# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^5+z^4+z^2+1,z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a,b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a,b)$ in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^5+z^4+z^2+1,z+1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a,b)$-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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**Theorem 1.1.** *When* $(a,b)=(z^5+z^4+z^2+1,z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{12}+z^{10}+z^9+z^7+z^5+z^4+z^2+z+1,\\
p_1(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{10}+z^4+z^3+z,\\
p_2(z)&=z^{18}+z^8+z^6+1,\\
p_3(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{10}+z^4+z^3+z,\\
p_4(z)&=z^{16}+z^{14}+z^9+z^7+z^6+z^5+z^4+z+1,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{14}+z^9+z^8+z^7+z^6+z^5+z^2+z+1,\\
p_1(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{10}+z^4+z^3+z,\\
p_2(z)&=z^{18}+z^8+z^6+1,\\
p_3(z)&=z^{17}+z^{16}+z^{15}+z^{14}+z^{10}+z^4+z^3+z,\\
p_4(z)&=z^{16}+z^{12}+z^{10}+z^9+z^8+z^7+z^5+z+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)=x^{\deg(t_{2^n-1})}&\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)=x^{\deg(\bar t_{2^n-1})}&\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2\times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 540 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity $\mu$, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 660 and substitute it for $y$ in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] &= M^e[:6u] + x^{3u}M^e[:3u] + x^{3u}M^e[:6u] + x^{6u}M^e[:3u] + x^{6u}M^e[:6u]\\ &\quad + x^{7u}M^e[:5u] + x^{11u}M^e[:u] + (x^{6u}+x^{9u})R^e,\\ W^e[:12u] &= W^e[:6u] + x^{3u}W^e[:3u] + x^{3u}W^e[:6u] + x^{6u}W^e[:3u] + x^{6u}W^e[:6u]\\ &\quad + x^{7u}W^e[:5u] + x^{11u}W^e[:u] + (x^{6u}+x^{9u})R^e. \end{aligned}$$

*For all integers $k \ge 2$ and $u = 2^{2k}$, the following identities hold.*

$$M^o[:12u] = M^o[:6u] + x^{3u}M^o[:3u] + x^{3u}M^o[:6u] + x^{6u}M^o[:3u] + x^{6u}M^o[:6u]$$

$$+ x^{7u}M^o[:5u] + x^{11u}M^o[:u] + (x^{6u}+x^{9u})R^o,$$
$$W^o[:12u] = W^o[:6u] + x^{3u}W^o[:3u] + x^{3u}W^o[:6u] + x^{6u}W^o[:3u] + x^{6u}W^o[:6u]$$
$$+ x^{7u}W^o[:5u] + x^{11u}W^o[:u] + (x^{6u}+x^{9u})R^o.$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$T[:12u] = T[:6u] + x^{3u}T[:3u] + x^{3u}T[:6u] + x^{6u}T[:3u] + x^{6u}T[:6u] + x^{7u}T[:5u]$$
$$+ x^{11u}T[:u] + (x^{6u}+x^{9u}).$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$x^6u = x^{3u}T[3u:4u] + T[6u:7u],$$
$$0 = x^{7u}T[:u] + x^{3u}T[4u:5u] + T[7u:8u],$$
$$0 = x^{7u}T[u:2u] + x^{3u}T[5u:6u] + T[8u:9u],$$
$$x^9u = x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + T[9u:10u],$$
$$0 = x^{7u}T[3u:4u] + x^{6u}T[4u:5u] + T[10u:11u],$$
$$0 = x^{11u}T[:u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u] + T[11u:12u],$$

which can be rewritten as

$$0 = T[[11w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[11w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[11w]_2] + T[[110w]_2], \tag{2.13}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[1w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$1 = T[[10w]_2] + T[[11w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[11w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 845 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.19}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.20}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.21}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1001)), \tag{2.22}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.23}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$1 = \tau(A(s, 11)) + \tau(A(s, 110)), \tag{2.25}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 111)), \tag{2.26}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1000)), \tag{2.27}$$

$$1 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 1001)), \tag{2.28}$$

$$0 = \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.29}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{24}), E_{24}) = (A(i, 0^{12}), E_{12})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 12$, which turns out to be true. □

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^{3u} M^e[:3u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^{3u} W^e[:3u] + x^{6u} R^e.$$

*For all integer $k \ge 2$, and $u = 2^{2k}$,*

$$M_{2k+1} = M^o[:6u] + x^{3u} M^o[:3u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^{3u} W^o[:3u] + x^{6u} R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k} M_{2k} = \big(W^e[:6v] + x^{3v} W^e[:3v] + x^{6u} R^e\big) \times \big(M^e[:6v] + x^{3v} M^e[:3v] \\ + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is

$O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{12v}$, to

$$W^e[:12v]M^e[:12v] + x^{6v}W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.31) and $rhs$ as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^9 + x^8 + x^6, \\ q_1(x) &= x^{17} + x^{15} + x^{14} + x^8 + x^4 + x^3 + x^2 + x, \\ q_2(x) &= x^{18} + x^{12} + x^{10} + 1, \\ q_3(x) &= x^{17} + x^{15} + x^{14} + x^8 + x^4 + x^3 + x^2 + x, \\ q_4(x) &= x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} + x^9 + x^4 + x^2. \end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{84} + x^{74} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{54} + x^{52} + x^{50} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{28} + x^{22} + x^{18} + x^{16} + x^{12}, \\
p_3(x) &= x^{82} + x^{78} + x^{76} + x^{74} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} \\
&\quad + x^{20} + x^{18} + x^{10} + x^{8}, \\
p_6(x) &= x^{82} + x^{80} + x^{74} + x^{68} + x^{58} + x^{54} + x^{50} + x^{48} + x^{40} + x^{36} + x^{34} \\
&\quad + x^{30} + x^{24} + x^{20} + x^{16} + x^{14} + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{82} + x^{74} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{44} + x^{40} + x^{32} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{10} + x^{8} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{64} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{42} + x^{38} + x^{34} + x^{30} + x^{26} + x^{22} + x^{18} + x^{14} + x^{10} + x^{6} \\
&\quad + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{82} + x^{81} + x^{77} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} \\
&\quad + x^{64} + x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} \\
&\quad + x^{48} + x^{47} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33} \\
&\quad + x^{32} + x^{31} + x^{24}, \\
p_3(x) &= x^{87} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} \\
&\quad + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{23} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{10} + x^{9},\\
p_6(x) &= x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{73}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50}\\
&+ x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{27} + x^{24} + x^{22} + x^{21} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{93} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{35} + x^{34} + x^{33} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9} + x^{8} + x^{7}\\
&+ x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{96} + x^{95} + x^{92} + x^{76} + x^{75} + x^{71} + x^{67} + x^{63} + x^{60} + x^{48} + x^{47}\\
&+ x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{19} + x^{16} + x^{12} + x^{11} + x^{7}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{82} + x^{81} + x^{77} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65}\\
&+ x^{64} + x^{63} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{32} + x^{31} + x^{24},\\
p_3(x) &= x^{87} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71}\\
&+ x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34}\\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{23} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{73}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50}\\
&+ x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{27} + x^{24} + x^{22} + x^{21} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{93} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{35} + x^{34} + x^{33} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9} + x^{8} + x^{7}\\
&+ x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{96} + x^{95} + x^{92} + x^{76} + x^{75} + x^{71} + x^{67} + x^{63} + x^{60} + x^{48} + x^{47}
\end{aligned}
$$

