

PROOF OF THE CONTINUED FRACTION CONJECTURE FOR

$$(a, b) = (z^2 + z + 1, z^4 + z^3 + z^2 + z + 1)$$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b) -Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^2 + z + 1, z^4 + z^3 + z^2 + z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$(1.1) \quad \text{CF}(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\ddots}}}}$$

as the limit of the finite continued fractions

$$(1.2) \quad \text{CF}_n(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{\ddots + \frac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)).$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$(1.3) \quad c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \dots$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b) -Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$(1.4) \quad \text{CF}(a, b) := \text{CF}(\mathbf{t}).$$

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Theorem 1.1. *When $(a, b) = (z^2 + z + 1, z^4 + z^3 + z^2 + z + 1)$, the two power series $\text{CF}(a, b)$ and $\text{CF}(b, a)$ are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for $\text{CF}(a, b)$,

$$\begin{aligned} p_0(z) &= z^{18} + z^{15} + z^{14} + z^{12} + z^{10} + z^8 + z^7 + z^6 + z^3 + z^2 + 1, \\ p_1(z) &= z^{22} + z^{21} + z^{20} + z^{17} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^{10} + z^7 + z^5 + z^4 \\ &\quad + z^3, \\ p_2(z) &= z^{24} + z^{20} + z^{18} + z^{12} + z^{10} + z^6 + 1, \\ p_3(z) &= z^{22} + z^{21} + z^{20} + z^{17} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^{10} + z^7 + z^5 + z^4 \\ &\quad + z^3, \\ p_4(z) &= z^{20} + z^{15} + z^{14} + z^7 + z^3 + z^2 + 1, \end{aligned}$$

and for $\text{CF}(b, a)$,

$$\begin{aligned} p_0(z) &= z^{18} + z^{16} + z^{15} + z^{14} + z^{10} + z^8 + z^7 + z^6 + z^3 + z^2 + 1, \\ p_1(z) &= z^{22} + z^{21} + z^{20} + z^{17} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^{10} + z^7 + z^5 + z^4 \\ &\quad + z^3, \\ p_2(z) &= z^{24} + z^{20} + z^{18} + z^{12} + z^{10} + z^6 + 1, \\ p_3(z) &= z^{22} + z^{21} + z^{20} + z^{17} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^{10} + z^7 + z^5 + z^4 \\ &\quad + z^3, \\ p_4(z) &= z^{20} + z^{16} + z^{15} + z^{14} + z^{12} + z^7 + z^3 + z^2 + 1. \end{aligned}$$

2. PROOF

Define the sequences of polynomials $(P_n(z))_{n \geq 0}$ and $(Q_n(z))_{n \geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$(2.1) \quad \text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)).$$

As formula (2.1) is not efficient for calculating $\text{CF}_n(a, b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\ W_n(x) &= x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b, a) -Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer d_n . Hence,

$$(2.2) \quad \text{CF}_{2^{2n}-1}(a, b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}.$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\text{CF}(a, b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i, j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i, j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T, i, j)$ under certain initial conditions; the polynomials $\phi(T, i, j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T, i, j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i, j \leq 1$, the coefficients of the polynomial $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$) are the Padé-Hermite approximants of type

$$(150, 150, 150, 150, 150)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e, i, j)$ (resp. $\phi(M^o, i, j)$, $\phi(W^e, i, j)$, and $\phi(W^o, i, j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x, y) \in \mathbb{F}_2[x, y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o . The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$(2.3) \quad M^e = W^o \cdot M^o,$$

$$(2.4) \quad M^o = W^e \cdot M^e,$$

$$(2.5) \quad W^e = M^o \cdot W^o,$$

$$(2.6) \quad W^o = M^e \cdot W^e.$$

Proof. We give the proof of the identity

$$M_{0,0}^e = W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o,$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W_{0,0}^o M_{0,0}^o$ and $W_{0,1}^o M_{1,0}^o$. We know that

$$P(x, y) = \text{Res}_z (\phi(W^o, 0, 0)(x, z), z^{12} \cdot \phi(M^o, 0, 0)(x, y/z))$$

is an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W_{0,0}^o M_{0,0}^o$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We verify the first point directly. For the second point, we truncate $W_{0,0}^o M_{0,0}^o$ to order 540 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 540, which proves that $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o$. We find the minimal polynomial $\phi_1(x, y)$ of $W_{0,1}^o M_{1,0}^o$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We know that

$$S(x, y) = \text{Res}_z (\phi_0(x, z), \phi_1(x, y + z))$$

is an annihilating of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity μ , and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$. To see the last point, we truncate $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ to order 660 and substitute it for y in $Q(x, y)$. We get a series of valuation less than 660, and therefore $Q(x, y)$ is not an annihilating polynomial of $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$.

Finally, the first 16 terms of $M_{0,0}^e$ and $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. $\square$

Define

$$R^e = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R^o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:12u] &= M^e[:6u] + x^{2u} M^e[:4u] + x^{3u} M^e[:3u] + x^{2u} M^e[:6u] + x^{4u} M^e[:4u] \\ &\quad + x^{5u} M^e[:3u] + x^{3u} M^e[:6u] + x^{5u} M^e[:4u] + x^{6u} M^e[:3u] \\ &\quad + x^{4u} M^e[:6u] + x^{6u} M^e[:4u] + x^{7u} M^e[:3u] + x^{6u} M^e[:6u] \\ &\quad + x^{8u} M^e[:4u] + x^{9u} M^e[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u}) R^e, \end{aligned}$$

