]	1163	[1372, 1373]
27	[54, 55]	406	[46, 665]	785	[1097, 603]	1164	[1373, 1291]
28	[56, 57]	407	[666, 667]	786	[453, 136]	1165	[1374, 838]
29	[58, 59]	408	[668, 669]	787	[150, 138]	1166	[1375, 739]
30	[60, 61]	409	[670, 474]	788	[1124, 1125]	1167	[1376, 297]
31	[62, 63]	410	[597, 665]	789	[1102, 1126]	1168	[1377, 1378]
32	[64, 65]	411	[671, 672]	790	[1127, 470]	1169	[1014, 1379]
33	[66, 67]	412	[523, 516]	791	[126, 1128]	1170	[1306, 1380]
34	[68, 69]	413	[673, 674]	792	[1129, 1130]	1171	[1381, 285]

35	[70, 71]	414	[569, 675]	793	[226, 1131]	1172	[1277, 1324]
36	[72, 73]	415	[458, 676]	794	[1132, 631]	1173	[1382, 1383]
37	[74, 75]	416	[677, 678]	795	[1133, 176]	1174	[840, 294]
38	[76, 77]	417	[679, 680]	796	[1134, 695]	1175	[1287, 397]
39	[78, 79]	418	[681, 682]	797	[1135, 1136]	1176	[1384, 1032]
40	[80, 81]	419	[446, 683]	798	[1137, 660]	1177	[1019, 1355]
41	[82, 83]	420	[684, 685]	799	[1138, 681]	1178	[1385, 1386]
42	[84, 85]	421	[166, 686]	800	[1139, 1061]	1179	[1387, 1388]
43	[86, 87]	422	[593, 38]	801	[1140, 693]	1180	[1389, 716]
44	[63, 88]	423	[687, 688]	802	[1141, 183]	1181	[112, 1369]
45	[89, 90]	424	[132, 689]	803	[662, 629]	1182	[1390, 1020]
46	[91, 92]	425	[690, 441]	804	[175, 1142]	1183	[1391, 1307]
47	[93, 94]	426	[691, 166]	805	[627, 626]	1184	[1392, 935]
48	[95, 96]	427	[481, 692]	806	[627, 505]	1185	[1015, 64]
49	[97, 98]	428	[208, 145]	807	[622, 627]	1186	[1275, 1393]
50	[99, 100]	429	[693, 694]	808	[506, 196]	1187	[1341, 1338]
51	[101, 102]	430	[695, 696]	809	[1143, 1144]	1188	[1394, 379]
52	[103, 104]	431	[516, 697]	810	[1135, 637]	1189	[1318, 704]
53	[105, 106]	432	[228, 698]	811	[531, 649]	1190	[846, 1395]
54	[107, 108]	433	[609, 194]	812	[1145, 522]	1191	[1396, 924]
55	[109, 110]	434	[140, 699]	813	[536, 1146]	1192	[795, 922]
56	[111, 112]	435	[700, 521]	814	[538, 1083]	1193	[779, 1397]
57	[113, 114]	436	[189, 701]	815	[1147, 8]	1194	[1398, 1029]
58	[115, 116]	437	[24, 702]	816	[1086, 1148]	1195	[816, 1399]
59	[117, 118]	438	[434, 703]	817	[1149, 511]	1196	[959, 99]
60	[119, 120]	439	[704, 29]	818	[534, 464]	1197	[990, 1400]
61	[121, 23]	440	[705, 706]	819	[48, 683]	1198	[1401, 303]
62	[7, 122]	441	[384, 707]	820	[168, 1150]	1199	[1388, 900]
63	[123, 34]	442	[708, 709]	821	[1151, 172]	1200	[1402, 1403]
64	[124, 125]	443	[710, 711]	822	[477, 166]	1201	[409, 1036]
65	[126, 127]	444	[712, 713]	823	[1152, 646]	1202	[250, 279]
66	[128, 129]	445	[714, 715]	824	[1153, 1154]	1203	[1294, 1280]
67	[130, 131]	446	[716, 717]	825	[496, 200]	1204	[276, 893]
68	[132, 133]	447	[94, 718]	826	[560, 49]	1205	[1404, 1405]
69	[134, 135]	448	[719, 720]	827	[650, 214]	1206	[1310, 1406]
70	[136, 137]	449	[721, 722]	828	[205, 1155]	1207	[1407, 306]
71	[138, 139]	450	[319, 723]	829	[1156, 1157]	1208	[894, 241]
72	[140, 141]	451	[724, 725]	830	[193, 610]	1209	[1002, 1408]
73	[142, 143]	452	[726, 727]	831	[490, 1158]	1210	[311, 714]
74	[144, 145]	453	[100, 265]	832	[1159, 226]	1211	[1409, 841]
75	[146, 147]	454	[728, 729]	833	[1122, 53]	1212	[413, 929]
76	[148, 149]	455	[730, 731]	834	[601, 1160]	1213	[1410, 1292]
77	[150, 151]	456	[732, 267]	835	[1161, 1162]	1214	[965, 861]
78	[152, 153]	457	[733, 734]	836	[571, 1088]	1215	[1411, 423]