$$+ x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{19} + x^{16} + x^{12} + x^{11} + x^{7} + x^{3} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{102} + x^{100} + x^{88} + x^{86} + x^{78} + x^{72} + x^{70} + x^{64} + x^{60} \\
&\quad + x^{58} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36}, \\
p_3(x) &= x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{88} + x^{76} + x^{70} + x^{64} + x^{56} \\
&\quad + x^{54} + x^{44} + x^{38} + x^{34} + x^{32} + x^{24} + x^{22} + x^{20} + x^{14} + x^{10} + x^{6}, \\
p_6(x) &= x^{106} + x^{104} + x^{102} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{72} + x^{70} + x^{64} \\
&\quad + x^{60} + x^{56} + x^{54} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} + x^{28} \\
&\quad + x^{26} + x^{24} + x^{22} + x^{18} + x^{14} + x^{6} + x^{2} + 1, \\
p_9(x) &= x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{84} + x^{82} + x^{80} + x^{76} + x^{70} + x^{68} \\
&\quad + x^{64} + x^{60} + x^{54} + x^{52} + x^{50} + x^{46} + x^{42} + x^{34} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{24} + x^{12} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{108} + x^{104} + x^{92} + x^{88} + x^{68} + x^{64} + x^{44} + x^{40} + x^{36} + x^{32} + x^{20} \\
&\quad + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{92} + x^{88} + x^{72} + x^{70} + x^{66} + x^{62} \\
&\quad + x^{60} + x^{52} + x^{48} + x^{46} + x^{34} + x^{30} + x^{26} + x^{24} + x^{20} + x^{12}, \\
p_3(x) &= x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{88} + x^{84} + x^{82} + x^{80} + x^{76} \\
&\quad + x^{72} + x^{70} + x^{68} + x^{62} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{38} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{14} + x^{12} + x^{8}, \\
p_6(x) &= x^{106} + x^{100} + x^{96} + x^{92} + x^{90} + x^{88} + x^{84} + x^{72} + x^{70} + x^{68} + x^{62} \\
&\quad + x^{58} + x^{50} + x^{46} + x^{42} + x^{40} + x^{38} + x^{34} + x^{30} + x^{26} + x^{18} + x^{14} \\
&\quad + x^{12} + x^{10} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{106} + x^{104} + x^{102} + x^{100} + x^{96} + x^{84} + x^{82} + x^{80} + x^{76} + x^{70} + x^{68} \\
&\quad + x^{64} + x^{60} + x^{54} + x^{52} + x^{50} + x^{46} + x^{42} + x^{34} + x^{32} + x^{30} + x^{28} \\
&\quad + x^{24} + x^{12} + x^{6} + x^{2}, \\
p_{12}(x) &= x^{108} + x^{104} + x^{92} + x^{88} + x^{68} + x^{64} + x^{44} + x^{40} + x^{36} + x^{32} + x^{20} \\
&\quad + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{83} + x^{77} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57} \\
&\quad + x^{56} + x^{54} + x^{53} + x^{52} + x^{47} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{33} + x^{32} + x^{31} + x^{26} + x^{25} + x^{23} + x^{21} + x^{18},\\
p_3(x) &= x^{87} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71}\\
&+ x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34}\\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{23} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{73}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50}\\
&+ x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{27} + x^{24} + x^{22} + x^{21} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{93} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{35} + x^{34} + x^{33} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9} + x^{8} + x^{7}\\
&+ x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{96} + x^{95} + x^{92} + x^{76} + x^{75} + x^{71} + x^{67} + x^{63} + x^{60} + x^{48} + x^{47}\\
&+ x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{19} + x^{16} + x^{12} + x^{11} + x^{7}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{83} + x^{77} + x^{75}\\
&+ x^{74} + x^{72} + x^{70} + x^{69} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{57}\\
&+ x^{56} + x^{54} + x^{53} + x^{52} + x^{47} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37}\\
&+ x^{33} + x^{32} + x^{31} + x^{26} + x^{25} + x^{23} + x^{21} + x^{18},\\
p_3(x) &= x^{87} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{71}\\
&+ x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{55} + x^{54}\\
&+ x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34}\\
&+ x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{23} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{73}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{64} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50}\\
&+ x^{46} + x^{45} + x^{41} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{27} + x^{24} + x^{22} + x^{21} + x^{18} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{93} + x^{92} + x^{90} + x^{89} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{75}\\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{58}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{35} + x^{34} + x^{33} + x^{29}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{17} + x^{16} + x^{14} + x^{13} + x^{9} + x^{8} + x^{7}\\
&+ x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{96} + x^{95} + x^{92} + x^{76} + x^{75} + x^{71} + x^{67} + x^{63} + x^{60} + x^{48} + x^{47}\\
&+ x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{19} + x^{16} + x^{12} + x^{11} + x^{7}\\
&+ x^{3} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{60} + x^{56} + x^{52}\\
&+ x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24},\\
p_3(x) &= x^{82} + x^{74} + x^{72} + x^{68} + x^{64} + x^{56} + x^{46} + x^{44} + x^{38} + x^{32} + x^{30}\\
&+ x^{28} + x^{26} + x^{20} + x^{18} + x^{6},\\
p_6(x) &= x^{82} + x^{72} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{54} + x^{50} + x^{48} + x^{44}\\
&+ x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{22} + x^{16} + x^{14} + x^{12} + x^{8}\\
&+ x^{2} + 1,\\
p_9(x) &= x^{82} + x^{74} + x^{68} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{44} + x^{40} + x^{32}\\
&+ x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{14} + x^{10} + x^{8} + x^{6} + x^{2},\\
p_{12}(x) &= x^{84} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{64} + x^{52} + x^{50} + x^{48}\\
&+ x^{46} + x^{42} + x^{38} + x^{34} + x^{30} + x^{26} + x^{22} + x^{18} + x^{14} + x^{10} + x^{6}\\
&+ x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59}\\
&+ x^{54} + x^{53} + x^{50} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34}\\
&+ x^{30} + x^{29} + x^{28} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_3(x) &= x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{80} + x^{79} + x^{75}\\
&+ x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{64} + x^{61} + x^{58} + x^{55} + x^{53} + x^{52}\\
&+ x^{50} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{39} + x^{35} + x^{34} + x^{32} + x^{30}\\
&+ x^{26} + x^{24} + x^{22} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{8},\\
p_6(x) &= x^{94} + x^{93} + x^{92} + x^{90} + x^{87} + x^{86} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78}\\
&+ x^{74} + x^{73} + x^{63} + x^{61} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{35} + x^{31} + x^{30} + x^{29}\\
&+ x^{27} + x^{25} + x^{24} + x^{19} + x^{18} + x^{16} + x^{9} + x^{6} + x^{4} + x^{3} + 1,\\
p_9(x) &= x^{92} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{76} + x^{72} + x^{71} + x^{68} + x^{60}
\end{aligned}
$$

$$+ x^{59} + x^{56} + x^{52} + x^{51} + x^{47} + x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} + x^{19} + x^{15} + x^{12} + x^{8} + x^{7} + x^{4},$$

$$p_{12}(x) = x^{96} + x^{95} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{75} + x^{72} + x^{68} + x^{67} + x^{63} + x^{59} + x^{56} + x^{52} + x^{51} + x^{48} + x^{16} + x^{15} + x^{11} + x^{8} + x^{4} + x^{3} + 1,$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$p_0(x) = x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{96} + x^{95} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{80} + x^{79} + x^{77} + x^{76} + x^{75} + x^{69} + x^{67} + x^{66} + x^{64} + x^{58} + x^{56} + x^{55} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} + x^{42} + x^{41} + x^{38} + x^{36} + x^{35} + x^{29} + x^{27} + x^{26} + x^{24},$$

$$p_3(x) = x^{99} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{86} + x^{85} + x^{84} + x^{80} + x^{78} + x^{76} + x^{73} + x^{71} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{56} + x^{52} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{30} + x^{27} + x^{26} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9},$$

$$p_6(x) = x^{102} + x^{98} + x^{96} + x^{94} + x^{88} + x^{86} + x^{74} + x^{70} + x^{68} + x^{62} + x^{58} + x^{50} + x^{48} + x^{44} + x^{38} + x^{28} + x^{22} + x^{20} + x^{8} + x^{6},$$

$$p_9(x) = x^{105} + x^{103} + x^{102} + x^{96} + x^{95} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{64} + x^{63} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^{8} + x^{7} + x^{4} + x^{3},$$

$$p_{12}(x) = x^{108} + x^{100} + x^{96} + x^{92} + x^{88} + x^{84} + x^{76} + x^{72} + x^{68} + x^{56} + x^{48} + x^{28} + x^{20} + x^{16} + x^{8} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(M^o, 1, 0)$:

$$p_0(x) = x^{92} + x^{89} + x^{87} + x^{84} + x^{83} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{69} + x^{66} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{25} + x^{23} + x^{22} + x^{20} + x^{18},$$

$$p_3(x) = x^{83} + x^{81} + x^{80} + x^{77} + x^{75} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{64} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}$$

