PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^3+z+1, z^3+z^2+z+1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^3+z+1, z^3+z^2+z+1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z), t_1(z), t_2(z), \ldots) = (a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b) = (z^3+z+1, z^3+z^2+z+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{18}+z^{17}+z^{16}+z^{13}+z^{11}+z^{10}+z^8+z^7+z^6+z^5+z^3+1,\\
p_1(z) &= z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^9+z^7+z^6+z^5+z^2,\\
p_2(z) &= z^{24}+z^{20}+z^{16}+z^{14}+z^{12}+z^{10}+z^4+1,\\
p_3(z) &= z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^9+z^7+z^6+z^5+z^2,\\
p_4(z) &= z^{18}+z^{17}+z^{14}+z^{13}+z^{11}+z^7+z^6+z^5+z^4+z^3+1,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{18}+z^{17}+z^{16}+z^{13}+z^{11}+z^{10}+z^7+z^6+z^5+z^3+1,\\
p_1(z) &= z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^9+z^7+z^6+z^5+z^2,\\
p_2(z) &= z^{24}+z^{20}+z^{16}+z^{14}+z^{12}+z^{10}+z^4+1,\\
p_3(z) &= z^{20}+z^{19}+z^{18}+z^{15}+z^{13}+z^9+z^7+z^6+z^5+z^2,\\
p_4(z) &= z^{18}+z^{17}+z^{14}+z^{13}+z^{11}+z^8+z^7+z^6+z^5+z^4+z^3+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})} \begin{pmatrix} \bar t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar t_{2^n-2})} \begin{pmatrix} \bar t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
&\qquad \cdots \times x^{\deg(\bar t_0)} \begin{pmatrix} \bar t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x,y) = \operatorname{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 540 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 540, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x,y) = \operatorname{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 660 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 660, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 24 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] = {} & M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^u M^e[:6u] + x^{2u} M^e[:5u] \\ & + x^{4u} M^e[:3u] + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{5u} M^e[:3u] \\ & + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] + x^{7u} M^e[:3u] + x^{6u} M^e[:6u] \\ & + (x^{6u} + x^{7u} + x^{8u} + x^{10u}) R^e, \\ W^e[:12u] = {} & W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^u W^e[:6u] + x^{2u} W^e[:5u] \\ & + x^{4u} W^e[:3u] + x^{2u} W^e[:6u] + x^{3u} W^e[:5u] + x^{5u} W^e[:3u] \end{aligned}$$

$$+ x^{4u}W^e[:6u] + x^{5u}W^e[:5u] + x^{7u}W^e[:3u] + x^{6u}W^e[:6u]$$
$$+ (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^e.$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^u M^o[:5u] + x^{3u}M^o[:3u] + x^u M^o[:6u] + x^{2u}M^o[:5u]\\
&\quad + x^{4u}M^o[:3u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{5u}M^o[:3u]\\
&\quad + x^{4u}M^o[:6u] + x^{5u}M^o[:5u] + x^{7u}M^o[:3u] + x^{6u}M^o[:6u]\\
&\quad + (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^o,\\
W^o[:12u] &= W^o[:6u] + x^u W^o[:5u] + x^{3u}W^o[:3u] + x^u W^o[:6u] + x^{2u}W^o[:5u]\\
&\quad + x^{4u}W^o[:3u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{5u}W^o[:3u]\\
&\quad + x^{4u}W^o[:6u] + x^{5u}W^o[:5u] + x^{7u}W^o[:3u] + x^{6u}W^o[:6u]\\
&\quad + (x^{6u} + x^{7u} + x^{8u} + x^{10u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] &= T[:6u] + x^u T[:5u] + x^{3u}T[:3u] + x^u T[:6u] + x^{2u}T[:5u] + x^{4u}T[:3u]\\
&\quad + x^{2u}T[:6u] + x^{3u}T[:5u] + x^{5u}T[:3u] + x^{4u}T[:6u] + x^{5u}T[:5u]\\
&\quad + x^{7u}T[:3u] + x^{6u}T[:6u] + (x^{6u} + x^{7u} + x^{8u} + x^{10u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
x^6u &= x^{6u}T[:u] + x^{3u}T[3u:4u] + x^u T[5u:6u] + T[6u:7u],\\
x^7u &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u]\\
&\quad + x^{2u}T[5u:6u] + T[7u:8u],\\
x^8u &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u]\\
&\quad + T[8u:9u],\\
0 &= x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u] + x^{4u}T[5u:6u]\\
&\quad + T[9u:10u],\\
x^1 0u &= x^{6u}T[4u:5u] + T[10u:11u],\\
0 &= x^{6u}T[5u:6u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.7}$$
$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[111w]_2],\end{aligned} \tag{2.8}$$
$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.9}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.10}$$
$$0 = T[[100w]_2] + T[[1010w]_2], \tag{2.11}$$

$$0 = T[[101w]_2] + T[[1011w]_2], \tag{2.12}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$1 = T[[w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2], \tag{2.13}$$

$$1 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \tag{2.14}$$

$$1 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$1 = T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1566 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.19}$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.20}$$

$$0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.21}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.22}$$

$$0 = \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.23}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.24}$$

for all $s \in E_{2k}$, and

$$1 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 110)), \tag{2.25}$$

$$1 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 111)), \tag{2.26}$$

$$1 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 1000)), \tag{2.27}$$

$$0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 101)) + \tau(A(s, 1001)), \tag{2.28}$$

$$1 = \tau(A(s, 100)) + \tau(A(s, 1010)), \tag{2.29}$$

$$0 = \tau(A(s, 101)) + \tau(A(s, 1011)), \tag{2.30}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{24}), E_{24}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 12$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:6u] + x^u M^e[:5u] + x^{3u} M^e[:3u] + x^{6u} R^e,$$
$$W_{2k} = W^e[:6u] + x^u W^e[:5u] + x^{3u} W^e[:3u] + x^{6u} R^e.$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:6u] + x^u M^o[:5u] + x^{3u} M^o[:3u] + x^{6u} R^o,$$
$$W_{2k+1} = W^o[:6u] + x^u W^o[:5u] + x^{3u} W^o[:3u] + x^{6u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(W^e[:6v] + x^v W^e[:5v] + x^{3v} W^e[:3v] + x^{6u} R^e\big) \times \big(M^e[:6v] \\ + x^v M^e[:5v] + x^{3v} M^e[:3v] + x^{6u} R^e\big); \end{aligned} \tag{2.31}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v} R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v}, to

$$W^e[:12v]M^e[:12v] + x^{2v} W^e[:10v]M^e[:10v] + x^{6v} W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.31) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{12}, b_1, ..., b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$

$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{24} + x^{21} + x^{19} + x^{18} + x^{17} + x^{16} + x^{14} + x^{13} + x^{11} + x^8 + x^7 + x^6,$$
$$q_1(x) = x^{22} + x^{19} + x^{18} + x^{17} + x^{15} + x^{11} + x^9 + x^6 + x^5 + x^4,$$
$$q_2(x) = x^{24} + x^{20} + x^{14} + x^{12} + x^{10} + x^8 + x^4 + 1,$$
$$q_3(x) = x^{22} + x^{19} + x^{18} + x^{17} + x^{15} + x^{11} + x^9 + x^6 + x^5 + x^4,$$
$$q_4(x) = x^{24} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{13} + x^{11} + x^{10} + x^7 + x^6.$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 186, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 24 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned} p_0(x) = {} & x^{126} + x^{124} + x^{118} + x^{116} + x^{108} + x^{100} + x^{98} + x^{92} + x^{88} + x^{86} \\ & + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{60} + x^{48} + x^{46} \\ & + x^{44} + x^{42} + x^{38} + x^{36}, \end{aligned}$$
$$\begin{aligned} p_3(x) = {} & x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{100} \\ & + x^{96} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{78} + x^{74} + x^{70} + x^{60} + x^{56} \\ & + x^{54} + x^{50} + x^{42} + x^{40} + x^{38} + x^{30} + x^{28} + x^{26} + x^{20}, \end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{120} + x^{116} + x^{112} + x^{110} + x^{106} + x^{104} + x^{96} + x^{92} + x^{88} + x^{80} \\
&\quad + x^{76} + x^{72} + x^{70} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{34} + x^{32} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{16} + x^{10} + x^6 + x^4 + 1, \\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{106} + x^{104} + x^{100} + x^{96} + x^{94} \\
&\quad + x^{92} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{66} + x^{60} + x^{56} + x^{50} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{22} + x^{16} + x^{10} + x^8, \\
p_{12}(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} \\
&\quad + x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} + x^{80} + x^{62} + x^{60} + x^{56} \\
&\quad + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{14} + x^{12} \\
&\quad + x^8 + x^6 + x^4 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110} \\
&\quad + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} \\
&\quad + x^{90} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{73} + x^{70} + x^{66} \\
&\quad + x^{62} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36}, \\
p_3(x) &= x^{108} + x^{104} + x^{98} + x^{96} + x^{94} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} \\
&\quad + x^{58} + x^{54} + x^{52} + x^{50} + x^{46} + x^{40} + x^{36}, \\
p_6(x) &= x^{114} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99} + x^{97} + x^{94} \\
&\quad + x^{93} + x^{92} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{51} + x^{49} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25} \\
&\quad + x^{24}, \\
p_9(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{92} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{24} + x^{20} + x^{18} + x^{14} + x^{12}, \\
p_{12}(x) &= x^{126} + x^{123} + x^{121} + x^{120} + x^{114} + x^{111} + x^{109} + x^{108} + x^{106} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{89} + x^{88} + x^{82} \\
&\quad + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} \\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{43} + x^{41} + x^{40} + x^{34} + x^{31} \\
&\quad + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^7 + x^6 + x^5 + x^4
\end{aligned}$$

