PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z+1,z^5+z^2+z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z+1,z^5+z^2+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z+1, z^5+z^2+z+1)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{16}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^6+z^5+z^4+z^3+1,\\
p_1(z)&=z^{19}+z^{18}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^2,\\
p_2(z)&=z^{20}+z^{18}+z^{16}+z^{14}+z^{12}+1,\\
p_3(z)&=z^{19}+z^{18}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^2,\\
p_4(z)&=z^{18}+z^{16}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^6+z^5+z^3+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{14}+z^{13}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^3+1,\\
p_1(z)&=z^{19}+z^{18}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^2,\\
p_2(z)&=z^{20}+z^{18}+z^{16}+z^{14}+z^{12}+1,\\
p_3(z)&=z^{19}+z^{18}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^7+z^2,\\
p_4(z)&=z^{18}+z^{13}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z) \right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x, y+z) \right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{5u} M^e[:u] + x^u M^e[:6u] \\ & + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{6u} M^e[:u] + x^{3u} M^e[:6u] \\ & + x^{4u} M^e[:5u] + x^{5u} M^e[:4u] + x^{8u} M^e[:u] + x^{4u} M^e[:6u] \\ & + x^{5u} M^e[:5u] + x^{6u} M^e[:4u] + x^{9u} M^e[:u] + x^{5u} M^e[:6u] \\ & + x^{6u} M^e[:5u] + x^{7u} M^e[:4u] + x^{10u} M^e[:u] + x^{8u} M^e[:4u] \\ & + x^{9u} M^e[:3u] + x^{10u} M^e[:2u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{6u} + x^{7u} + x^{9u} + x^{10u} + x^{11u})R^e,\\
W^e[:12u] &= W^e[:6u] + x^uW^e[:5u] + x^{2u}W^e[:4u] + x^{5u}W^e[:u] + x^uW^e[:6u]\\
&+ x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{6u}W^e[:u] + x^{3u}W^e[:6u]\\
&+ x^{4u}W^e[:5u] + x^{5u}W^e[:4u] + x^{8u}W^e[:u] + x^{4u}W^e[:6u]\\
&+ x^{5u}W^e[:5u] + x^{6u}W^e[:4u] + x^{9u}W^e[:u] + x^{5u}W^e[:6u]\\
&+ x^{6u}W^e[:5u] + x^{7u}W^e[:4u] + x^{10u}W^e[:u] + x^{8u}W^e[:4u]\\
&+ x^{9u}W^e[:3u] + x^{10u}W^e[:2u]\\
&+ (x^{6u} + x^{7u} + x^{9u} + x^{10u} + x^{11u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^uM^o[:5u] + x^{2u}M^o[:4u] + x^{5u}M^o[:u] + x^uM^o[:6u]\\
&+ x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{6u}M^o[:u] + x^{3u}M^o[:6u]\\
&+ x^{4u}M^o[:5u] + x^{5u}M^o[:4u] + x^{8u}M^o[:u] + x^{4u}M^o[:6u]\\
&+ x^{5u}M^o[:5u] + x^{6u}M^o[:4u] + x^{9u}M^o[:u] + x^{5u}M^o[:6u]\\
&+ x^{6u}M^o[:5u] + x^{7u}M^o[:4u] + x^{10u}M^o[:u] + x^{8u}M^o[:4u]\\
&+ x^{9u}M^o[:3u] + x^{10u}M^o[:2u]\\
&+ (x^{6u} + x^{7u} + x^{9u} + x^{10u} + x^{11u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^uW^o[:5u] + x^{2u}W^o[:4u] + x^{5u}W^o[:u] + x^uW^o[:6u]\\
&+ x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{6u}W^o[:u] + x^{3u}W^o[:6u]\\
&+ x^{4u}W^o[:5u] + x^{5u}W^o[:4u] + x^{8u}W^o[:u] + x^{4u}W^o[:6u]\\
&+ x^{5u}W^o[:5u] + x^{6u}W^o[:4u] + x^{9u}W^o[:u] + x^{5u}W^o[:6u]\\
&+ x^{6u}W^o[:5u] + x^{7u}W^o[:4u] + x^{10u}W^o[:u] + x^{8u}W^o[:4u]\\
&+ x^{9u}W^o[:3u] + x^{10u}W^o[:2u]\\
&+ (x^{6u} + x^{7u} + x^{9u} + x^{10u} + x^{11u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:12u] &= T[:6u] + x^uT[:5u] + x^{2u}T[:4u] + x^{5u}T[:u] + x^uT[:6u] + x^{2u}T[:5u]\\
&+ x^{3u}T[:4u] + x^{6u}T[:u] + x^{3u}T[:6u] + x^{4u}T[:5u] + x^{5u}T[:4u]\\
&+ x^{8u}T[:u] + x^{4u}T[:6u] + x^{5u}T[:5u] + x^{6u}T[:4u] + x^{9u}T[:u]\\
&+ x^{5u}T[:6u] + x^{6u}T[:5u] + x^{7u}T[:4u] + x^{10u}T[:u] + x^{8u}T[:4u]\\
&+ x^{9u}T[:3u] + x^{10u}T[:2u] + (x^{6u} + x^{7u} + x^{9u} + x^{10u} + x^{11u}).
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$
\begin{aligned}
x^6u &= x^{6u}T[:u] + x^{5u}T[u:2u] + x^{2u}T[4u:5u] + x^uT[5u:6u]\\
&+ T[6u:7u],
\end{aligned}
$$

$$\begin{aligned}
x^7u &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + T[7u:8u],\\
0 &= x^{7u}T[u:2u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] + T[8u:9u],\\
x^9u &= x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{4u}T[5u:6u] + T[9u:10u],\\
x^10u &= x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u]\\
&\quad + x^{5u}T[5u:6u] + T[10u:11u],\\
x^11u &= x^{10u}T[u:2u] + x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$
$$\begin{aligned}0 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1010w]_2],\end{aligned} \tag{2.11}$$
$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$
$$1 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$
$$1 = T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$\begin{aligned}1 &= T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1010w]_2],\end{aligned} \tag{2.17}$$
$$1 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1513 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101))\\ &\quad + \tau(A(s,110)),\end{aligned} \tag{2.19}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,111)), \tag{2.20}$$
$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.21}$$
$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.22}$$
$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)),\end{aligned} \tag{2.23}$$
$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 1 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)), \end{aligned} \tag{2.25}$$

$$1 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,111)), \tag{2.26}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.27}$$

$$1 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.28}$$

$$\begin{aligned} 1 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.29}$$

$$1 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{22}), E_{22}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \le k \le 11$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^u M^e[:5u] + x^{2u} M^e[:4u] + x^{5u} M^e[:u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^u W^e[:5u] + x^{2u} W^e[:4u] + x^{5u} W^e[:u] + x^{6u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^u M^o[:5u] + x^{2u} M^o[:4u] + x^{5u} M^o[:u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^u W^o[:5u] + x^{2u} W^o[:4u] + x^{5u} W^o[:u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:6v] + x^v W^e[:5v] + x^{2v} W^e[:4v] + x^{5v} W^e[:v] + x^{6u} R^e\big) \times \big(\\ &M^e[:6v] + x^v M^e[:5v] + x^{2v} M^e[:4v] + x^{5v} M^e[:v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$\begin{aligned} W^e[:12v]M^e[:12v] + x^{2v} W^e[:10v] M^e[:10v] + x^{4v} W^e[:8v] M^e[:8v] \\ + x^{10v} W^e[:2v] M^e[:2v]. \end{aligned}$$

For all $n \le 12$, replace the occurrences of $W^e[: n \cdot v]$ and $M^e[: n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^4, \\ q_1(x) &= x^{18} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^2 + x, \\ q_2(x) &= x^{20} + x^8 + x^6 + x^4 + x^2 + 1, \\ q_3(x) &= x^{18} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^2 + x, \\ q_4(x) &= x^{20} + x^{17} + x^{15} + x^{14} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^4 + x^2. \end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate

$f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{110} + x^{96} + x^{90} + x^{88} + x^{78} + x^{70} + x^{68} + x^{66} + x^{54} \\
&\quad + x^{52} + x^{48} + x^{46} + x^{34} + x^{28} + x^{26} + x^{22} + x^{20} + x^{16} + x^{12}, \\
p_3(x) &= x^{116} + x^{110} + x^{108} + x^{104} + x^{100} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} \\
&\quad + x^{84} + x^{82} + x^{80} + x^{74} + x^{68} + x^{64} + x^{58} + x^{46} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{28} + x^{24} + x^{20} + x^{14} + x^{8}, \\
p_6(x) &= x^{114} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{94} + x^{90} + x^{88} \\
&\quad + x^{84} + x^{74} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{50} + x^{46} + x^{44} + x^{40} \\
&\quad + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{12} + x^{10} + x^{6} \\
&\quad + x^{2} + 1, \\
p_9(x) &= x^{116} + x^{112} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{82} + x^{68} + x^{64} \\
&\quad + x^{60} + x^{58} + x^{48} + x^{46} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{22} + x^{20} + x^{16} + x^{14} + x^{10} + x^{2}, \\
p_{12}(x) &= x^{120} + x^{112} + x^{96} + x^{88} + x^{80} + x^{72} + x^{48} + x^{24} + x^{16} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{99} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{81} \\
&\quad + x^{73} + x^{70} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{50} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{31} \\
&\quad + x^{29} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_3(x) &= x^{90} + x^{88} + x^{87} + x^{85} + x^{78} + x^{75} + x^{68} + x^{65} + x^{50} + x^{48} + x^{47} \\
&\quad + x^{45} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} + x^{19} \\
&\quad + x^{12} + x^{9}, \\
p_6(x) &= x^{96} + x^{94} + x^{93} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{37} + x^{36} \\
&\quad + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{16} \\
&\quad + x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6},
\end{aligned}$$

$$\begin{aligned}
p_9(x) = {} & x^{102} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} \\
& + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} \\
& + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} \\
& + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} \\
& + x^{39} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} \\
& + x^{11} + x^{8} + x^{6} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85} \\
& + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} \\
& + x^{60} + x^{58} + x^{57} + x^{56} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{38} + x^{37} \\
& + x^{34} + x^{33} + x^{30} + x^{29} + x^{26} + x^{25} + x^{22} + x^{21} + x^{18} + x^{17} + x^{14} \\
& + x^{13} + x^{12} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{108} + x^{106} + x^{105} + x^{99} + x^{93} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} + x^{81} \\
& + x^{73} + x^{70} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{53} \\
& + x^{52} + x^{51} + x^{50} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{31} \\
& + x^{29} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_3(x) = {} & x^{90} + x^{88} + x^{87} + x^{85} + x^{78} + x^{75} + x^{68} + x^{65} + x^{50} + x^{48} + x^{47} \\
& + x^{45} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} + x^{19} \\
& + x^{12} + x^{9}, \\
p_6(x) = {} & x^{96} + x^{94} + x^{93} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{74} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} \\
& + x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{37} + x^{36} \\
& + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{16} \\
& + x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{102} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} \\
& + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} \\
& + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57} \\
& + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} \\
& + x^{39} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13} \\
& + x^{11} + x^{8} + x^{6} + x^{5} + x^{3}, \\
p_{12}(x) = {} & x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85} \\
& + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} \\
& + x^{60} + x^{58} + x^{57} + x^{56} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{38} + x^{37} \\
& + x^{34} + x^{33} + x^{30} + x^{29} + x^{26} + x^{25} + x^{22} + x^{21} + x^{18} + x^{17} + x^{14}
\end{aligned}$$