79	[154, 155]	458	[282, 735]	837	[1058, 151]	1216	[1412, 982]
80	[156, 157]	459	[720, 736]	838	[1163, 1164]	1217	[709, 109]
81	[158, 159]	460	[737, 738]	839	[1165, 476]	1218	[889, 344]
82	[160, 161]	461	[739, 740]	840	[173, 611]	1219	[1326, 388]
83	[151, 162]	462	[298, 741]	841	[1166, 546]	1220	[1315, 994]
84	[163, 164]	463	[742, 743]	842	[476, 1167]	1221	[1413, 391]
85	[142, 165]	464	[744, 433]	843	[1070, 1076]	1222	[1403, 886]
86	[166, 167]	465	[430, 745]	844	[1168, 518]	1223	[272, 1414]
87	[168, 169]	466	[746, 747]	845	[450, 1160]	1224	[775, 420]
88	[122, 170]	467	[748, 354]	846	[684, 468]	1225	[412, 1381]
89	[171, 172]	468	[110, 749]	847	[700, 1169]	1226	[1415, 912]
90	[173, 174]	469	[750, 721]	848	[1170, 1171]	1227	[1416, 1268]
91	[175, 176]	470	[751, 752]	849	[1172, 516]	1228	[1269, 1352]
92	[177, 178]	471	[753, 754]	850	[687, 10]	1229	[1417, 962]
93	[179, 180]	472	[755, 756]	851	[1173, 482]	1230	[1405, 1312]
94	[181, 182]	473	[757, 758]	852	[12, 592]	1231	[1418, 1419]
95	[183, 184]	474	[759, 760]	853	[698, 1174]	1232	[944, 1420]
96	[185, 186]	475	[761, 762]	854	[652, 218]	1233	[1421, 1276]
97	[187, 188]	476	[763, 105]	855	[1072, 556]	1234	[738, 905]
98	[189, 190]	477	[266, 764]	856	[1078, 1175]	1235	[1422, 753]
99	[191, 192]	478	[765, 766]	857	[1176, 1177]	1236	[1423, 1278]
100	[193, 194]	479	[767, 326]	858	[1178, 669]	1237	[1424, 814]
101	[195, 196]	480	[415, 768]	859	[1179, 122]	1238	[1425, 1047]
102	[197, 198]	481	[769, 76]	860	[1180, 591]	1239	[910, 1387]
103	[199, 200]	482	[770, 771]	861	[478, 167]	1240	[917, 823]
104	[201, 202]	483	[772, 773]	862	[1104, 1128]	1241	[1426, 796]
105	[203, 204]	484	[774, 775]	863	[201, 700]	1242	[798, 1045]
106	[205, 206]	485	[776, 777]	864	[47, 446]	1243	[968, 1423]
107	[207, 208]	486	[778, 779]	865	[658, 537]	1244	[1427, 1428]
108	[209, 210]	487	[780, 245]	866	[1178, 1181]	1245	[866, 1009]
109	[211, 212]	488	[781, 782]	867	[1182, 220]	1246	[1429, 283]
110	[213, 214]	489	[783, 287]	868	[542, 1070]	1247	[978, 945]
111	[215, 216]	490	[784, 785]	869	[1095, 548]	1248	[869, 980]
112	[217, 218]	491	[786, 787]	870	[202, 1169]	1249	[1430, 852]
113	[219, 220]	492	[788, 235]	871	[551, 1183]	1250	[815, 312]
114	[221, 32]	493	[258, 789]	872	[1184, 192]	1251	[417, 247]
115	[222, 223]	494	[790, 791]	873	[610, 469]	1252	[1383, 1431]
116	[224, 225]	495	[792, 793]	874	[611, 524]	1253	[1360, 1432]
117	[226, 227]	496	[794, 795]	875	[1091, 1158]	1254	[1433, 19]
118	[228, 229]	497	[796, 429]	876	[1185, 1186]	1255	[1434, 1435]
119	[55, 230]	498	[73, 72]	877	[494, 697]	1256	[1320, 1296]
120	[231, 232]	499	[797, 798]	878	[1187, 500]	1257	[1329, 237]
121	[233, 234]	500	[799, 800]	879	[1188, 199]	1258	[1436, 1025]
122	[235, 236]	501	[801, 802]	880	[642, 1189]	1259	[715, 958]

123	[237, 238]	502	[803, 804]	881	[443, 1190]	1260	[1378, 1284]
124	[239, 240]	503	[805, 277]	882	[474, 689]	1261	[407, 1356]
125	[241, 242]	504	[278, 806]	883	[1191, 1192]	1262	[1043, 1413]
126	[243, 244]	505	[807, 101]	884	[37, 594]	1263	[1437, 1438]
127	[245, 246]	506	[369, 808]	885	[123, 1193]	1264	[426, 1370]
128	[247, 248]	507	[809, 790]	886	[1194, 1195]	1265	[427, 1402]
129	[249, 250]	508	[810, 811]	887	[1152, 480]	1266	[1439, 405]
130	[251, 252]	509	[365, 812]	888	[1196, 183]	1267	[1440, 1433]
131	[253, 254]	510	[813, 395]	889	[128, 634]	1268	[1106, 1239]
132	[255, 256]	511	[814, 815]	890	[1080, 485]	1269	[456, 639]
133	[257, 258]	512	[754, 816]	891	[641, 1124]	1270	[1160, 193]
134	[259, 260]	513	[817, 74]	892	[664, 130]	1271	[1202, 1223]
135	[261, 262]	514	[330, 324]	893	[624, 624]	1272	[199, 497]
136	[263, 264]	515	[818, 819]	894	[165, 449]	1273	[1267, 199]
137	[265, 266]	516	[756, 820]	895	[1059, 1197]	1274	[454, 574]
138	[267, 268]	517	[821, 822]	896	[1188, 496]	1275	[32, 1212]
139	[269, 270]	518	[823, 824]	897	[1187, 544]	1276	[1196, 515]
140	[271, 272]	519	[825, 826]	898	[1198, 210]	1277	[1441, 43]
141	[273, 274]	520	[827, 828]	899	[655, 1167]	1278	[130, 125]
142	[275, 276]	521	[829, 830]	900	[8, 1199]	1279	[471, 1150]
143	[277, 278]	522	[20, 831]	901	[1168, 1200]	1280	[1442, 132]
144	[279, 280]	523	[832, 792]	902	[1156, 520]	1281	[673, 1130]
145	[281, 282]	524	[833, 834]	903	[187, 550]	1282	[478, 686]
146	[283, 284]	525	[307, 835]	904	[1170, 607]	1283	[439, 1443]
147	[285, 286]	526	[836, 837]	905	[1201, 543]	1284	[39, 1099]
148	[287, 288]	527	[838, 784]	906	[33, 1193]	1285	[576, 1234]
149	[289, 290]	528	[839, 840]	907	[1202, 1065]	1286	[1165, 655]
150	[291, 269]	529	[353, 827]	908	[647, 659]	1287	[1444, 592]
151	[292, 293]	530	[841, 810]	909	[1200, 651]	1288	[1445, 178]
152	[294, 17]	531	[842, 421]	910	[654, 1203]	1289	[1228, 1144]
153	[295, 296]	532	[843, 844]	911	[657, 536]	1290	[515, 489]
154	[297, 298]	533	[845, 846]	912	[55, 680]	1291	[179, 1085]
155	[299, 300]	534	[847, 848]	913	[648, 532]	1292	[1446, 174]
156	[301, 302]	535	[849, 850]	914	[626, 1204]	1293	[46, 1447]
157	[303, 304]	536	[804, 851]	915	[1204, 198]	1294	[1258, 1177]
158	[244, 305]	537	[852, 755]	916	[1205, 1155]	1295	[1448, 226]
159	[306, 307]	538	[853, 113]	917	[1132, 1206]	1296	[1448, 656]
160	[308, 309]	539	[854, 855]	918	[467, 685]	1297	[1248, 1443]
161	[310, 311]	540	[856, 857]	919	[1207, 693]	1298	[1259, 681]
162	[312, 313]	541	[858, 849]	920	[1208, 124]	1299	[535, 48]
163	[314, 315]	542	[859, 860]	921	[644, 1209]	1300	[483, 1263]
164	[316, 317]	543	[861, 862]	922	[1194, 1154]	1301	[1089, 1183]
165	[318, 319]	544	[863, 864]	923	[1133, 1142]	1302	[1217, 1119]
166	[320, 321]	545	[865, 866]	924	[1210, 224]	1303	[154, 1197]