$$+ x^3 + x + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{115} + x^{112} + x^{111} + x^{110}\\
&\quad + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95}\\
&\quad + x^{90} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{73} + x^{70} + x^{66}\\
&\quad + x^{62} + x^{61} + x^{60} + x^{57} + x^{55} + x^{53} + x^{52} + x^{49} + x^{47} + x^{46} + x^{44}\\
&\quad + x^{42} + x^{41} + x^{40} + x^{37} + x^{36},\\
p_3(x) &= x^{108} + x^{104} + x^{98} + x^{96} + x^{94} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68}\\
&\quad + x^{58} + x^{54} + x^{52} + x^{50} + x^{46} + x^{40} + x^{36},\\
p_6(x) &= x^{114} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99} + x^{97} + x^{94}\\
&\quad + x^{93} + x^{92} + x^{78} + x^{75} + x^{74} + x^{73} + x^{71} + x^{68} + x^{67} + x^{64} + x^{63}\\
&\quad + x^{62} + x^{60} + x^{58} + x^{57} + x^{56} + x^{54} + x^{51} + x^{49} + x^{45} + x^{44} + x^{43}\\
&\quad + x^{41} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{26} + x^{25}\\
&\quad + x^{24},\\
p_9(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98}\\
&\quad + x^{92} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{46}\\
&\quad + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{24} + x^{20} + x^{18} + x^{14} + x^{12},\\
p_{12}(x) &= x^{126} + x^{123} + x^{121} + x^{120} + x^{114} + x^{111} + x^{109} + x^{108} + x^{106} + x^{103}\\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{89} + x^{88} + x^{82}\\
&\quad + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}\\
&\quad + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{43} + x^{41} + x^{40} + x^{34} + x^{31}\\
&\quad + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4}\\
&\quad + x^3 + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{116} + x^{114} + x^{108} + x^{106} + x^{96} + x^{94} + x^{92}\\
&\quad + x^{88} + x^{86} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} + x^{66} + x^{54} + x^{46} + x^{44}\\
&\quad + x^{42} + x^{36},\\
p_3(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{110} + x^{108} + x^{104} + x^{98} + x^{94} + x^{88}\\
&\quad + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{70} + x^{66} + x^{64} + x^{62} + x^{54} + x^{52}\\
&\quad + x^{50} + x^{48} + x^{46} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} \\
&\quad + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{56} + x^{54} \\
&\quad + x^{46} + x^{44} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{26} + x^{18} + x^{14} + x^{8} \\
&\quad + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{122} + x^{120} + x^{116} + x^{112} + x^{108} + x^{100} + x^{96} + x^{90} + x^{86} + x^{84} \\
&\quad + x^{78} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{30} + x^{26} + x^{24} + x^{22} + x^{18} + x^{14} + x^{12} + x^{8}, \\
p_{12}(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{102} + x^{100} \\
&\quad + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{72} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{122} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} \\
&\quad + x^{96} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{68} + x^{64} + x^{56} \\
&\quad + x^{52} + x^{50} + x^{48} + x^{44} + x^{40} + x^{36}, \\
p_3(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{108} + x^{98} + x^{94} + x^{88} \\
&\quad + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{64} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{40} + x^{38} + x^{22} + x^{20}, \\
p_6(x) &= x^{120} + x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} \\
&\quad + x^{82} + x^{80} + x^{72} + x^{66} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} + x^{44} \\
&\quad + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{16} + x^{14} + x^{8} \\
&\quad + x^{6} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{122} + x^{120} + x^{116} + x^{112} + x^{108} + x^{100} + x^{96} + x^{90} + x^{86} + x^{84} \\
&\quad + x^{78} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{54} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{30} + x^{26} + x^{24} + x^{22} + x^{18} + x^{14} + x^{12} + x^{8}, \\
p_{12}(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{102} + x^{100} \\
&\quad + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{72} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} \\
&\quad + x^{107} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} + x^{92} + x^{91} + x^{89} + x^{88}
\end{aligned}$$

$$
\begin{aligned}
&+x^{87}+x^{85}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{74}+x^{73}+x^{71}\\
&+x^{70}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{57}+x^{55}\\
&+x^{53}+x^{50}+x^{48}+x^{47}+x^{46}+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36},\\
p_3(x)&=x^{108}+x^{104}+x^{98}+x^{96}+x^{94}+x^{82}+x^{78}+x^{76}+x^{74}+x^{70}+x^{68}\\
&+x^{58}+x^{54}+x^{52}+x^{50}+x^{46}+x^{40}+x^{36},\\
p_6(x)&=x^{114}+x^{111}+x^{109}+x^{106}+x^{105}+x^{104}+x^{102}+x^{99}+x^{97}+x^{94}\\
&+x^{93}+x^{92}+x^{78}+x^{75}+x^{74}+x^{73}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}\\
&+x^{62}+x^{60}+x^{58}+x^{57}+x^{56}+x^{54}+x^{51}+x^{49}+x^{45}+x^{44}+x^{43}\\
&+x^{41}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}+x^{31}+x^{30}+x^{28}+x^{26}+x^{25}\\
&+x^{24},\\
p_9(x)&=x^{120}+x^{116}+x^{114}+x^{112}+x^{110}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}\\
&+x^{92}+x^{82}+x^{80}+x^{78}+x^{76}+x^{70}+x^{68}+x^{66}+x^{60}+x^{56}+x^{46}\\
&+x^{44}+x^{42}+x^{40}+x^{36}+x^{34}+x^{32}+x^{24}+x^{20}+x^{18}+x^{14}+x^{12},\\
p_{12}(x)&=x^{126}+x^{123}+x^{121}+x^{120}+x^{114}+x^{111}+x^{109}+x^{108}+x^{106}+x^{103}\\
&+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{94}+x^{91}+x^{89}+x^{88}+x^{82}\\
&+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}\\
&+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{58}+x^{55}+x^{54}\\
&+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{46}+x^{43}+x^{41}+x^{40}+x^{34}+x^{31}\\
&+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}\\
&+x^{19}+x^{18}+x^{17}+x^{16}+x^{15}+x^{13}+x^{12}+x^{10}+x^{7}+x^{6}+x^{5}+x^{4}\\
&+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x)&=x^{126}+x^{123}+x^{122}+x^{121}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{110}\\
&+x^{107}+x^{104}+x^{103}+x^{102}+x^{100}+x^{97}+x^{92}+x^{91}+x^{89}+x^{88}\\
&+x^{87}+x^{85}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{74}+x^{73}+x^{71}\\
&+x^{70}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}+x^{61}+x^{60}+x^{57}+x^{55}\\
&+x^{53}+x^{50}+x^{48}+x^{47}+x^{46}+x^{44}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36},\\
p_3(x)&=x^{108}+x^{104}+x^{98}+x^{96}+x^{94}+x^{82}+x^{78}+x^{76}+x^{74}+x^{70}+x^{68}\\
&+x^{58}+x^{54}+x^{52}+x^{50}+x^{46}+x^{40}+x^{36},\\
p_6(x)&=x^{114}+x^{111}+x^{109}+x^{106}+x^{105}+x^{104}+x^{102}+x^{99}+x^{97}+x^{94}\\
&+x^{93}+x^{92}+x^{78}+x^{75}+x^{74}+x^{73}+x^{71}+x^{68}+x^{67}+x^{64}+x^{63}\\
&+x^{62}+x^{60}+x^{58}+x^{57}+x^{56}+x^{54}+x^{51}+x^{49}+x^{45}+x^{44}+x^{43}\\
&+x^{41}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}+x^{31}+x^{30}+x^{28}+x^{26}+x^{25}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{24},\\
p_9(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98}\\
&+ x^{92} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66} + x^{60} + x^{56} + x^{46}\\
&+ x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{24} + x^{20} + x^{18} + x^{14} + x^{12},\\
p_{12}(x) &= x^{126} + x^{123} + x^{121} + x^{120} + x^{114} + x^{111} + x^{109} + x^{108} + x^{106} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{89} + x^{88} + x^{82}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} + x^{54}\\
&+ x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{43} + x^{41} + x^{40} + x^{34} + x^{31}\\
&+ x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4}\\
&+ x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{112} + x^{108} + x^{104} + x^{100} + x^{90} + x^{88} + x^{86} + x^{82}\\
&+ x^{78} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{52} + x^{46} + x^{44}\\
&+ x^{42} + x^{40} + x^{38} + x^{36},\\
p_3(x) &= x^{122} + x^{120} + x^{116} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{90}\\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{70} + x^{62} + x^{56} + x^{52} + x^{48}\\
&+ x^{38} + x^{34} + x^{32} + x^{24} + x^{20},\\
p_6(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{108} + x^{104} + x^{102} + x^{96} + x^{94} + x^{90}\\
&+ x^{86} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{52}\\
&+ x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{30} + x^{24} + x^{20}\\
&+ x^{10} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{106} + x^{104} + x^{100} + x^{96} + x^{94}\\
&+ x^{92} + x^{84} + x^{82} + x^{80} + x^{76} + x^{72} + x^{70} + x^{66} + x^{60} + x^{56} + x^{50}\\
&+ x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{22} + x^{16} + x^{10} + x^{8},\\
p_{12}(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102}\\
&+ x^{100} + x^{96} + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} + x^{80} + x^{62} + x^{60} + x^{56}\\
&+ x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{32} + x^{14} + x^{12}\\
&+ x^{8} + x^{6} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
p_0(x) = x^{126} + x^{123} + x^{121} + x^{117} + x^{116} + x^{114} + x^{112} + x^{110} + x^{109} + x^{106}
$$

$$+ x^{102} + x^{101} + x^{99} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{85} + x^{84} + x^{82}$$
$$+ x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{61}$$
$$+ x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{45} + x^{42}$$
$$+ x^{41} + x^{38} + x^{37} + x^{36},$$

$$\begin{aligned}
p_3(x) = {} & x^{118} + x^{115} + x^{113} + x^{109} + x^{108} + x^{106} + x^{104} + x^{102} + x^{101} + x^{98} \\
& + x^{94} + x^{93} + x^{90} + x^{89} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{75} + x^{74} \\
& + x^{72} + x^{70} + x^{62} + x^{59} + x^{58} + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{45} \\
& + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{30} + x^{29} \\
& + x^{28},
\end{aligned}$$

$$\begin{aligned}
p_6(x) = {} & x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{105} \\
& + x^{102} + x^{98} + x^{97} + x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} \\
& + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} \\
& + x^{62} + x^{61} + x^{57} + x^{56} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{39} \\
& + x^{38} + x^{37} + x^{33} + x^{32} + x^{29} + x^{25} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} \\
& + x^{15} + x^{14} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

$$\begin{aligned}
p_9(x) = {} & x^{118} + x^{115} + x^{113} + x^{112} + x^{86} + x^{83} + x^{81} + x^{80} + x^{70} + x^{67} + x^{65} \\
& + x^{64} + x^{38} + x^{35} + x^{33} + x^{32} + x^{22} + x^{19} + x^{17} + x^{16},
\end{aligned}$$

$$\begin{aligned}
p_{12}(x) = {} & x^{126} + x^{123} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} \\
& + x^{109} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} \\
& + x^{94} + x^{91} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
& + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} \\
& + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} \\
& + x^{43} + x^{41} + x^{40} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} \\
& + x^{12} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{132} + x^{126} + x^{125} + x^{123} + x^{120} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} \\
& + x^{107} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} \\
& + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} \\
& + x^{73} + x^{71} + x^{69} + x^{68} + x^{61} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{50} \\
& + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{36},
\end{aligned}$$