$$\begin{aligned}
W^e[:12u] &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] \\
&\quad + x^{5u}W^e[:3u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{6u}W^e[:3u] \\
&\quad + x^{4u}W^e[:6u] + x^{6u}W^e[:4u] + x^{7u}W^e[:3u] + x^{6u}W^e[:6u] \\
&\quad + x^{8u}W^e[:4u] + x^{9u}W^e[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^e.
\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}
M^o[:12u] &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] \\
&\quad + x^{5u}M^o[:3u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{6u}M^o[:3u] \\
&\quad + x^{4u}M^o[:6u] + x^{6u}M^o[:4u] + x^{7u}M^o[:3u] + x^{6u}M^o[:6u] \\
&\quad + x^{8u}M^o[:4u] + x^{9u}M^o[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^o, \\
W^o[:12u] &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] \\
&\quad + x^{5u}W^o[:3u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{6u}W^o[:3u] \\
&\quad + x^{4u}W^o[:6u] + x^{6u}W^o[:4u] + x^{7u}W^o[:3u] + x^{6u}W^o[:6u] \\
&\quad + x^{8u}W^o[:4u] + x^{9u}W^o[:3u] + (x^{6u} + x^{8u} + x^{9u} + x^{10u})R^o.
\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k 's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M_{0,0}^o$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}
T[:12u] &= T[:6u] + x^{2u}T[:4u] + x^{3u}T[:3u] + x^{2u}T[:6u] + x^{4u}T[:4u] + x^{5u}T[:3u] \\
&\quad + x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{4u}T[:6u] + x^{6u}T[:4u] \\
&\quad + x^{7u}T[:3u] + x^{6u}T[:6u] + x^{8u}T[:4u] + x^{9u}T[:3u] \\
&\quad + (x^{6u} + x^{8u} + x^{9u} + x^{10u}).
\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 6 parts:

$$\begin{aligned}
x^6u &= x^{6u}T[:u] + x^{3u}T[3u:4u] + x^{2u}T[4u:5u] + T[6u:7u], \\
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] \\
&\quad + T[7u:8u], \\
x^8u &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] \\
&\quad + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + T[8u:9u], \\
x^9u &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{4u}T[5u:6u] \\
&\quad + T[9u:10u], \\
x^{10}u &= x^{9u}T[u:2u] + x^{8u}T[2u:3u] + x^{6u}T[4u:5u] + T[10u:11u], \\
0 &= x^{9u}T[2u:3u] + x^{8u}T[3u:4u] + x^{6u}T[5u:6u] + T[11u:12u],
\end{aligned}$$

which can be rewritten as

$$(2.7) \quad 0 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2],$$

$$\begin{aligned}
(2.8) \quad 0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2], \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]
\end{aligned}$$

$$(2.9) \quad + T[[101w]_2] + T[[1000w]_2],$$

$$(2.10) \quad 0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1001w]_2],$$

$$(2.11) \quad 0 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2],$$

$$(2.12) \quad 0 = T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2],$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$(2.13) \quad 1 = T[[w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2],$$

$$(2.14) \quad 0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[111w]_2],$$

$$1 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2]$$

$$(2.15) \quad + T[[101w]_2] + T[[1000w]_2],$$

$$(2.16) \quad 1 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[1001w]_2],$$

$$(2.17) \quad 1 = T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1010w]_2],$$

$$(2.18) \quad 0 = T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1011w]_2],$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 252 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s , and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.18) can be written as

$$(2.19) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 110)),$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101))$$

$$(2.20) \quad + \tau(A(s, 111)),$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100))$$

$$(2.21) \quad + \tau(A(s, 101)) + \tau(A(s, 1000)),$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 101))$$

$$(2.22) \quad + \tau(A(s, 1001)),$$

$$(2.23) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1010)),$$

$$(2.24) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)),$$

for all $s \in E_{2k}$, and

$$(2.25) \quad 1 = \tau(A(s, \epsilon)) + \tau(A(s, 11)) + \tau(A(s, 100)) + \tau(A(s, 110)),$$

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101))$$

$$(2.26) \quad + \tau(A(s, 111)),$$

$$1 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 100))$$

$$(2.27) \quad + \tau(A(s, 101)) + \tau(A(s, 1000)),$$

$$1 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 101))$$

$$(2.28) \quad + \tau(A(s, 1001)),$$

$$(2.29) \quad 1 = \tau(A(s, 1)) + \tau(A(s, 10)) + \tau(A(s, 100)) + \tau(A(s, 1010)),$$

$$(2.30) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 11)) + \tau(A(s, 101)) + \tau(A(s, 1011)),$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{16}), E_{16}) = (A(i, 0^{12}), E_{12})$, so that we only have to verify that identities (2.19) through (2.30) hold for $2 \leq k \leq 8$, which turns out to be true. $\square$

In the following lemma, we express M_{2k} , M_{2k+1} , W_{2k} , and W_{2k+1} in terms of M^e , M^o , W^e , and W^o .

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:6u] + x^{2u}M^e[:4u] + x^{3u}M^e[:3u] + x^{6u}R^e, \\ W_{2k} &= W^e[:6u] + x^{2u}W^e[:4u] + x^{3u}W^e[:3u] + x^{6u}R^e. \end{aligned}$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:6u] + x^{2u}M^o[:4u] + x^{3u}M^o[:3u] + x^{6u}R^o, \\ W_{2k+1} &= W^o[:6u] + x^{2u}W^o[:4u] + x^{3u}W^o[:3u] + x^{6u}R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k} , W_{2k} , M_{2k+1} , and W_{2k+1} . For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$(2.31) \quad \begin{aligned} W_{2k}M_{2k} &= (W^e[:6v] + x^{2v}W^e[:4v] + x^{3v}W^e[:3v] + x^{6v}R^e) \times (M^e[:6v] \\ &\quad + x^{2v}M^e[:4v] + x^{3v}M^e[:3v] + x^{6v}R^e); \end{aligned}$$