$$+ x^{13} + x^{12} + x^8 + x^6 + x^5 + x^2 + x + 1,$$

and the initial terms are [0, 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{92} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64} + x^{62} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{34} + x^{26} + x^{24}, \\
p_3(x) &= x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{54} + x^{44} + x^{36} + x^{34} + x^{32} + x^{22} + x^{14} + x^8 + x^6, \\
p_6(x) &= x^{90} + x^{86} + x^{84} + x^{80} + x^{72} + x^{70} + x^{66} + x^{62} + x^{58} + x^{50} + x^{48} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{36} + x^{28} + x^{26} + x^{20} + x^{16} + x^{12} + x^4 + 1, \\
p_9(x) &= x^{92} + x^{86} + x^{78} + x^{74} + x^{72} + x^{66} + x^{62} + x^{60} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{38} + x^{36} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{10} + x^6 \\
&\quad + x^4 + x^2, \\
p_{12}(x) &= x^{96} + x^{92} + x^{90} + x^{84} + x^{82} + x^{76} + x^{74} + x^{68} + x^{66} + x^{64} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{36} + x^{34} + x^{28} + x^{26} + x^{20} + x^{18} + x^{12} + x^{10} + x^4 \\
&\quad + x^2 + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{96} + x^{92} + x^{88} + x^{84} + x^{80} + x^{78} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{60} + x^{56} + x^{52} + x^{46} + x^{36} + x^{32} + x^{30} + x^{28} + x^{16} + x^{14} \\
&\quad + x^{12}, \\
p_3(x) &= x^{92} + x^{88} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{66} + x^{60} + x^{54} \\
&\quad + x^{52} + x^{50} + x^{46} + x^{44} + x^{36} + x^{34} + x^{26} + x^{22} + x^{20} + x^{12} + x^8, \\
p_6(x) &= x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{64} \\
&\quad + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} + x^{44} + x^{42} + x^{40} + x^{38} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{18} + x^{12} + x^{10} + 1, \\
p_9(x) &= x^{92} + x^{86} + x^{78} + x^{74} + x^{72} + x^{66} + x^{62} + x^{60} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{38} + x^{36} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{10} + x^6 \\
&\quad + x^4 + x^2, \\
p_{12}(x) &= x^{96} + x^{92} + x^{90} + x^{84} + x^{82} + x^{76} + x^{74} + x^{68} + x^{66} + x^{64} + x^{48} \\
&\quad + x^{44} + x^{42} + x^{36} + x^{34} + x^{28} + x^{26} + x^{20} + x^{18} + x^{12} + x^{10} + x^4 \\
&\quad + x^2 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{99} + x^{96} + x^{94} + x^{91} + x^{87} + x^{86} + x^{85} + x^{83} \\
&\quad + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65} \\
&\quad + x^{62} + x^{61} + x^{56} + x^{55} + x^{51} + x^{49} + x^{48} + x^{46} + x^{43} + x^{38} + x^{37}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{34} + x^{32} + x^{31} + x^{29} + x^{26} + x^{24},\\
p_3(x) &= x^{90} + x^{88} + x^{87} + x^{85} + x^{78} + x^{75} + x^{68} + x^{65} + x^{50} + x^{48} + x^{47}\\
&+ x^{45} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} + x^{19}\\
&+ x^{12} + x^{9},\\
p_6(x) &= x^{96} + x^{94} + x^{93} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{74} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56}\\
&+ x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{37} + x^{36}\\
&+ x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{102} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72}\\
&+ x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{61} + x^{60} + x^{58} + x^{57}\\
&+ x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{20} + x^{17} + x^{16} + x^{14} + x^{13}\\
&+ x^{11} + x^{8} + x^{6} + x^{5} + x^{3},\\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{86} + x^{85}\\
&+ x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64}\\
&+ x^{60} + x^{58} + x^{57} + x^{56} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{38} + x^{37}\\
&+ x^{34} + x^{33} + x^{30} + x^{29} + x^{26} + x^{25} + x^{22} + x^{21} + x^{18} + x^{17} + x^{14}\\
&+ x^{13} + x^{12} + x^{8} + x^{6} + x^{5} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{99} + x^{96} + x^{94} + x^{91} + x^{87} + x^{86} + x^{85} + x^{83}\\
&+ x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{73} + x^{71} + x^{70} + x^{69} + x^{67} + x^{65}\\
&+ x^{62} + x^{61} + x^{56} + x^{55} + x^{51} + x^{49} + x^{48} + x^{46} + x^{43} + x^{38} + x^{37}\\
&+ x^{34} + x^{32} + x^{31} + x^{29} + x^{26} + x^{24},\\
p_3(x) &= x^{90} + x^{88} + x^{87} + x^{85} + x^{78} + x^{75} + x^{68} + x^{65} + x^{50} + x^{48} + x^{47}\\
&+ x^{45} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{28} + x^{25} + x^{22} + x^{19}\\
&+ x^{12} + x^{9},\\
p_6(x) &= x^{96} + x^{94} + x^{93} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77}\\
&+ x^{74} + x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56}\\
&+ x^{55} + x^{54} + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{41} + x^{37} + x^{36}\\
&+ x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{102} + x^{100} + x^{99} + x^{97} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{75}+x^{74}+x^{72}\\
&+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{64}+x^{61}+x^{60}+x^{58}+x^{57}\\
&+x^{55}+x^{54}+x^{52}+x^{51}+x^{49}+x^{48}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}\\
&+x^{39}+x^{37}+x^{34}+x^{31}+x^{28}+x^{25}+x^{20}+x^{17}+x^{16}+x^{14}+x^{13}\\
&+x^{11}+x^{8}+x^{6}+x^{5}+x^{3},\\
p_{12}(x)=\ &x^{108}+x^{106}+x^{105}+x^{104}+x^{96}+x^{94}+x^{93}+x^{90}+x^{89}+x^{86}+x^{85}\\
&+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}+x^{72}+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}\\
&+x^{60}+x^{58}+x^{57}+x^{56}+x^{48}+x^{46}+x^{45}+x^{42}+x^{41}+x^{38}+x^{37}\\
&+x^{34}+x^{33}+x^{30}+x^{29}+x^{26}+x^{25}+x^{22}+x^{21}+x^{18}+x^{17}+x^{14}\\
&+x^{13}+x^{12}+x^{8}+x^{6}+x^{5}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x)=\ &x^{120}+x^{114}+x^{112}+x^{110}+x^{100}+x^{98}+x^{96}+x^{94}+x^{90}+x^{88}+x^{86}\\
&+x^{80}+x^{78}+x^{76}+x^{66}+x^{58}+x^{54}+x^{52}+x^{48}+x^{44}+x^{40}+x^{38}\\
&+x^{36},\\
p_3(x)=\ &x^{116}+x^{110}+x^{104}+x^{98}+x^{94}+x^{92}+x^{90}+x^{84}+x^{82}+x^{80}+x^{74}\\
&+x^{68}+x^{64}+x^{60}+x^{58}+x^{50}+x^{48}+x^{46}+x^{40}+x^{30}+x^{28}+x^{26}\\
&+x^{24}+x^{22}+x^{18}+x^{6},\\
p_6(x)=\ &x^{114}+x^{106}+x^{102}+x^{100}+x^{94}+x^{92}+x^{90}+x^{88}+x^{86}+x^{74}+x^{66}\\
&+x^{64}+x^{62}+x^{58}+x^{56}+x^{46}+x^{42}+x^{40}+x^{38}+x^{36}+x^{32}+x^{28}\\
&+x^{22}+x^{18}+x^{14}+x^{12}+x^{10}+x^{4}+x^{2}+1,\\
p_9(x)=\ &x^{116}+x^{112}+x^{106}+x^{104}+x^{100}+x^{98}+x^{96}+x^{94}+x^{82}+x^{68}+x^{64}\\
&+x^{60}+x^{58}+x^{48}+x^{46}+x^{40}+x^{38}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}\\
&+x^{22}+x^{20}+x^{16}+x^{14}+x^{10}+x^{2},\\
p_{12}(x)=\ &x^{120}+x^{112}+x^{96}+x^{88}+x^{80}+x^{72}+x^{48}+x^{24}+x^{16}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)=\ &x^{108}+x^{106}+x^{105}+x^{104}+x^{102}+x^{100}+x^{99}+x^{97}+x^{96}+x^{95}\\
&+x^{93}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}+x^{78}\\
&+x^{73}+x^{72}+x^{71}+x^{70}+x^{63}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}\\
&+x^{52}+x^{48}+x^{47}+x^{43}+x^{40}+x^{39}+x^{38}+x^{34}+x^{32}+x^{31}+x^{30}\\
&+x^{29}+x^{28}+x^{26}+x^{25}+x^{22}+x^{21}+x^{20}+x^{15}+x^{13}+x^{12},\\
p_3(x)=\ &x^{100}+x^{98}+x^{97}+x^{96}+x^{94}+x^{92}+x^{91}+x^{89}+x^{87}+x^{86}+x^{84}\\
&+x^{82}+x^{81}+x^{78}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{67}\\
&+x^{66}+x^{65}+x^{63}+x^{59}+x^{57}+x^{56}+x^{49}+x^{48}+x^{42}+x^{39}+x^{35}
\end{aligned}
$$

$$+ x^{32} + x^{29} + x^{28} + x^{24} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} + x^{9} + x^{8},$$

$$p_6(x) = x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{61} + x^{60} + x^{57} + x^{55} + x^{52} + x^{51} + x^{49} + x^{43} + x^{40} + x^{37} + x^{33} + x^{32} + x^{29} + x^{25} + x^{23} + x^{22} + x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^{9} + x^{7} + x^{6} + x^{4} + x^{2} + x + 1,$$

$$p_9(x) = x^{100} + x^{98} + x^{97} + x^{96} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} + x^{72} + x^{70} + x^{69} + x^{68} + x^{52} + x^{50} + x^{49} + x^{48} + x^{40} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{24} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{8} + x^{6} + x^{5} + x^{4},$$

$$p_{12}(x) = x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{80} + x^{78} + x^{77} + x^{76} + x^{68} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58} + x^{57} + x^{56} + x^{52} + x^{50} + x^{49} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{32} + x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14} + x^{13} + x^{12} + x^{4} + x^{2} + x + 1,$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0].