167	[322, 323]	546	[867, 868]	925	[1124, 1189]	1304	[501, 582]
168	[324, 325]	547	[401, 316]	926	[601, 1056]	1305	[633, 129]
169	[326, 327]	548	[762, 869]	927	[1211, 633]	1306	[41, 1190]
170	[16, 328]	549	[870, 871]	928	[643, 443]	1307	[556, 1101]
171	[329, 330]	550	[872, 873]	929	[43, 1212]	1308	[134, 215]
172	[331, 332]	551	[874, 875]	930	[1213, 135]	1309	[1449, 558]
173	[333, 334]	552	[876, 877]	931	[160, 682]	1310	[1251, 509]
174	[335, 336]	553	[878, 879]	932	[598, 466]	1311	[1198, 1450]
175	[337, 338]	554	[880, 881]	933	[158, 1123]	1312	[1451, 1075]
176	[280, 339]	555	[882, 883]	934	[1214, 570]	1313	[1452, 1164]
177	[340, 341]	556	[67, 884]	935	[498, 549]	1314	[1216, 1098]
178	[248, 342]	557	[885, 398]	936	[1215, 1089]	1315	[1134, 224]
179	[114, 343]	558	[886, 887]	937	[1216, 39]	1316	[1161, 59]
180	[344, 345]	559	[888, 889]	938	[691, 478]	1317	[1453, 1135]
181	[346, 292]	560	[703, 350]	939	[1173, 692]	1318	[156, 1454]
182	[313, 347]	561	[341, 890]	940	[1153, 1195]	1319	[215, 1081]
183	[348, 349]	562	[891, 892]	941	[1217, 526]	1320	[668, 1181]
184	[350, 351]	563	[893, 750]	942	[1218, 182]	1321	[1084, 180]
185	[352, 353]	564	[288, 894]	943	[1219, 1220]	1322	[1114, 1220]
186	[354, 355]	565	[895, 896]	944	[619, 534]	1323	[610, 142]
187	[356, 357]	566	[102, 376]	945	[1137, 232]	1324	[1236, 555]
188	[358, 359]	567	[75, 897]	946	[1221, 1087]	1325	[1229, 1209]
189	[360, 361]	568	[898, 400]	947	[122, 1199]	1326	[630, 1206]
190	[362, 363]	569	[899, 730]	948	[1077, 1222]	1327	[1455, 1127]
191	[364, 365]	570	[900, 901]	949	[1160, 609]	1328	[1207, 1451]
192	[332, 366]	571	[834, 902]	950	[1064, 1223]	1329	[1456, 443]
193	[367, 275]	572	[903, 904]	951	[1092, 1186]	1330	[636, 1136]
194	[368, 369]	573	[905, 906]	952	[1073, 557]	1331	[1457, 642]
195	[370, 371]	574	[881, 907]	953	[621, 559]	1332	[632, 1127]
196	[372, 373]	575	[908, 898]	954	[505, 1204]	1333	[1458, 1089]
197	[374, 375]	576	[909, 910]	955	[1204, 503]	1334	[1213, 215]
198	[376, 370]	577	[877, 911]	956	[463, 1109]	1335	[1145, 485]
199	[377, 378]	578	[912, 913]	957	[138, 162]	1336	[1114, 1459]
200	[379, 380]	579	[914, 915]	958	[600, 699]	1337	[510, 1460]
201	[85, 253]	580	[897, 916]	959	[566, 51]	1338	[1079, 577]
202	[381, 382]	581	[729, 757]	960	[547, 1096]	1339	[1247, 459]
203	[383, 384]	582	[378, 917]	961	[1214, 675]	1340	[1452, 1461]
204	[104, 385]	583	[802, 732]	962	[1224, 444]	1341	[679, 230]
205	[386, 387]	584	[359, 918]	963	[1225, 1170]	1342	[681, 161]
206	[388, 389]	585	[919, 920]	964	[45, 597]	1343	[574, 1110]
207	[390, 281]	586	[921, 91]	965	[1143, 1226]	1344	[1462, 1170]
208	[391, 392]	587	[922, 923]	966	[519, 1157]	1345	[1463, 1265]
209	[393, 394]	588	[924, 925]	967	[1227, 1228]	1346	[1180, 473]
210	[395, 396]	589	[926, 774]	968	[1208, 130]	1347	[206, 1464]