Call this expression *lhs*. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{12v}R^o$. Therefore we only need to prove that their difference is $O(x^{12v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{12v} , to

$$W^e[:12v]M^e[:12v] + x^{4v}W^e[:8v]M^e[:8v] + x^{6v}W^e[:6v]M^e[:6v].$$

For all $n \leq 12$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.31) and *rhs* as polynomials in $\mathbb{F}_2[a_1, \dots, a_{12}, b_1, \dots, b_{12}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e , because in the expressions concerned, the products of W^e -terms and M^e -terms are always in the

same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{12})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\text{CF}(a, b)$; for $\text{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \leq j, k \leq 1$,

$$\begin{aligned}\lim_{n \rightarrow \infty} M_{2n, j, k} &= M_{j, k}^e, \\ \lim_{n \rightarrow \infty} M_{2n+1, j, k} &= M_{j, k}^o, \\ \lim_{n \rightarrow \infty} W_{2n, j, k} &= W_{j, k}^e, \\ \lim_{n \rightarrow \infty} W_{2n+1, j, k} &= W_{j, k}^o.\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\text{CF}(a, b) = \frac{M_{0,1}^e(x)}{M_{0,0}^e(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M_{0,1}^e$ and $M_{0,0}^e$. Therefore

$$P(x, y) = \text{Res}_t (\phi(M^e, 0, 1)(x, t), y^{12} \phi(M^e, 0, 1)(x, t/y))$$

is an annihilating polynomial of $f(x) = M_{0,1}^e/M_{0,0}^e$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}q_0(x) &= x^{24} + x^{22} + x^{21} + x^{18} + x^{17} + x^{16} + x^{14} + x^{12} + x^{10} + x^9 + x^6, \\ q_1(x) &= x^{21} + x^{20} + x^{19} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^4 + x^3 \\ &\quad + x^2, \\ q_2(x) &= x^{24} + x^{18} + x^{14} + x^{12} + x^6 + x^4 + 1, \\ q_3(x) &= x^{21} + x^{20} + x^{19} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^4 + x^3 \\ &\quad + x^2, \\ q_4(x) &= x^{24} + x^{22} + x^{21} + x^{17} + x^{10} + x^9 + x^4.\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 186, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 186, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\text{CF}(a, b) = f(1/z)$. $\square$

3. ANNEXE

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{130} + x^{122} + x^{120} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96} \\
&\quad + x^{94} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{68} + x^{56} + x^{54} + x^{50} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{36} + x^{32} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{126} + x^{120} + x^{114} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{80} + x^{76} + x^{74} + x^{72} + x^{64} + x^{56} + x^{50} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18}, \\
p_6(x) &= x^{128} + x^{116} + x^{114} + x^{108} + x^{106} + x^{104} + x^{98} + x^{96} + x^{90} + x^{88} \\
&\quad + x^{84} + x^{78} + x^{72} + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{40} + x^{34} + x^{30} + x^{28} + x^{24} + x^{14} + x^{12} + x^8 + x^2 + 1, \\
p_9(x) &= x^{126} + x^{122} + x^{120} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} \\
&\quad + x^{100} + x^{96} + x^{94} + x^{88} + x^{86} + x^{80} + x^{78} + x^{72} + x^{70} + x^{64} + x^{62} \\
&\quad + x^{56} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^4, \\
p_{12}(x) &= x^{132} + x^{130} + x^{128} + x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{108} \\
&\quad + x^{106} + x^{104} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} \\
&\quad + x^{16} + x^{12} + x^{10} + x^8 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{111} + x^{110} + x^{108} \\
&\quad + x^{105} + x^{103} + x^{102} + x^{100} + x^{97} + x^{96} + x^{95} + x^{88} + x^{87} + x^{85} \\
&\quad + x^{84} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{61} + x^{56} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{32} + x^{30}, \\
p_3(x) &= x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{69} + x^{68} + x^{67} + x^{66} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{56} + x^{55} + x^{54} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{65} + x^{63} + x^{62} \\
&\quad + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{34} + x^{30} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{18} + x^{17} + x^{15} + x^{12}, \\
p_9(x) &= x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{93} + x^{92}
\end{aligned}$$

$$\begin{aligned}
& + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{74} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} \\
& + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{25} + x^{24} + x^{23} + x^{22} + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{111} + x^{107} \\
& + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} \\
& + x^{90} + x^{88} + x^{86} + x^{84} + x^{83} + x^{79} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} \\
& + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} \\
& + x^{15} + x^{11} + x^{10} + x^8 + x^6 + x^4 + x^3 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 1]$.

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = & x^{126} + x^{124} + x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{111} + x^{110} + x^{108} \\
& + x^{105} + x^{103} + x^{102} + x^{100} + x^{97} + x^{96} + x^{95} + x^{88} + x^{87} + x^{85} \\
& + x^{84} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{70} + x^{68} + x^{67} + x^{66} \\
& + x^{65} + x^{64} + x^{63} + x^{61} + x^{56} + x^{54} + x^{51} + x^{50} + x^{47} + x^{46} + x^{44} \\
& + x^{43} + x^{42} + x^{41} + x^{38} + x^{37} + x^{32} + x^{30}, \\
p_3(x) = & x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
& + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{69} + x^{68} + x^{67} + x^{66} + x^{59} + x^{58} \\
& + x^{57} + x^{56} + x^{55} + x^{54} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} \\
& + x^{38} + x^{37} + x^{36} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_6(x) = & x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} \\
& + x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{65} + x^{63} + x^{62} \\
& + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{38} \\
& + x^{37} + x^{36} + x^{34} + x^{30} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} \\
& + x^{18} + x^{17} + x^{15} + x^{12}, \\
p_9(x) = & x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{93} + x^{92} \\
& + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{74} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} \\
& + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{25} + x^{24} + x^{23} + x^{22} + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{111} + x^{107} \\
& + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} \\
& + x^{90} + x^{88} + x^{86} + x^{84} + x^{83} + x^{79} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} \\
& + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} \\
& + x^{15} + x^{11} + x^{10} + x^8 + x^6 + x^4 + x^3 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1]$.