For $\phi(M^o, 0, 1)$:

$$p_0(x) = x^{102} + x^{100} + x^{96} + x^{95} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{81} + x^{80} + x^{79} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{61} + x^{58} + x^{56} + x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18},$$

$$p_3(x) = x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66} + x^{65} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{10} + x^{9},$$

$$p_6(x) = x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{54} + x^{46} + x^{42} + x^{40} + x^{34} + x^{30} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} + x^{6},$$

$$p_9(x) = x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{48} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{12} + x^{11} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{3},$$

$$p_{12}(x) = x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66}$$

$$+ x^{64} + x^{54} + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{26} + x^{24} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2} + 1,$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{120} + x^{116} + x^{114} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} \\
& + x^{102} + x^{101} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{87} + x^{86} + x^{82} + x^{81} \\
& + x^{80} + x^{78} + x^{77} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} \\
& + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
& + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{28} \\
& + x^{27} + x^{25} + x^{24}, \\
p_3(x) = {} & x^{102} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{85} + x^{84} + x^{82} + x^{81} \\
& + x^{80} + x^{79} + x^{76} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{57} \\
& + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} + x^{46} + x^{45} + x^{44} + x^{42} + x^{39} + x^{37} \\
& + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24} + x^{23} + x^{20} + x^{15} + x^{14} \\
& + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{108} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68} \\
& + x^{66} + x^{52} + x^{44} + x^{42} + x^{28} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6}, \\
p_9(x) = {} & x^{114} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96} \\
& + x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80} \\
& + x^{79} + x^{78} + x^{74} + x^{73} + x^{72} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{57} \\
& + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} \\
& + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} \\
& + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{16} + x^{15} + x^{14} + x^{10} + x^{9} \\
& + x^{8} + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{120} + x^{104} + x^{96} + x^{92} + x^{88} + x^{84} + x^{76} + x^{68} + x^{64} + x^{56} + x^{48} \\
& + x^{44} + x^{36} + x^{28} + x^{24} + x^{20} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{90} \\
& + x^{87} + x^{86} + x^{85} + x^{84} + x^{78} + x^{75} + x^{73} + x^{71} + x^{70} + x^{65} + x^{64} \\
& + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{45} + x^{44} \\
& + x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{33} + x^{30}, \\
p_3(x) = {} & x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{87} + x^{85} + x^{82} + x^{81} \\
& + x^{80} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{64} \\
& + x^{63} + x^{59} + x^{57} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{21} + x^{20}\\
&+ x^{19} + x^{17} + x^{13} + x^{12},\\
p_6(x) &= x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{91} + x^{90} + x^{87} + x^{86} + x^{80} + x^{78}\\
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63}\\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{51} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24}\\
&+ x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{9} + x^{6} + x^{5} + x^{2}\\
&+ x + 1,\\
p_9(x) &= x^{100} + x^{98} + x^{97} + x^{96} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} + x^{72}\\
&+ x^{70} + x^{69} + x^{68} + x^{52} + x^{50} + x^{49} + x^{48} + x^{40} + x^{38} + x^{37} + x^{34}\\
&+ x^{33} + x^{32} + x^{24} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{8} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{90} + x^{89}\\
&+ x^{88} + x^{80} + x^{78} + x^{77} + x^{76} + x^{68} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58}\\
&+ x^{57} + x^{56} + x^{52} + x^{50} + x^{49} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{32}\\
&+ x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14} + x^{13} + x^{12} + x^{4} + x^{2} + x\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0].
For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{100} + x^{99} + x^{97} + x^{96} + x^{95}\\
&+ x^{93} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79} + x^{78}\\
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{53}\\
&+ x^{52} + x^{48} + x^{47} + x^{43} + x^{40} + x^{39} + x^{38} + x^{34} + x^{32} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{15} + x^{13} + x^{12},\\
p_3(x) &= x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84}\\
&+ x^{82} + x^{81} + x^{78} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{63} + x^{59} + x^{57} + x^{56} + x^{49} + x^{48} + x^{42} + x^{39} + x^{35}\\
&+ x^{32} + x^{29} + x^{28} + x^{24} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{9} + x^{8},\\
p_6(x) &= x^{102} + x^{99} + x^{98} + x^{95} + x^{94} + x^{93} + x^{91} + x^{89} + x^{88} + x^{87} + x^{84}\\
&+ x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65}\\
&+ x^{61} + x^{60} + x^{57} + x^{55} + x^{52} + x^{51} + x^{49} + x^{43} + x^{40} + x^{37} + x^{33}\\
&+ x^{32} + x^{29} + x^{25} + x^{23} + x^{22} + x^{18} + x^{17} + x^{16} + x^{12} + x^{10} + x^{9}\\
&+ x^{7} + x^{6} + x^{4} + x^{2} + x + 1,\\
p_9(x) &= x^{100} + x^{98} + x^{97} + x^{96} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} + x^{72}\\
&+ x^{70} + x^{69} + x^{68} + x^{52} + x^{50} + x^{49} + x^{48} + x^{40} + x^{38} + x^{37} + x^{34}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{33} + x^{32} + x^{24} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{8} + x^{6} + x^{5} + x^{4},\\
p_{12}(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{90} + x^{89}\\
&+ x^{88} + x^{80} + x^{78} + x^{77} + x^{76} + x^{68} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58}\\
&+ x^{57} + x^{56} + x^{52} + x^{50} + x^{49} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{32}\\
&+ x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14} + x^{13} + x^{12} + x^{4} + x^{2} + x\\
&+ 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0].
For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{120} + x^{116} + x^{114} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{98} + x^{96} + x^{94} + x^{90} + x^{87} + x^{86} + x^{82} + x^{81}\\
&+ x^{80} + x^{78} + x^{77} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62}\\
&+ x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49}\\
&+ x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{28}\\
&+ x^{27} + x^{25} + x^{24},\\
p_3(x) &= x^{102} + x^{97} + x^{96} + x^{93} + x^{92} + x^{91} + x^{89} + x^{85} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{76} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{57}\\
&+ x^{56} + x^{53} + x^{52} + x^{51} + x^{49} + x^{46} + x^{45} + x^{44} + x^{42} + x^{39} + x^{37}\\
&+ x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{27} + x^{24} + x^{23} + x^{20} + x^{15} + x^{14}\\
&+ x^{12} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{108} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{78} + x^{76} + x^{72} + x^{70} + x^{68}\\
&+ x^{66} + x^{52} + x^{44} + x^{42} + x^{28} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8} + x^{6},\\
p_9(x) &= x^{114} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} + x^{101} + x^{99} + x^{98} + x^{96}\\
&+ x^{95} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{80}\\
&+ x^{79} + x^{78} + x^{74} + x^{73} + x^{72} + x^{68} + x^{67} + x^{66} + x^{61} + x^{60} + x^{57}\\
&+ x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28}\\
&+ x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{16} + x^{15} + x^{14} + x^{10} + x^{9}\\
&+ x^{8} + x^{4} + x^{3},\\
p_{12}(x) &= x^{120} + x^{104} + x^{96} + x^{92} + x^{88} + x^{84} + x^{76} + x^{68} + x^{64} + x^{56} + x^{48}\\
&+ x^{44} + x^{36} + x^{28} + x^{24} + x^{20} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0].
For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{102} + x^{100} + x^{96} + x^{95} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{81}\\
&+ x^{80} + x^{79} + x^{73} + x^{71} + x^{69} + x^{68} + x^{67} + x^{64} + x^{61} + x^{58} + x^{56}\\
&+ x^{53} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{38} + x^{37} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{24} + x^{23}\\
&+ x^{22} + x^{21} + x^{18},\\
p_3(x) &= x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{66}\\
&+ x^{65} + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{17} + x^{16} + x^{15} + x^{13} + x^{10}\\
&+ x^{9},\\
p_6(x) &= x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{60}\\
&+ x^{56} + x^{54} + x^{46} + x^{42} + x^{40} + x^{34} + x^{30} + x^{22} + x^{20} + x^{18} + x^{16}\\
&+ x^{14} + x^{6},\\
p_9(x) &= x^{96} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80}\\
&+ x^{78} + x^{76} + x^{75} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} + x^{48} + x^{46}\\
&+ x^{44} + x^{43} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{28}\\
&+ x^{27} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{14} + x^{12} + x^{11}\\
&+ x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{102} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66}\\
&+ x^{64} + x^{54} + x^{52} + x^{50} + x^{48} + x^{42} + x^{40} + x^{26} + x^{24} + x^{10} + x^{8}\\
&+ x^{6} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{96} + x^{95} + x^{94} + x^{90}\\
&+ x^{87} + x^{86} + x^{85} + x^{84} + x^{78} + x^{75} + x^{73} + x^{71} + x^{70} + x^{65} + x^{64}\\
&+ x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{51} + x^{48} + x^{45} + x^{44}\\
&+ x^{42} + x^{41} + x^{40} + x^{39} + x^{36} + x^{34} + x^{33} + x^{30},\\
p_3(x) &= x^{100} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{87} + x^{85} + x^{82} + x^{81}\\
&+ x^{80} + x^{78} + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{64}\\
&+ x^{63} + x^{59} + x^{57} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{40} + x^{39} + x^{37} + x^{36} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{21} + x^{20}\\
&+ x^{19} + x^{17} + x^{13} + x^{12},\\
p_6(x) &= x^{102} + x^{100} + x^{99} + x^{97} + x^{95} + x^{91} + x^{90} + x^{87} + x^{86} + x^{80} + x^{78}\\
&+ x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{64} + x^{63}\\
&+ x^{62} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{51} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{43} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24}\\
&+ x^{23} + x^{22} + x^{20} + x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{9} + x^{6} + x^{5} + x^{2}\\
&+ x + 1,\\
p_9(x) &= x^{100} + x^{98} + x^{97} + x^{96} + x^{88} + x^{86} + x^{85} + x^{82} + x^{81} + x^{80} + x^{72}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{70} + x^{69} + x^{68} + x^{52} + x^{50} + x^{49} + x^{48} + x^{40} + x^{38} + x^{37} + x^{34} \\
&+ x^{33} + x^{32} + x^{24} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{8} + x^{6} + x^{5} + x^{4}, \\
p_{12}(x) = {}& x^{108} + x^{106} + x^{105} + x^{104} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{90} + x^{89} \\
&+ x^{88} + x^{80} + x^{78} + x^{77} + x^{76} + x^{68} + x^{66} + x^{65} + x^{64} + x^{60} + x^{58} \\
&+ x^{57} + x^{56} + x^{52} + x^{50} + x^{49} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{32} \\
&+ x^{30} + x^{29} + x^{26} + x^{25} + x^{24} + x^{16} + x^{14} + x^{13} + x^{12} + x^{4} + x^{2} + x \\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	379	[182, 652]	758	[1115, 1116]	1137	[943, 1356]
1	[3, 4]	380	[653, 654]	759	[1117, 1118]	1138	[1357, 300]
2	[5, 6]	381	[561, 655]	760	[144, 59]	1139	[1338, 812]
3	[7, 8]	382	[656, 657]	761	[1090, 1119]	1140	[104, 956]
4	[9, 10]	383	[658, 590]	762	[463, 168]	1141	[339, 1358]
5	[11, 12]	384	[659, 561]	763	[641, 1120]	1142	[418, 873]
6	[13, 14]	385	[660, 661]	764	[1121, 151]	1143	[1359, 1360]
7	[15, 16]	386	[662, 585]	765	[663, 588]	1144	[1044, 1361]
8	[17, 18]	387	[663, 11]	766	[1122, 136]	1145	[1362, 1363]
9	[19, 20]	388	[467, 596]	767	[1083, 648]	1146	[1364, 246]
10	[18, 21]	389	[664, 665]	768	[529, 440]	1147	[788, 1365]
11	[22, 23]	390	[666, 667]	769	[52, 1123]	1148	[1366, 85]
12	[24, 25]	391	[208, 668]	770	[607, 58]	1149	[694, 1367]
13	[26, 27]	392	[669, 670]	771	[1068, 1090]	1150	[1295, 1368]
14	[28, 29]	393	[209, 669]	772	[619, 39]	1151	[945, 797]
15	[30, 31]	394	[671, 671]	773	[123, 430]	1152	[1369, 266]
16	[32, 33]	395	[672, 491]	774	[130, 1124]	1153	[1370, 1371]
17	[34, 35]	396	[207, 672]	775	[1125, 120]	1154	[879, 1372]
18	[36, 37]	397	[206, 670]	776	[205, 1126]	1155	[951, 1373]
19	[38, 39]	398	[490, 209]	777	[219, 1127]	1156	[1374, 1031]
20	[40, 41]	399	[668, 206]	778	[497, 1088]	1157	[1311, 1375]
21	[42, 43]	400	[671, 673]	779	[490, 668]	1158	[1376, 1312]
22	[44, 45]	401	[669, 207]	780	[147, 444]	1159	[994, 1017]
23	[46, 47]	402	[211, 490]	781	[628, 1061]	1160	[1377, 1378]
24	[48, 49]	403	[674, 675]	782	[430, 1128]	1161	[1379, 1380]
25	[50, 51]	404	[200, 192]	783	[432, 1129]	1162	[320, 101]
26	[52, 53]	405	[676, 498]	784	[1130, 1131]	1163	[1030, 1285]
27	[54, 55]	406	[677, 224]	785	[602, 122]	1164	[929, 369]
28	[56, 57]	407	[678, 679]	786	[1065, 151]	1165	[390, 1381]
29	[58, 59]	408	[680, 504]	787	[609, 1098]	1166	[1382, 1383]