211	[397, 95]	590	[927, 928]	969	[1229, 645]	1348	[1240, 1465]
212	[398, 399]	591	[929, 930]	970	[1155, 1230]	1349	[209, 1450]
213	[400, 401]	592	[931, 932]	971	[584, 1088]	1350	[172, 223]
214	[402, 403]	593	[933, 934]	972	[1231, 1115]	1351	[1466, 158]
215	[92, 404]	594	[935, 936]	973	[1111, 588]	1352	[1117, 1232]
216	[405, 115]	595	[28, 119]	974	[148, 444]	1353	[1467, 587]
217	[406, 407]	596	[937, 938]	975	[60, 1232]	1354	[1163, 1461]
218	[408, 409]	597	[939, 940]	976	[1233, 617]	1355	[1172, 494]
219	[410, 411]	598	[238, 941]	977	[56, 1234]	1356	[670, 132]
220	[394, 412]	599	[942, 843]	978	[529, 1235]	1357	[1068, 700]
221	[396, 413]	600	[943, 944]	979	[1236, 615]	1358	[1468, 189]
222	[414, 415]	601	[375, 367]	980	[507, 452]	1359	[1192, 1469]
223	[416, 417]	602	[945, 946]	981	[1221, 1148]	1360	[217, 653]
224	[418, 419]	603	[947, 948]	982	[1237, 1107]	1361	[1225, 606]
225	[420, 25]	604	[949, 358]	983	[172, 514]	1362	[1470, 529]
226	[421, 422]	605	[764, 333]	984	[1127, 36]	1363	[206, 1230]
227	[423, 424]	606	[116, 950]	985	[517, 1200]	1364	[1115, 1465]
228	[425, 426]	607	[951, 952]	986	[1238, 462]	1365	[585, 617]
229	[262, 427]	608	[953, 863]	987	[1237, 1239]	1366	[1219, 1459]
230	[108, 428]	609	[954, 372]	988	[203, 1121]	1367	[1449, 621]
231	[429, 430]	610	[955, 318]	989	[1240, 1116]	1368	[1215, 551]
232	[431, 432]	611	[956, 957]	990	[589, 234]	1369	[1159, 656]
233	[433, 434]	612	[958, 959]	991	[1118, 1219]	1370	[1451, 694]
234	[435, 436]	613	[960, 961]	992	[1231, 1240]	1371	[1249, 536]
235	[437, 438]	614	[962, 963]	993	[207, 144]	1372	[1265, 1469]
236	[439, 440]	615	[363, 964]	994	[1228, 1226]	1373	[540, 1203]
237	[441, 442]	616	[4, 295]	995	[656, 1131]	1374	[1466, 462]
238	[443, 42]	617	[824, 794]	996	[1241, 1222]	1375	[1265, 1471]
239	[444, 445]	618	[938, 965]	997	[1242, 1235]	1376	[1472, 157]
240	[446, 447]	619	[830, 966]	998	[698, 1243]	1377	[1473, 39]
241	[448, 164]	620	[967, 968]	999	[1244, 1245]	1378	[1446, 611]
242	[449, 450]	621	[969, 969]	1000	[1126, 1246]	1379	[194, 469]
243	[144, 451]	622	[806, 368]	1001	[58, 1162]	1380	[185, 614]
244	[211, 452]	623	[623, 623]	1002	[1247, 676]	1381	[1218, 1474]
245	[453, 49]	624	[723, 954]	1003	[124, 131]	1382	[502, 1475]
246	[454, 455]	625	[371, 374]	1004	[1248, 440]	1383	[233, 483]
247	[456, 457]	626	[373, 805]	1005	[1151, 222]	1384	[1222, 1175]
248	[458, 459]	627	[722, 914]	1006	[1210, 695]	1385	[1126, 1257]
249	[460, 461]	628	[970, 971]	1007	[628, 663]	1386	[671, 1245]
250	[462, 159]	629	[30, 972]	1008	[1249, 658]	1387	[1179, 8]
251	[463, 464]	630	[973, 974]	1009	[1250, 556]	1388	[1129, 674]
252	[465, 466]	631	[975, 976]	1010	[587, 1112]	1389	[642, 1125]
253	[467, 468]	632	[351, 872]	1011	[1138, 160]	1390	[229, 1243]
254	[469, 143]	633	[835, 781]	1012	[1251, 1252]	1391	[181, 1474]

255	[35, 470]	634	[977, 978]	1013	[1113, 1219]	1392	[558, 1264]
256	[471, 169]	635	[979, 726]	1014	[1108, 1090]	1393	[1476, 1126]
257	[472, 473]	636	[980, 981]	1015	[14, 442]	1394	[1477, 633]
258	[474, 133]	637	[982, 383]	1016	[229, 1174]	1395	[579, 601]
259	[475, 476]	638	[983, 984]	1017	[1176, 1253]	1396	[475, 655]
260	[477, 478]	639	[793, 985]	1018	[1078, 1254]	1397	[1476, 1103]
261	[479, 480]	640	[868, 942]	1019	[1255, 667]	1398	[502, 583]
262	[481, 482]	641	[986, 987]	1020	[1255, 1256]	1399	[1467, 1111]
263	[483, 484]	642	[961, 988]	1021	[1250, 1073]	1400	[1468, 1061]
264	[485, 486]	643	[323, 744]	1022	[1147, 122]	1401	[135, 216]
265	[487, 488]	644	[989, 990]	1023	[1103, 1257]	1402	[1445, 1478]
266	[194, 142]	645	[741, 991]	1024	[666, 1256]	1403	[177, 1478]
267	[183, 489]	646	[946, 992]	1025	[1258, 1253]	1404	[1140, 1451]
268	[490, 491]	647	[993, 817]	1026	[14, 595]	1405	[1455, 35]
269	[492, 493]	648	[994, 776]	1027	[596, 46]	1406	[590, 1111]
270	[494, 495]	649	[995, 996]	1028	[1241, 1078]	1407	[1105, 55]
271	[496, 497]	650	[997, 331]	1029	[1259, 160]	1408	[1266, 1115]
272	[498, 187]	651	[892, 899]	1030	[661, 465]	1409	[1479, 567]
273	[499, 500]	652	[998, 999]	1031	[518, 1260]	1410	[13, 441]
274	[501, 502]	653	[857, 291]	1032	[219, 1261]	1411	[1201, 1070]
275	[197, 503]	654	[1000, 867]	1033	[512, 1262]	1412	[1479, 602]
276	[504, 195]	655	[1001, 1002]	1034	[234, 1263]	1413	[659, 1262]
277	[505, 197]	656	[1003, 1004]	1035	[606, 1171]	1414	[1462, 606]
278	[504, 506]	657	[1005, 1006]	1036	[1185, 1093]	1415	[1472, 1454]
279	[507, 212]	658	[851, 1007]	1037	[543, 1071]	1416	[1470, 1242]
280	[508, 509]	659	[120, 273]	1038	[621, 1264]	1417	[582, 1475]
281	[510, 511]	660	[88, 1008]	1039	[1205, 206]	1418	[1480, 1228]
282	[512, 513]	661	[1009, 4]	1040	[1191, 1265]	1419	[1141, 515]
283	[222, 514]	662	[1010, 1011]	1041	[9, 688]	1420	[1139, 189]
284	[515, 184]	663	[1012, 1013]	1042	[1238, 158]	1421	[1457, 1124]
285	[516, 495]	664	[242, 1014]	1043	[1166, 1069]	1422	[1149, 1460]
286	[517, 518]	665	[920, 1015]	1044	[51, 566]	1423	[1098, 40]
287	[469, 165]	666	[1016, 1017]	1045	[508, 1252]	1424	[1200, 1260]
288	[519, 520]	667	[1017, 728]	1046	[1266, 1240]	1425	[1481, 144]
289	[202, 521]	668	[1018, 1019]	1047	[677, 461]	1426	[690, 14]
290	[522, 486]	669	[1020, 1021]	1048	[1242, 530]	1427	[1222, 1254]
291	[523, 494]	670	[1022, 951]	1049	[136, 50]	1428	[1244, 672]
292	[174, 524]	671	[1023, 1024]	1050	[465, 599]	1429	[1444, 437]
293	[525, 526]	672	[1025, 832]	1051	[163, 565]	1430	[1473, 1098]
294	[476, 527]	673	[1026, 759]	1052	[1267, 496]	1431	[640, 465]
295	[8, 170]	674	[424, 1027]	1053	[234, 484]	1432	[1442, 474]
296	[528, 229]	675	[270, 1028]	1054	[1061, 701]	1433	[1456, 41]
297	[529, 530]	676	[1029, 1030]	1055	[492, 1120]	1434	[597, 1447]
298	[531, 532]	677	[1031, 858]	1056	[808, 807]	1435	[1182, 1261]