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{102} + x^{98} + x^{94} + x^{92} + x^{90} + x^{78} + x^{70} \\
&\quad + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{48} + x^{46} + x^{40} + x^{38} + x^{36}, \\
p_3(x) &= x^{114} + x^{104} + x^{96} + x^{94} + x^{90} + x^{86} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} \\
&\quad + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{32} \\
&\quad + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{12}, \\
p_6(x) &= x^{116} + x^{110} + x^{108} + x^{100} + x^{98} + x^{84} + x^{82} + x^{74} + x^{66} + x^{62} + x^{58} \\
&\quad + x^{56} + x^{54} + x^{50} + x^{48} + x^{44} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} \\
&\quad + x^{28} + x^{26} + x^{18} + x^{14} + x^8 + x^6 + x^2 + 1, \\
p_9(x) &= x^{114} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{54} + x^{52} \\
&\quad + x^{48} + x^{46} + x^{44} + x^{36} + x^{34} + x^{30} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} \\
&\quad + x^4, \\
p_{12}(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{96} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^6 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0]$.

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{92} \\
&\quad + x^{90} + x^{80} + x^{72} + x^{62} + x^{60} + x^{52} + x^{50} + x^{44} + x^{40} + x^{34} + x^{32} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{24}, \\
p_3(x) &= x^{114} + x^{104} + x^{102} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} \\
&\quad + x^{70} + x^{62} + x^{60} + x^{42} + x^{30} + x^{28} + x^{24} + x^{22} + x^{18}, \\
p_6(x) &= x^{116} + x^{110} + x^{106} + x^{102} + x^{100} + x^{96} + x^{84} + x^{82} + x^{76} + x^{72} + x^{68} \\
&\quad + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{44} + x^{36} + x^{30} + x^{28} + x^{26} \\
&\quad + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^6 + x^2 + 1, \\
p_9(x) &= x^{114} + x^{110} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{54} + x^{52} \\
&\quad + x^{48} + x^{46} + x^{44} + x^{36} + x^{34} + x^{30} + x^{26} + x^{24} + x^{20} + x^{16} + x^{14} \\
&\quad + x^4, \\
p_{12}(x) &= x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{96} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{66} \\
&\quad + x^{64} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{12} + x^{10} + x^6 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0]$.

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{111} + x^{110} + x^{99} \\
&\quad + x^{98} + x^{94} + x^{91} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} \\
&\quad + x^{69} + x^{66} + x^{64} + x^{62} + x^{59} + x^{57} + x^{55} + x^{53} + x^{47} + x^{46} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{35} + x^{34} + x^{30}, \\
p_3(x) &= x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{69} + x^{68} + x^{67} + x^{66} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{56} + x^{55} + x^{54} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{65} + x^{63} + x^{62} \\
&\quad + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{34} + x^{30} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{18} + x^{17} + x^{15} + x^{12}, \\
p_9(x) &= x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} \\
&\quad + x^{76} + x^{75} + x^{74} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{37} + x^{36} + x^{35} \\
&\quad + x^{34} + x^{25} + x^{24} + x^{23} + x^{22} + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{111} + x^{107} \\
&\quad + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{83} + x^{79} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} \\
&\quad + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} \\
&\quad + x^{15} + x^{11} + x^{10} + x^8 + x^6 + x^4 + x^3 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0]$.

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{111} + x^{110} + x^{99} \\
&\quad + x^{98} + x^{94} + x^{91} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{72} \\
&\quad + x^{69} + x^{66} + x^{64} + x^{62} + x^{59} + x^{57} + x^{55} + x^{53} + x^{47} + x^{46} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{35} + x^{34} + x^{30}, \\
p_3(x) &= x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{69} + x^{68} + x^{67} + x^{66} + x^{59} + x^{58} \\
&\quad + x^{57} + x^{56} + x^{55} + x^{54} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{108} + x^{105} + x^{103} + x^{102} + x^{99} + x^{98} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87}
\end{aligned}$$

$$\begin{aligned}
& + x^{85} + x^{84} + x^{83} + x^{78} + x^{77} + x^{74} + x^{73} + x^{71} + x^{65} + x^{63} + x^{62} \\
& + x^{59} + x^{58} + x^{53} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{38} \\
& + x^{37} + x^{36} + x^{34} + x^{30} + x^{27} + x^{26} + x^{25} + x^{23} + x^{22} + x^{21} + x^{20} \\
& + x^{18} + x^{17} + x^{15} + x^{12}, \\
p_9(x) = & x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{93} + x^{92} \\
& + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} \\
& + x^{76} + x^{75} + x^{74} + x^{65} + x^{64} + x^{63} + x^{62} + x^{53} + x^{52} + x^{51} + x^{50} \\
& + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{37} + x^{36} + x^{35} \\
& + x^{34} + x^{25} + x^{24} + x^{23} + x^{22} + x^9 + x^8 + x^7 + x^6, \\
p_{12}(x) = & x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{111} + x^{107} \\
& + x^{106} + x^{104} + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} \\
& + x^{90} + x^{88} + x^{86} + x^{84} + x^{83} + x^{79} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} \\
& + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} \\
& + x^{15} + x^{11} + x^{10} + x^8 + x^6 + x^4 + x^3 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0]$.