30	[60, 61]	409	[466, 572]	788	[1132, 484]	1167	[67, 1384]
31	[62, 63]	410	[527, 215]	789	[1133, 1134]	1168	[1385, 884]
32	[64, 65]	411	[681, 682]	790	[173, 592]	1169	[374, 708]
33	[66, 67]	412	[660, 57]	791	[1076, 225]	1170	[1378, 279]
34	[68, 69]	413	[683, 684]	792	[616, 1135]	1171	[833, 1386]
35	[70, 71]	414	[661, 516]	793	[1136, 640]	1172	[1387, 753]
36	[72, 73]	415	[685, 488]	794	[1137, 177]	1173	[1388, 1343]
37	[74, 75]	416	[438, 533]	795	[1138, 170]	1174	[1389, 1390]
38	[76, 77]	417	[624, 686]	796	[656, 1119]	1175	[890, 375]
39	[78, 79]	418	[687, 31]	797	[1071, 1139]	1176	[410, 1391]
40	[20, 80]	419	[688, 653]	798	[175, 1140]	1177	[1392, 312]
41	[81, 82]	420	[689, 424]	799	[1141, 569]	1178	[1306, 1393]
42	[83, 84]	421	[690, 691]	800	[1142, 599]	1179	[983, 907]
43	[85, 86]	422	[8, 692]	801	[673, 671]	1180	[1394, 1395]
44	[87, 88]	423	[693, 694]	802	[209, 206]	1181	[346, 835]
45	[89, 90]	424	[695, 696]	803	[1085, 491]	1182	[322, 926]
46	[91, 92]	425	[697, 698]	804	[1143, 1144]	1183	[259, 982]
47	[93, 94]	426	[699, 700]	805	[666, 665]	1184	[1396, 871]
48	[95, 96]	427	[701, 702]	806	[1145, 1146]	1185	[1381, 844]
49	[97, 98]	428	[703, 116]	807	[1124, 1147]	1186	[61, 893]
50	[99, 100]	429	[704, 705]	808	[526, 172]	1187	[976, 1397]
51	[101, 102]	430	[706, 707]	809	[167, 464]	1188	[964, 1398]
52	[103, 104]	431	[708, 709]	810	[551, 1140]	1189	[872, 1399]
53	[105, 106]	432	[710, 711]	811	[1148, 1149]	1190	[1400, 789]
54	[107, 108]	433	[712, 713]	812	[1150, 1151]	1191	[796, 1401]
55	[109, 110]	434	[714, 715]	813	[1066, 13]	1192	[1360, 70]
56	[111, 112]	435	[716, 717]	814	[664, 667]	1193	[723, 1402]
57	[113, 114]	436	[718, 719]	815	[1152, 1153]	1194	[1403, 1023]
58	[115, 93]	437	[720, 721]	816	[1154, 118]	1195	[1317, 1404]
59	[116, 117]	438	[722, 723]	817	[451, 1155]	1196	[272, 840]
60	[7, 118]	439	[724, 725]	818	[1156, 158]	1197	[1405, 358]
61	[119, 120]	440	[726, 727]	819	[1157, 691]	1198	[1046, 1405]
62	[121, 122]	441	[728, 729]	820	[1158, 1103]	1199	[1345, 4]
63	[123, 37]	442	[730, 731]	821	[1159, 127]	1200	[1406, 1407]
64	[124, 125]	443	[732, 733]	822	[1160, 1161]	1201	[1356, 1408]
65	[126, 127]	444	[734, 735]	823	[1162, 1163]	1202	[892, 1318]
66	[128, 129]	445	[736, 252]	824	[581, 1164]	1203	[705, 1409]
67	[130, 131]	446	[414, 737]	825	[530, 47]	1204	[715, 1410]
68	[132, 133]	447	[280, 738]	826	[546, 482]	1205	[314, 1411]
69	[57, 134]	448	[739, 319]	827	[529, 139]	1206	[1412, 968]
70	[135, 136]	449	[740, 741]	828	[141, 606]	1207	[1373, 1309]
71	[137, 138]	450	[742, 743]	829	[1165, 1166]	1208	[80, 1291]
72	[139, 140]	451	[744, 745]	830	[44, 166]	1209	[927, 1413]
73	[141, 142]	452	[746, 747]	831	[171, 1087]	1210	[1414, 1415]

74	[143, 144]	453	[267, 748]	832	[1167, 512]	1211	[21, 1324]
75	[145, 146]	454	[749, 750]	833	[1168, 1075]	1212	[1413, 1416]
76	[147, 148]	455	[751, 752]	834	[680, 1169]	1213	[324, 845]
77	[149, 150]	456	[753, 754]	835	[604, 439]	1214	[1353, 862]
78	[151, 152]	457	[755, 66]	836	[1114, 472]	1215	[1054, 1289]
79	[153, 154]	458	[756, 757]	837	[539, 1170]	1216	[1417, 1389]
80	[155, 156]	459	[383, 758]	838	[588, 12]	1217	[852, 1418]
81	[157, 158]	460	[759, 354]	839	[510, 1171]	1218	[1294, 904]
82	[159, 160]	461	[760, 761]	840	[1172, 634]	1219	[1419, 373]
83	[161, 162]	462	[762, 349]	841	[1173, 1174]	1220	[1420, 1421]
84	[163, 164]	463	[763, 703]	842	[1175, 653]	1221	[416, 1422]
85	[165, 166]	464	[764, 765]	843	[467, 1176]	1222	[1404, 1423]
86	[167, 168]	465	[766, 767]	844	[1177, 446]	1223	[953, 989]
87	[169, 170]	466	[768, 769]	845	[691, 1146]	1224	[894, 1424]
88	[171, 172]	467	[770, 771]	846	[653, 1178]	1225	[1375, 236]
89	[173, 174]	468	[772, 406]	847	[509, 682]	1226	[960, 886]
90	[175, 176]	469	[773, 774]	848	[1179, 1180]	1227	[1425, 911]
91	[177, 178]	470	[775, 759]	849	[1080, 1181]	1228	[955, 1426]
92	[179, 180]	471	[776, 99]	850	[1177, 649]	1229	[1331, 1346]
93	[181, 182]	472	[777, 778]	851	[503, 1169]	1230	[864, 1376]
94	[183, 11]	473	[270, 779]	852	[1160, 539]	1231	[828, 72]
95	[180, 184]	474	[110, 391]	853	[1182, 217]	1232	[1427, 1370]
96	[185, 186]	475	[780, 781]	854	[1183, 1107]	1233	[253, 980]
97	[187, 188]	476	[782, 783]	855	[1110, 1086]	1234	[1428, 1429]
98	[189, 190]	477	[745, 784]	856	[1095, 1184]	1235	[1358, 1330]
99	[191, 192]	478	[785, 786]	857	[1109, 198]	1236	[1430, 1431]
100	[193, 174]	479	[787, 740]	858	[1185, 587]	1237	[1008, 1278]
101	[194, 195]	480	[288, 766]	859	[31, 1186]	1238	[301, 1420]
102	[196, 197]	481	[767, 788]	860	[1136, 1187]	1239	[815, 699]
103	[198, 199]	482	[789, 790]	861	[191, 201]	1240	[1384, 1432]
104	[200, 201]	483	[791, 792]	862	[1188, 647]	1241	[1422, 1433]
105	[202, 203]	484	[793, 794]	863	[482, 1189]	1242	[1434, 1435]
106	[204, 205]	485	[795, 796]	864	[1190, 1191]	1243	[765, 865]
107	[206, 207]	486	[797, 798]	865	[151, 13]	1244	[1418, 837]
108	[208, 209]	487	[799, 800]	866	[1192, 1193]	1245	[23, 83]
109	[210, 208]	488	[255, 387]	867	[1142, 1117]	1246	[743, 966]
110	[211, 208]	489	[725, 394]	868	[1194, 223]	1247	[1386, 888]
111	[212, 213]	490	[779, 801]	869	[456, 433]	1248	[1410, 1436]
112	[214, 215]	491	[802, 803]	870	[1106, 656]	1249	[422, 1414]
113	[216, 217]	492	[804, 805]	871	[1195, 1196]	1250	[924, 936]
114	[218, 219]	493	[806, 807]	872	[1197, 195]	1251	[1372, 1412]
115	[220, 221]	494	[29, 808]	873	[468, 1131]	1252	[874, 1428]
116	[222, 223]	495	[809, 810]	874	[478, 1198]	1253	[999, 1437]
117	[224, 225]	496	[811, 345]	875	[1199, 683]	1254	[112, 1296]