$$\begin{aligned}
p_3(x) = {} & x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97} \\
& + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{79}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{62} + x^{59}\\
&+ x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{43} + x^{41}\\
&+ x^{38} + x^{37} + x^{36},\\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{104} + x^{98} + x^{94} + x^{92} + x^{84} + x^{78} + x^{76}\\
&+ x^{74} + x^{70} + x^{68} + x^{64} + x^{58} + x^{54} + x^{46} + x^{42} + x^{36} + x^{34} + x^{28}\\
&+ x^{26} + x^{24},\\
p_9(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111}\\
&+ x^{109} + x^{108} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82}\\
&+ x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68}\\
&+ x^{66} + x^{63} + x^{61} + x^{60} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{37}\\
&+ x^{36} + x^{34} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23}\\
&+ x^{21} + x^{20} + x^{18} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{132} + x^{128} + x^{124} + x^{120} + x^{116} + x^{108} + x^{104} + x^{92} + x^{88} + x^{80}\\
&+ x^{64} + x^{60} + x^{56} + x^{44} + x^{40} + x^{32} + x^{16} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{117} + x^{110} + x^{108}\\
&+ x^{107} + x^{105} + x^{104} + x^{101} + x^{97} + x^{95} + x^{94} + x^{91} + x^{87} + x^{84}\\
&+ x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{71} + x^{69} + x^{66} + x^{64} + x^{62}\\
&+ x^{61} + x^{60} + x^{57} + x^{55} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{43} + x^{41}\\
&+ x^{40} + x^{39} + x^{36},\\
p_3(x) &= x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100}\\
&+ x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72}\\
&+ x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52}\\
&+ x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36},\\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92} + x^{84} + x^{78} + x^{74}\\
&+ x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{40} + x^{34} + x^{32} + x^{30}\\
&+ x^{26} + x^{24},\\
p_9(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{94} + x^{91} + x^{90} + x^{89}\\
&+ x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78}\\
&+ x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}\\
&+x^{17}+x^{16}+x^{15}+x^{13}+x^{12},\\
p_{12}(x)&=x^{132}+x^{128}+x^{120}+x^{112}+x^{108}+x^{100}+x^{88}+x^{84}+x^{76}+x^{72}\\
&+x^{60}+x^{52}+x^{40}+x^{36}+x^{28}+x^{24}+x^{12}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)&=x^{126}+x^{123}+x^{121}+x^{118}+x^{117}+x^{116}+x^{115}+x^{113}+x^{111}+x^{107}\\
&+x^{104}+x^{102}+x^{99}+x^{98}+x^{97}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}+x^{89}\\
&+x^{86}+x^{82}+x^{81}+x^{79}+x^{76}+x^{74}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}\\
&+x^{65}+x^{63}+x^{62}+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}+x^{52}+x^{50}\\
&+x^{49}+x^{48}+x^{47}+x^{45}+x^{43}+x^{42}+x^{40}+x^{38}+x^{37}+x^{36},\\
p_3(x)&=x^{118}+x^{115}+x^{113}+x^{110}+x^{109}+x^{108}+x^{107}+x^{105}+x^{103}+x^{102}\\
&+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{89}+x^{85}+x^{83}+x^{82}+x^{81}+x^{79}\\
&+x^{77}+x^{76}+x^{75}+x^{72}+x^{71}+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}+x^{62}\\
&+x^{60}+x^{59}+x^{58}+x^{57}+x^{55}+x^{54}+x^{53}+x^{52}+x^{49}+x^{48}+x^{47}\\
&+x^{46}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}+x^{35}+x^{34}+x^{32}+x^{30}\\
&+x^{29}+x^{28},\\
p_6(x)&=x^{120}+x^{117}+x^{116}+x^{111}+x^{109}+x^{107}+x^{106}+x^{105}+x^{102}+x^{97}\\
&+x^{96}+x^{95}+x^{93}+x^{92}+x^{91}+x^{89}+x^{87}+x^{86}+x^{80}+x^{79}+x^{78}\\
&+x^{76}+x^{73}+x^{68}+x^{67}+x^{66}+x^{64}+x^{62}+x^{59}+x^{57}+x^{56}+x^{55}\\
&+x^{53}+x^{49}+x^{47}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}+x^{38}+x^{37}+x^{35}\\
&+x^{34}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{24}+x^{23}+x^{18}+x^{16}\\
&+x^{15}+x^{14}+x^{10}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x+1,\\
p_9(x)&=x^{118}+x^{115}+x^{113}+x^{112}+x^{86}+x^{83}+x^{81}+x^{80}+x^{70}+x^{67}+x^{65}\\
&+x^{64}+x^{38}+x^{35}+x^{33}+x^{32}+x^{22}+x^{19}+x^{17}+x^{16},\\
p_{12}(x)&=x^{126}+x^{123}+x^{121}+x^{120}+x^{118}+x^{115}+x^{114}+x^{113}+x^{112}+x^{111}\\
&+x^{109}+x^{108}+x^{106}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}\\
&+x^{94}+x^{91}+x^{89}+x^{88}+x^{86}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}\\
&+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{66}+x^{63}\\
&+x^{61}+x^{60}+x^{58}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{46}\\
&+x^{43}+x^{41}+x^{40}+x^{38}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}\\
&+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{15}+x^{13}\\
&+x^{12}+x^{10}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{126} + x^{123} + x^{121} + x^{117} + x^{116} + x^{114} + x^{112} + x^{110} + x^{109} + x^{106} \\
& + x^{102} + x^{101} + x^{99} + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{85} + x^{84} + x^{82} \\
& + x^{79} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} + x^{61} \\
& + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{45} + x^{42} \\
& + x^{41} + x^{38} + x^{37} + x^{36}, \\
p_3(x) = {} & x^{118} + x^{115} + x^{113} + x^{109} + x^{108} + x^{106} + x^{104} + x^{102} + x^{101} + x^{98} \\
& + x^{94} + x^{93} + x^{90} + x^{89} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{75} + x^{74} \\
& + x^{72} + x^{70} + x^{62} + x^{59} + x^{58} + x^{56} + x^{54} + x^{50} + x^{49} + x^{48} + x^{45} \\
& + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{30} + x^{29} \\
& + x^{28}, \\
p_6(x) = {} & x^{120} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{105} \\
& + x^{102} + x^{98} + x^{97} + x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} \\
& + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} \\
& + x^{62} + x^{61} + x^{57} + x^{56} + x^{52} + x^{49} + x^{48} + x^{47} + x^{46} + x^{43} + x^{39} \\
& + x^{38} + x^{37} + x^{33} + x^{32} + x^{29} + x^{25} + x^{23} + x^{22} + x^{19} + x^{18} + x^{17} \\
& + x^{15} + x^{14} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) = {} & x^{118} + x^{115} + x^{113} + x^{112} + x^{86} + x^{83} + x^{81} + x^{80} + x^{70} + x^{67} + x^{65} \\
& + x^{64} + x^{38} + x^{35} + x^{33} + x^{32} + x^{22} + x^{19} + x^{17} + x^{16}, \\
p_{12}(x) = {} & x^{126} + x^{123} + x^{121} + x^{120} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} \\
& + x^{109} + x^{108} + x^{106} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} \\
& + x^{94} + x^{91} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
& + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{66} + x^{63} \\
& + x^{61} + x^{60} + x^{58} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} \\
& + x^{43} + x^{41} + x^{40} + x^{38} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} \\
& + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{15} + x^{13} \\
& + x^{12} + x^{10} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{132} + x^{126} + x^{125} + x^{124} + x^{123} + x^{120} + x^{118} + x^{117} + x^{110} + x^{108} \\
& + x^{107} + x^{105} + x^{104} + x^{101} + x^{97} + x^{95} + x^{94} + x^{91} + x^{87} + x^{84} \\
& + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{71} + x^{69} + x^{66} + x^{64} + x^{62} \\
& + x^{61} + x^{60} + x^{57} + x^{55} + x^{54} + x^{52} + x^{50} + x^{46} + x^{44} + x^{43} + x^{41} \\
& + x^{40} + x^{39} + x^{36},
\end{aligned}$$

$$
\begin{aligned}
p_3(x) &= x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{106} + x^{104} + x^{102} + x^{101} + x^{100}\\
&\quad + x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{80} + x^{79} + x^{76} + x^{75} + x^{74} + x^{72}\\
&\quad + x^{70} + x^{69} + x^{68} + x^{66} + x^{63} + x^{62} + x^{61} + x^{59} + x^{56} + x^{55} + x^{52}\\
&\quad + x^{51} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{40} + x^{38} + x^{37} + x^{36},\\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92} + x^{84} + x^{78} + x^{74}\\
&\quad + x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{54} + x^{52} + x^{40} + x^{34} + x^{32} + x^{30}\\
&\quad + x^{26} + x^{24},\\
p_9(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&\quad + x^{114} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{94} + x^{91} + x^{90} + x^{89}\\
&\quad + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78}\\
&\quad + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&\quad + x^{66} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40}\\
&\quad + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&\quad + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}\\
&\quad + x^{17} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{132} + x^{128} + x^{120} + x^{112} + x^{108} + x^{100} + x^{88} + x^{84} + x^{76} + x^{72}\\
&\quad + x^{60} + x^{52} + x^{40} + x^{36} + x^{28} + x^{24} + x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{126} + x^{125} + x^{123} + x^{120} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109}\\
&\quad + x^{107} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92}\\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74}\\
&\quad + x^{73} + x^{71} + x^{69} + x^{68} + x^{61} + x^{57} + x^{56} + x^{55} + x^{52} + x^{51} + x^{50}\\
&\quad + x^{49} + x^{48} + x^{46} + x^{45} + x^{44} + x^{41} + x^{39} + x^{36},\\
p_3(x) &= x^{114} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102} + x^{100} + x^{97}\\
&\quad + x^{96} + x^{95} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{79}\\
&\quad + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{64} + x^{62} + x^{59}\\
&\quad + x^{58} + x^{57} + x^{55} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{43} + x^{41}\\
&\quad + x^{38} + x^{37} + x^{36},\\
p_6(x) &= x^{120} + x^{116} + x^{114} + x^{112} + x^{104} + x^{98} + x^{94} + x^{92} + x^{84} + x^{78} + x^{76}\\
&\quad + x^{74} + x^{70} + x^{68} + x^{64} + x^{58} + x^{54} + x^{46} + x^{42} + x^{36} + x^{34} + x^{28}\\
&\quad + x^{26} + x^{24},\\
p_9(x) &= x^{126} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111}\\
&\quad + x^{109} + x^{108} + x^{94} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{85} + x^{84} + x^{82}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}\\
&+x^{66}+x^{63}+x^{61}+x^{60}+x^{46}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}\\
&+x^{36}+x^{34}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}\\
&+x^{21}+x^{20}+x^{18}+x^{15}+x^{13}+x^{12},\\
p_{12}(x)=&\,x^{132}+x^{128}+x^{124}+x^{120}+x^{116}+x^{108}+x^{104}+x^{92}+x^{88}+x^{80}\\
&+x^{64}+x^{60}+x^{56}+x^{44}+x^{40}+x^{32}+x^{16}+x^{12}+x^{8}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{126}+x^{123}+x^{121}+x^{118}+x^{117}+x^{116}+x^{115}+x^{113}+x^{111}+x^{107}\\
&+x^{104}+x^{102}+x^{99}+x^{98}+x^{97}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}+x^{89}\\
&+x^{86}+x^{82}+x^{81}+x^{79}+x^{76}+x^{74}+x^{72}+x^{70}+x^{69}+x^{68}+x^{67}\\
&+x^{65}+x^{63}+x^{62}+x^{61}+x^{60}+x^{57}+x^{56}+x^{55}+x^{54}+x^{52}+x^{50}\\
&+x^{49}+x^{48}+x^{47}+x^{45}+x^{43}+x^{42}+x^{40}+x^{38}+x^{37}+x^{36},\\
p_3(x)=&\,x^{118}+x^{115}+x^{113}+x^{110}+x^{109}+x^{108}+x^{107}+x^{105}+x^{103}+x^{102}\\
&+x^{97}+x^{96}+x^{93}+x^{92}+x^{91}+x^{89}+x^{85}+x^{83}+x^{82}+x^{81}+x^{79}\\
&+x^{77}+x^{76}+x^{75}+x^{72}+x^{71}+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}+x^{62}\\
&+x^{60}+x^{59}+x^{58}+x^{57}+x^{55}+x^{54}+x^{53}+x^{52}+x^{49}+x^{48}+x^{47}\\
&+x^{46}+x^{44}+x^{42}+x^{41}+x^{40}+x^{39}+x^{37}+x^{35}+x^{34}+x^{32}+x^{30}\\
&+x^{29}+x^{28},\\
p_6(x)=&\,x^{120}+x^{117}+x^{116}+x^{111}+x^{109}+x^{107}+x^{106}+x^{105}+x^{102}+x^{97}\\
&+x^{96}+x^{95}+x^{93}+x^{92}+x^{91}+x^{89}+x^{87}+x^{86}+x^{80}+x^{79}+x^{78}\\
&+x^{76}+x^{73}+x^{68}+x^{67}+x^{66}+x^{64}+x^{62}+x^{59}+x^{57}+x^{56}+x^{55}\\
&+x^{53}+x^{49}+x^{47}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}+x^{38}+x^{37}+x^{35}\\
&+x^{34}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{24}+x^{23}+x^{18}+x^{16}\\
&+x^{15}+x^{14}+x^{10}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x+1,\\
p_9(x)=&\,x^{118}+x^{115}+x^{113}+x^{112}+x^{86}+x^{83}+x^{81}+x^{80}+x^{70}+x^{67}+x^{65}\\
&+x^{64}+x^{38}+x^{35}+x^{33}+x^{32}+x^{22}+x^{19}+x^{17}+x^{16},\\
p_{12}(x)=&\,x^{126}+x^{123}+x^{121}+x^{120}+x^{118}+x^{115}+x^{114}+x^{113}+x^{112}+x^{111}\\
&+x^{109}+x^{108}+x^{106}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}\\
&+x^{94}+x^{91}+x^{89}+x^{88}+x^{86}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}\\
&+x^{77}+x^{76}+x^{75}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{66}+x^{63}\\
&+x^{61}+x^{60}+x^{58}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{46}\\
&+x^{43}+x^{41}+x^{40}+x^{38}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}\\
&+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{15}+x^{13}
\end{aligned}
$$