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = & x^{132} + x^{130} + x^{122} + x^{118} + x^{114} + x^{112} + x^{110} + x^{106} + x^{100} + x^{98} \\
& + x^{94} + x^{92} + x^{88} + x^{86} + x^{84} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} \\
& + x^{64} + x^{58} + x^{54} + x^{52} + x^{50} + x^{44} + x^{38} + x^{36}, \\
p_3(x) = & x^{126} + x^{120} + x^{108} + x^{106} + x^{100} + x^{98} + x^{96} + x^{90} + x^{88} + x^{84} + x^{80} \\
& + x^{76} + x^{72} + x^{62} + x^{58} + x^{56} + x^{52} + x^{48} + x^{46} + x^{34} + x^{18} + x^{12}, \\
p_6(x) = & x^{128} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{106} + x^{90} + x^{86} + x^{82} \\
& + x^{72} + x^{70} + x^{66} + x^{54} + x^{52} + x^{42} + x^{40} + x^{36} + x^{34} + x^{24} + x^{16} \\
& + x^{10} + x^2 + 1, \\
p_9(x) = & x^{126} + x^{122} + x^{120} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} \\
& + x^{100} + x^{96} + x^{94} + x^{88} + x^{86} + x^{80} + x^{78} + x^{72} + x^{70} + x^{64} + x^{62} \\
& + x^{56} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} \\
& + x^{24} + x^{22} + x^{20} + x^{16} + x^{10} + x^4, \\
p_{12}(x) = & x^{132} + x^{130} + x^{128} + x^{124} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{108} \\
& + x^{106} + x^{104} + x^{92} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} \\
& + x^{68} + x^{66} + x^{64} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} \\
& + x^{16} + x^{12} + x^{10} + x^8 + x^4 + x^2 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0]$.

For $\phi(M^o, 0, 0)$:

$$p_0(x) = x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{117} + x^{112} + x^{110} + x^{109} + x^{108}$$

$$\begin{aligned}
& + x^{107} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} \\
& + x^{89} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{74} + x^{73} + x^{71} \\
& + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{58} + x^{54} + x^{53} + x^{49} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{40} + x^{33} + x^{31} + x^{27} + x^{26} + x^{24}, \\
p_3(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85} + x^{82} + x^{80} + x^{78} \\
& + x^{76} + x^{75} + x^{74} + x^{67} + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} \\
& + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{45} + x^{43} + x^{41} + x^{40} + x^{38} + x^{36} \\
& + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{20} + x^{19} \\
& + x^{18} + x^{16}, \\
p_6(x) &= x^{122} + x^{119} + x^{117} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{101} + x^{97} \\
& + x^{94} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} \\
& + x^{76} + x^{74} + x^{73} + x^{72} + x^{67} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
& + x^{54} + x^{51} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{33} + x^{30} + x^{29} + x^{27} \\
& + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^6 + x^4 + x^3 + x^2 \\
& + 1, \\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{107} + x^{106} + x^{104} + x^{82} + x^{80} + x^{79} + x^{75} \\
& + x^{74} + x^{72} + x^{18} + x^{16} + x^{15} + x^{11} + x^{10} + x^8, \\
p_{12}(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} \\
& + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} \\
& + x^{84} + x^{83} + x^{82} + x^{80} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} \\
& + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^6 + x^4 + x^3 + x^2 \\
& + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0]$.

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{128} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108} \\
& + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{92} + x^{91} \\
& + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{74} + x^{67} + x^{66} \\
& + x^{60} + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} + x^{36} + x^{35} \\
& + x^{34} + x^{33} + x^{30}, \\
p_3(x) &= x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{81} \\
& + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{66} + x^{65} + x^{64} + x^{62} \\
& + x^{60} + x^{58} + x^{55} + x^{53} + x^{50} + x^{48} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} \\
& + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{28} + x^{26} + x^{23} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{90} + x^{88} + x^{80} + x^{76} + x^{72} + x^{66}
\end{aligned}$$

$$\begin{aligned}
& + x^{62} + x^{58} + x^{50} + x^{48} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{22} + x^{16} \\
& + x^{14} + x^{12}, \\
p_9(x) &= x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} \\
& + x^{103} + x^{102} + x^{87} + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\
& + x^{73} + x^{71} + x^{70} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} \\
& + x^{12} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{128} + x^{124} + x^{116} + x^{112} + x^{104} + x^{100} + x^{92} + x^{84} + x^{80} + x^{72} \\
& + x^{68} + x^{64} + x^{32} + x^{28} + x^{20} + x^{16} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1]$.

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{127} + x^{125} + x^{123} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} \\
& + x^{112} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{96} \\
& + x^{95} + x^{93} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{75} \\
& + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{63} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} \\
& + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} \\
& + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{109} + x^{108} + x^{107} + x^{104} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{89} \\
& + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} \\
& + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} \\
& + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46} + x^{43} + x^{41} + x^{38} + x^{36} \\
& + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{118} + x^{116} + x^{114} + x^{110} + x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{88} \\
& + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{54} + x^{52} \\
& + x^{48} + x^{46} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{127} + x^{126} + x^{124} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
& + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{103} \\
& + x^{102} + x^{95} + x^{94} + x^{92} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
& + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{71} + x^{70} + x^{31} \\
& + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} \\
& + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^9 + x^7 + x^6, \\
p_{12}(x) &= x^{136} + x^{132} + x^{116} + x^{108} + x^{96} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{40} \\
& + x^{36} + x^{20} + x^{12} + x^8 + x^4 + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1]$.