$$
\begin{aligned}
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{86} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{64} + x^{52} + x^{50} + x^{42}\\
&+ x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{18} + x^{8} + x^{6},\\
p_9(x) &= x^{89} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76}\\
&+ x^{74} + x^{72} + x^{71} + x^{68} + x^{67} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36}\\
&+ x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{10} + x^{8} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{64} + x^{60} + x^{56} + x^{48} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{81} + x^{79} + x^{77} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{63} + x^{62}\\
&+ x^{61} + x^{58} + x^{57} + x^{54} + x^{47} + x^{45} + x^{43} + x^{42} + x^{39} + x^{36} + x^{35}\\
&+ x^{32} + x^{30},\\
p_3(x) &= x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{77}\\
&+ x^{75} + x^{74} + x^{70} + x^{68} + x^{67} + x^{66} + x^{56} + x^{53} + x^{52} + x^{51} + x^{47}\\
&+ x^{43} + x^{42} + x^{40} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{27}\\
&+ x^{26} + x^{25} + x^{20} + x^{19} + x^{15} + x^{14} + x^{12},\\
p_6(x) &= x^{94} + x^{93} + x^{91} + x^{89} + x^{84} + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76}\\
&+ x^{74} + x^{69} + x^{68} + x^{67} + x^{66} + x^{61} + x^{58} + x^{56} + x^{55} + x^{53} + x^{52}\\
&+ x^{50} + x^{48} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{19} + x^{15} + x^{12} + x^{10} + x^{8} + x^{7} + x^{3}\\
&+ 1,\\
p_9(x) &= x^{92} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{76} + x^{72} + x^{71} + x^{68} + x^{60}\\
&+ x^{59} + x^{56} + x^{52} + x^{51} + x^{47} + x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23}\\
&+ x^{19} + x^{15} + x^{12} + x^{8} + x^{7} + x^{4},\\
p_{12}(x) &= x^{96} + x^{95} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{75} + x^{72} + x^{68} + x^{67}\\
&+ x^{63} + x^{59} + x^{56} + x^{52} + x^{51} + x^{48} + x^{16} + x^{15} + x^{11} + x^{8} + x^{4} + x^{3}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{96} + x^{95} + x^{94} + x^{93} + x^{90} + x^{87} + x^{86} + x^{84} + x^{83} + x^{82} + x^{81}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{67} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59}\\
&+ x^{54} + x^{53} + x^{50} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{38} + x^{37} + x^{34}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{30} + x^{29} + x^{28} + x^{17} + x^{16} + x^{15} + x^{14} + x^{12},\\
p_3(x) &= x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{80} + x^{79} + x^{75}\\
&+ x^{72} + x^{71} + x^{70} + x^{68} + x^{65} + x^{64} + x^{61} + x^{58} + x^{55} + x^{53} + x^{52}\\
&+ x^{50} + x^{49} + x^{46} + x^{44} + x^{43} + x^{41} + x^{39} + x^{35} + x^{34} + x^{32} + x^{30}\\
&+ x^{26} + x^{24} + x^{22} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{8},\\
p_6(x) &= x^{94} + x^{93} + x^{92} + x^{90} + x^{87} + x^{86} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78}\\
&+ x^{74} + x^{73} + x^{63} + x^{61} + x^{59} + x^{58} + x^{55} + x^{54} + x^{51} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{35} + x^{31} + x^{30} + x^{29}\\
&+ x^{27} + x^{25} + x^{24} + x^{19} + x^{18} + x^{16} + x^{9} + x^{6} + x^{4} + x^{3} + 1,\\
p_9(x) &= x^{92} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{76} + x^{72} + x^{71} + x^{68} + x^{60}\\
&+ x^{59} + x^{56} + x^{52} + x^{51} + x^{47} + x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23}\\
&+ x^{19} + x^{15} + x^{12} + x^{8} + x^{7} + x^{4},\\
p_{12}(x) &= x^{96} + x^{95} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{75} + x^{72} + x^{68} + x^{67}\\
&+ x^{63} + x^{59} + x^{56} + x^{52} + x^{51} + x^{48} + x^{16} + x^{15} + x^{11} + x^{8} + x^{4} + x^{3}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{92} + x^{89} + x^{87} + x^{84} + x^{83} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{69}\\
&+ x^{66} + x^{63} + x^{60} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{47}\\
&+ x^{45} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31}\\
&+ x^{29} + x^{25} + x^{23} + x^{22} + x^{20} + x^{18},\\
p_3(x) &= x^{83} + x^{81} + x^{80} + x^{77} + x^{75} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{64}\\
&+ x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{45} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}\\
&+ x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{10} + x^{9},\\
p_6(x) &= x^{86} + x^{80} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{64} + x^{52} + x^{50} + x^{42}\\
&+ x^{38} + x^{36} + x^{34} + x^{30} + x^{28} + x^{24} + x^{18} + x^{8} + x^{6},\\
p_9(x) &= x^{89} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76}\\
&+ x^{74} + x^{72} + x^{71} + x^{68} + x^{67} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36}\\
&+ x^{34} + x^{33} + x^{32} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21}\\
&+ x^{20} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{12} + x^{10} + x^{8} + x^{7} + x^{4} + x^{3},\\
p_{12}(x) &= x^{92} + x^{88} + x^{80} + x^{76} + x^{72} + x^{64} + x^{60} + x^{56} + x^{48} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0].

For $\phi(W^o,1,0)$:

$$
\begin{aligned}
p_0(x) &= x^{108}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{96}+x^{95}+x^{91}+x^{89}\\
&\quad +x^{88}+x^{87}+x^{86}+x^{85}+x^{80}+x^{79}+x^{77}+x^{76}+x^{75}+x^{69}+x^{67}\\
&\quad +x^{66}+x^{64}+x^{58}+x^{56}+x^{55}+x^{52}+x^{51}+x^{50}+x^{47}+x^{46}+x^{44}\\
&\quad +x^{42}+x^{41}+x^{38}+x^{36}+x^{35}+x^{29}+x^{27}+x^{26}+x^{24},\\
p_3(x) &= x^{99}+x^{97}+x^{96}+x^{95}+x^{92}+x^{91}+x^{89}+x^{86}+x^{85}+x^{84}+x^{80}\\
&\quad +x^{78}+x^{76}+x^{73}+x^{71}+x^{67}+x^{66}+x^{65}+x^{64}+x^{62}+x^{60}+x^{59}\\
&\quad +x^{56}+x^{52}+x^{50}+x^{48}+x^{45}+x^{41}+x^{40}+x^{39}+x^{38}+x^{36}+x^{34}\\
&\quad +x^{33}+x^{30}+x^{27}+x^{26}+x^{21}+x^{20}+x^{18}+x^{17}+x^{15}+x^{14}+x^{13}\\
&\quad +x^{12}+x^{11}+x^{10}+x^{9},\\
p_6(x) &= x^{102}+x^{98}+x^{96}+x^{94}+x^{88}+x^{86}+x^{74}+x^{70}+x^{68}+x^{62}+x^{58}\\
&\quad +x^{50}+x^{48}+x^{44}+x^{38}+x^{28}+x^{22}+x^{20}+x^{8}+x^{6},\\
p_9(x) &= x^{105}+x^{103}+x^{102}+x^{96}+x^{95}+x^{85}+x^{83}+x^{82}+x^{81}+x^{79}+x^{78}\\
&\quad +x^{77}+x^{76}+x^{74}+x^{73}+x^{72}+x^{70}+x^{68}+x^{67}+x^{64}+x^{63}+x^{57}\\
&\quad +x^{55}+x^{54}+x^{53}+x^{51}+x^{50}+x^{49}+x^{48}+x^{46}+x^{45}+x^{44}+x^{42}\\
&\quad +x^{41}+x^{40}+x^{38}+x^{37}+x^{36}+x^{34}+x^{33}+x^{32}+x^{30}+x^{29}+x^{28}\\
&\quad +x^{26}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{17}+x^{15}+x^{14}+x^{13}+x^{12}\\
&\quad +x^{10}+x^{8}+x^{7}+x^{4}+x^{3},\\
p_{12}(x) &= x^{108}+x^{100}+x^{96}+x^{92}+x^{88}+x^{84}+x^{76}+x^{72}+x^{68}+x^{56}+x^{48}\\
&\quad +x^{28}+x^{20}+x^{16}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1].