$$+ x^{12} + x^{10} + x^7 + x^6 + x^5 + x^4 + x^3 + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	392	[629, 32]	784	[1017, 163]	1176	[325, 1244]
1	[3, 4]	393	[630, 631]	785	[493, 991]	1177	[1276, 985]
2	[5, 6]	394	[35, 632]	786	[202, 1118]	1178	[1430, 1327]
3	[7, 8]	395	[209, 130]	787	[168, 1119]	1179	[242, 394]
4	[9, 10]	396	[616, 10]	788	[1120, 454]	1180	[1431, 1432]
5	[11, 12]	397	[419, 633]	789	[501, 155]	1181	[402, 1334]
6	[13, 14]	398	[634, 211]	790	[1111, 1084]	1182	[1316, 1411]
7	[15, 16]	399	[520, 635]	791	[1034, 1121]	1183	[1428, 1433]
8	[17, 18]	400	[213, 636]	792	[1122, 1123]	1184	[845, 904]
9	[19, 20]	401	[207, 428]	793	[1092, 1124]	1185	[244, 1434]
10	[21, 22]	402	[131, 207]	794	[1125, 519]	1186	[16, 1435]
11	[23, 24]	403	[637, 638]	795	[186, 569]	1187	[1436, 943]
12	[25, 26]	404	[511, 432]	796	[568, 1102]	1188	[1368, 115]
13	[27, 28]	405	[639, 640]	797	[543, 1059]	1189	[1376, 873]
14	[29, 30]	406	[641, 642]	798	[195, 1126]	1190	[715, 854]
15	[31, 32]	407	[643, 498]	799	[1127, 517]	1191	[1437, 1365]
16	[33, 34]	408	[557, 644]	800	[1128, 204]	1192	[839, 73]
17	[35, 36]	409	[645, 646]	801	[1129, 1098]	1193	[1438, 1322]
18	[37, 38]	410	[14, 647]	802	[177, 1130]	1194	[1439, 1347]
19	[39, 40]	411	[648, 41]	803	[222, 632]	1195	[1323, 388]
20	[41, 42]	412	[649, 650]	804	[447, 1131]	1196	[1440, 1275]
21	[43, 44]	413	[651, 652]	805	[1027, 220]	1197	[957, 931]
22	[45, 46]	414	[653, 654]	806	[209, 223]	1198	[783, 696]
23	[47, 48]	415	[655, 656]	807	[1132, 1133]	1199	[1441, 1442]
24	[49, 50]	416	[657, 658]	808	[1134, 1116]	1200	[1443, 1439]
25	[51, 52]	417	[659, 660]	809	[56, 223]	1201	[380, 803]
26	[53, 54]	418	[661, 662]	810	[1135, 1030]	1202	[311, 1444]
27	[55, 56]	419	[663, 376]	811	[153, 1136]	1203	[936, 1445]
28	[57, 58]	420	[664, 665]	812	[520, 1137]	1204	[389, 1446]
29	[59, 60]	421	[666, 667]	813	[423, 1138]	1205	[1447, 384]
30	[61, 62]	422	[668, 669]	814	[1139, 635]	1206	[708, 1448]
31	[63, 64]	423	[670, 671]	815	[577, 200]	1207	[1449, 709]
32	[65, 66]	424	[672, 673]	816	[516, 1140]	1208	[1450, 893]
33	[67, 68]	425	[674, 675]	817	[526, 1089]	1209	[1451, 1354]
34	[69, 70]	426	[676, 677]	818	[482, 1141]	1210	[338, 804]
35	[71, 72]	427	[678, 247]	819	[1142, 647]	1211	[1452, 1407]

36	[73, 74]	428	[679, 680]	820	[1143, 567]	1212	[872, 265]
37	[75, 76]	429	[680, 240]	821	[1133, 1144]	1213	[1453, 844]
38	[76, 77]	430	[231, 681]	822	[1029, 522]	1214	[1291, 1454]
39	[78, 25]	431	[682, 683]	823	[593, 1145]	1215	[798, 1455]
40	[79, 80]	432	[77, 684]	824	[11, 1017]	1216	[1456, 1457]
41	[81, 82]	433	[685, 686]	825	[1146, 1147]	1217	[297, 749]
42	[83, 84]	434	[687, 688]	826	[1148, 615]	1218	[68, 277]
43	[85, 86]	435	[689, 690]	827	[1149, 1150]	1219	[940, 1458]
44	[87, 88]	436	[122, 691]	828	[483, 1118]	1220	[1459, 706]
45	[89, 90]	437	[692, 693]	829	[57, 609]	1221	[836, 1249]
46	[91, 92]	438	[694, 695]	830	[128, 1140]	1222	[1460, 1349]
47	[93, 94]	439	[696, 697]	831	[1086, 1151]	1223	[746, 1373]
48	[95, 96]	440	[698, 699]	832	[590, 442]	1224	[1461, 1319]
49	[97, 98]	441	[700, 328]	833	[1152, 1153]	1225	[1348, 304]
50	[99, 100]	442	[701, 335]	834	[1154, 1155]	1226	[1462, 1254]
51	[101, 102]	443	[702, 703]	835	[1152, 630]	1227	[1463, 1464]
52	[103, 104]	444	[303, 687]	836	[607, 1156]	1228	[1331, 1366]
53	[105, 106]	445	[691, 83]	837	[1157, 1026]	1229	[897, 1465]
54	[107, 108]	446	[704, 705]	838	[1020, 144]	1230	[1298, 800]
55	[109, 110]	447	[706, 678]	839	[1158, 11]	1231	[1466, 719]
56	[111, 112]	448	[707, 708]	840	[14, 8]	1232	[1435, 851]
57	[113, 114]	449	[709, 710]	841	[1159, 1000]	1233	[1467, 1279]
58	[115, 116]	450	[711, 712]	842	[1148, 564]	1234	[1468, 1469]
59	[117, 118]	451	[713, 714]	843	[449, 1011]	1235	[984, 1470]
60	[119, 120]	452	[665, 715]	844	[1023, 1160]	1236	[1471, 1381]
61	[121, 122]	453	[716, 306]	845	[32, 1044]	1237	[1472, 1290]
62	[123, 124]	454	[717, 718]	846	[1077, 439]	1238	[764, 1259]
63	[125, 126]	455	[719, 720]	847	[1161, 570]	1239	[31, 637]
64	[33, 127]	456	[721, 722]	848	[1162, 449]	1240	[1232, 1206]
65	[128, 129]	457	[280, 723]	849	[1163, 469]	1241	[431, 512]
66	[130, 131]	458	[724, 725]	850	[585, 1164]	1242	[435, 642]
67	[132, 133]	459	[726, 727]	851	[509, 529]	1243	[1237, 1052]
68	[134, 135]	460	[728, 729]	852	[1165, 1166]	1244	[645, 1062]
69	[136, 137]	461	[730, 731]	853	[1167, 1168]	1245	[1063, 1092]
70	[138, 139]	462	[732, 733]	854	[644, 1094]	1246	[1100, 1156]
71	[140, 141]	463	[734, 735]	855	[556, 558]	1247	[1074, 650]
72	[142, 143]	464	[736, 737]	856	[1012, 1169]	1248	[990, 34]
73	[144, 145]	465	[738, 739]	857	[619, 1170]	1249	[566, 593]
74	[146, 147]	466	[363, 740]	858	[601, 461]	1250	[1156, 1473]
75	[130, 148]	467	[741, 742]	859	[1149, 519]	1251	[1082, 1136]
76	[149, 55]	468	[743, 744]	860	[502, 1171]	1252	[1474, 1197]
77	[148, 150]	469	[745, 252]	861	[515, 1088]	1253	[1475, 1181]
78	[151, 53]	470	[26, 746]	862	[487, 418]	1254	[1476, 1477]
79	[152, 153]	471	[747, 748]	863	[1132, 1172]	1255	[1198, 625]