299	[533, 534]	678	[1032, 859]	1057	[1268, 243]	1436	[1103, 1246]
300	[535, 446]	679	[811, 1012]	1058	[740, 885]	1437	[1184, 580]
301	[536, 537]	680	[987, 1033]	1059	[1033, 1269]	1438	[165, 579]
302	[538, 539]	681	[1034, 1035]	1060	[902, 1270]	1439	[1211, 128]
303	[540, 541]	682	[334, 949]	1061	[284, 1271]	1440	[1227, 1143]
304	[542, 543]	683	[1036, 1037]	1062	[1272, 402]	1441	[1295, 362]
305	[499, 544]	684	[1038, 299]	1063	[392, 1273]	1442	[1432, 257]
306	[545, 546]	685	[707, 1039]	1064	[725, 1001]	1443	[1482, 1483]
307	[547, 548]	686	[918, 926]	1065	[974, 1274]	1444	[1379, 1437]
308	[549, 550]	687	[925, 1003]	1066	[343, 1275]	1445	[1484, 1482]
309	[551, 552]	688	[952, 1040]	1067	[1276, 788]	1446	[1395, 833]
310	[487, 553]	689	[1041, 117]	1068	[1270, 829]	1447	[90, 983]
311	[143, 449]	690	[1042, 1043]	1069	[302, 1277]	1448	[1485, 876]
312	[554, 555]	691	[1044, 322]	1070	[1278, 1279]	1449	[1486, 888]
313	[556, 557]	692	[1045, 1046]	1071	[850, 21]	1450	[1386, 1411]
314	[558, 559]	693	[735, 975]	1072	[1280, 1281]	1451	[934, 895]
315	[560, 136]	694	[1047, 1048]	1073	[1282, 1283]	1452	[1487, 854]
316	[213, 561]	695	[1049, 1050]	1074	[1284, 874]	1453	[1488, 80]
317	[562, 502]	696	[1051, 845]	1075	[1285, 933]	1454	[999, 1485]
318	[198, 563]	697	[906, 882]	1076	[860, 947]	1455	[1438, 751]
319	[503, 563]	698	[923, 803]	1077	[1286, 1287]	1456	[1323, 84]
320	[149, 564]	699	[988, 1052]	1078	[389, 1288]	1457	[1489, 1340]
321	[151, 139]	700	[1053, 1054]	1079	[1289, 1290]	1458	[855, 1391]
322	[448, 565]	701	[1055, 733]	1080	[1291, 778]	1459	[1332, 1389]
323	[566, 566]	702	[48, 447]	1081	[1292, 346]	1460	[1435, 93]
324	[567, 568]	703	[450, 1056]	1082	[1293, 1294]	1461	[1428, 1022]
325	[569, 570]	704	[567, 1057]	1083	[1024, 1295]	1462	[1321, 1041]
326	[571, 572]	705	[604, 549]	1084	[1296, 746]	1463	[6, 1490]
327	[573, 574]	706	[1058, 138]	1085	[984, 1297]	1464	[1006, 66]
328	[32, 44]	707	[1059, 155]	1086	[1048, 763]	1465	[1419, 1289]
329	[575, 567]	708	[460, 678]	1087	[1298, 1299]	1466	[1491, 1337]
330	[576, 57]	709	[602, 1057]	1088	[1300, 1301]	1467	[1492, 1327]
331	[152, 577]	710	[49, 137]	1089	[1290, 1302]	1468	[304, 1055]
332	[578, 204]	711	[455, 581]	1090	[1303, 1304]	1469	[928, 1350]
333	[143, 579]	712	[572, 1060]	1091	[428, 414]	1470	[1493, 249]
334	[191, 580]	713	[1061, 190]	1092	[1305, 1306]	1471	[1490, 1026]
335	[444, 564]	714	[1062, 1063]	1093	[1307, 1308]	1472	[1494, 408]
336	[574, 581]	715	[1056, 609]	1094	[305, 1309]	1473	[1495, 1313]
337	[582, 583]	716	[174, 612]	1095	[785, 1310]	1474	[328, 1496]
338	[584, 572]	717	[1064, 1065]	1096	[1311, 797]	1475	[822, 1305]
339	[585, 586]	718	[180, 1066]	1097	[1312, 255]	1476	[1497, 1429]
340	[587, 588]	719	[617, 1067]	1098	[325, 770]	1477	[1498, 977]
341	[589, 483]	720	[1068, 202]	1099	[1313, 997]	1478	[981, 1499]
342	[590, 587]	721	[503, 504]	1100	[26, 1044]	1479	[1500, 1415]