For $\phi(M^o, 1, 1)$:

$$p_0(x) = x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{117} + x^{114} + x^{111} + x^{110} + x^{109}$$

$$\begin{aligned}
& + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} \\
& + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{73} + x^{69} \\
& + x^{67} + x^{66} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} \\
& + x^{50} + x^{48} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{36}, \\
p_3(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} \\
& + x^{97} + x^{95} + x^{94} + x^{92} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} \\
& + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\
& + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{40} \\
& + x^{39} + x^{37} + x^{36} + x^{33} + x^{29} + x^{28} + x^{27} + x^{23} + x^{20}, \\
p_6(x) &= x^{122} + x^{119} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} + x^{105} + x^{103} \\
& + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{79} \\
& + x^{78} + x^{72} + x^{71} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} \\
& + x^{53} + x^{52} + x^{49} + x^{47} + x^{45} + x^{42} + x^{40} + x^{35} + x^{33} + x^{32} + x^{30} \\
& + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{12} \\
& + x^{11} + x^{10} + x^8 + x^6 + x^4 + x^3 + x^2 + 1, \\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{107} + x^{106} + x^{104} + x^{82} + x^{80} + x^{79} + x^{75} \\
& + x^{74} + x^{72} + x^{18} + x^{16} + x^{15} + x^{11} + x^{10} + x^8, \\
p_{12}(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} \\
& + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} \\
& + x^{84} + x^{83} + x^{82} + x^{80} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} \\
& + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^6 + x^4 + x^3 + x^2 \\
& + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0]$.

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{117} + x^{112} + x^{110} + x^{109} + x^{108} \\
& + x^{107} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} \\
& + x^{89} + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{74} + x^{73} + x^{71} \\
& + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{58} + x^{54} + x^{53} + x^{49} + x^{45} + x^{43} \\
& + x^{42} + x^{41} + x^{40} + x^{33} + x^{31} + x^{27} + x^{26} + x^{24}, \\
p_3(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} \\
& + x^{97} + x^{95} + x^{94} + x^{93} + x^{91} + x^{87} + x^{86} + x^{85} + x^{82} + x^{80} + x^{78} \\
& + x^{76} + x^{75} + x^{74} + x^{67} + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{58} + x^{55} \\
& + x^{54} + x^{53} + x^{51} + x^{50} + x^{49} + x^{45} + x^{43} + x^{41} + x^{40} + x^{38} + x^{36} \\
& + x^{34} + x^{33} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{22} + x^{20} + x^{19} \\
& + x^{18} + x^{16},
\end{aligned}$$

$$\begin{aligned}
p_6(x) &= x^{122} + x^{119} + x^{117} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{101} + x^{97} \\
&\quad + x^{94} + x^{91} + x^{90} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} \\
&\quad + x^{76} + x^{74} + x^{73} + x^{72} + x^{67} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{54} + x^{51} + x^{44} + x^{43} + x^{42} + x^{40} + x^{37} + x^{33} + x^{30} + x^{29} + x^{27} \\
&\quad + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^6 + x^4 + x^3 + x^2 \\
&\quad + 1, \\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{107} + x^{106} + x^{104} + x^{82} + x^{80} + x^{79} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{18} + x^{16} + x^{15} + x^{11} + x^{10} + x^8, \\
p_{12}(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} \\
&\quad + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{80} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} \\
&\quad + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^6 + x^4 + x^3 + x^2 \\
&\quad + 1,
\end{aligned}$$

and the initial terms are $[1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0]$.

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{136} + x^{127} + x^{125} + x^{123} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{113} \\
&\quad + x^{112} + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{96} \\
&\quad + x^{95} + x^{93} + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{83} + x^{82} + x^{80} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{69} + x^{67} + x^{63} + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{54} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} \\
&\quad + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{30}, \\
p_3(x) &= x^{109} + x^{108} + x^{107} + x^{104} + x^{101} + x^{99} + x^{96} + x^{94} + x^{93} + x^{92} + x^{89} \\
&\quad + x^{88} + x^{86} + x^{85} + x^{84} + x^{83} + x^{81} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} \\
&\quad + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} \\
&\quad + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46} + x^{43} + x^{41} + x^{38} + x^{36} \\
&\quad + x^{33} + x^{31} + x^{28} + x^{26} + x^{23} + x^{20} + x^{19} + x^{18}, \\
p_6(x) &= x^{118} + x^{116} + x^{114} + x^{110} + x^{104} + x^{102} + x^{98} + x^{94} + x^{92} + x^{88} \\
&\quad + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{54} + x^{52} \\
&\quad + x^{48} + x^{46} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{127} + x^{126} + x^{124} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} + x^{103} \\
&\quad + x^{102} + x^{95} + x^{94} + x^{92} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
&\quad + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{71} + x^{70} + x^{31} \\
&\quad + x^{30} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} \\
&\quad + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^9 + x^7 + x^6,
\end{aligned}$$

$$p_{12}(x) = x^{136} + x^{132} + x^{116} + x^{108} + x^{96} + x^{84} + x^{76} + x^{72} + x^{68} + x^{64} + x^{40} \\ + x^{36} + x^{20} + x^{12} + x^8 + x^4 + 1,$$

and the initial terms are $[0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1]$.

For $\phi(W^o, 1, 0)$:

$$p_0(x) = x^{128} + x^{119} + x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108} \\ + x^{106} + x^{103} + x^{102} + x^{101} + x^{99} + x^{97} + x^{94} + x^{93} + x^{92} + x^{91} \\ + x^{88} + x^{87} + x^{86} + x^{84} + x^{83} + x^{80} + x^{79} + x^{77} + x^{74} + x^{67} + x^{66} \\ + x^{60} + x^{56} + x^{53} + x^{51} + x^{47} + x^{46} + x^{43} + x^{41} + x^{40} + x^{36} + x^{35} \\ + x^{34} + x^{33} + x^{30},$$

$$p_3(x) = x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{81} \\ + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{70} + x^{66} + x^{65} + x^{64} + x^{62} \\ + x^{60} + x^{58} + x^{55} + x^{53} + x^{50} + x^{48} + x^{45} + x^{43} + x^{41} + x^{39} + x^{38} \\ + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{28} + x^{26} + x^{23} + x^{20} + x^{19} + x^{18},$$