80	[154, 155]	472	[749, 750]	864	[37, 37]	1256	[1004, 590]
81	[43, 156]	473	[723, 751]	865	[1120, 527]	1257	[1194, 1478]
82	[157, 158]	474	[752, 378]	866	[613, 1173]	1258	[448, 165]
83	[159, 160]	475	[753, 754]	867	[596, 1174]	1259	[437, 628]
84	[161, 162]	476	[751, 755]	868	[1094, 479]	1260	[513, 1479]
85	[12, 163]	477	[345, 756]	869	[1094, 590]	1261	[134, 144]
86	[164, 165]	478	[757, 758]	870	[644, 1004]	1262	[1480, 1176]
87	[166, 167]	479	[759, 760]	871	[560, 560]	1263	[1069, 211]
88	[168, 169]	480	[96, 408]	872	[557, 559]	1264	[479, 442]
89	[162, 170]	481	[761, 762]	873	[524, 1174]	1265	[1060, 477]
90	[171, 172]	482	[763, 764]	874	[1175, 1176]	1266	[465, 42]
91	[166, 173]	483	[765, 765]	875	[126, 1177]	1267	[628, 162]
92	[174, 175]	484	[370, 766]	876	[1178, 1179]	1268	[1481, 1482]
93	[176, 177]	485	[767, 768]	877	[1180, 1181]	1269	[1219, 544]
94	[178, 179]	486	[319, 769]	878	[1182, 1042]	1270	[1031, 518]
95	[180, 181]	487	[770, 105]	879	[1023, 1055]	1271	[570, 572]
96	[182, 183]	488	[729, 771]	880	[1158, 40]	1272	[1051, 467]
97	[184, 185]	489	[772, 773]	881	[462, 156]	1273	[1483, 996]
98	[186, 187]	490	[774, 775]	882	[438, 1078]	1274	[1480, 1484]
99	[188, 189]	491	[776, 777]	883	[139, 1183]	1275	[1024, 219]
100	[190, 191]	492	[778, 779]	884	[999, 1138]	1276	[1172, 1144]
101	[192, 193]	493	[780, 405]	885	[517, 1032]	1277	[139, 1107]
102	[194, 137]	494	[781, 782]	886	[185, 1141]	1278	[1103, 636]
103	[195, 196]	495	[294, 783]	887	[443, 1006]	1279	[617, 215]
104	[197, 198]	496	[784, 785]	888	[1101, 1000]	1280	[1001, 1485]
105	[199, 200]	497	[786, 787]	889	[1184, 1185]	1281	[1486, 634]
106	[201, 202]	498	[788, 789]	890	[1010, 450]	1282	[59, 537]
107	[203, 204]	499	[790, 791]	891	[1049, 993]	1283	[1153, 631]
108	[205, 206]	500	[792, 793]	892	[647, 1186]	1284	[1025, 1235]
109	[207, 208]	501	[794, 795]	893	[1187, 620]	1285	[576, 580]
110	[55, 209]	502	[100, 796]	894	[1188, 1189]	1286	[604, 1229]
111	[208, 210]	503	[797, 798]	895	[1052, 468]	1287	[1221, 571]
112	[131, 150]	504	[72, 799]	896	[1190, 1191]	1288	[1487, 205]
113	[211, 212]	505	[800, 801]	897	[50, 1019]	1289	[1180, 1488]
114	[213, 214]	506	[94, 802]	898	[192, 1169]	1290	[1202, 1477]
115	[10, 215]	507	[803, 109]	899	[994, 457]	1291	[194, 227]
116	[216, 217]	508	[804, 243]	900	[473, 160]	1292	[1127, 1031]
117	[218, 219]	509	[805, 117]	901	[1192, 473]	1293	[1475, 1488]
118	[220, 221]	510	[806, 119]	902	[1193, 1129]	1294	[1225, 1479]
119	[222, 36]	511	[807, 808]	903	[1194, 1195]	1295	[519, 1139]
120	[223, 148]	512	[809, 810]	904	[1196, 1179]	1296	[1155, 1489]
121	[224, 225]	513	[811, 736]	905	[1197, 1166]	1297	[1082, 1490]
122	[226, 159]	514	[254, 19]	906	[1058, 583]	1298	[1491, 1129]
123	[136, 227]	515	[812, 813]	907	[1193, 1097]	1299	[1492, 533]

124	[228, 229]	516	[814, 815]	908	[530, 434]	1300	[1054, 1160]
125	[230, 231]	517	[688, 816]	909	[1198, 471]	1301	[606, 61]
126	[232, 233]	518	[817, 818]	910	[1199, 1058]	1302	[1154, 1116]
127	[234, 235]	519	[819, 820]	911	[1064, 1124]	1303	[1006, 1115]
128	[236, 237]	520	[735, 231]	912	[1200, 1197]	1304	[1178, 1493]
129	[238, 239]	521	[816, 809]	913	[1128, 1201]	1305	[532, 1494]
130	[240, 241]	522	[821, 822]	914	[1202, 1203]	1306	[451, 539]
131	[120, 242]	523	[823, 824]	915	[1065, 1204]	1307	[1091, 1085]
132	[243, 244]	524	[825, 826]	916	[183, 558]	1308	[586, 623]
133	[245, 246]	525	[827, 828]	917	[1017, 1205]	1309	[993, 1016]
134	[247, 248]	526	[829, 830]	918	[1071, 1206]	1310	[628, 554]
135	[249, 250]	527	[831, 832]	919	[494, 1168]	1311	[1495, 1043]
136	[251, 93]	528	[833, 834]	920	[567, 1145]	1312	[996, 422]
137	[252, 95]	529	[112, 747]	921	[161, 554]	1313	[12, 1205]
138	[253, 254]	530	[835, 836]	922	[1207, 603]	1314	[205, 1496]
139	[255, 256]	531	[276, 837]	923	[1208, 1010]	1315	[180, 1151]
140	[257, 258]	532	[779, 91]	924	[1047, 133]	1316	[469, 1018]
141	[259, 260]	533	[838, 839]	925	[1209, 1064]	1317	[1491, 1097]
142	[261, 262]	534	[375, 99]	926	[1061, 589]	1318	[1064, 1093]
143	[263, 264]	535	[840, 841]	927	[1002, 477]	1319	[155, 1171]
144	[265, 266]	536	[842, 843]	928	[47, 623]	1320	[1497, 1119]
145	[267, 268]	537	[284, 806]	929	[480, 182]	1321	[1487, 553]
146	[269, 270]	538	[844, 845]	930	[647, 453]	1322	[1035, 1498]
147	[271, 272]	539	[290, 846]	931	[540, 186]	1323	[1499, 625]
148	[273, 274]	540	[847, 342]	932	[1143, 593]	1324	[1073, 492]
149	[275, 276]	541	[848, 842]	933	[550, 605]	1325	[467, 1500]
150	[241, 275]	542	[849, 850]	934	[482, 1210]	1326	[1501, 415]
151	[277, 107]	543	[250, 726]	935	[1211, 996]	1327	[1134, 1155]
152	[278, 279]	544	[851, 852]	936	[1013, 1212]	1328	[648, 465]
153	[22, 280]	545	[828, 853]	937	[1213, 60]	1329	[454, 618]
154	[281, 282]	546	[854, 855]	938	[1133, 1214]	1330	[641, 436]
155	[283, 284]	547	[856, 103]	939	[1139, 1137]	1331	[1085, 1173]
156	[282, 285]	548	[857, 858]	940	[142, 1215]	1332	[1006, 1502]
157	[286, 287]	549	[859, 860]	941	[1000, 1216]	1333	[528, 510]
158	[270, 288]	550	[861, 862]	942	[1217, 1218]	1334	[1135, 486]
159	[289, 290]	551	[863, 763]	943	[1163, 476]	1335	[595, 534]
160	[291, 292]	552	[864, 17]	944	[1219, 179]	1336	[1068, 634]
161	[293, 294]	553	[865, 320]	945	[1220, 1221]	1337	[1166, 1478]
162	[295, 296]	554	[768, 866]	946	[458, 1212]	1338	[625, 472]
163	[24, 297]	555	[327, 867]	947	[1222, 622]	1339	[1497, 169]
164	[298, 299]	556	[755, 868]	948	[1223, 1215]	1340	[525, 50]
165	[300, 301]	557	[855, 869]	949	[1224, 202]	1341	[1503, 1123]
166	[302, 303]	558	[870, 871]	950	[1096, 13]	1342	[585, 1057]
167	[304, 305]	559	[336, 872]	951	[1165, 1194]	1343	[546, 183]

168	[306, 307]	560	[718, 664]	952	[1045, 640]	1344	[1187, 1170]
169	[308, 309]	561	[561, 561]	953	[1225, 514]	1345	[1233, 1014]
170	[310, 311]	562	[873, 874]	954	[1226, 1079]	1346	[1166, 1195]
171	[312, 313]	563	[875, 876]	955	[144, 425]	1347	[1227, 54]
172	[314, 315]	564	[877, 878]	956	[584, 588]	1348	[476, 470]
173	[316, 317]	565	[683, 879]	957	[500, 1012]	1349	[617, 1067]
174	[318, 319]	566	[880, 661]	958	[591, 1204]	1350	[1217, 1105]
175	[320, 321]	567	[881, 882]	959	[1155, 1117]	1351	[185, 1210]
176	[322, 323]	568	[883, 884]	960	[201, 483]	1352	[1200, 1165]
177	[324, 325]	569	[885, 886]	961	[1227, 582]	1353	[1223, 143]
178	[326, 327]	570	[887, 888]	962	[1190, 1228]	1354	[203, 1201]
179	[328, 329]	571	[889, 654]	963	[550, 1229]	1355	[1504, 491]
180	[330, 257]	572	[695, 890]	964	[1150, 1139]	1356	[1022, 1054]
181	[331, 332]	573	[891, 892]	965	[1230, 504]	1357	[1505, 521]
182	[333, 334]	574	[893, 894]	966	[1231, 527]	1358	[1081, 153]
183	[335, 336]	575	[714, 895]	967	[61, 221]	1359	[1114, 1075]
184	[337, 338]	576	[896, 897]	968	[1232, 1072]	1360	[503, 612]
185	[339, 340]	577	[898, 899]	969	[1069, 140]	1361	[1481, 1121]
186	[341, 232]	578	[739, 717]	970	[426, 1109]	1362	[1028, 521]
187	[342, 343]	579	[900, 5]	971	[1142, 8]	1363	[1153, 1506]
188	[344, 345]	580	[901, 902]	972	[992, 1015]	1364	[1104, 1218]
189	[346, 347]	581	[903, 412]	973	[1156, 508]	1365	[589, 1130]
190	[348, 349]	582	[904, 905]	974	[1233, 198]	1366	[1123, 1228]
191	[350, 351]	583	[866, 906]	975	[1234, 630]	1367	[1147, 1507]
192	[352, 353]	584	[907, 908]	976	[218, 126]	1368	[8, 1186]
193	[354, 355]	585	[815, 909]	977	[1157, 1235]	1369	[153, 1490]
194	[356, 357]	586	[910, 911]	978	[1123, 1191]	1370	[1015, 1110]
195	[358, 359]	587	[912, 913]	979	[1236, 1034]	1371	[498, 1080]
196	[360, 361]	588	[914, 774]	980	[162, 1036]	1372	[1508, 1126]
197	[362, 363]	589	[915, 870]	981	[471, 626]	1373	[599, 598]
198	[364, 365]	590	[916, 333]	982	[1039, 1076]	1374	[621, 1509]
199	[355, 366]	591	[917, 918]	983	[1237, 467]	1375	[547, 1498]
200	[367, 368]	592	[860, 919]	984	[578, 1216]	1376	[1224, 483]
201	[369, 370]	593	[882, 324]	985	[219, 1177]	1377	[38, 38]
202	[28, 371]	594	[920, 921]	986	[639, 1046]	1378	[1172, 1214]
203	[372, 373]	595	[922, 331]	987	[1182, 1238]	1379	[1150, 520]
204	[374, 375]	596	[923, 924]	988	[634, 140]	1380	[1003, 478]
205	[376, 377]	597	[925, 926]	989	[1239, 256]	1381	[551, 13]
206	[378, 379]	598	[760, 915]	990	[1240, 891]	1382	[1510, 491]
207	[233, 380]	599	[927, 928]	991	[1241, 1242]	1383	[1199, 543]
208	[274, 27]	600	[929, 881]	992	[1243, 238]	1384	[1003, 1511]
209	[381, 273]	601	[930, 920]	993	[1244, 738]	1385	[1196, 1493]
210	[66, 111]	602	[931, 97]	994	[1245, 23]	1386	[1129, 1512]
211	[382, 64]	603	[834, 932]	995	[869, 831]	1387	[1146, 651]