118	[226, 227]	497	[812, 813]	876	[1183, 542]	1255	[1297, 1377]
119	[228, 229]	498	[814, 815]	877	[1148, 1200]	1256	[94, 1382]
120	[230, 231]	499	[816, 817]	878	[637, 1201]	1257	[1039, 1438]
121	[232, 233]	500	[818, 819]	879	[1154, 8]	1258	[995, 1430]
122	[234, 235]	501	[820, 821]	880	[1150, 1082]	1259	[243, 1000]
123	[236, 237]	502	[822, 823]	881	[1202, 1116]	1260	[1351, 1439]
124	[238, 239]	503	[824, 105]	882	[685, 1203]	1261	[1440, 984]
125	[240, 241]	504	[825, 826]	883	[518, 675]	1262	[713, 1441]
126	[242, 243]	505	[827, 828]	884	[677, 1076]	1263	[1441, 240]
127	[244, 245]	506	[803, 398]	885	[1204, 1205]	1264	[711, 335]
128	[246, 247]	507	[829, 830]	886	[599, 1118]	1265	[1442, 1443]
129	[248, 249]	508	[831, 832]	887	[639, 1187]	1266	[397, 1308]
130	[250, 251]	509	[65, 833]	888	[1206, 621]	1267	[1444, 1406]
131	[252, 253]	510	[834, 720]	889	[1207, 535]	1268	[311, 1445]
132	[254, 255]	511	[835, 836]	890	[567, 1098]	1269	[1446, 831]
133	[256, 257]	512	[239, 389]	891	[600, 1102]	1270	[698, 1024]
134	[258, 259]	513	[837, 838]	892	[31, 1208]	1271	[350, 1447]
135	[260, 261]	514	[839, 777]	893	[118, 692]	1272	[1448, 273]
136	[262, 263]	515	[840, 310]	894	[1209, 188]	1273	[1449, 289]
137	[264, 265]	516	[841, 306]	895	[185, 1210]	1274	[998, 1052]
138	[266, 267]	517	[842, 843]	896	[1211, 624]	1275	[915, 1355]
139	[268, 109]	518	[844, 76]	897	[1209, 1212]	1276	[1450, 739]
140	[269, 270]	519	[378, 814]	898	[622, 1135]	1277	[1074, 1217]
141	[271, 272]	520	[845, 846]	899	[1213, 586]	1278	[658, 1451]
142	[273, 274]	521	[847, 848]	900	[1214, 432]	1279	[1204, 553]
143	[275, 276]	522	[721, 849]	901	[691, 1215]	1280	[1094, 1165]
144	[277, 278]	523	[850, 338]	902	[1216, 1072]	1281	[1092, 1265]
145	[279, 280]	524	[299, 61]	903	[538, 1161]	1282	[1452, 518]
146	[281, 282]	525	[851, 852]	904	[1168, 1217]	1283	[1453, 132]
147	[283, 284]	526	[853, 854]	905	[1218, 561]	1284	[1454, 521]
148	[285, 286]	527	[855, 856]	906	[1179, 563]	1285	[46, 531]
149	[287, 288]	528	[88, 857]	907	[1219, 1200]	1286	[439, 139]
150	[289, 290]	529	[395, 269]	908	[1185, 1220]	1287	[549, 433]
151	[291, 292]	530	[858, 816]	909	[598, 1117]	1288	[1454, 435]
152	[293, 294]	531	[82, 360]	910	[1221, 448]	1289	[1266, 473]
153	[284, 295]	532	[859, 860]	911	[494, 643]	1290	[1227, 508]
154	[296, 297]	533	[861, 366]	912	[572, 1176]	1291	[41, 1208]
155	[298, 299]	534	[862, 863]	913	[448, 1222]	1292	[1455, 1456]
156	[300, 301]	535	[700, 864]	914	[687, 41]	1293	[177, 1201]
157	[302, 303]	536	[865, 866]	915	[1194, 577]	1294	[499, 478]
158	[304, 305]	537	[867, 868]	916	[440, 140]	1295	[137, 50]
159	[306, 307]	538	[27, 841]	917	[583, 1174]	1296	[54, 1266]
160	[308, 309]	539	[869, 870]	918	[1223, 1138]	1297	[548, 455]
161	[310, 311]	540	[871, 872]	919	[145, 202]	1298	[38, 1242]

162	[312, 304]	541	[873, 874]	920	[1188, 1224]	1299	[1157, 1145]
163	[313, 314]	542	[875, 876]	921	[217, 1105]	1300	[452, 35]
164	[315, 316]	543	[877, 827]	922	[521, 1225]	1301	[1273, 1113]
165	[317, 318]	544	[878, 879]	923	[1122, 437]	1302	[1243, 1110]
166	[319, 320]	545	[409, 880]	924	[431, 1192]	1303	[1223, 169]
167	[321, 322]	546	[810, 881]	925	[1156, 1064]	1304	[1457, 435]
168	[323, 324]	547	[729, 882]	926	[9, 33]	1305	[460, 1097]
169	[325, 326]	548	[883, 710]	927	[1100, 1226]	1306	[48, 1084]
170	[327, 328]	549	[884, 714]	928	[179, 578]	1307	[1458, 426]
171	[329, 330]	550	[386, 811]	929	[678, 502]	1308	[207, 1085]
172	[331, 332]	551	[826, 885]	930	[1216, 576]	1309	[1145, 1215]
173	[333, 275]	552	[886, 887]	931	[128, 1062]	1310	[1082, 1459]
174	[245, 334]	553	[888, 889]	932	[1227, 1228]	1311	[474, 604]
175	[335, 24]	554	[337, 890]	933	[492, 1166]	1312	[1190, 1144]
176	[336, 337]	555	[891, 892]	934	[1229, 1230]	1313	[1082, 1460]
177	[338, 339]	556	[893, 894]	935	[595, 627]	1314	[1182, 1104]
178	[340, 341]	557	[895, 896]	936	[129, 1231]	1315	[1175, 1461]
179	[342, 343]	558	[257, 897]	937	[1192, 1129]	1316	[682, 1462]
180	[344, 262]	559	[898, 899]	938	[1232, 1233]	1317	[589, 1451]
181	[345, 346]	560	[900, 901]	939	[1132, 1234]	1318	[638, 1463]
182	[347, 348]	561	[902, 903]	940	[1077, 562]	1319	[1207, 426]
183	[349, 350]	562	[904, 356]	941	[631, 1127]	1320	[1261, 1264]
184	[343, 351]	563	[905, 906]	942	[1073, 1080]	1321	[648, 1464]
185	[352, 353]	564	[907, 756]	943	[625, 574]	1322	[121, 603]
186	[354, 355]	565	[908, 909]	944	[1235, 1236]	1323	[1456, 1465]
187	[356, 357]	566	[326, 910]	945	[1232, 1237]	1324	[37, 1128]
188	[358, 359]	567	[798, 296]	946	[1238, 523]	1325	[43, 1466]
189	[360, 361]	568	[911, 89]	947	[1239, 1240]	1326	[1272, 221]
190	[362, 363]	569	[912, 913]	948	[636, 1241]	1327	[213, 213]
191	[364, 365]	570	[914, 915]	949	[155, 1155]	1328	[216, 1104]
192	[366, 367]	571	[916, 917]	950	[34, 453]	1329	[582, 1180]
193	[368, 342]	572	[918, 919]	951	[690, 1145]	1330	[1152, 651]
194	[369, 370]	573	[71, 920]	952	[619, 1242]	1331	[688, 1461]
195	[371, 372]	574	[856, 762]	953	[126, 613]	1332	[1104, 1275]
196	[373, 374]	575	[921, 922]	954	[581, 684]	1333	[1165, 493]
197	[375, 376]	576	[923, 264]	955	[542, 1108]	1334	[180, 579]
198	[377, 378]	577	[278, 924]	956	[487, 1203]	1335	[1206, 559]
199	[379, 380]	578	[925, 773]	957	[489, 210]	1336	[1244, 637]
200	[381, 382]	579	[926, 927]	958	[1230, 1126]	1337	[423, 1245]
201	[86, 383]	580	[928, 929]	959	[1243, 198]	1338	[1121, 1066]
202	[384, 385]	581	[930, 931]	960	[32, 10]	1339	[153, 1467]
203	[237, 386]	582	[719, 932]	961	[1058, 648]	1340	[139, 54]
204	[387, 388]	583	[933, 861]	962	[552, 1205]	1341	[641, 1468]
205	[389, 390]	584	[808, 934]	963	[1244, 177]	1342	[1469, 512]