$$p_6(x) = x^{110} + x^{108} + x^{106} + x^{102} + x^{98} + x^{90} + x^{88} + x^{80} + x^{76} + x^{72} + x^{66} \\ + x^{62} + x^{58} + x^{50} + x^{48} + x^{40} + x^{38} + x^{34} + x^{32} + x^{30} + x^{22} + x^{16} \\ + x^{14} + x^{12},$$

$$p_9(x) = x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{105} \\ + x^{103} + x^{102} + x^{87} + x^{86} + x^{84} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} \\ + x^{73} + x^{71} + x^{70} + x^{23} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} \\ + x^{12} + x^9 + x^7 + x^6,$$

$$p_{12}(x) = x^{128} + x^{124} + x^{116} + x^{112} + x^{104} + x^{100} + x^{92} + x^{84} + x^{80} + x^{72} \\ + x^{68} + x^{64} + x^{32} + x^{28} + x^{20} + x^{16} + x^8 + x^4 + 1,$$

and the initial terms are $[0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1]$.

For $\phi(W^o, 1, 1)$:

$$p_0(x) = x^{126} + x^{124} + x^{123} + x^{120} + x^{119} + x^{117} + x^{114} + x^{111} + x^{110} + x^{109} \\ + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} \\ + x^{87} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{73} + x^{69} \\ + x^{67} + x^{66} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{51} \\ + x^{50} + x^{48} + x^{46} + x^{45} + x^{42} + x^{40} + x^{39} + x^{36},$$

$$p_3(x) = x^{114} + x^{112} + x^{111} + x^{110} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} \\ + x^{97} + x^{95} + x^{94} + x^{92} + x^{89} + x^{86} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} \\ + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} \\ + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{44} + x^{40} \\ + x^{39} + x^{37} + x^{36} + x^{33} + x^{29} + x^{28} + x^{27} + x^{23} + x^{20},$$

$$p_6(x) = x^{122} + x^{119} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{107} + x^{105} + x^{103}$$

$$\begin{aligned}
& + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{79} \\
& + x^{78} + x^{72} + x^{71} + x^{67} + x^{65} + x^{63} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} \\
& + x^{53} + x^{52} + x^{49} + x^{47} + x^{45} + x^{42} + x^{40} + x^{35} + x^{33} + x^{32} + x^{30} \\
& + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{12} \\
& + x^{11} + x^{10} + x^8 + x^6 + x^4 + x^3 + x^2 + 1, \\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{107} + x^{106} + x^{104} + x^{82} + x^{80} + x^{79} + x^{75} \\
& + x^{74} + x^{72} + x^{18} + x^{16} + x^{15} + x^{11} + x^{10} + x^8, \\
p_{12}(x) &= x^{126} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{116} + x^{115} + x^{114} + x^{112} \\
& + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{94} + x^{92} + x^{91} + x^{90} + x^{88} + x^{86} \\
& + x^{84} + x^{83} + x^{82} + x^{80} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{30} + x^{28} \\
& + x^{27} + x^{26} + x^{24} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^6 + x^4 + x^3 + x^2 \\
& + 1,
\end{aligned}$$

and the initial terms are $[0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0]$.

Below is the transition function and output function of an 2-automaton that generates $T = M_{0,0}^0$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	51	[66, 81]	102	[152, 144]	153	[190, 191]	204	[62, 165]
1	[3, 4]	52	[82, 83]	103	[153, 154]	154	[136, 192]	205	[224, 80]
2	[5, 6]	53	[59, 84]	104	[152, 75]	155	[182, 193]	206	[62, 166]
3	[7, 8]	54	[72, 76]	105	[23, 150]	156	[183, 194]	207	[148, 67]
4	[9, 10]	55	[85, 13]	106	[67, 155]	157	[157, 157]	208	[56, 77]
5	[11, 12]	56	[86, 87]	107	[67, 81]	158	[195, 9]	209	[56, 6]
6	[13, 14]	57	[88, 89]	108	[156, 66]	159	[196, 197]	210	[155, 81]
7	[15, 16]	58	[90, 91]	109	[157, 148]	160	[198, 199]	211	[6, 165]
8	[17, 18]	59	[92, 93]	110	[66, 157]	161	[200, 201]	212	[76, 78]
9	[19, 20]	60	[94, 95]	111	[62, 20]	162	[202, 203]	213	[225, 221]
10	[21, 22]	61	[96, 97]	112	[77, 63]	163	[204, 205]	214	[155, 67]
11	[23, 24]	62	[98, 99]	113	[158, 159]	164	[206, 207]	215	[28, 78]
12	[25, 24]	63	[100, 101]	114	[6, 22]	165	[87, 208]	216	[21, 146]
13	[26, 27]	64	[102, 103]	115	[64, 160]	166	[30, 209]	217	[225, 223]
14	[5, 28]	65	[104, 105]	116	[148, 155]	167	[210, 211]	218	[226, 8]
15	[29, 30]	66	[106, 107]	117	[82, 69]	168	[212, 213]	219	[227, 228]
16	[31, 32]	67	[108, 109]	118	[142, 58]	169	[214, 215]	220	[229, 230]
17	[33, 34]	68	[110, 111]	119	[143, 5]	170	[216, 217]	221	[231, 232]
18	[35, 36]	69	[112, 113]	120	[161, 61]	171	[218, 72]	222	[233, 234]
19	[37, 38]	70	[114, 115]	121	[143, 56]	172	[143, 150]	223	[235, 236]
20	[39, 40]	71	[116, 117]	122	[155, 156]	173	[73, 61]	224	[237, 238]
21	[41, 42]	72	[118, 43]	123	[162, 153]	174	[149, 148]	225	[239, 240]
22	[43, 44]	73	[119, 98]	124	[150, 28]	175	[162, 170]	226	[141, 150]
23	[45, 37]	74	[120, 121]	125	[26, 144]	176	[57, 24]	227	[241, 60]