For $\phi(W^o,1,1)$:

$$
\begin{aligned}
p_0(x) &= x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}\\
&\quad +x^{84}+x^{81}+x^{79}+x^{77}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}+x^{63}+x^{62}\\
&\quad +x^{61}+x^{58}+x^{57}+x^{54}+x^{47}+x^{45}+x^{43}+x^{42}+x^{39}+x^{36}+x^{35}\\
&\quad +x^{32}+x^{30},\\
p_3(x) &= x^{92}+x^{91}+x^{90}+x^{89}+x^{86}+x^{84}+x^{83}+x^{81}+x^{80}+x^{79}+x^{77}\\
&\quad +x^{75}+x^{74}+x^{70}+x^{68}+x^{67}+x^{66}+x^{56}+x^{53}+x^{52}+x^{51}+x^{47}\\
&\quad +x^{43}+x^{42}+x^{40}+x^{37}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{27}\\
&\quad +x^{26}+x^{25}+x^{20}+x^{19}+x^{15}+x^{14}+x^{12},\\
p_6(x) &= x^{94}+x^{93}+x^{91}+x^{89}+x^{84}+x^{83}+x^{81}+x^{80}+x^{78}+x^{77}+x^{76}\\
&\quad +x^{74}+x^{69}+x^{68}+x^{67}+x^{66}+x^{61}+x^{58}+x^{56}+x^{55}+x^{53}+x^{52}\\
&\quad +x^{50}+x^{48}+x^{42}+x^{40}+x^{38}+x^{37}+x^{36}+x^{33}+x^{32}+x^{31}+x^{29}\\
&\quad +x^{27}+x^{25}+x^{24}+x^{22}+x^{21}+x^{19}+x^{15}+x^{12}+x^{10}+x^{8}+x^{7}+x^{3}\\
&\quad +1,
\end{aligned}
$$

$$\begin{aligned} p_9(x) &= x^{92} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{76} + x^{72} + x^{71} + x^{68} + x^{60} \\ &\quad + x^{59} + x^{56} + x^{52} + x^{51} + x^{47} + x^{43} + x^{39} + x^{35} + x^{31} + x^{27} + x^{23} \\ &\quad + x^{19} + x^{15} + x^{12} + x^{8} + x^{7} + x^{4}, \\ p_{12}(x) &= x^{96} + x^{95} + x^{91} + x^{88} + x^{84} + x^{83} + x^{79} + x^{75} + x^{72} + x^{68} + x^{67} \\ &\quad + x^{63} + x^{59} + x^{56} + x^{52} + x^{51} + x^{48} + x^{16} + x^{15} + x^{11} + x^{8} + x^{4} + x^{3} \\ &\quad + 1, \end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 170 | [301, 302] | 340 | [438, 525] | 510 | [552, 351] | 680 | [810, 527] |
| 1 | [3, 4] | 171 | [303, 96] | 341 | [50, 526] | 511 | [257, 644] | 681 | [726, 496] |
| 2 | [5, 6] | 172 | [104, 304] | 342 | [420, 192] | 512 | [628, 672] | 682 | [773, 811] |
| 3 | [7, 8] | 173 | [273, 305] | 343 | [527, 528] | 513 | [243, 673] | 683 | [393, 812] |
| 4 | [9, 10] | 174 | [306, 307] | 344 | [517, 529] | 514 | [674, 262] | 684 | [695, 759] |
| 5 | [11, 12] | 175 | [101, 308] | 345 | [157, 530] | 515 | [331, 242] | 685 | [756, 532] |
| 6 | [13, 14] | 176 | [250, 88] | 346 | [531, 207] | 516 | [266, 574] | 686 | [813, 702] |
| 7 | [15, 16] | 177 | [309, 310] | 347 | [532, 183] | 517 | [111, 555] | 687 | [814, 303] |
| 8 | [17, 18] | 178 | [311, 16] | 348 | [193, 533] | 518 | [351, 606] | 688 | [593, 253] |
| 9 | [19, 20] | 179 | [312, 313] | 349 | [534, 535] | 519 | [575, 316] | 689 | [815, 619] |
| 10 | [21, 22] | 180 | [120, 103] | 350 | [536, 125] | 520 | [595, 101] | 690 | [109, 309] |
| 11 | [23, 24] | 181 | [314, 315] | 351 | [537, 2] | 521 | [675, 623] | 691 | [816, 587] |
| 12 | [25, 26] | 182 | [4, 316] | 352 | [538, 539] | 522 | [360, 303] | 692 | [109, 273] |
| 13 | [27, 28] | 183 | [116, 232] | 353 | [62, 385] | 523 | [355, 671] | 693 | [229, 817] |
| 14 | [29, 30] | 184 | [94, 317] | 354 | [415, 540] | 524 | [676, 627] | 694 | [572, 660] |
| 15 | [31, 32] | 185 | [85, 318] | 355 | [541, 516] | 525 | [677, 603] | 695 | [641, 563] |
| 16 | [33, 34] | 186 | [319, 320] | 356 | [509, 510] | 526 | [228, 302] | 696 | [117, 648] |
| 17 | [35, 36] | 187 | [321, 322] | 357 | [542, 543] | 527 | [25, 320] | 697 | [816, 627] |
| 18 | [37, 38] | 188 | [240, 323] | 358 | [544, 545] | 528 | [678, 118] | 698 | [648, 573] |
| 19 | [39, 40] | 189 | [100, 324] | 359 | [535, 392] | 529 | [574, 4] | 699 | [286, 358] |
| 20 | [41, 42] | 190 | [325, 326] | 360 | [453, 373] | 530 | [656, 567] | 700 | [259, 818] |
| 21 | [43, 44] | 191 | [256, 327] | 361 | [413, 546] | 531 | [679, 680] | 701 | [819, 820] |
| 22 | [45, 46] | 192 | [119, 67] | 362 | [547, 262] | 532 | [681, 94] | 702 | [665, 351] |
| 23 | [47, 48] | 193 | [328, 329] | 363 | [548, 549] | 533 | [267, 575] | 703 | [596, 102] |
| 24 | [49, 50] | 194 | [66, 274] | 364 | [349, 339] | 534 | [682, 245] | 704 | [324, 244] |
| 25 | [51, 52] | 195 | [330, 233] | 365 | [115, 550] | 535 | [647, 90] | 705 | [235, 285] |
| 26 | [53, 54] | 196 | [331, 93] | 366 | [551, 92] | 536 | [583, 346] | 706 | [338, 350] |
| 27 | [55, 56] | 197 | [332, 333] | 367 | [552, 553] | 537 | [234, 16] | 707 | [681, 559] |
| 28 | [57, 58] | 198 | [265, 334] | 368 | [554, 555] | 538 | [323, 562] | 708 | [111, 632] |
| 29 | [59, 60] | 199 | [120, 23] | 369 | [354, 556] | 539 | [658, 683] | 709 | [283, 270] |
| 30 | [61, 62] | 200 | [88, 304] | 370 | [359, 557] | 540 | [606, 684] | 710 | [677, 292] |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 31 | [63, 64] | 201 | [335, 336] | 371 | [558, 559] | 541 | [669, 111] | 711 | [247, 582] |
| 32 | [65, 66] | 202 | [238, 337] | 372 | [64, 288] | 542 | [21, 683] | 712 | [685, 246] |
| 33 | [67, 23] | 203 | [338, 339] | 373 | [560, 561] | 543 | [685, 668] | 713 | [659, 255] |
| 34 | [68, 69] | 204 | [267, 4] | 374 | [241, 562] | 544 | [671, 334] | 714 | [360, 347] |
| 35 | [70, 71] | 205 | [340, 341] | 375 | [285, 100] | 545 | [627, 588] | 715 | [609, 357] |
| 36 | [72, 73] | 206 | [64, 332] | 376 | [17, 563] | 546 | [236, 686] | 716 | [676, 587] |
| 37 | [74, 75] | 207 | [342, 343] | 377 | [281, 343] | 547 | [687, 170] | 717 | [614, 658] |
| 38 | [76, 77] | 208 | [258, 84] | 378 | [564, 565] | 548 | [150, 688] | 718 | [357, 287] |
| 39 | [78, 79] | 209 | [344, 73] | 379 | [5, 566] | 549 | [546, 213] | 719 | [645, 668] |
| 40 | [80, 81] | 210 | [345, 346] | 380 | [567, 70] | 550 | [689, 690] | 720 | [821, 261] |
| 41 | [82, 83] | 211 | [347, 96] | 381 | [568, 569] | 551 | [691, 433] | 721 | [309, 305] |
| 42 | [84, 85] | 212 | [348, 326] | 382 | [248, 570] | 552 | [692, 693] | 722 | [74, 605] |
| 43 | [86, 87] | 213 | [349, 350] | 383 | [96, 571] | 553 | [694, 695] | 723 | [321, 653] |
| 44 | [88, 89] | 214 | [351, 352] | 384 | [572, 255] | 554 | [444, 696] | 724 | [822, 286] |
| 45 | [90, 91] | 215 | [353, 19] | 385 | [118, 573] | 555 | [697, 698] | 725 | [345, 328] |
| 46 | [92, 93] | 216 | [263, 354] | 386 | [574, 575] | 556 | [417, 699] | 726 | [282, 567] |
| 47 | [94, 95] | 217 | [355, 265] | 387 | [576, 577] | 557 | [700, 701] | 727 | [646, 262] |
| 48 | [96, 97] | 218 | [112, 120] | 388 | [103, 251] | 558 | [702, 703] | 728 | [675, 823] |
| 49 | [98, 99] | 219 | [319, 26] | 389 | [323, 578] | 559 | [704, 705] | 729 | [588, 824] |
| 50 | [81, 100] | 220 | [356, 257] | 390 | [290, 579] | 560 | [706, 492] | 730 | [643, 267] |
| 51 | [101, 102] | 221 | [272, 309] | 391 | [580, 104] | 561 | [482, 707] | 731 | [90, 354] |
| 52 | [67, 103] | 222 | [357, 358] | 392 | [581, 241] | 562 | [708, 709] | 732 | [346, 666] |
| 53 | [104, 89] | 223 | [359, 360] | 393 | [275, 582] | 563 | [710, 13] | 733 | [674, 647] |
| 54 | [105, 106] | 224 | [260, 361] | 394 | [583, 328] | 564 | [711, 194] | 734 | [825, 566] |
| 55 | [98, 107] | 225 | [362, 363] | 395 | [584, 250] | 565 | [712, 481] | 735 | [241, 578] |
| 56 | [108, 109] | 226 | [364, 364] | 396 | [327, 585] | 566 | [472, 710] | 736 | [232, 826] |
| 57 | [110, 111] | 227 | [365, 366] | 397 | [586, 300] | 567 | [713, 363] | 737 | [70, 271] |
| 58 | [112, 67] | 228 | [367, 368] | 398 | [343, 570] | 568 | [130, 437] | 738 | [29, 652] |
| 59 | [113, 114] | 229 | [369, 370] | 399 | [587, 588] | 569 | [714, 715] | 739 | [827, 686] |
| 60 | [115, 116] | 230 | [371, 151] | 400 | [270, 71] | 570 | [716, 398] | 740 | [647, 263] |
| 61 | [117, 118] | 231 | [372, 161] | 401 | [279, 589] | 571 | [717, 718] | 741 | [558, 94] |
| 62 | [119, 120] | 232 | [373, 35] | 402 | [590, 591] | 572 | [459, 123] | 742 | [263, 91] |
| 63 | [121, 122] | 233 | [171, 374] | 403 | [592, 565] | 573 | [218, 198] | 743 | [294, 649] |
| 64 | [123, 124] | 234 | [56, 170] | 404 | [593, 594] | 574 | [719, 720] | 744 | [633, 629] |
| 65 | [125, 41] | 235 | [375, 188] | 405 | [595, 596] | 575 | [407, 721] | 745 | [634, 664] |
| 66 | [126, 127] | 236 | [376, 375] | 406 | [597, 324] | 576 | [722, 723] | 746 | [72, 828] |
| 67 | [128, 49] | 237 | [377, 378] | 407 | [559, 317] | 577 | [169, 724] | 747 | [829, 294] |
| 68 | [129, 130] | 238 | [379, 153] | 408 | [598, 557] | 578 | [382, 33] | 748 | [660, 617] |
| 69 | [131, 132] | 239 | [380, 381] | 409 | [599, 600] | 579 | [725, 726] | 749 | [352, 830] |
| 70 | [133, 134] | 240 | [58, 382] | 410 | [575, 601] | 580 | [727, 728] | 750 | [252, 594] |
| 71 | [135, 136] | 241 | [383, 384] | 411 | [602, 336] | 581 | [144, 452] | 751 | [668, 275] |
| 72 | [137, 138] | 242 | [385, 386] | 412 | [291, 603] | 582 | [402, 729] | 752 | [671, 657] |
| 73 | [139, 140] | 243 | [387, 388] | 413 | [604, 605] | 583 | [528, 730] | 753 | [84, 818] |
| 74 | [141, 142] | 244 | [189, 389] | 414 | [553, 606] | 584 | [731, 538] | 754 | [356, 327] |