206	[391, 392]	585	[392, 724]	964	[1117, 612]	1343	[1267, 1096]
207	[393, 107]	586	[781, 935]	965	[689, 1245]	1344	[1158, 686]
208	[394, 395]	587	[936, 937]	966	[1100, 1246]	1345	[537, 1076]
209	[396, 397]	588	[938, 809]	967	[1247, 1248]	1346	[454, 1470]
210	[398, 399]	589	[90, 939]	968	[1249, 1248]	1347	[1461, 654]
211	[400, 401]	590	[404, 695]	969	[1250, 1185]	1348	[50, 1471]
212	[402, 396]	591	[940, 941]	970	[429, 1226]	1349	[1456, 125]
213	[401, 402]	592	[942, 943]	971	[623, 1158]	1350	[42, 1163]
214	[403, 404]	593	[944, 945]	972	[118, 1251]	1351	[556, 1463]
215	[303, 405]	594	[69, 946]	973	[1252, 615]	1352	[510, 1462]
216	[406, 407]	595	[947, 336]	974	[618, 38]	1353	[1195, 636]
217	[408, 409]	596	[769, 916]	975	[500, 620]	1354	[1472, 1141]
218	[332, 410]	597	[265, 948]	976	[1247, 1253]	1355	[500, 1198]
219	[411, 412]	598	[909, 949]	977	[1221, 480]	1356	[430, 447]
220	[413, 414]	599	[950, 951]	978	[1161, 1254]	1357	[1112, 1274]
221	[415, 416]	600	[952, 953]	979	[1060, 629]	1358	[1143, 1191]
222	[417, 418]	601	[421, 900]	980	[635, 1196]	1359	[519, 1089]
223	[63, 419]	602	[954, 955]	981	[1069, 611]	1360	[1269, 1233]
224	[420, 421]	603	[956, 722]	982	[507, 1228]	1361	[1091, 1235]
225	[4, 422]	604	[957, 268]	983	[55, 662]	1362	[143, 58]
226	[423, 424]	605	[958, 22]	984	[1180, 1255]	1363	[1457, 521]
227	[425, 160]	606	[365, 959]	985	[1256, 1165]	1364	[214, 528]
228	[426, 427]	607	[821, 960]	986	[498, 1257]	1365	[1098, 1473]
229	[428, 429]	608	[294, 293]	987	[1180, 564]	1366	[1110, 199]
230	[36, 430]	609	[961, 962]	988	[1258, 1092]	1367	[1469, 1239]
231	[431, 432]	610	[328, 793]	989	[501, 679]	1368	[1197, 1474]
232	[433, 434]	611	[370, 963]	990	[613, 1259]	1369	[1088, 1257]
233	[435, 436]	612	[949, 964]	991	[1218, 1077]	1370	[563, 1255]
234	[135, 437]	613	[965, 230]	992	[511, 631]	1371	[1452, 674]
235	[438, 162]	614	[966, 967]	993	[1141, 476]	1372	[1101, 601]
236	[439, 440]	615	[968, 969]	994	[1063, 8]	1373	[1162, 43]
237	[441, 442]	616	[970, 97]	995	[1081, 1151]	1374	[683, 1164]
238	[443, 444]	617	[353, 971]	996	[1115, 1260]	1375	[1173, 584]
239	[445, 446]	618	[910, 806]	997	[1261, 1134]	1376	[1475, 594]
240	[37, 447]	619	[935, 377]	998	[644, 545]	1377	[585, 506]
241	[448, 449]	620	[817, 972]	999	[1249, 1253]	1378	[1276, 1123]
242	[450, 429]	621	[973, 974]	1000	[429, 1246]	1379	[1453, 525]
243	[451, 156]	622	[975, 976]	1001	[1250, 586]	1380	[550, 146]
244	[452, 453]	623	[977, 978]	1002	[443, 148]	1381	[1475, 1059]
245	[425, 454]	624	[363, 979]	1003	[682, 1171]	1382	[182, 1181]
246	[440, 54]	625	[980, 981]	1004	[633, 1262]	1383	[11, 1271]
247	[441, 455]	626	[982, 983]	1005	[586, 1220]	1384	[160, 1470]
248	[456, 457]	627	[984, 985]	1006	[1263, 124]	1385	[124, 1465]
249	[458, 171]	628	[986, 938]	1007	[1133, 1264]	1386	[127, 1259]

250	[459, 138]	629	[987, 988]	1008	[1172, 1262]	1387	[190, 1468]
251	[460, 164]	630	[989, 990]	1009	[1238, 519]	1388	[1230, 1476]
252	[461, 462]	631	[991, 992]	1010	[1235, 1265]	1389	[1107, 543]
253	[463, 464]	632	[917, 111]	1011	[461, 515]	1390	[1167, 1239]
254	[220, 465]	633	[993, 994]	1012	[1258, 1235]	1391	[1266, 662]
255	[466, 467]	634	[227, 995]	1013	[140, 1266]	1392	[1092, 1236]
256	[468, 469]	635	[305, 996]	1014	[1267, 1184]	1393	[1196, 1241]
257	[127, 470]	636	[997, 998]	1015	[610, 1070]	1394	[681, 510]
258	[471, 472]	637	[979, 850]	1016	[533, 1268]	1395	[1078, 1107]
259	[473, 474]	638	[990, 999]	1017	[1130, 469]	1396	[565, 1139]
260	[475, 476]	639	[1000, 315]	1018	[580, 683]	1397	[1477, 38]
261	[477, 478]	640	[96, 1001]	1019	[218, 631]	1398	[1478, 1210]
262	[159, 454]	641	[1002, 804]	1020	[1219, 1149]	1399	[547, 1189]
263	[479, 480]	642	[1003, 987]	1021	[213, 212]	1400	[1239, 513]
264	[481, 482]	643	[1004, 787]	1022	[1202, 1260]	1401	[517, 674]
265	[483, 484]	644	[1005, 928]	1023	[1159, 613]	1402	[568, 1473]
266	[485, 486]	645	[786, 977]	1024	[55, 473]	1403	[535, 427]
267	[487, 488]	646	[117, 371]	1025	[1199, 581]	1404	[483, 1234]
268	[489, 211]	647	[887, 1006]	1026	[56, 661]	1405	[41, 1186]
269	[210, 490]	648	[1007, 1008]	1027	[672, 489]	1406	[219, 632]
270	[491, 211]	649	[382, 1009]	1028	[646, 1224]	1407	[204, 1230]
271	[492, 493]	650	[1010, 1011]	1029	[512, 1240]	1408	[574, 1467]
272	[494, 495]	651	[233, 1012]	1030	[1269, 1237]	1409	[573, 153]
273	[202, 496]	652	[932, 1013]	1031	[435, 1225]	1410	[1478, 186]
274	[497, 498]	653	[1014, 923]	1032	[1256, 492]	1411	[560, 641]
275	[499, 500]	654	[1015, 1016]	1033	[523, 520]	1412	[8, 1251]
276	[501, 502]	655	[1017, 791]	1034	[522, 519]	1413	[1252, 1479]
277	[503, 504]	656	[1018, 1019]	1035	[668, 669]	1414	[614, 1479]
278	[505, 129]	657	[1020, 1021]	1036	[650, 1153]	1415	[1213, 1185]
279	[506, 439]	658	[696, 1022]	1037	[454, 1270]	1416	[1472, 475]
280	[507, 508]	659	[1023, 775]	1038	[588, 1271]	1417	[525, 133]
281	[509, 510]	660	[1024, 258]	1039	[1080, 652]	1418	[1214, 1192]
282	[511, 219]	661	[1025, 1026]	1040	[40, 31]	1419	[674, 1480]
283	[512, 513]	662	[1027, 957]	1041	[426, 536]	1420	[1066, 152]
284	[514, 515]	663	[235, 1028]	1042	[540, 1124]	1421	[480, 1222]
285	[57, 516]	664	[1029, 1030]	1043	[433, 541]	1422	[194, 1474]
286	[517, 518]	665	[1031, 1032]	1044	[661, 134]	1423	[138, 51]
287	[519, 520]	666	[1033, 1007]	1045	[477, 500]	1424	[1137, 637]
288	[514, 462]	667	[318, 1034]	1046	[1125, 1111]	1425	[205, 1476]
289	[521, 436]	668	[399, 393]	1047	[1272, 465]	1426	[1229, 205]
290	[522, 523]	669	[801, 1035]	1048	[505, 1062]	1427	[1138, 630]
291	[524, 525]	670	[727, 802]	1049	[1273, 1274]	1428	[187, 1212]
292	[458, 526]	671	[108, 1027]	1050	[190, 1120]	1429	[1263, 1456]
293	[527, 528]	672	[1035, 400]	1051	[574, 154]	1430	[160, 1270]

294	[506, 529]	673	[673, 673]	1052	[161, 533]	1431	[1151, 1460]
295	[530, 531]	674	[1036, 1036]	1053	[217, 1275]	1432	[131, 1481]
296	[532, 49]	675	[1037, 1038]	1054	[1276, 53]	1433	[1140, 1482]
297	[533, 534]	676	[774, 1039]	1055	[1067, 1138]	1434	[1062, 1231]
298	[535, 536]	677	[1040, 891]	1056	[608, 171]	1435	[1163, 1466]
299	[537, 224]	678	[1041, 914]	1057	[1277, 95]	1436	[1483, 1141]
300	[538, 539]	679	[819, 1042]	1058	[1278, 1279]	1437	[1211, 1158]
301	[540, 131]	680	[1043, 1044]	1059	[761, 1280]	1438	[1124, 1481]
302	[457, 541]	681	[1045, 1046]	1060	[1281, 997]	1439	[1455, 124]
303	[542, 543]	682	[1047, 1048]	1061	[922, 1282]	1440	[144, 1484]
304	[544, 545]	683	[1049, 942]	1062	[1283, 1284]	1441	[479, 448]
305	[546, 547]	684	[868, 352]	1063	[229, 327]	1442	[539, 1254]
306	[474, 506]	685	[1050, 1045]	1064	[1285, 250]	1443	[1461, 1178]
307	[548, 442]	686	[843, 1051]	1065	[1286, 331]	1444	[58, 1484]
308	[549, 457]	687	[800, 256]	1066	[1287, 1288]	1445	[176, 1485]
309	[550, 202]	688	[1052, 1015]	1067	[261, 16]	1446	[1077, 655]
310	[551, 176]	689	[1053, 285]	1068	[1289, 1290]	1447	[568, 1099]
311	[552, 553]	690	[1013, 1054]	1069	[1291, 895]	1448	[457, 434]
312	[485, 554]	691	[1055, 1056]	1070	[357, 1292]	1449	[132, 1057]
313	[224, 555]	692	[16, 859]	1071	[1293, 1294]	1450	[518, 1480]
314	[556, 557]	693	[525, 1057]	1072	[882, 1295]	1451	[733, 1004]
315	[558, 559]	694	[146, 496]	1073	[1296, 1297]	1452	[334, 1037]
316	[560, 190]	695	[544, 645]	1074	[1298, 242]	1453	[92, 1277]
317	[561, 562]	696	[1058, 196]	1075	[1299, 842]	1454	[1021, 718]
318	[563, 564]	697	[140, 55]	1076	[1300, 764]	1455	[823, 1486]
319	[565, 566]	698	[605, 142]	1077	[1301, 323]	1456	[77, 362]
320	[567, 568]	699	[593, 1059]	1078	[738, 877]	1457	[1391, 1364]
321	[475, 569]	700	[1060, 1061]	1079	[747, 1302]	1458	[263, 701]
322	[570, 180]	701	[1062, 626]	1080	[1303, 1304]	1459	[939, 1487]
323	[571, 572]	702	[480, 449]	1081	[367, 1305]	1460	[1305, 1488]
324	[573, 574]	703	[1063, 118]	1082	[830, 234]	1461	[1011, 1396]
325	[575, 576]	704	[157, 1064]	1083	[748, 1306]	1462	[784, 1334]
326	[571, 467]	705	[1065, 1066]	1084	[860, 1307]	1463	[967, 1489]
327	[222, 577]	706	[1067, 169]	1085	[1308, 726]	1464	[1279, 1490]
328	[578, 579]	707	[1068, 656]	1086	[1309, 1310]	1465	[388, 1491]
329	[580, 581]	708	[459, 50]	1087	[757, 1311]	1466	[84, 1492]
330	[582, 563]	709	[1069, 1070]	1088	[1312, 1002]	1467	[981, 1493]
331	[583, 584]	710	[165, 45]	1089	[849, 1313]	1468	[813, 1494]
332	[473, 585]	711	[1071, 566]	1090	[1314, 28]	1469	[880, 340]
333	[586, 587]	712	[575, 1072]	1091	[1042, 912]	1470	[1290, 1340]
334	[183, 588]	713	[1073, 182]	1092	[805, 238]	1471	[758, 1495]
335	[589, 590]	714	[1074, 1075]	1093	[901, 5]	1472	[1409, 1442]
336	[591, 554]	715	[1076, 555]	1094	[1315, 78]	1473	[920, 1496]
337	[193, 592]	716	[659, 1077]	1095	[1316, 734]	1474	[897, 1497]