343	[472, 591]	722	[622, 625]	1101	[1281, 68]	1480	[1501, 1354]
344	[592, 438]	723	[563, 195]	1102	[1314, 1315]	1481	[1502, 416]
345	[593, 594]	724	[545, 1069]	1103	[1288, 939]	1482	[1224, 149]
346	[441, 595]	725	[208, 451]	1104	[81, 1316]	1483	[1458, 551]
347	[596, 597]	726	[1070, 1071]	1105	[1317, 1318]	1484	[1155, 1464]
348	[149, 445]	727	[528, 698]	1106	[1319, 1320]	1485	[1441, 32]
349	[598, 599]	728	[1072, 1073]	1107	[411, 1321]	1486	[1477, 128]
350	[600, 141]	729	[1074, 493]	1108	[1322, 705]	1487	[658, 1146]
351	[449, 601]	730	[695, 225]	1109	[966, 1323]	1488	[1503, 659]
352	[602, 568]	731	[613, 186]	1110	[1324, 431]	1489	[54, 679]
353	[146, 603]	732	[693, 1075]	1111	[1325, 1326]	1490	[12, 437]
354	[604, 187]	733	[543, 1076]	1112	[1327, 809]	1491	[1504, 636]
355	[573, 455]	734	[1077, 1078]	1113	[1328, 1329]	1492	[1480, 1143]
356	[572, 605]	735	[1079, 153]	1114	[404, 261]	1493	[1503, 512]
357	[606, 607]	736	[1080, 522]	1115	[1007, 1325]	1494	[1192, 1471]
358	[608, 488]	737	[135, 1081]	1116	[260, 880]	1495	[1505, 1242]
359	[609, 610]	738	[1082, 539]	1117	[1330, 78]	1496	[1463, 1192]
360	[611, 612]	739	[1082, 1083]	1118	[1331, 1332]	1497	[171, 222]
361	[613, 614]	740	[1084, 1085]	1119	[315, 1333]	1498	[1505, 529]
362	[554, 615]	741	[1086, 1087]	1120	[718, 1334]	1499	[1233, 586]
363	[180, 616]	742	[1088, 605]	1121	[1011, 1335]	1500	[1506, 547]
364	[617, 618]	743	[1089, 552]	1122	[1336, 786]	1501	[575, 602]
365	[619, 463]	744	[1062, 1090]	1123	[1337, 1330]	1502	[1506, 1095]
366	[620, 621]	745	[1091, 491]	1124	[1338, 1339]	1503	[1507, 708]
367	[198, 504]	746	[1092, 1093]	1125	[1340, 1341]	1504	[1508, 1349]
368	[506, 622]	747	[1085, 616]	1126	[887, 377]	1505	[1509, 1422]
369	[563, 506]	748	[586, 618]	1127	[1342, 1343]	1506	[1510, 1376]
370	[623, 624]	749	[620, 558]	1128	[338, 1344]	1507	[1453, 636]
371	[625, 626]	750	[195, 622]	1129	[380, 360]	1508	[1511, 547]
372	[196, 627]	751	[52, 1094]	1130	[747, 1345]	1509	[1511, 1095]
373	[625, 505]	752	[579, 450]	1131	[1346, 995]	1510	[1481, 208]
374	[624, 623]	753	[1095, 1096]	1132	[1347, 121]	1511	[1512, 1311]
375	[626, 197]	754	[1097, 147]	1133	[1348, 799]	1512	[1504, 1135]
376	[196, 625]	755	[1098, 1099]	1134	[752, 1051]		
377	[628, 629]	756	[592, 1100]	1135	[913, 878]		
378	[630, 631]	757	[1073, 1101]	1136	[1349, 960]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	253	0	506	1	759	0	1012	0	1265	1
1	0	254	0	507	0	760	1	1013	1	1266	1
2	0	255	1	508	1	761	1	1014	0	1267	1
3	0	256	0	509	0	762	0	1015	1	1268	0
4	0	257	1	510	1	763	0	1016	1	1269	1

5	0	258	0	511	0	764	0	1017	0	1270	0
6	0	259	1	512	0	765	0	1018	1	1271	0
7	0	260	1	513	0	766	1	1019	0	1272	1
8	1	261	1	514	1	767	1	1020	0	1273	1
9	0	262	1	515	0	768	1	1021	1	1274	0
10	1	263	1	516	1	769	1	1022	1	1275	0
11	0	264	0	517	1	770	0	1023	1	1276	1
12	0	265	0	518	0	771	0	1024	1	1277	1
13	0	266	1	519	0	772	1	1025	1	1278	1
14	1	267	1	520	0	773	0	1026	1	1279	0
15	0	268	0	521	1	774	1	1027	1	1280	0
16	1	269	1	522	1	775	1	1028	1	1281	1
17	1	270	0	523	0	776	0	1029	1	1282	0
18	0	271	0	524	0	777	1	1030	1	1283	1
19	0	272	1	525	1	778	0	1031	0	1284	0
20	1	273	1	526	0	779	0	1032	0	1285	1
21	1	274	0	527	1	780	0	1033	0	1286	0
22	0	275	1	528	0	781	1	1034	0	1287	1
23	0	276	0	529	1	782	1	1035	0	1288	1
24	0	277	1	530	1	783	0	1036	1	1289	1
25	0	278	0	531	0	784	0	1037	0	1290	0
26	1	279	0	532	1	785	0	1038	1	1291	0
27	0	280	1	533	1	786	0	1039	0	1292	0
28	0	281	1	534	0	787	0	1040	1	1293	1
29	1	282	0	535	1	788	1	1041	0	1294	1
30	1	283	1	536	1	789	1	1042	1	1295	1
31	0	284	0	537	0	790	0	1043	1	1296	1
32	0	285	1	538	0	791	0	1044	0	1297	0
33	1	286	1	539	0	792	0	1045	1	1298	1
34	1	287	0	540	1	793	1	1046	1	1299	1
35	1	288	0	541	0	794	0	1047	0	1300	1
36	0	289	1	542	0	795	0	1048	0	1301	0
37	0	290	1	543	0	796	0	1049	1	1302	0
38	0	291	0	544	1	797	1	1050	1	1303	1
39	0	292	0	545	0	798	0	1051	0	1304	0
40	1	293	1	546	1	799	0	1052	1	1305	0
41	1	294	0	547	1	800	1	1053	0	1306	1
42	0	295	1	548	0	801	0	1054	0	1307	1
43	1	296	0	549	1	802	0	1055	1	1308	1
44	0	297	1	550	0	803	0	1056	1	1309	0
45	0	298	0	551	1	804	1	1057	0	1310	0
46	1	299	1	552	1	805	1	1058	1	1311	1
47	0	300	1	553	0	806	1	1059	0	1312	0
48	1	301	1	554	0	807	1	1060	1	1313	0