212	[383, 298]	604	[810, 341]	996	[1246, 390]	1388	[1008, 1185]
213	[384, 367]	605	[933, 934]	997	[1247, 1248]	1389	[1161, 1221]
214	[385, 386]	606	[935, 936]	998	[1249, 1250]	1390	[1234, 1153]
215	[20, 387]	607	[937, 938]	999	[720, 657]	1391	[1174, 1106]
216	[388, 389]	608	[939, 940]	1000	[409, 289]	1392	[1513, 1050]
217	[390, 391]	609	[941, 942]	1001	[1251, 817]	1393	[9, 617]
218	[392, 393]	610	[943, 944]	1002	[1252, 1253]	1394	[630, 1506]
219	[394, 395]	611	[945, 786]	1003	[1254, 1255]	1395	[1106, 1183]
220	[90, 396]	612	[946, 947]	1004	[871, 1256]	1396	[1222, 1509]
221	[397, 398]	613	[948, 949]	1005	[1257, 1258]	1397	[651, 1507]
222	[399, 400]	614	[950, 951]	1006	[684, 381]	1398	[146, 1485]
223	[401, 402]	615	[952, 953]	1007	[1259, 350]	1399	[1147, 652]
224	[403, 404]	616	[954, 955]	1008	[973, 300]	1400	[643, 1079]
225	[405, 406]	617	[846, 249]	1009	[908, 1260]	1401	[473, 1053]
226	[407, 291]	618	[744, 269]	1010	[733, 1261]	1402	[1514, 633]
227	[408, 409]	619	[956, 957]	1011	[1262, 1263]	1403	[138, 1106]
228	[410, 411]	620	[868, 958]	1012	[958, 1264]	1404	[1079, 499]
229	[412, 413]	621	[959, 823]	1013	[1265, 278]	1405	[1515, 228]
230	[414, 60]	622	[960, 281]	1014	[756, 369]	1406	[1514, 420]
231	[415, 416]	623	[82, 360]	1015	[266, 352]	1407	[1495, 225]
232	[417, 418]	624	[961, 962]	1016	[1266, 1267]	1408	[444, 1502]
233	[223, 131]	625	[796, 65]	1017	[114, 1268]	1409	[1208, 449]
234	[419, 420]	626	[963, 964]	1018	[1269, 896]	1410	[1159, 578]
235	[421, 422]	627	[965, 295]	1019	[1270, 1271]	1411	[1075, 167]
236	[211, 141]	628	[966, 310]	1020	[1272, 267]	1412	[1167, 495]
237	[423, 424]	629	[686, 967]	1021	[654, 728]	1413	[1192, 159]
238	[135, 425]	630	[968, 4]	1022	[1273, 1274]	1414	[1516, 637]
239	[426, 427]	631	[899, 969]	1023	[785, 271]	1415	[572, 998]
240	[428, 149]	632	[307, 970]	1024	[1275, 1276]	1416	[1517, 1085]
241	[210, 429]	633	[832, 745]	1025	[1277, 1278]	1417	[1226, 498]
242	[429, 209]	634	[971, 259]	1026	[1279, 1280]	1418	[1174, 139]
243	[219, 430]	635	[972, 973]	1027	[1281, 397]	1419	[1518, 990]
244	[431, 432]	636	[671, 974]	1028	[1282, 1283]	1420	[1486, 1069]
245	[433, 434]	637	[837, 374]	1029	[1284, 75]	1421	[1503, 1190]
246	[435, 436]	638	[64, 312]	1030	[248, 1285]	1422	[1197, 1194]
247	[437, 161]	639	[975, 976]	1031	[748, 302]	1423	[1231, 454]
248	[135, 145]	640	[18, 977]	1032	[1286, 1287]	1424	[611, 1075]
249	[438, 439]	641	[978, 87]	1033	[1288, 1289]	1425	[1034, 1482]
250	[157, 440]	642	[411, 907]	1034	[1290, 960]	1426	[535, 1090]
251	[441, 178]	643	[979, 790]	1035	[906, 1291]	1427	[553, 1496]
252	[442, 182]	644	[353, 916]	1036	[1292, 689]	1428	[1007, 539]
253	[443, 444]	645	[980, 968]	1037	[1293, 1294]	1429	[587, 585]
254	[445, 41]	646	[981, 982]	1038	[795, 1295]	1430	[1519, 1054]
255	[164, 446]	647	[371, 401]	1039	[260, 346]	1431	[1520, 1133]

256	[447, 416]	648	[983, 984]	1040	[1296, 674]	1432	[1505, 1029]
257	[448, 446]	649	[985, 986]	1041	[1297, 885]	1433	[507, 1473]
258	[449, 450]	650	[986, 987]	1042	[1261, 1298]	1434	[1041, 1238]
259	[451, 452]	651	[287, 838]	1043	[1253, 1299]	1435	[32, 638]
260	[8, 453]	652	[102, 988]	1044	[1300, 318]	1436	[441, 1219]
261	[454, 455]	653	[989, 521]	1045	[1301, 1302]	1437	[588, 1164]
262	[456, 457]	654	[990, 127]	1046	[110, 283]	1438	[1521, 1113]
263	[458, 459]	655	[171, 991]	1047	[1303, 1304]	1439	[1083, 1112]
264	[460, 461]	656	[992, 993]	1048	[404, 1305]	1440	[1520, 1172]
265	[462, 44]	657	[994, 995]	1049	[391, 1266]	1441	[1122, 1190]
266	[463, 464]	658	[996, 997]	1050	[1302, 1306]	1442	[477, 1511]
267	[465, 466]	659	[571, 998]	1051	[1307, 743]	1443	[151, 1227]
268	[467, 468]	660	[999, 424]	1052	[1308, 308]	1444	[1508, 196]
269	[469, 470]	661	[1000, 579]	1053	[1309, 1310]	1445	[1522, 1189]
270	[471, 472]	662	[1001, 147]	1054	[1311, 383]	1446	[1040, 175]
271	[226, 473]	663	[1002, 1003]	1055	[1274, 1312]	1447	[1523, 517]
272	[39, 11]	664	[1004, 479]	1056	[1313, 1247]	1448	[1492, 1494]
273	[210, 55]	665	[1005, 646]	1057	[1304, 1314]	1449	[1524, 651]
274	[208, 149]	666	[565, 1006]	1058	[1315, 759]	1450	[1211, 421]
275	[149, 429]	667	[1007, 452]	1059	[850, 1316]	1451	[1525, 1504]
276	[429, 56]	668	[1008, 1009]	1060	[1317, 757]	1452	[1524, 1147]
277	[474, 205]	669	[1010, 1011]	1061	[926, 1318]	1453	[1516, 32]
278	[475, 476]	670	[1012, 193]	1062	[1319, 1320]	1454	[421, 997]
279	[477, 478]	671	[597, 600]	1063	[1321, 245]	1455	[1522, 1526]
280	[479, 480]	672	[1013, 459]	1064	[1322, 1323]	1456	[567, 594]
281	[481, 482]	673	[197, 1014]	1065	[1324, 255]	1457	[1052, 1500]
282	[483, 484]	674	[1015, 1016]	1066	[256, 403]	1458	[1499, 471]
283	[485, 486]	675	[40, 1017]	1067	[773, 1325]	1459	[1501, 447]
284	[487, 488]	676	[476, 1018]	1068	[1326, 67]	1460	[537, 1021]
285	[489, 490]	677	[526, 1019]	1069	[1327, 1328]	1461	[1220, 570]
286	[491, 492]	678	[1020, 135]	1070	[909, 1329]	1462	[1521, 504]
287	[493, 172]	679	[150, 428]	1071	[1250, 952]	1463	[1476, 1203]
288	[494, 495]	680	[428, 210]	1072	[1289, 1330]	1464	[1523, 1031]
289	[496, 464]	681	[60, 1021]	1073	[660, 1331]	1465	[1221, 572]
290	[496, 497]	682	[1022, 1023]	1074	[1332, 1333]	1466	[542, 1058]
291	[41, 466]	683	[1024, 126]	1075	[1334, 861]	1467	[1116, 1489]
292	[498, 499]	684	[1025, 1026]	1076	[1335, 1336]	1468	[1515, 1050]
293	[500, 192]	685	[1027, 61]	1077	[1337, 655]	1469	[1175, 1484]
294	[501, 502]	686	[1028, 1029]	1078	[1338, 1339]	1470	[1108, 427]
295	[503, 52]	687	[38, 37]	1079	[725, 1340]	1471	[1525, 1510]
296	[504, 505]	688	[485, 1030]	1080	[884, 961]	1472	[1527, 53]
297	[48, 506]	689	[48, 624]	1081	[1341, 1342]	1473	[938, 362]
298	[507, 508]	690	[1031, 1032]	1082	[727, 1343]	1474	[1528, 914]
299	[509, 510]	691	[627, 161]	1083	[1329, 69]	1475	[1529, 1246]

300	[511, 512]	692	[1033, 1034]	1084	[1344, 1345]	1476	[1530, 123]
301	[513, 514]	693	[1035, 548]	1085	[693, 1292]	1477	[1444, 1531]
302	[515, 191]	694	[554, 1036]	1086	[1346, 1240]	1478	[1424, 261]
303	[516, 129]	695	[649, 1037]	1087	[1347, 1348]	1479	[1413, 1288]
304	[517, 518]	696	[502, 1038]	1088	[1349, 1350]	1480	[1532, 293]
305	[519, 520]	697	[1039, 490]	1089	[98, 1351]	1481	[349, 1447]
306	[521, 522]	698	[155, 1038]	1090	[86, 16]	1482	[299, 1533]
307	[452, 523]	699	[1040, 555]	1091	[1352, 950]	1483	[1534, 668]
308	[154, 502]	700	[474, 553]	1092	[1353, 730]	1484	[334, 1315]
309	[524, 138]	701	[561, 560]	1093	[1354, 1355]	1485	[1328, 1321]
310	[525, 526]	702	[629, 637]	1094	[1343, 701]	1486	[1535, 382]
311	[527, 455]	703	[220, 62]	1095	[775, 933]	1487	[1536, 1375]
312	[126, 430]	704	[433, 531]	1096	[1356, 1357]	1488	[395, 1284]
313	[528, 529]	705	[1041, 1042]	1097	[1358, 1359]	1489	[1537, 772]
314	[530, 531]	706	[224, 1043]	1098	[1360, 1361]	1490	[379, 1530]
315	[532, 533]	707	[637, 1044]	1099	[731, 672]	1491	[1538, 1360]
316	[534, 187]	708	[1045, 1046]	1100	[1362, 1363]	1492	[962, 1270]
317	[535, 536]	709	[1047, 1048]	1101	[272, 900]	1493	[1377, 829]
318	[537, 538]	710	[1049, 1015]	1102	[955, 1364]	1494	[1410, 700]
319	[539, 523]	711	[456, 995]	1103	[1365, 711]	1495	[1433, 1385]
320	[540, 534]	712	[1050, 229]	1104	[1366, 1338]	1496	[400, 1539]
321	[541, 535]	713	[1051, 1052]	1105	[888, 880]	1497	[892, 1309]
322	[542, 543]	714	[159, 1053]	1106	[826, 1367]	1498	[366, 1540]
323	[178, 544]	715	[581, 46]	1107	[4, 1368]	1499	[1541, 963]
324	[45, 545]	716	[1054, 1055]	1108	[1369, 981]	1500	[813, 1542]
325	[480, 546]	717	[1056, 440]	1109	[841, 1317]	1501	[1543, 385]
326	[547, 548]	718	[182, 556]	1110	[1370, 1371]	1502	[1248, 1460]
327	[549, 550]	719	[588, 1057]	1111	[802, 1372]	1503	[359, 1437]
328	[551, 552]	720	[1058, 1059]	1112	[1373, 1374]	1504	[928, 778]
329	[553, 206]	721	[1060, 1003]	1113	[1375, 1376]	1505	[1432, 821]
330	[554, 170]	722	[1061, 177]	1114	[321, 316]	1506	[1469, 1544]
331	[202, 484]	723	[1005, 1062]	1115	[879, 875]	1507	[1406, 1545]
332	[174, 555]	724	[1063, 1064]	1116	[977, 1377]	1508	[1546, 1308]
333	[556, 557]	725	[610, 1012]	1117	[1378, 707]	1509	[80, 1281]
334	[558, 559]	726	[442, 546]	1118	[677, 1379]	1510	[1547, 1335]
335	[183, 557]	727	[1065, 592]	1119	[106, 922]	1511	[313, 1548]
336	[560, 561]	728	[521, 1066]	1120	[722, 1380]	1512	[1278, 354]
337	[562, 447]	729	[10, 1067]	1121	[1381, 1382]	1513	[1549, 121]
338	[563, 564]	730	[549, 604]	1122	[1383, 1384]	1514	[1550, 753]
339	[565, 444]	731	[1068, 1069]	1123	[361, 929]	1515	[1551, 1552]
340	[566, 567]	732	[491, 1070]	1124	[1385, 1386]	1516	[1553, 1300]
341	[568, 189]	733	[1071, 1072]	1125	[1387, 1388]	1517	[902, 1293]
342	[534, 569]	734	[1073, 1070]	1126	[1256, 21]	1518	[1554, 234]
343	[570, 571]	735	[1074, 1037]	1127	[1389, 1286]	1519	[1336, 1344]