338	[593, 594]	717	[1078, 542]	1096	[790, 1317]	1475	[1498, 1320]
339	[595, 150]	718	[471, 1079]	1097	[1318, 1319]	1476	[1486, 1499]
340	[572, 596]	719	[212, 212]	1098	[1320, 415]	1477	[1350, 1434]
341	[43, 597]	720	[181, 1080]	1099	[962, 1321]	1478	[972, 1335]
342	[598, 599]	721	[1081, 1082]	1100	[1322, 1299]	1479	[1398, 1500]
343	[600, 601]	722	[1083, 196]	1101	[1323, 1040]	1480	[978, 1501]
344	[602, 603]	723	[532, 1084]	1102	[903, 1315]	1481	[251, 1502]
345	[585, 604]	724	[491, 210]	1103	[1324, 1325]	1482	[1028, 1503]
346	[605, 606]	725	[670, 672]	1104	[1326, 822]	1483	[231, 782]
347	[607, 144]	726	[1085, 489]	1105	[407, 313]	1484	[276, 975]
348	[608, 526]	727	[670, 1085]	1106	[1327, 1020]	1485	[885, 1504]
349	[609, 568]	728	[198, 1086]	1107	[1328, 1329]	1486	[1161, 1170]
350	[610, 611]	729	[642, 495]	1108	[247, 1327]	1487	[162, 534]
351	[599, 612]	730	[526, 1087]	1109	[419, 379]	1488	[636, 1505]
352	[613, 470]	731	[676, 1088]	1110	[1330, 780]	1489	[1483, 475]
353	[614, 615]	732	[523, 1089]	1111	[931, 1331]	1490	[1451, 1506]
354	[616, 617]	733	[591, 486]	1112	[1332, 751]	1491	[432, 1193]
355	[618, 619]	734	[1090, 657]	1113	[295, 961]	1492	[162, 1268]
356	[478, 620]	735	[1091, 1092]	1114	[1333, 1014]	1493	[1196, 1505]
357	[558, 621]	736	[498, 1093]	1115	[1334, 898]	1494	[1163, 597]
358	[622, 617]	737	[1094, 492]	1116	[1335, 1336]	1495	[590, 1506]
359	[623, 624]	738	[604, 529]	1117	[1337, 1338]	1496	[1451, 1507]
360	[625, 153]	739	[1095, 1096]	1118	[754, 970]	1497	[1458, 535]
361	[129, 626]	740	[163, 1097]	1119	[307, 768]	1498	[1088, 1093]
362	[149, 627]	741	[1098, 1099]	1120	[361, 1339]	1499	[1151, 1459]
363	[628, 629]	742	[450, 1100]	1121	[1340, 26]	1500	[1477, 619]
364	[169, 630]	743	[1101, 1102]	1122	[1341, 867]	1501	[131, 1147]
365	[631, 632]	744	[624, 1103]	1123	[941, 1342]	1502	[138, 1471]
366	[633, 634]	745	[428, 1100]	1124	[1343, 825]	1503	[176, 1482]
367	[635, 636]	746	[1104, 1105]	1125	[1344, 1345]	1504	[482, 1508]
368	[637, 178]	747	[146, 203]	1126	[1346, 1347]	1505	[996, 1509]
369	[578, 184]	748	[481, 547]	1127	[836, 1286]	1506	[1022, 1510]
370	[638, 557]	749	[524, 132]	1128	[73, 697]	1507	[1368, 1511]
371	[639, 640]	750	[1106, 1090]	1129	[709, 1348]	1508	[881, 1512]
372	[189, 641]	751	[1107, 1108]	1130	[1349, 742]	1509	[547, 1508]
373	[642, 643]	752	[1109, 1110]	1131	[1048, 1350]	1510	[196, 1464]
374	[644, 645]	753	[570, 578]	1132	[351, 1351]	1511	[590, 1507]
375	[646, 647]	754	[119, 1111]	1133	[1352, 730]	1512	[1140, 1485]
376	[648, 197]	755	[1112, 1113]	1134	[100, 1353]		
377	[445, 649]	756	[662, 474]	1135	[792, 1354]		
378	[650, 651]	757	[1114, 1079]	1136	[1355, 973]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	253	1	506	1	759	1	1012	0	1265	1
1	0	254	0	507	1	760	1	1013	0	1266	1
2	0	255	0	508	0	761	1	1014	0	1267	0
3	0	256	1	509	0	762	1	1015	0	1268	0
4	0	257	1	510	0	763	1	1016	0	1269	0
5	0	258	0	511	0	764	0	1017	0	1270	1
6	0	259	1	512	0	765	1	1018	0	1271	0
7	0	260	0	513	1	766	1	1019	1	1272	1
8	0	261	1	514	0	767	1	1020	0	1273	0
9	0	262	1	515	0	768	0	1021	0	1274	0
10	0	263	0	516	1	769	0	1022	0	1275	1
11	0	264	1	517	1	770	1	1023	1	1276	1
12	1	265	1	518	1	771	1	1024	0	1277	0
13	0	266	1	519	0	772	1	1025	1	1278	0
14	1	267	1	520	1	773	1	1026	1	1279	1
15	0	268	0	521	0	774	0	1027	0	1280	1
16	1	269	0	522	0	775	1	1028	0	1281	1
17	0	270	1	523	1	776	0	1029	0	1282	0
18	0	271	0	524	1	777	1	1030	0	1283	0
19	0	272	0	525	1	778	1	1031	1	1284	0
20	0	273	1	526	1	779	0	1032	0	1285	0
21	0	274	1	527	1	780	0	1033	1	1286	1
22	0	275	0	528	0	781	0	1034	0	1287	0
23	0	276	1	529	0	782	1	1035	0	1288	0
24	1	277	1	530	1	783	1	1036	0	1289	1
25	0	278	0	531	1	784	0	1037	1	1290	0
26	0	279	1	532	0	785	1	1038	1	1291	1
27	1	280	1	533	0	786	1	1039	0	1292	1
28	1	281	0	534	1	787	0	1040	0	1293	0
29	0	282	0	535	1	788	0	1041	0	1294	0
30	0	283	0	536	1	789	0	1042	1	1295	1
31	0	284	0	537	1	790	0	1043	0	1296	1
32	1	285	0	538	1	791	1	1044	1	1297	1
33	1	286	1	539	1	792	1	1045	1	1298	0
34	0	287	0	540	1	793	0	1046	1	1299	1
35	0	288	0	541	1	794	0	1047	1	1300	1
36	0	289	0	542	1	795	1	1048	0	1301	0
37	0	290	0	543	1	796	0	1049	0	1302	1
38	0	291	1	544	1	797	0	1050	0	1303	0
39	1	292	0	545	0	798	1	1051	1	1304	1
40	0	293	1	546	0	799	1	1052	0	1305	1
41	1	294	1	547	1	800	1	1053	0	1306	1
42	0	295	1	548	1	801	0	1054	1	1307	0

43	1	296	0	549	0	802	1	1055	1	1308	1
44	0	297	0	550	0	803	1	1056	1	1309	0
45	0	298	1	551	0	804	0	1057	0	1310	0
46	0	299	1	552	0	805	1	1058	0	1311	1
47	0	300	1	553	0	806	0	1059	1	1312	1
48	1	301	1	554	1	807	0	1060	1	1313	0
49	1	302	1	555	1	808	1	1061	0	1314	1
50	0	303	1	556	1	809	0	1062	0	1315	1
51	0	304	1	557	0	810	0	1063	1	1316	1
52	0	305	0	558	1	811	1	1064	0	1317	1
53	1	306	1	559	0	812	1	1065	1	1318	0
54	1	307	1	560	1	813	0	1066	0	1319	1
55	0	308	0	561	1	814	0	1067	1	1320	1
56	1	309	0	562	1	815	1	1068	1	1321	0
57	0	310	0	563	0	816	0	1069	1	1322	0
58	0	311	0	564	0	817	0	1070	1	1323	0
59	0	312	1	565	1	818	0	1071	0	1324	0
60	0	313	0	566	1	819	1	1072	0	1325	1
61	0	314	1	567	1	820	1	1073	1	1326	1
62	0	315	1	568	0	821	1	1074	0	1327	0
63	1	316	1	569	0	822	0	1075	1	1328	0
64	1	317	1	570	1	823	1	1076	1	1329	1
65	0	318	0	571	1	824	1	1077	0	1330	1
66	1	319	1	572	0	825	1	1078	0	1331	0
67	0	320	1	573	1	826	0	1079	0	1332	1
68	0	321	0	574	1	827	0	1080	0	1333	1
69	0	322	1	575	0	828	0	1081	0	1334	1
70	0	323	1	576	1	829	1	1082	0	1335	0
71	1	324	1	577	0	830	0	1083	1	1336	1
72	0	325	0	578	0	831	0	1084	0	1337	1
73	0	326	1	579	0	832	0	1085	1	1338	0
74	0	327	0	580	0	833	1	1086	0	1339	0
75	1	328	0	581	1	834	0	1087	1	1340	0
76	0	329	0	582	1	835	0	1088	1	1341	1
77	0	330	1	583	0	836	1	1089	0	1342	1
78	1	331	0	584	1	837	1	1090	1	1343	0
79	0	332	1	585	0	838	1	1091	1	1344	1
80	1	333	0	586	0	839	0	1092	1	1345	1
81	1	334	0	587	1	840	0	1093	1	1346	1
82	1	335	1	588	1	841	1	1094	1	1347	1
83	0	336	0	589	1	842	1	1095	1	1348	0
84	0	337	1	590	1	843	1	1096	0	1349	0
85	1	338	0	591	0	844	1	1097	0	1350	0
86	0	339	1	592	1	845	1	1098	1	1351	1