49	0	302	0	555	0	808	1	1061	0	1314	1
50	1	303	1	556	1	809	0	1062	1	1315	0
51	0	304	0	557	0	810	1	1063	0	1316	0
52	1	305	1	558	0	811	0	1064	1	1317	1
53	0	306	0	559	1	812	1	1065	0	1318	1
54	0	307	1	560	1	813	1	1066	1	1319	0
55	1	308	1	561	0	814	0	1067	1	1320	1
56	0	309	1	562	1	815	1	1068	0	1321	1
57	0	310	0	563	1	816	0	1069	0	1322	0
58	1	311	1	564	0	817	0	1070	1	1323	0
59	1	312	0	565	0	818	0	1071	1	1324	1
60	1	313	1	566	1	819	1	1072	0	1325	1
61	1	314	0	567	1	820	1	1073	0	1326	1
62	0	315	1	568	1	821	1	1074	0	1327	0
63	0	316	1	569	1	822	1	1075	1	1328	1
64	0	317	1	570	1	823	0	1076	0	1329	0
65	0	318	0	571	0	824	1	1077	0	1330	0
66	1	319	1	572	0	825	0	1078	1	1331	0
67	1	320	1	573	1	826	1	1079	1	1332	1
68	1	321	0	574	0	827	0	1080	0	1333	0
69	1	322	1	575	0	828	1	1081	0	1334	0
70	1	323	1	576	1	829	1	1082	1	1335	1
71	1	324	1	577	0	830	0	1083	1	1336	0
72	0	325	1	578	1	831	0	1084	1	1337	1
73	1	326	0	579	0	832	0	1085	0	1338	1
74	0	327	1	580	0	833	0	1086	0	1339	1
75	1	328	0	581	0	834	0	1087	1	1340	0
76	0	329	0	582	1	835	0	1088	1	1341	0
77	0	330	1	583	0	836	0	1089	0	1342	0
78	0	331	0	584	1	837	1	1090	1	1343	0
79	1	332	1	585	1	838	1	1091	1	1344	1
80	1	333	1	586	0	839	0	1092	0	1345	0
81	1	334	1	587	0	840	1	1093	1	1346	0
82	1	335	0	588	1	841	1	1094	1	1347	0
83	0	336	0	589	0	842	0	1095	0	1348	0
84	0	337	1	590	1	843	1	1096	1	1349	0
85	1	338	1	591	1	844	0	1097	0	1350	0
86	1	339	1	592	1	845	1	1098	1	1351	0
87	1	340	0	593	1	846	1	1099	0	1352	0
88	0	341	0	594	1	847	0	1100	1	1353	0
89	0	342	1	595	0	848	1	1101	1	1354	1
90	1	343	1	596	1	849	1	1102	1	1355	1
91	1	344	1	597	0	850	1	1103	1	1356	1
92	0	345	1	598	0	851	0	1104	1	1357	0

93	0	346	0	599	1	852	0	1105	1	1358	0
94	0	347	1	600	1	853	0	1106	0	1359	0
95	1	348	1	601	0	854	0	1107	1	1360	1
96	0	349	0	602	0	855	0	1108	0	1361	0
97	0	350	1	603	0	856	1	1109	0	1362	1
98	1	351	1	604	0	857	0	1110	1	1363	0
99	1	352	0	605	0	858	0	1111	1	1364	1
100	0	353	1	606	0	859	0	1112	0	1365	1
101	0	354	0	607	0	860	0	1113	1	1366	1
102	1	355	1	608	1	861	0	1114	0	1367	0
103	1	356	0	609	1	862	1	1115	1	1368	1
104	1	357	0	610	0	863	1	1116	1	1369	0
105	0	358	1	611	1	864	0	1117	0	1370	0
106	1	359	1	612	1	865	0	1118	0	1371	0
107	0	360	1	613	1	866	0	1119	1	1372	1
108	0	361	1	614	1	867	1	1120	1	1373	1
109	1	362	0	615	1	868	0	1121	0	1374	0
110	1	363	1	616	0	869	0	1122	0	1375	1
111	0	364	1	617	1	870	1	1123	1	1376	0
112	1	365	0	618	0	871	1	1124	1	1377	0
113	0	366	1	619	0	872	0	1125	0	1378	0
114	0	367	0	620	1	873	0	1126	0	1379	1
115	1	368	1	621	1	874	1	1127	0	1380	0
116	0	369	1	622	1	875	1	1128	1	1381	1
117	1	370	0	623	0	876	1	1129	0	1382	0
118	1	371	0	624	1	877	0	1130	0	1383	1
119	1	372	0	625	0	878	0	1131	0	1384	0
120	1	373	0	626	0	879	0	1132	0	1385	0
121	1	374	1	627	1	880	0	1133	0	1386	1
122	0	375	0	628	1	881	0	1134	0	1387	0
123	0	376	0	629	1	882	0	1135	1	1388	0
124	0	377	1	630	1	883	1	1136	0	1389	0
125	1	378	1	631	1	884	0	1137	0	1390	1
126	0	379	1	632	1	885	0	1138	0	1391	0
127	0	380	0	633	0	886	0	1139	1	1392	0
128	1	381	1	634	0	887	0	1140	0	1393	0
129	1	382	1	635	1	888	1	1141	0	1394	0
130	1	383	0	636	0	889	1	1142	0	1395	0
131	0	384	0	637	1	890	0	1143	0	1396	1
132	1	385	0	638	0	891	1	1144	1	1397	0
133	1	386	1	639	1	892	1	1145	1	1398	0
134	1	387	1	640	0	893	1	1146	1	1399	0
135	1	388	0	641	1	894	0	1147	1	1400	0
136	1	389	1	642	0	895	0	1148	0	1401	1