344	[572, 573]	736	[1075, 173]	1128	[1390, 1362]	1520	[309, 1402]
345	[199, 574]	737	[489, 1076]	1129	[357, 364]	1521	[1555, 314]
346	[452, 575]	738	[1077, 1078]	1130	[323, 797]	1522	[822, 1456]
347	[576, 217]	739	[463, 497]	1131	[874, 1391]	1523	[1556, 698]
348	[140, 212]	740	[1079, 1080]	1132	[1392, 946]	1524	[867, 263]
349	[577, 574]	741	[1081, 1082]	1133	[924, 1393]	1525	[1557, 776]
350	[578, 579]	742	[49, 526]	1134	[976, 1394]	1526	[789, 1556]
351	[216, 580]	743	[1083, 1084]	1135	[1395, 948]	1527	[1558, 704]
352	[581, 545]	744	[1085, 614]	1136	[1342, 978]	1528	[1517, 613]
353	[546, 556]	745	[1086, 181]	1137	[820, 930]	1529	[1519, 1023]
354	[53, 582]	746	[177, 1087]	1138	[1372, 1396]	1530	[589, 1087]
355	[543, 583]	747	[608, 58]	1139	[1388, 1397]	1531	[460, 602]
356	[584, 585]	748	[190, 1088]	1140	[782, 1398]	1532	[1559, 178]
357	[586, 48]	749	[50, 1089]	1141	[1258, 1399]	1533	[1097, 1512]
358	[587, 588]	750	[603, 1090]	1142	[1357, 716]	1534	[1162, 1010]
359	[176, 589]	751	[1056, 158]	1143	[1400, 1401]	1535	[1518, 33]
360	[590, 480]	752	[1091, 613]	1144	[1402, 1403]	1536	[1527, 1227]
361	[591, 592]	753	[475, 469]	1145	[1401, 1404]	1537	[415, 1131]
362	[60, 538]	754	[1092, 1093]	1146	[1405, 1406]	1538	[1236, 1481]
363	[593, 594]	755	[559, 1094]	1147	[1407, 972]	1539	[504, 1560]
364	[595, 186]	756	[1095, 604]	1148	[1408, 1241]	1540	[1188, 1526]
365	[596, 138]	757	[1096, 552]	1149	[1409, 1311]	1541	[1125, 1150]
366	[597, 598]	758	[1097, 1098]	1150	[808, 1410]	1542	[1113, 1560]
367	[599, 600]	759	[559, 1004]	1151	[1411, 1412]	1543	[1513, 228]
368	[601, 602]	760	[558, 644]	1152	[365, 857]	1544	[1050, 1561]
369	[541, 603]	761	[1099, 990]	1153	[762, 1413]	1545	[228, 1561]
370	[604, 605]	762	[132, 1048]	1154	[1414, 1378]	1546	[1559, 1219]
371	[417, 488]	763	[1100, 507]	1155	[830, 679]	1547	[152, 1082]
372	[606, 220]	764	[1101, 578]	1156	[862, 864]	1548	[1510, 1073]
373	[607, 507]	765	[188, 1102]	1157	[1415, 1353]	1549	[1099, 33]
374	[56, 130]	766	[603, 536]	1158	[1416, 326]	1550	[1209, 1092]
375	[608, 609]	767	[571, 573]	1159	[1417, 941]	1551	[562, 415]
376	[610, 192]	768	[1103, 214]	1160	[262, 1418]	1552	[1184, 1009]
377	[611, 166]	769	[1104, 1105]	1161	[1419, 889]	1553	[1213, 537]
378	[51, 612]	770	[1106, 1107]	1162	[1420, 1262]	1554	[1483, 421]
379	[613, 614]	771	[1108, 1109]	1163	[1421, 1269]	1555	[1474, 1165]
380	[148, 207]	772	[1029, 1066]	1164	[913, 1422]	1556	[1207, 535]
381	[150, 208]	773	[993, 1110]	1165	[1423, 1424]	1557	[1230, 1113]
382	[563, 615]	774	[1095, 550]	1166	[237, 1425]	1558	[1562, 1510]
383	[616, 617]	775	[184, 482]	1167	[681, 1370]	1559	[1563, 877]
384	[527, 618]	776	[1111, 1112]	1168	[949, 1420]	1560	[876, 1564]
385	[619, 620]	777	[1113, 505]	1169	[944, 676]	1561	[1552, 1537]
386	[621, 622]	778	[1114, 166]	1170	[1264, 81]	1562	[1565, 1463]
387	[40, 12]	779	[623, 506]	1171	[690, 1426]	1563	[1562, 1504]

388	[623, 624]	780	[444, 1115]	1172	[918, 901]	1564	[1504, 1073]
389	[625, 626]	781	[1116, 1117]	1173	[1333, 1427]	1565	[1033, 1481]
390	[445, 465]	782	[539, 575]	1174	[373, 1428]		
391	[627, 628]	783	[481, 185]	1175	[1429, 849]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	262	0	524	0	786	0	1048	1	1310	0
1	0	263	1	525	1	787	1	1049	0	1311	0
2	0	264	1	526	0	788	1	1050	0	1312	1
3	0	265	1	527	1	789	0	1051	0	1313	0
4	0	266	0	528	1	790	1	1052	1	1314	0
5	0	267	1	529	1	791	1	1053	1	1315	0
6	1	268	0	530	1	792	1	1054	0	1316	1
7	0	269	1	531	0	793	1	1055	1	1317	0
8	1	270	1	532	1	794	0	1056	0	1318	0
9	0	271	0	533	0	795	0	1057	1	1319	0
10	0	272	0	534	1	796	0	1058	0	1320	0
11	0	273	0	535	1	797	1	1059	0	1321	1
12	0	274	1	536	1	798	0	1060	0	1322	0
13	1	275	1	537	1	799	0	1061	1	1323	0
14	1	276	0	538	1	800	0	1062	0	1324	0
15	0	277	0	539	1	801	1	1063	1	1325	0
16	1	278	1	540	0	802	1	1064	0	1326	1
17	1	279	1	541	0	803	0	1065	0	1327	1
18	0	280	1	542	0	804	1	1066	1	1328	1
19	0	281	1	543	1	805	0	1067	1	1329	0
20	0	282	1	544	0	806	0	1068	1	1330	1
21	0	283	0	545	1	807	1	1069	1	1331	0
22	1	284	1	546	0	808	1	1070	1	1332	0
23	0	285	0	547	1	809	1	1071	1	1333	1
24	0	286	1	548	1	810	1	1072	0	1334	1
25	0	287	1	549	1	811	1	1073	0	1335	1
26	1	288	1	550	0	812	0	1074	0	1336	1
27	1	289	1	551	1	813	1	1075	1	1337	1
28	0	290	1	552	0	814	1	1076	1	1338	0
29	1	291	0	553	1	815	0	1077	1	1339	0
30	1	292	1	554	0	816	1	1078	0	1340	1
31	0	293	1	555	1	817	0	1079	0	1341	0
32	0	294	0	556	1	818	1	1080	0	1342	0
33	1	295	1	557	1	819	0	1081	0	1343	0
34	0	296	0	558	0	820	0	1082	0	1344	0
35	1	297	0	559	1	821	0	1083	0	1345	1
36	1	298	0	560	1	822	0	1084	0	1346	1

37	0	299	0	561	0	823	0	1085	0	1347	0
38	1	300	1	562	0	824	0	1086	1	1348	0
39	0	301	1	563	0	825	0	1087	0	1349	1
40	1	302	0	564	0	826	1	1088	1	1350	0
41	0	303	1	565	0	827	1	1089	0	1351	0
42	1	304	0	566	1	828	1	1090	0	1352	0
43	0	305	0	567	1	829	0	1091	0	1353	1
44	0	306	1	568	0	830	0	1092	1	1354	1
45	1	307	0	569	0	831	1	1093	1	1355	0
46	0	308	1	570	1	832	0	1094	0	1356	0
47	0	309	0	571	0	833	1	1095	0	1357	0
48	0	310	1	572	1	834	0	1096	0	1358	0
49	0	311	1	573	0	835	1	1097	0	1359	0
50	1	312	0	574	0	836	0	1098	1	1360	1
51	0	313	1	575	1	837	1	1099	1	1361	0
52	0	314	1	576	0	838	0	1100	1	1362	1
53	1	315	1	577	0	839	1	1101	0	1363	1
54	1	316	1	578	0	840	1	1102	1	1364	1
55	1	317	1	579	0	841	1	1103	0	1365	0
56	1	318	1	580	1	842	1	1104	1	1366	1
57	0	319	1	581	0	843	0	1105	0	1367	0
58	0	320	0	582	0	844	1	1106	1	1368	1
59	1	321	0	583	1	845	0	1107	0	1369	1
60	0	322	0	584	1	846	1	1108	1	1370	0
61	1	323	1	585	0	847	0	1109	1	1371	1
62	0	324	1	586	1	848	0	1110	0	1372	1
63	0	325	1	587	0	849	0	1111	1	1373	1
64	1	326	1	588	1	850	0	1112	1	1374	0
65	0	327	1	589	0	851	0	1113	1	1375	1
66	0	328	1	590	0	852	0	1114	0	1376	1
67	1	329	1	591	1	853	0	1115	1	1377	1
68	1	330	0	592	1	854	0	1116	1	1378	1
69	0	331	0	593	0	855	1	1117	1	1379	1
70	1	332	1	594	1	856	1	1118	0	1380	0
71	1	333	1	595	1	857	1	1119	0	1381	1
72	0	334	0	596	1	858	0	1120	1	1382	1
73	1	335	0	597	0	859	1	1121	1	1383	1
74	1	336	1	598	0	860	1	1122	1	1384	0
75	0	337	0	599	1	861	0	1123	1	1385	0
76	1	338	0	600	1	862	1	1124	0	1386	1
77	0	339	0	601	0	863	1	1125	0	1387	0
78	0	340	1	602	0	864	0	1126	1	1388	1
79	1	341	0	603	0	865	1	1127	0	1389	0
80	1	342	1	604	1	866	1	1128	0	1390	0