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|---|---|---|---|---|---|---|---|---|---|
| 75 | [143, 144] | 245 | [38, 390] | 415 | [607, 608] | 585 | [403, 732] | 755 | [825, 6] |
| 76 | [145, 146] | 246 | [391, 122] | 416 | [609, 286] | 586 | [146, 215] | 756 | [239, 614] |
| 77 | [147, 148] | 247 | [392, 393] | 417 | [247, 78] | 587 | [733, 734] | 757 | [613, 82] |
| 78 | [149, 150] | 248 | [54, 394] | 418 | [610, 101] | 588 | [735, 736] | 758 | [273, 310] |
| 79 | [151, 152] | 249 | [395, 396] | 419 | [611, 612] | 589 | [737, 419] | 759 | [660, 64] |
| 80 | [153, 154] | 250 | [397, 168] | 420 | [613, 312] | 590 | [738, 739] | 760 | [829, 624] |
| 81 | [42, 155] | 251 | [2, 365] | 421 | [614, 21] | 591 | [740, 741] | 761 | [110, 554] |
| 82 | [156, 157] | 252 | [398, 399] | 422 | [615, 307] | 592 | [742, 743] | 762 | [259, 85] |
| 83 | [158, 159] | 253 | [400, 219] | 423 | [254, 107] | 593 | [447, 744] | 763 | [4, 601] |
| 84 | [160, 135] | 254 | [172, 401] | 424 | [616, 322] | 594 | [745, 156] | 764 | [564, 633] |
| 85 | [161, 162] | 255 | [180, 402] | 425 | [255, 617] | 595 | [405, 486] | 765 | [661, 600] |
| 86 | [163, 164] | 256 | [210, 403] | 426 | [618, 619] | 596 | [746, 541] | 766 | [822, 357] |
| 87 | [136, 165] | 257 | [404, 405] | 427 | [619, 620] | 597 | [747, 717] | 767 | [330, 826] |
| 88 | [166, 167] | 258 | [406, 407] | 428 | [354, 621] | 598 | [748, 479] | 768 | [615, 239] |
| 89 | [168, 169] | 259 | [408, 409] | 429 | [622, 623] | 599 | [749, 750] | 769 | [559, 95] |
| 90 | [170, 171] | 260 | [410, 411] | 430 | [108, 272] | 600 | [14, 412] | 770 | [325, 631] |
| 91 | [155, 172] | 261 | [412, 413] | 431 | [618, 295] | 601 | [720, 751] | 771 | [332, 289] |
| 92 | [173, 174] | 262 | [414, 415] | 432 | [550, 330] | 602 | [752, 753] | 772 | [586, 115] |
| 93 | [175, 176] | 263 | [416, 417] | 433 | [346, 329] | 603 | [754, 475] | 773 | [818, 667] |
| 94 | [177, 178] | 264 | [418, 158] | 434 | [293, 624] | 604 | [755, 756] | 774 | [275, 78] |
| 95 | [179, 180] | 265 | [419, 420] | 435 | [347, 625] | 605 | [757, 691] | 775 | [637, 115] |
| 96 | [181, 182] | 266 | [421, 9] | 436 | [626, 250] | 606 | [693, 758] | 776 | [302, 640] |
| 97 | [183, 184] | 267 | [422, 423] | 437 | [27, 337] | 607 | [739, 722] | 777 | [625, 571] |
| 98 | [185, 186] | 268 | [424, 129] | 438 | [604, 75] | 608 | [759, 760] | 778 | [324, 673] |
| 99 | [187, 187] | 269 | [425, 426] | 439 | [262, 90] | 609 | [524, 544] | 779 | [682, 114] |
| 100 | [188, 189] | 270 | [427, 428] | 440 | [301, 229] | 610 | [761, 716] | 780 | [588, 672] |
| 101 | [190, 191] | 271 | [429, 430] | 441 | [82, 313] | 611 | [487, 762] | 781 | [602, 831] |
| 102 | [192, 193] | 272 | [431, 432] | 442 | [627, 628] | 612 | [763, 507] | 782 | [626, 580] |
| 103 | [194, 195] | 273 | [34, 11] | 443 | [629, 630] | 613 | [220, 764] | 783 | [306, 239] |
| 104 | [196, 197] | 274 | [433, 214] | 444 | [348, 631] | 614 | [154, 45] | 784 | [91, 556] |
| 105 | [198, 199] | 275 | [434, 435] | 445 | [100, 243] | 615 | [60, 747] | 785 | [76, 579] |
| 106 | [200, 201] | 276 | [436, 175] | 446 | [311, 108] | 616 | [765, 485] | 786 | [581, 323] |
| 107 | [202, 203] | 277 | [437, 438] | 447 | [554, 632] | 617 | [766, 483] | 787 | [353, 66] |
| 108 | [204, 205] | 278 | [439, 440] | 448 | [633, 634] | 618 | [461, 179] | 788 | [335, 831] |
| 109 | [32, 206] | 279 | [441, 442] | 449 | [635, 636] | 619 | [767, 768] | 789 | [116, 330] |
| 110 | [207, 208] | 280 | [443, 444] | 450 | [91, 621] | 620 | [703, 473] | 790 | [624, 663] |
| 111 | [209, 210] | 281 | [191, 395] | 451 | [637, 300] | 621 | [216, 769] | 791 | [81, 597] |
| 112 | [211, 47] | 282 | [445, 133] | 452 | [257, 585] | 622 | [729, 770] | 792 | [832, 650] |
| 113 | [44, 212] | 283 | [446, 429] | 453 | [565, 629] | 623 | [203, 37] | 793 | [818, 318] |
| 114 | [213, 202] | 284 | [208, 447] | 454 | [638, 639] | 624 | [771, 772] | 794 | [622, 823] |
| 115 | [214, 181] | 285 | [448, 449] | 455 | [229, 640] | 625 | [701, 773] | 795 | [342, 248] |
| 116 | [215, 216] | 286 | [450, 451] | 456 | [641, 18] | 626 | [774, 166] | 796 | [584, 580] |
| 117 | [217, 218] | 287 | [452, 453] | 457 | [557, 347] | 627 | [775, 8] | 797 | [610, 596] |
| 118 | [219, 220] | 288 | [454, 410] | 458 | [548, 299] | 628 | [776, 777] | 798 | [592, 633] |