24	[46, 39]	75	[122, 123]	126	[150, 16]	177	[26, 151]	228	[83, 242]
25	[47, 48]	76	[8, 124]	127	[143, 61]	178	[24, 19]	229	[66, 156]
26	[49, 50]	77	[125, 126]	128	[58, 6]	179	[163, 144]	230	[19, 63]
27	[51, 52]	78	[127, 128]	129	[163, 75]	180	[142, 61]	231	[16, 22]
28	[53, 54]	79	[129, 130]	130	[83, 164]	181	[157, 156]	232	[158, 168]
29	[55, 56]	80	[131, 132]	131	[163, 27]	182	[156, 148]	233	[25, 74]
30	[57, 58]	81	[133, 134]	132	[161, 58]	183	[67, 149]	234	[74, 21]
31	[59, 60]	82	[135, 136]	133	[81, 67]	184	[161, 72]	235	[16, 20]
32	[61, 62]	83	[137, 138]	134	[81, 81]	185	[57, 72]	236	[167, 159]
33	[28, 63]	84	[139, 140]	135	[149, 149]	186	[149, 66]	237	[243, 18]
34	[64, 65]	85	[141, 5]	136	[28, 165]	187	[169, 153]	238	[153, 244]
35	[66, 67]	86	[142, 56]	137	[77, 166]	188	[28, 166]	239	[149, 155]
36	[68, 69]	87	[143, 72]	138	[167, 168]	189	[219, 160]	240	[6, 78]
37	[70, 71]	88	[57, 74]	139	[155, 157]	190	[21, 165]	241	[245, 246]
38	[72, 62]	89	[17, 144]	140	[169, 170]	191	[220, 221]	242	[203, 247]
39	[73, 5]	90	[145, 5]	141	[171, 53]	192	[149, 156]	243	[248, 249]
40	[74, 19]	91	[73, 24]	142	[172, 125]	193	[156, 157]	244	[250, 251]
41	[70, 75]	92	[62, 146]	143	[173, 31]	194	[81, 149]	245	[6, 20]
42	[61, 76]	93	[79, 147]	144	[174, 175]	195	[148, 148]	246	[219, 65]
43	[57, 56]	94	[148, 149]	145	[176, 177]	196	[76, 222]	247	[66, 155]
44	[58, 77]	95	[68, 83]	146	[12, 178]	197	[220, 223]	248	[19, 222]
45	[73, 72]	96	[145, 150]	147	[179, 180]	198	[152, 71]	249	[224, 147]
46	[23, 74]	97	[25, 5]	148	[181, 182]	199	[145, 74]	250	[28, 146]
47	[25, 56]	98	[17, 151]	149	[134, 183]	200	[25, 58]	251	[155, 66]
48	[59, 71]	99	[5, 16]	150	[184, 185]	201	[70, 60]		
49	[19, 78]	100	[73, 150]	151	[186, 187]	202	[148, 66]		
50	[79, 80]	101	[24, 21]	152	[188, 189]	203	[62, 63]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	37	0	74	1	111	1	148	0	185	1	222	1
1	0	38	0	75	1	112	0	149	0	186	0	223	1
2	0	39	0	76	1	113	0	150	1	187	1	224	0
3	0	40	1	77	0	114	0	151	0	188	1	225	0
4	0	41	0	78	0	115	1	152	1	189	0	226	0
5	0	42	1	79	1	116	0	153	0	190	0	227	0
6	0	43	1	80	1	117	0	154	1	191	1	228	0
7	0	44	1	81	0	118	0	155	1	192	0	229	1
8	1	45	0	82	0	119	0	156	1	193	1	230	0
9	0	46	0	83	0	120	1	157	0	194	0	231	1
10	0	47	1	84	1	121	0	158	0	195	0	232	0
11	0	48	1	85	0	122	1	159	1	196	1	233	1
12	1	49	0	86	0	123	0	160	1	197	1	234	1
13	0	50	1	87	0	124	1	161	1	198	1	235	1

14	0	51	1	88	1	125	0	162	0	199	1	236	1
15	0	52	0	89	1	126	1	163	1	200	1	237	0
16	1	53	1	90	1	127	0	164	1	201	0	238	0
17	1	54	0	91	0	128	1	165	0	202	0	239	0
18	1	55	0	92	1	129	1	166	1	203	1	240	0
19	0	56	0	93	1	130	0	167	1	204	1	241	0
20	0	57	1	94	0	131	1	168	1	205	0	242	1
21	0	58	1	95	1	132	1	169	1	206	1	243	0
22	1	59	1	96	1	133	0	170	0	207	0	244	1
23	0	60	0	97	1	134	0	171	0	208	0	245	0
24	0	61	1	98	1	135	0	172	0	209	0	246	0
25	1	62	1	99	0	136	1	173	0	210	1	247	1
26	0	63	0	100	0	137	0	174	0	211	0	248	0
27	1	64	1	101	0	138	1	175	0	212	1	249	0
28	1	65	1	102	1	139	1	176	1	213	0	250	1
29	0	66	1	103	0	140	1	177	0	214	1	251	1
30	1	67	1	104	1	141	0	178	0	215	1		
31	1	68	1	105	0	142	0	179	1	216	0		
32	1	69	0	106	1	143	0	180	0	217	0		
33	1	70	0	107	1	144	0	181	0	218	0		
34	1	71	0	108	1	145	1	182	1	219	0		
35	1	72	0	109	0	146	1	183	1	220	1		
36	1	73	0	110	1	147	1	184	1	221	1		

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