81	0	343	1	605	0	867	1	1129	1	1391	0
82	1	344	1	606	1	868	0	1130	1	1392	1
83	1	345	1	607	0	869	0	1131	0	1393	0
84	1	346	0	608	1	870	0	1132	1	1394	0
85	0	347	0	609	1	871	1	1133	0	1395	1
86	0	348	1	610	0	872	1	1134	1	1396	1
87	0	349	0	611	1	873	0	1135	1	1397	1
88	1	350	0	612	1	874	0	1136	0	1398	1
89	1	351	1	613	1	875	0	1137	0	1399	0
90	0	352	0	614	0	876	1	1138	1	1400	0
91	0	353	0	615	1	877	0	1139	1	1401	0
92	1	354	1	616	1	878	1	1140	1	1402	0
93	0	355	1	617	1	879	1	1141	1	1403	1
94	1	356	1	618	0	880	1	1142	0	1404	0
95	0	357	1	619	1	881	1	1143	0	1405	0
96	1	358	0	620	0	882	0	1144	0	1406	0
97	0	359	0	621	0	883	0	1145	0	1407	0
98	0	360	0	622	0	884	0	1146	0	1408	1
99	1	361	1	623	1	885	0	1147	0	1409	1
100	1	362	0	624	0	886	0	1148	1	1410	1
101	0	363	0	625	0	887	1	1149	1	1411	1
102	1	364	1	626	0	888	0	1150	1	1412	0
103	0	365	1	627	0	889	0	1151	1	1413	1
104	0	366	0	628	0	890	1	1152	1	1414	0
105	1	367	1	629	1	891	0	1153	1	1415	1
106	0	368	0	630	0	892	0	1154	0	1416	1
107	1	369	0	631	1	893	0	1155	0	1417	1
108	0	370	1	632	0	894	1	1156	1	1418	0
109	1	371	0	633	0	895	1	1157	1	1419	0
110	1	372	1	634	0	896	0	1158	1	1420	0
111	1	373	0	635	1	897	1	1159	1	1421	0
112	1	374	1	636	0	898	0	1160	0	1422	1
113	0	375	1	637	1	899	1	1161	0	1423	0
114	1	376	0	638	1	900	0	1162	0	1424	1
115	0	377	1	639	0	901	1	1163	0	1425	1
116	1	378	0	640	0	902	1	1164	0	1426	1
117	1	379	1	641	1	903	0	1165	0	1427	1
118	0	380	0	642	1	904	0	1166	1	1428	1
119	0	381	0	643	0	905	1	1167	0	1429	0
120	1	382	0	644	0	906	0	1168	1	1430	1
121	1	383	1	645	1	907	1	1169	0	1431	0
122	0	384	1	646	1	908	1	1170	1	1432	0
123	0	385	1	647	0	909	1	1171	1	1433	0
124	1	386	0	648	1	910	1	1172	1	1434	0

125	0	387	1	649	1	911	0	1173	1	1435	0
126	0	388	1	650	0	912	0	1174	0	1436	0
127	1	389	0	651	1	913	0	1175	0	1437	1
128	0	390	0	652	1	914	1	1176	1	1438	1
129	0	391	0	653	0	915	0	1177	1	1439	0
130	0	392	1	654	0	916	0	1178	1	1440	0
131	1	393	0	655	0	917	1	1179	0	1441	1
132	1	394	1	656	1	918	1	1180	0	1442	1
133	0	395	0	657	1	919	1	1181	1	1443	0
134	1	396	1	658	1	920	1	1182	1	1444	1
135	0	397	1	659	0	921	1	1183	1	1445	0
136	0	398	0	660	0	922	1	1184	0	1446	0
137	0	399	0	661	1	923	1	1185	0	1447	1
138	1	400	1	662	0	924	0	1186	1	1448	0
139	0	401	1	663	1	925	0	1187	0	1449	1
140	1	402	1	664	1	926	1	1188	1	1450	1
141	0	403	1	665	0	927	1	1189	1	1451	0
142	0	404	1	666	0	928	0	1190	0	1452	1
143	1	405	0	667	1	929	1	1191	1	1453	0
144	1	406	1	668	1	930	0	1192	1	1454	0
145	1	407	0	669	1	931	0	1193	1	1455	0
146	1	408	1	670	1	932	0	1194	0	1456	1
147	0	409	1	671	0	933	0	1195	0	1457	1
148	0	410	1	672	0	934	1	1196	0	1458	0
149	1	411	1	673	0	935	1	1197	1	1459	1
150	0	412	1	674	0	936	0	1198	1	1460	1
151	0	413	1	675	1	937	0	1199	1	1461	1
152	1	414	0	676	0	938	0	1200	0	1462	1
153	1	415	0	677	0	939	1	1201	0	1463	0
154	1	416	1	678	0	940	0	1202	1	1464	1
155	0	417	0	679	0	941	1	1203	0	1465	0
156	1	418	1	680	0	942	0	1204	0	1466	0
157	1	419	1	681	0	943	0	1205	1	1467	1
158	1	420	1	682	0	944	0	1206	1	1468	0
159	1	421	0	683	0	945	1	1207	1	1469	0
160	0	422	1	684	0	946	1	1208	1	1470	1
161	1	423	1	685	0	947	1	1209	0	1471	0
162	1	424	0	686	1	948	1	1210	0	1472	1
163	0	425	0	687	1	949	1	1211	1	1473	0
164	0	426	0	688	0	950	0	1212	1	1474	1
165	1	427	0	689	0	951	0	1213	0	1475	1
166	0	428	0	690	1	952	1	1214	1	1476	0
167	0	429	0	691	0	953	0	1215	0	1477	1
168	1	430	0	692	1	954	1	1216	1	1478	1

169	1	431	0	693	0	955	1	1217	0	1479	1
170	1	432	0	694	0	956	1	1218	1	1480	1
171	0	433	0	695	1	957	1	1219	0	1481	0
172	1	434	1	696	1	958	1	1220	1	1482	0
173	1	435	0	697	1	959	0	1221	0	1483	0
174	1	436	0	698	0	960	0	1222	1	1484	0
175	0	437	1	699	0	961	0	1223	1	1485	1
176	0	438	0	700	0	962	0	1224	1	1486	0
177	1	439	1	701	0	963	0	1225	0	1487	1
178	1	440	0	702	1	964	1	1226	1	1488	0
179	1	441	0	703	0	965	0	1227	0	1489	0
180	0	442	0	704	0	966	0	1228	0	1490	1
181	0	443	1	705	0	967	1	1229	1	1491	0
182	1	444	1	706	1	968	0	1230	0	1492	0
183	0	445	0	707	1	969	1	1231	0	1493	1
184	0	446	0	708	1	970	0	1232	0	1494	1
185	0	447	1	709	0	971	0	1233	1	1495	0
186	0	448	1	710	0	972	1	1234	0	1496	1
187	1	449	0	711	0	973	1	1235	0	1497	0
188	1	450	0	712	0	974	1	1236	0	1498	0
189	0	451	0	713	0	975	0	1237	1	1499	0
190	1	452	0	714	1	976	1	1238	0	1500	1
191	0	453	0	715	0	977	1	1239	0	1501	1
192	0	454	0	716	0	978	1	1240	0	1502	0
193	1	455	1	717	0	979	0	1241	0	1503	0
194	1	456	0	718	1	980	1	1242	0	1504	0
195	0	457	1	719	1	981	1	1243	1	1505	0
196	0	458	1	720	0	982	1	1244	1	1506	0
197	0	459	0	721	0	983	1	1245	1	1507	0
198	1	460	1	722	1	984	0	1246	1	1508	1
199	1	461	1	723	0	985	1	1247	0	1509	1
200	1	462	1	724	1	986	0	1248	0	1510	1
201	0	463	0	725	0	987	1	1249	1	1511	1
202	0	464	1	726	0	988	0	1250	1	1512	0
203	1	465	1	727	0	989	0	1251	0	1513	1
204	1	466	0	728	1	990	0	1252	1	1514	0
205	0	467	0	729	0	991	0	1253	1	1515	0
206	0	468	0	730	1	992	1	1254	0	1516	0
207	1	469	1	731	1	993	1	1255	1	1517	1
208	1	470	1	732	1	994	1	1256	1	1518	0
209	0	471	1	733	1	995	0	1257	0	1519	1
210	0	472	1	734	0	996	1	1258	1	1520	0
211	0	473	0	735	0	997	0	1259	1	1521	1
212	1	474	0	736	1	998	1	1260	1	1522	0

213	1	475	1	737	0	999	0	1261	1	1523	1
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215	0	477	1	739	0	1001	0	1263	1	1525	0
216	1	478	0	740	0	1002	1	1264	1	1526	0
217	0	479	1	741	0	1003	0	1265	0	1527	1
218	1	480	1	742	0	1004	1	1266	1	1528	1
219	1	481	1	743	0	1005	0	1267	0	1529	1
220	0	482	1	744	0	1006	0	1268	0	1530	0
221	1	483	1	745	1	1007	1	1269	0	1531	1
222	0	484	1	746	1	1008	1	1270	1	1532	1
223	1	485	0	747	1	1009	1	1271	1	1533	0
224	1	486	1	748	1	1010	1	1272	0	1534	0
225	0	487	1	749	1	1011	1	1273	0	1535	0
226	0	488	0	750	0	1012	1	1274	1	1536	1
227	1	489	0	751	0	1013	0	1275	0	1537	0
228	1	490	0	752	0	1014	0	1276	1	1538	0
229	1	491	1	753	1	1015	0	1277	0	1539	0
230	0	492	0	754	1	1016	1	1278	0	1540	1
231	0	493	1	755	1	1017	1	1279	1	1541	0
232	0	494	1	756	0	1018	0	1280	0	1542	1
233	1	495	0	757	0	1019	1	1281	0	1543	1
234	1	496	1	758	0	1020	0	1282	1	1544	0
235	0	497	0	759	1	1021	0	1283	1	1545	1
236	0	498	1	760	0	1022	0	1284	0	1546	1
237	1	499	1	761	1	1023	1	1285	0	1547	1
238	0	500	1	762	1	1024	0	1286	1	1548	1
239	0	501	0	763	1	1025	0	1287	0	1549	1
240	0	502	1	764	0	1026	1	1288	1	1550	0
241	0	503	1	765	1	1027	0	1289	0	1551	0
242	0	504	0	766	0	1028	1	1290	1	1552	0
243	1	505	0	767	0	1029	0	1291	1	1553	0
244	0	506	1	768	0	1030	0	1292	0	1554	0
245	0	507	0	769	1	1031	1	1293	1	1555	1
246	0	508	1	770	1	1032	1	1294	0	1556	1
247	1	509	0	771	1	1033	1	1295	0	1557	0
248	0	510	0	772	0	1034	1	1296	0	1558	1
249	0	511	1	773	1	1035	0	1297	0	1559	1
250	1	512	1	774	0	1036	0	1298	0	1560	1
251	0	513	1	775	0	1037	1	1299	0	1561	0
252	0	514	0	776	1	1038	0	1300	0	1562	1
253	1	515	0	777	1	1039	1	1301	1	1563	1
254	0	516	1	778	0	1040	0	1302	0	1564	0
255	0	517	0	779	1	1041	0	1303	0	1565	1
256	1	518	0	780	1	1042	1	1304	1		

257	1	519	0	781	1	1043	1	1305	1		
258	0	520	0	782	1	1044	0	1306	0		
259	0	521	1	783	1	1045	1	1307	0		
260	1	522	0	784	1	1046	1	1308	1		
261	0	523	0	785	1	1047	0	1309	1		

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