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|---|---|---|---|---|---|---|---|---|---|
| 119 | [221, 222] | 289 | [124, 455] | 459 | [307, 230] | 629 | [222, 778] | 799 | [815, 295] |
| 120 | [223, 224] | 290 | [456, 457] | 460 | [642, 312] | 630 | [779, 131] | 800 | [617, 332] |
| 121 | [225, 85] | 291 | [458, 459] | 461 | [643, 574] | 631 | [780, 781] | 801 | [832, 106] |
| 122 | [226, 227] | 292 | [460, 461] | 462 | [327, 644] | 632 | [782, 504] | 802 | [827, 237] |
| 123 | [228, 229] | 293 | [148, 462] | 463 | [625, 97] | 633 | [783, 779] | 803 | [303, 625] |
| 124 | [230, 21] | 294 | [463, 464] | 464 | [645, 246] | 634 | [743, 784] | 804 | [565, 634] |
| 125 | [231, 84] | 295 | [465, 466] | 465 | [582, 297] | 635 | [785, 424] | 805 | [295, 620] |
| 126 | [232, 233] | 296 | [467, 448] | 466 | [646, 647] | 636 | [495, 786] | 806 | [679, 612] |
| 127 | [234, 108] | 297 | [435, 468] | 467 | [648, 284] | 637 | [142, 766] | 807 | [550, 232] |
| 128 | [235, 81] | 298 | [469, 470] | 468 | [624, 649] | 638 | [525, 706] | 808 | [599, 662] |
| 129 | [236, 237] | 299 | [471, 472] | 469 | [634, 630] | 639 | [430, 787] | 809 | [553, 352] |
| 130 | [238, 28] | 300 | [473, 474] | 470 | [105, 650] | 640 | [368, 414] | 810 | [352, 684] |
| 131 | [239, 230] | 301 | [475, 476] | 471 | [651, 652] | 641 | [488, 788] | 811 | [611, 680] |
| 132 | [240, 241] | 302 | [477, 478] | 472 | [616, 653] | 642 | [508, 491] | 812 | [302, 817] |
| 133 | [92, 242] | 303 | [479, 480] | 473 | [587, 628] | 643 | [789, 790] | 813 | [678, 648] |
| 134 | [243, 244] | 304 | [481, 482] | 474 | [654, 655] | 644 | [730, 51] | 814 | [1, 181] |
| 135 | [113, 245] | 305 | [483, 484] | 475 | [656, 283] | 645 | [457, 434] | 815 | [798, 467] |
| 136 | [246, 247] | 306 | [485, 371] | 476 | [265, 657] | 646 | [791, 32] | 816 | [833, 427] |
| 137 | [248, 249] | 307 | [176, 43] | 477 | [658, 22] | 647 | [545, 387] | 817 | [834, 700] |
| 138 | [250, 104] | 308 | [486, 487] | 478 | [659, 660] | 648 | [792, 761] | 818 | [790, 835] |
| 139 | [23, 251] | 309 | [381, 488] | 479 | [551, 331] | 649 | [462, 204] | 819 | [723, 785] |
| 140 | [252, 253] | 310 | [206, 489] | 480 | [617, 288] | 650 | [793, 746] | 820 | [836, 837] |
| 141 | [254, 99] | 311 | [490, 431] | 481 | [661, 662] | 651 | [794, 791] | 821 | [540, 838] |
| 142 | [255, 64] | 312 | [140, 217] | 482 | [285, 597] | 652 | [795, 796] | 822 | [796, 708] |
| 143 | [256, 257] | 313 | [491, 441] | 483 | [597, 243] | 653 | [797, 798] | 823 | [390, 802] |
| 144 | [258, 259] | 314 | [492, 163] | 484 | [294, 663] | 654 | [164, 738] | 824 | [423, 714] |
| 145 | [260, 261] | 315 | [493, 494] | 485 | [16, 272] | 655 | [751, 799] | 825 | [512, 839] |
| 146 | [262, 263] | 316 | [8, 495] | 486 | [231, 259] | 656 | [800, 776] | 826 | [690, 840] |
| 147 | [264, 265] | 317 | [496, 380] | 487 | [596, 308] | 657 | [764, 752] | 827 | [841, 490] |
| 148 | [266, 267] | 318 | [363, 439] | 488 | [629, 664] | 658 | [205, 749] | 828 | [842, 792] |
| 149 | [268, 269] | 319 | [159, 497] | 489 | [66, 20] | 659 | [522, 689] | 829 | [378, 834] |
| 150 | [270, 271] | 320 | [498, 404] | 490 | [300, 550] | 660 | [386, 454] | 830 | [224, 836] |
| 151 | [272, 273] | 321 | [499, 500] | 491 | [665, 553] | 661 | [182, 801] | 831 | [843, 531] |
| 152 | [19, 274] | 322 | [501, 367] | 492 | [651, 30] | 662 | [802, 471] | 832 | [396, 844] |
| 153 | [246, 275] | 323 | [502, 503] | 493 | [557, 303] | 663 | [476, 391] | 833 | [230, 658] |
| 154 | [276, 92] | 324 | [449, 469] | 494 | [276, 331] | 664 | [728, 425] | 834 | [343, 249] |
| 155 | [277, 278] | 325 | [504, 505] | 495 | [16, 109] | 665 | [803, 196] | 835 | [582, 79] |
| 156 | [279, 280] | 326 | [506, 209] | 496 | [328, 666] | 666 | [530, 137] | 836 | [307, 614] |
| 157 | [281, 248] | 327 | [507, 508] | 497 | [283, 70] | 667 | [409, 493] | 837 | [65, 19] |
| 158 | [282, 283] | 328 | [201, 509] | 498 | [85, 667] | 668 | [709, 149] | 838 | [344, 828] |
| 159 | [118, 284] | 329 | [510, 511] | 499 | [298, 549] | 669 | [696, 697] | 839 | [670, 280] |
| 160 | [80, 285] | 330 | [474, 512] | 500 | [668, 247] | 670 | [732, 804] | 840 | [619, 296] |
| 161 | [286, 287] | 331 | [513, 514] | 501 | [669, 554] | 671 | [781, 418] | 841 | [821, 361] |
| 162 | [288, 289] | 332 | [122, 185] | 502 | [288, 333] | 672 | [734, 740] | 842 | [628, 824] |