87	0	340	0	593	0	846	0	1099	0	1352	0
88	0	341	1	594	0	847	0	1100	0	1353	1
89	0	342	0	595	1	848	0	1101	0	1354	1
90	1	343	1	596	0	849	0	1102	1	1355	0
91	0	344	1	597	1	850	1	1103	0	1356	1
92	0	345	0	598	0	851	1	1104	1	1357	1
93	0	346	1	599	0	852	0	1105	0	1358	0
94	0	347	1	600	1	853	1	1106	0	1359	0
95	1	348	1	601	0	854	1	1107	0	1360	0
96	0	349	0	602	1	855	1	1108	0	1361	1
97	1	350	0	603	1	856	1	1109	0	1362	0
98	0	351	0	604	0	857	0	1110	1	1363	1
99	0	352	0	605	1	858	1	1111	1	1364	0
100	1	353	0	606	0	859	0	1112	1	1365	1
101	0	354	1	607	1	860	0	1113	1	1366	1
102	1	355	1	608	1	861	0	1114	1	1367	1
103	0	356	1	609	0	862	1	1115	1	1368	1
104	1	357	1	610	0	863	0	1116	0	1369	1
105	1	358	0	611	0	864	1	1117	1	1370	0
106	1	359	1	612	1	865	1	1118	0	1371	0
107	1	360	0	613	0	866	0	1119	1	1372	0
108	1	361	1	614	0	867	1	1120	1	1373	1
109	0	362	0	615	0	868	1	1121	0	1374	0
110	1	363	0	616	1	869	1	1122	1	1375	1
111	1	364	0	617	0	870	0	1123	0	1376	1
112	0	365	0	618	1	871	1	1124	0	1377	0
113	0	366	1	619	1	872	1	1125	1	1378	1
114	1	367	0	620	0	873	1	1126	1	1379	0
115	0	368	1	621	1	874	1	1127	1	1380	0
116	0	369	0	622	0	875	1	1128	0	1381	1
117	0	370	0	623	1	876	1	1129	1	1382	1
118	1	371	1	624	0	877	1	1130	0	1383	0
119	0	372	0	625	0	878	1	1131	0	1384	0
120	0	373	1	626	1	879	0	1132	0	1385	1
121	0	374	0	627	1	880	1	1133	0	1386	1
122	0	375	0	628	0	881	0	1134	1	1387	0
123	1	376	0	629	1	882	0	1135	1	1388	1
124	1	377	0	630	1	883	1	1136	0	1389	0
125	0	378	0	631	0	884	0	1137	0	1390	0
126	0	379	1	632	0	885	1	1138	1	1391	1
127	1	380	0	633	1	886	0	1139	0	1392	1
128	1	381	1	634	0	887	1	1140	1	1393	0
129	1	382	0	635	0	888	0	1141	1	1394	1
130	0	383	0	636	1	889	1	1142	1	1395	0

131	1	384	1	637	1	890	1	1143	0	1396	1
132	0	385	0	638	0	891	1	1144	1	1397	0
133	1	386	0	639	1	892	0	1145	0	1398	1
134	0	387	1	640	0	893	1	1146	0	1399	1
135	0	388	1	641	1	894	0	1147	0	1400	1
136	1	389	0	642	1	895	0	1148	1	1401	1
137	1	390	1	643	1	896	0	1149	0	1402	0
138	1	391	1	644	0	897	0	1150	1	1403	1
139	0	392	0	645	1	898	0	1151	1	1404	1
140	0	393	1	646	0	899	1	1152	1	1405	1
141	0	394	1	647	1	900	1	1153	0	1406	1
142	1	395	0	648	0	901	1	1154	0	1407	1
143	0	396	1	649	0	902	1	1155	0	1408	1
144	1	397	1	650	0	903	1	1156	0	1409	1
145	1	398	0	651	1	904	1	1157	1	1410	1
146	0	399	0	652	0	905	0	1158	1	1411	1
147	0	400	1	653	0	906	0	1159	1	1412	0
148	0	401	0	654	0	907	0	1160	0	1413	1
149	0	402	1	655	0	908	1	1161	0	1414	0
150	0	403	0	656	0	909	0	1162	1	1415	1
151	1	404	1	657	0	910	1	1163	0	1416	1
152	1	405	0	658	0	911	0	1164	0	1417	1
153	0	406	0	659	1	912	0	1165	1	1418	1
154	0	407	0	660	0	913	1	1166	1	1419	0
155	1	408	0	661	1	914	1	1167	0	1420	0
156	1	409	0	662	0	915	1	1168	1	1421	0
157	1	410	1	663	1	916	1	1169	0	1422	0
158	1	411	1	664	0	917	0	1170	1	1423	1
159	1	412	0	665	1	918	0	1171	1	1424	0
160	0	413	0	666	1	919	1	1172	0	1425	0
161	0	414	1	667	0	920	1	1173	1	1426	0
162	1	415	0	668	0	921	0	1174	0	1427	1
163	0	416	1	669	0	922	0	1175	1	1428	1
164	1	417	0	670	0	923	1	1176	1	1429	0
165	1	418	1	671	1	924	0	1177	1	1430	0
166	1	419	0	672	0	925	0	1178	1	1431	1
167	0	420	0	673	0	926	0	1179	0	1432	1
168	1	421	0	674	0	927	0	1180	1	1433	1
169	0	422	0	675	1	928	0	1181	1	1434	0
170	0	423	1	676	0	929	0	1182	1	1435	0
171	0	424	1	677	0	930	1	1183	1	1436	0
172	0	425	0	678	0	931	1	1184	1	1437	0
173	0	426	0	679	1	932	0	1185	1	1438	0
174	0	427	0	680	0	933	0	1186	0	1439	1

175	1	428	1	681	1	934	0	1187	1	1440	1
176	0	429	1	682	1	935	1	1188	1	1441	0
177	0	430	1	683	0	936	1	1189	1	1442	1
178	0	431	0	684	1	937	0	1190	1	1443	1
179	0	432	1	685	0	938	1	1191	0	1444	0
180	1	433	0	686	1	939	0	1192	0	1445	0
181	0	434	0	687	1	940	0	1193	0	1446	0
182	1	435	1	688	0	941	0	1194	1	1447	0
183	0	436	0	689	0	942	1	1195	1	1448	1
184	1	437	0	690	0	943	0	1196	0	1449	0
185	0	438	1	691	1	944	0	1197	1	1450	1
186	1	439	1	692	1	945	1	1198	1	1451	0
187	1	440	1	693	1	946	1	1199	1	1452	0
188	0	441	0	694	0	947	1	1200	1	1453	0
189	0	442	1	695	1	948	1	1201	1	1454	0
190	0	443	1	696	0	949	1	1202	0	1455	1
191	0	444	1	697	0	950	0	1203	1	1456	0
192	1	445	0	698	1	951	0	1204	1	1457	1
193	1	446	1	699	0	952	1	1205	1	1458	0
194	0	447	1	700	1	953	0	1206	0	1459	0
195	1	448	1	701	0	954	1	1207	1	1460	1
196	1	449	0	702	0	955	1	1208	1	1461	1
197	0	450	0	703	1	956	1	1209	0	1462	0
198	0	451	0	704	1	957	0	1210	0	1463	1
199	1	452	1	705	1	958	1	1211	0	1464	1
200	1	453	1	706	1	959	1	1212	1	1465	1
201	0	454	1	707	1	960	1	1213	1	1466	0
202	1	455	0	708	0	961	0	1214	1	1467	1
203	0	456	1	709	1	962	0	1215	1	1468	0
204	1	457	1	710	1	963	1	1216	1	1469	1
205	0	458	0	711	0	964	1	1217	0	1470	0
206	1	459	0	712	0	965	0	1218	0	1471	1
207	1	460	1	713	1	966	0	1219	0	1472	1
208	1	461	1	714	0	967	1	1220	0	1473	1
209	1	462	1	715	1	968	0	1221	1	1474	0
210	0	463	1	716	1	969	0	1222	1	1475	1
211	1	464	0	717	0	970	1	1223	0	1476	0
212	1	465	1	718	0	971	1	1224	0	1477	0
213	0	466	0	719	1	972	1	1225	1	1478	1
214	0	467	1	720	0	973	1	1226	1	1479	1
215	1	468	1	721	0	974	1	1227	0	1480	0
216	0	469	1	722	1	975	0	1228	1	1481	1
217	0	470	1	723	0	976	1	1229	0	1482	0
218	1	471	0	724	1	977	1	1230	1	1483	0

219	1	472	1	725	0	978	0	1231	0	1484	1
220	0	473	1	726	1	979	1	1232	1	1485	1
221	0	474	1	727	0	980	0	1233	1	1486	0
222	0	475	0	728	0	981	1	1234	1	1487	1
223	1	476	1	729	1	982	1	1235	0	1488	1
224	0	477	1	730	1	983	0	1236	0	1489	0
225	0	478	1	731	0	984	1	1237	0	1490	0
226	1	479	0	732	1	985	0	1238	1	1491	1
227	0	480	0	733	0	986	0	1239	1	1492	1
228	0	481	1	734	1	987	1	1240	0	1493	0
229	1	482	0	735	1	988	0	1241	0	1494	0
230	0	483	1	736	0	989	1	1242	0	1495	1
231	0	484	0	737	1	990	0	1243	1	1496	0
232	0	485	1	738	0	991	0	1244	1	1497	0
233	1	486	0	739	1	992	0	1245	0	1498	1
234	0	487	1	740	0	993	1	1246	0	1499	1
235	1	488	0	741	1	994	1	1247	1	1500	0
236	1	489	0	742	0	995	0	1248	1	1501	1
237	0	490	0	743	0	996	1	1249	0	1502	1
238	1	491	1	744	0	997	1	1250	0	1503	0
239	0	492	0	745	1	998	0	1251	0	1504	0
240	0	493	0	746	1	999	0	1252	1	1505	1
241	1	494	0	747	0	1000	1	1253	0	1506	0
242	0	495	0	748	1	1001	0	1254	0	1507	1
243	0	496	1	749	1	1002	1	1255	1	1508	0
244	1	497	1	750	0	1003	1	1256	0	1509	1
245	0	498	0	751	0	1004	1	1257	0	1510	1
246	1	499	0	752	0	1005	0	1258	0	1511	1
247	0	500	0	753	1	1006	0	1259	0	1512	1
248	1	501	1	754	0	1007	0	1260	1		
249	0	502	0	755	1	1008	0	1261	1		
250	0	503	1	756	0	1009	1	1262	1		
251	1	504	1	757	1	1010	0	1263	0		
252	1	505	0	758	1	1011	1	1264	0		

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