137	0	390	0	643	1	896	0	1149	0	1402	1
138	1	391	1	644	0	897	0	1150	1	1403	0
139	1	392	0	645	0	898	1	1151	1	1404	0
140	0	393	0	646	1	899	1	1152	0	1405	0
141	1	394	0	647	0	900	1	1153	1	1406	1
142	1	395	0	648	1	901	0	1154	1	1407	1
143	1	396	0	649	0	902	1	1155	1	1408	1
144	0	397	1	650	0	903	0	1156	1	1409	1
145	1	398	0	651	1	904	1	1157	1	1410	0
146	1	399	1	652	0	905	1	1158	1	1411	1
147	1	400	1	653	0	906	1	1159	0	1412	1
148	0	401	1	654	0	907	0	1160	0	1413	1
149	1	402	1	655	1	908	0	1161	0	1414	1
150	0	403	1	656	0	909	1	1162	0	1415	0
151	0	404	0	657	1	910	0	1163	1	1416	1
152	0	405	1	658	0	911	1	1164	1	1417	1
153	1	406	1	659	1	912	1	1165	0	1418	0
154	1	407	1	660	0	913	1	1166	1	1419	0
155	1	408	1	661	1	914	0	1167	0	1420	1
156	1	409	1	662	0	915	0	1168	0	1421	0
157	1	410	0	663	0	916	0	1169	0	1422	0
158	1	411	1	664	1	917	0	1170	1	1423	1
159	0	412	0	665	0	918	0	1171	1	1424	1
160	1	413	1	666	1	919	1	1172	1	1425	1
161	0	414	1	667	0	920	0	1173	0	1426	1
162	0	415	0	668	1	921	0	1174	1	1427	0
163	0	416	0	669	0	922	0	1175	1	1428	0
164	1	417	0	670	1	923	0	1176	0	1429	1
165	0	418	0	671	1	924	1	1177	0	1430	0
166	1	419	0	672	1	925	1	1178	0	1431	0
167	1	420	1	673	1	926	0	1179	0	1432	0
168	1	421	1	674	1	927	1	1180	0	1433	0
169	0	422	1	675	0	928	1	1181	1	1434	0
170	1	423	1	676	1	929	1	1182	1	1435	1
171	0	424	1	677	0	930	0	1183	0	1436	1
172	0	425	1	678	0	931	1	1184	0	1437	0
173	1	426	0	679	0	932	0	1185	1	1438	0
174	0	427	1	680	1	933	1	1186	0	1439	1
175	1	428	1	681	0	934	0	1187	0	1440	1
176	1	429	1	682	1	935	1	1188	0	1441	1
177	0	430	1	683	1	936	1	1189	1	1442	0
178	0	431	1	684	1	937	1	1190	1	1443	1
179	0	432	1	685	0	938	0	1191	1	1444	1
180	1	433	1	686	0	939	0	1192	0	1445	1

181	0	434	0	687	1	940	1	1193	0	1446	0
182	1	435	0	688	0	941	0	1194	0	1447	1
183	1	436	1	689	0	942	1	1195	0	1448	1
184	1	437	0	690	1	943	1	1196	1	1449	0
185	0	438	0	691	0	944	0	1197	0	1450	1
186	0	439	1	692	1	945	0	1198	1	1451	0
187	0	440	0	693	1	946	1	1199	0	1452	0
188	1	441	0	694	0	947	0	1200	1	1453	1
189	1	442	1	695	1	948	0	1201	1	1454	0
190	0	443	0	696	0	949	0	1202	0	1455	0
191	1	444	0	697	1	950	1	1203	1	1456	0
192	1	445	1	698	0	951	0	1204	0	1457	0
193	0	446	0	699	0	952	0	1205	0	1458	0
194	1	447	0	700	0	953	1	1206	0	1459	1
195	0	448	1	701	1	954	1	1207	1	1460	1
196	0	449	1	702	1	955	0	1208	0	1461	0
197	1	450	1	703	1	956	1	1209	1	1462	1
198	0	451	0	704	1	957	1	1210	1	1463	0
199	1	452	1	705	0	958	1	1211	1	1464	1
200	1	453	0	706	1	959	1	1212	1	1465	0
201	1	454	0	707	0	960	1	1213	0	1466	0
202	1	455	1	708	1	961	0	1214	0	1467	0
203	0	456	1	709	0	962	1	1215	1	1468	0
204	1	457	0	710	0	963	0	1216	1	1469	1
205	1	458	0	711	1	964	0	1217	0	1470	1
206	0	459	0	712	0	965	0	1218	1	1471	0
207	0	460	1	713	0	966	0	1219	1	1472	0
208	1	461	1	714	1	967	1	1220	0	1473	0
209	0	462	0	715	1	968	0	1221	1	1474	0
210	0	463	1	716	0	969	1	1222	0	1475	1
211	1	464	1	717	1	970	1	1223	1	1476	0
212	0	465	1	718	1	971	1	1224	1	1477	0
213	1	466	0	719	1	972	0	1225	0	1478	1
214	1	467	0	720	0	973	1	1226	0	1479	1
215	0	468	1	721	1	974	0	1227	1	1480	0
216	1	469	0	722	1	975	1	1228	1	1481	1
217	1	470	1	723	1	976	0	1229	1	1482	1
218	1	471	0	724	0	977	0	1230	0	1483	0
219	0	472	1	725	1	978	1	1231	0	1484	1
220	0	473	0	726	1	979	1	1232	0	1485	1
221	0	474	0	727	0	980	0	1233	0	1486	0
222	1	475	1	728	0	981	1	1234	1	1487	0
223	0	476	0	729	0	982	1	1235	0	1488	1
224	0	477	1	730	1	983	0	1236	1	1489	0

225	1	478	0	731	1	984	0	1237	1	1490	0
226	1	479	1	732	1	985	1	1238	1	1491	0
227	1	480	0	733	0	986	1	1239	0	1492	0
228	1	481	1	734	0	987	1	1240	0	1493	1
229	1	482	0	735	1	988	0	1241	1	1494	0
230	0	483	1	736	0	989	0	1242	0	1495	0
231	1	484	1	737	1	990	0	1243	0	1496	0
232	1	485	0	738	1	991	0	1244	0	1497	0
233	1	486	0	739	1	992	0	1245	0	1498	0
234	0	487	0	740	1	993	0	1246	1	1499	0
235	0	488	1	741	0	994	1	1247	1	1500	1
236	1	489	0	742	1	995	0	1248	0	1501	0
237	0	490	0	743	0	996	1	1249	0	1502	1
238	0	491	0	744	1	997	0	1250	1	1503	1
239	0	492	1	745	1	998	0	1251	0	1504	0
240	0	493	0	746	0	999	0	1252	1	1505	0
241	1	494	0	747	0	1000	0	1253	1	1506	1
242	1	495	0	748	0	1001	1	1254	0	1507	1
243	0	496	0	749	1	1002	1	1255	0	1508	0
244	1	497	0	750	0	1003	0	1256	1	1509	0
245	0	498	1	751	1	1004	0	1257	0	1510	1
246	0	499	1	752	0	1005	1	1258	1	1511	0
247	1	500	0	753	0	1006	1	1259	1	1512	0
248	0	501	0	754	0	1007	1	1260	0		
249	1	502	0	755	1	1008	0	1261	1		
250	0	503	1	756	1	1009	1	1262	1		
251	1	504	0	757	0	1010	0	1263	0		
252	1	505	1	758	1	1011	0	1264	0		

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