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|---|---|---|---|---|---|---|---|---|---|
| 163 | [290, 77] | 333 | [432, 515] | 503 | [598, 360] | 673 | [480, 805] | 843 | [606, 830] |
| 164 | [291, 292] | 334 | [516, 517] | 504 | [312, 83] | 674 | [500, 416] | 844 | [567, 270] |
| 165 | [293, 294] | 335 | [511, 518] | 505 | [580, 88] | 675 | [388, 806] | | |
| 166 | [295, 296] | 336 | [519, 190] | 506 | [103, 24] | 676 | [807, 735] | | |
| 167 | [78, 297] | 337 | [520, 436] | 507 | [670, 589] | 677 | [808, 748] | | |
| 168 | [298, 299] | 338 | [521, 522] | 508 | [642, 82] | 678 | [394, 782] | | |
| 169 | [300, 116] | 339 | [523, 524] | 509 | [264, 671] | 679 | [698, 809] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 121 | 0 | 242 | 0 | 363 | 1 | 484 | 0 | 605 | 1 | 726 | 0 |
| 1 | 0 | 122 | 1 | 243 | 1 | 364 | 1 | 485 | 0 | 606 | 1 | 727 | 1 |
| 2 | 1 | 123 | 0 | 244 | 0 | 365 | 1 | 486 | 0 | 607 | 1 | 728 | 0 |
| 3 | 0 | 124 | 1 | 245 | 0 | 366 | 1 | 487 | 0 | 608 | 0 | 729 | 1 |
| 4 | 1 | 125 | 0 | 246 | 1 | 367 | 0 | 488 | 1 | 609 | 0 | 730 | 1 |
| 5 | 1 | 126 | 0 | 247 | 1 | 368 | 0 | 489 | 0 | 610 | 1 | 731 | 1 |
| 6 | 0 | 127 | 1 | 248 | 0 | 369 | 1 | 490 | 0 | 611 | 0 | 732 | 1 |
| 7 | 0 | 128 | 0 | 249 | 1 | 370 | 1 | 491 | 1 | 612 | 1 | 733 | 0 |
| 8 | 0 | 129 | 0 | 250 | 0 | 371 | 1 | 492 | 1 | 613 | 1 | 734 | 0 |
| 9 | 1 | 130 | 1 | 251 | 1 | 372 | 0 | 493 | 0 | 614 | 0 | 735 | 1 |
| 10 | 1 | 131 | 1 | 252 | 1 | 373 | 0 | 494 | 0 | 615 | 1 | 736 | 0 |
| 11 | 1 | 132 | 0 | 253 | 1 | 374 | 1 | 495 | 0 | 616 | 1 | 737 | 0 |
| 12 | 1 | 133 | 0 | 254 | 1 | 375 | 0 | 496 | 0 | 617 | 1 | 738 | 0 |
| 13 | 0 | 134 | 1 | 255 | 1 | 376 | 0 | 497 | 0 | 618 | 1 | 739 | 1 |
| 14 | 0 | 135 | 0 | 256 | 0 | 377 | 0 | 498 | 0 | 619 | 1 | 740 | 1 |
| 15 | 0 | 136 | 1 | 257 | 0 | 378 | 1 | 499 | 0 | 620 | 0 | 741 | 1 |
| 16 | 0 | 137 | 0 | 258 | 1 | 379 | 1 | 500 | 0 | 621 | 0 | 742 | 0 |
| 17 | 0 | 138 | 0 | 259 | 0 | 380 | 1 | 501 | 0 | 622 | 1 | 743 | 0 |
| 18 | 1 | 139 | 1 | 260 | 0 | 381 | 1 | 502 | 0 | 623 | 0 | 744 | 0 |
| 19 | 1 | 140 | 1 | 261 | 1 | 382 | 0 | 503 | 0 | 624 | 1 | 745 | 0 |
| 20 | 0 | 141 | 1 | 262 | 0 | 383 | 1 | 504 | 1 | 625 | 0 | 746 | 0 |
| 21 | 1 | 142 | 1 | 263 | 0 | 384 | 0 | 505 | 1 | 626 | 0 | 747 | 1 |
| 22 | 1 | 143 | 0 | 264 | 1 | 385 | 0 | 506 | 0 | 627 | 1 | 748 | 0 |
| 23 | 1 | 144 | 1 | 265 | 0 | 386 | 0 | 507 | 1 | 628 | 0 | 749 | 0 |
| 24 | 0 | 145 | 0 | 266 | 0 | 387 | 1 | 508 | 0 | 629 | 1 | 750 | 1 |
| 25 | 1 | 146 | 0 | 267 | 1 | 388 | 0 | 509 | 1 | 630 | 1 | 751 | 0 |
| 26 | 1 | 147 | 1 | 268 | 1 | 389 | 0 | 510 | 0 | 631 | 1 | 752 | 1 |
| 27 | 0 | 148 | 0 | 269 | 1 | 390 | 1 | 511 | 0 | 632 | 0 | 753 | 1 |
| 28 | 1 | 149 | 1 | 270 | 1 | 391 | 1 | 512 | 0 | 633 | 0 | 754 | 1 |
| 29 | 0 | 150 | 1 | 271 | 1 | 392 | 1 | 513 | 1 | 634 | 0 | 755 | 0 |
| 30 | 0 | 151 | 1 | 272 | 1 | 393 | 0 | 514 | 0 | 635 | 0 | 756 | 1 |
| 31 | 0 | 152 | 1 | 273 | 0 | 394 | 1 | 515 | 1 | 636 | 0 | 757 | 1 |
| 32 | 0 | 153 | 1 | 274 | 1 | 395 | 1 | 516 | 0 | 637 | 1 | 758 | 0 |
| 33 | 0 | 154 | 0 | 275 | 0 | 396 | 1 | 517 | 1 | 638 | 0 | 759 | 0 |

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 34 | 0 | 155 | 0 | 276 | 0 | 397 | 0 | 518 | 1 | 639 | 1 | 760 | 1 |
| 35 | 0 | 156 | 0 | 277 | 0 | 398 | 1 | 519 | 0 | 640 | 0 | 761 | 1 |
| 36 | 0 | 157 | 0 | 278 | 0 | 399 | 0 | 520 | 0 | 641 | 1 | 762 | 0 |
| 37 | 1 | 158 | 0 | 279 | 0 | 400 | 1 | 521 | 0 | 642 | 0 | 763 | 1 |
| 38 | 0 | 159 | 0 | 280 | 1 | 401 | 0 | 522 | 1 | 643 | 1 | 764 | 1 |
| 39 | 1 | 160 | 1 | 281 | 0 | 402 | 0 | 523 | 0 | 644 | 1 | 765 | 1 |
| 40 | 1 | 161 | 0 | 282 | 0 | 403 | 0 | 524 | 0 | 645 | 0 | 766 | 1 |
| 41 | 0 | 162 | 0 | 283 | 0 | 404 | 0 | 525 | 0 | 646 | 1 | 767 | 1 |
| 42 | 1 | 163 | 1 | 284 | 1 | 405 | 0 | 526 | 0 | 647 | 1 | 768 | 1 |
| 43 | 1 | 164 | 1 | 285 | 0 | 406 | 1 | 527 | 1 | 648 | 1 | 769 | 0 |
| 44 | 0 | 165 | 0 | 286 | 0 | 407 | 0 | 528 | 1 | 649 | 1 | 770 | 1 |
| 45 | 1 | 166 | 0 | 287 | 0 | 408 | 0 | 529 | 0 | 650 | 1 | 771 | 1 |
| 46 | 0 | 167 | 1 | 288 | 0 | 409 | 0 | 530 | 1 | 651 | 1 | 772 | 0 |
| 47 | 1 | 168 | 0 | 289 | 1 | 410 | 0 | 531 | 1 | 652 | 1 | 773 | 1 |
| 48 | 1 | 169 | 0 | 290 | 1 | 411 | 1 | 532 | 0 | 653 | 1 | 774 | 0 |
| 49 | 0 | 170 | 1 | 291 | 1 | 412 | 1 | 533 | 1 | 654 | 1 | 775 | 1 |
| 50 | 1 | 171 | 1 | 292 | 0 | 413 | 0 | 534 | 1 | 655 | 0 | 776 | 0 |
| 51 | 1 | 172 | 1 | 293 | 0 | 414 | 0 | 535 | 1 | 656 | 1 | 777 | 0 |
| 52 | 0 | 173 | 0 | 294 | 0 | 415 | 1 | 536 | 1 | 657 | 1 | 778 | 0 |
| 53 | 1 | 174 | 0 | 295 | 0 | 416 | 0 | 537 | 1 | 658 | 0 | 779 | 1 |
| 54 | 0 | 175 | 1 | 296 | 1 | 417 | 1 | 538 | 0 | 659 | 1 | 780 | 1 |
| 55 | 0 | 176 | 0 | 297 | 0 | 418 | 1 | 539 | 0 | 660 | 0 | 781 | 1 |
| 56 | 1 | 177 | 1 | 298 | 0 | 419 | 0 | 540 | 1 | 661 | 1 | 782 | 0 |
| 57 | 1 | 178 | 0 | 299 | 1 | 420 | 1 | 541 | 0 | 662 | 1 | 783 | 0 |
| 58 | 0 | 179 | 1 | 300 | 0 | 421 | 0 | 542 | 1 | 663 | 0 | 784 | 0 |
| 59 | 0 | 180 | 1 | 301 | 1 | 422 | 1 | 543 | 1 | 664 | 0 | 785 | 0 |
| 60 | 1 | 181 | 1 | 302 | 0 | 423 | 1 | 544 | 1 | 665 | 1 | 786 | 1 |
| 61 | 0 | 182 | 1 | 303 | 1 | 424 | 1 | 545 | 1 | 666 | 1 | 787 | 1 |
| 62 | 1 | 183 | 1 | 304 | 1 | 425 | 1 | 546 | 0 | 667 | 0 | 788 | 0 |
| 63 | 0 | 184 | 1 | 305 | 1 | 426 | 1 | 547 | 0 | 668 | 0 | 789 | 1 |
| 64 | 0 | 185 | 0 | 306 | 0 | 427 | 1 | 548 | 1 | 669 | 0 | 790 | 1 |
| 65 | 0 | 186 | 0 | 307 | 0 | 428 | 1 | 549 | 0 | 670 | 1 | 791 | 1 |
| 66 | 0 | 187 | 0 | 308 | 0 | 429 | 1 | 550 | 0 | 671 | 1 | 792 | 1 |
| 67 | 0 | 188 | 0 | 309 | 1 | 430 | 1 | 551 | 1 | 672 | 0 | 793 | 1 |
| 68 | 0 | 189 | 0 | 310 | 0 | 431 | 1 | 552 | 0 | 673 | 1 | 794 | 1 |
| 69 | 1 | 190 | 1 | 311 | 0 | 432 | 0 | 553 | 0 | 674 | 0 | 795 | 1 |
| 70 | 0 | 191 | 0 | 312 | 1 | 433 | 1 | 554 | 0 | 675 | 0 | 796 | 1 |
| 71 | 0 | 192 | 1 | 313 | 1 | 434 | 0 | 555 | 1 | 676 | 0 | 797 | 1 |
| 72 | 0 | 193 | 0 | 314 | 1 | 435 | 0 | 556 | 1 | 677 | 0 | 798 | 0 |
| 73 | 1 | 194 | 0 | 315 | 0 | 436 | 0 | 557 | 0 | 678 | 1 | 799 | 0 |
| 74 | 1 | 195 | 1 | 316 | 0 | 437 | 0 | 558 | 1 | 679 | 1 | 800 | 1 |
| 75 | 0 | 196 | 1 | 317 | 0 | 438 | 0 | 559 | 0 | 680 | 0 | 801 | 1 |
| 76 | 0 | 197 | 1 | 318 | 1 | 439 | 0 | 560 | 0 | 681 | 0 | 802 | 1 |
| 77 | 1 | 198 | 0 | 319 | 0 | 440 | 1 | 561 | 0 | 682 | 1 | 803 | 1 |

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 78 | 1 | 199 | 1 | 320 | 0 | 441 | 0 | 562 | 1 | 683 | 0 | 804 | 1 |
| 79 | 1 | 200 | 0 | 321 | 0 | 442 | 1 | 563 | 0 | 684 | 1 | 805 | 0 |
| 80 | 1 | 201 | 0 | 322 | 0 | 443 | 1 | 564 | 1 | 685 | 1 | 806 | 1 |
| 81 | 1 | 202 | 1 | 323 | 0 | 444 | 0 | 565 | 1 | 686 | 1 | 807 | 0 |
| 82 | 0 | 203 | 0 | 324 | 0 | 445 | 0 | 566 | 1 | 687 | 0 | 808 | 0 |
| 83 | 0 | 204 | 1 | 325 | 1 | 446 | 0 | 567 | 1 | 688 | 0 | 809 | 0 |
| 84 | 1 | 205 | 0 | 326 | 0 | 447 | 0 | 568 | 1 | 689 | 0 | 810 | 0 |
| 85 | 0 | 206 | 0 | 327 | 1 | 448 | 0 | 569 | 1 | 690 | 0 | 811 | 0 |
| 86 | 1 | 207 | 1 | 328 | 0 | 449 | 0 | 570 | 0 | 691 | 1 | 812 | 0 |
| 87 | 1 | 208 | 1 | 329 | 0 | 450 | 0 | 571 | 0 | 692 | 0 | 813 | 1 |
| 88 | 0 | 209 | 1 | 330 | 1 | 451 | 1 | 572 | 0 | 693 | 1 | 814 | 0 |
| 89 | 0 | 210 | 0 | 331 | 1 | 452 | 0 | 573 | 0 | 694 | 0 | 815 | 0 |
| 90 | 1 | 211 | 0 | 332 | 1 | 453 | 1 | 574 | 0 | 695 | 1 | 816 | 1 |
| 91 | 0 | 212 | 0 | 333 | 0 | 454 | 0 | 575 | 0 | 696 | 0 | 817 | 1 |
| 92 | 0 | 213 | 1 | 334 | 0 | 455 | 1 | 576 | 1 | 697 | 1 | 818 | 1 |
| 93 | 1 | 214 | 1 | 335 | 0 | 456 | 1 | 577 | 0 | 698 | 1 | 819 | 0 |
| 94 | 1 | 215 | 1 | 336 | 0 | 457 | 0 | 578 | 0 | 699 | 0 | 820 | 0 |
| 95 | 1 | 216 | 0 | 337 | 0 | 458 | 1 | 579 | 0 | 700 | 0 | 821 | 1 |
| 96 | 1 | 217 | 0 | 338 | 0 | 459 | 0 | 580 | 1 | 701 | 0 | 822 | 1 |
| 97 | 1 | 218 | 0 | 339 | 0 | 460 | 0 | 581 | 1 | 702 | 1 | 823 | 1 |
| 98 | 0 | 219 | 0 | 340 | 0 | 461 | 1 | 582 | 0 | 703 | 0 | 824 | 1 |
| 99 | 0 | 220 | 1 | 341 | 1 | 462 | 1 | 583 | 1 | 704 | 0 | 825 | 0 |
| 100 | 0 | 221 | 1 | 342 | 1 | 463 | 0 | 584 | 1 | 705 | 0 | 826 | 0 |
| 101 | 1 | 222 | 1 | 343 | 1 | 464 | 0 | 585 | 0 | 706 | 0 | 827 | 1 |
| 102 | 1 | 223 | 1 | 344 | 1 | 465 | 0 | 586 | 0 | 707 | 0 | 828 | 0 |
| 103 | 0 | 224 | 0 | 345 | 0 | 466 | 1 | 587 | 0 | 708 | 1 | 829 | 1 |
| 104 | 1 | 225 | 0 | 346 | 1 | 467 | 1 | 588 | 1 | 709 | 0 | 830 | 0 |
| 105 | 0 | 226 | 1 | 347 | 0 | 468 | 1 | 589 | 0 | 710 | 0 | 831 | 1 |
| 106 | 0 | 227 | 1 | 348 | 0 | 469 | 0 | 590 | 0 | 711 | 1 | 832 | 1 |
| 107 | 1 | 228 | 0 | 349 | 1 | 470 | 0 | 591 | 1 | 712 | 1 | 833 | 1 |
| 108 | 1 | 229 | 1 | 350 | 1 | 471 | 1 | 592 | 0 | 713 | 1 | 834 | 1 |
| 109 | 0 | 230 | 1 | 351 | 1 | 472 | 1 | 593 | 0 | 714 | 1 | 835 | 0 |
| 110 | 1 | 231 | 0 | 352 | 0 | 473 | 0 | 594 | 0 | 715 | 0 | 836 | 0 |
| 111 | 1 | 232 | 0 | 353 | 1 | 474 | 1 | 595 | 0 | 716 | 0 | 837 | 0 |
| 112 | 0 | 233 | 1 | 354 | 1 | 475 | 1 | 596 | 0 | 717 | 0 | 838 | 1 |
| 113 | 0 | 234 | 1 | 355 | 0 | 476 | 0 | 597 | 1 | 718 | 1 | 839 | 1 |
| 114 | 1 | 235 | 0 | 356 | 1 | 477 | 0 | 598 | 0 | 719 | 0 | 840 | 1 |
| 115 | 1 | 236 | 0 | 357 | 1 | 478 | 1 | 599 | 0 | 720 | 1 | 841 | 1 |
| 116 | 1 | 237 | 0 | 358 | 1 | 479 | 1 | 600 | 0 | 721 | 1 | 842 | 0 |
| 117 | 0 | 238 | 1 | 359 | 1 | 480 | 1 | 601 | 1 | 722 | 1 | 843 | 1 |
| 118 | 0 | 239 | 1 | 360 | 1 | 481 | 1 | 602 | 1 | 723 | 0 | 844 | 1 |
| 119 | 1 | 240 | 0 | 361 | 0 | 482 | 0 | 603 | 1 | 724 | 1 | | |
| 120 | 1 | 241 | 1 | 362 | 0 | 483 | 1 | 604 | 0 | 725 | 0 | | |

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