

# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR

$$(a, b) = (z^5, z^2)$$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the  $(a, b)$ -Thue-Morse sequence is algebraic over  $\mathbb{F}_2(x)$  for all pairs of distinct elements  $(a, b)$  in  $\mathbb{F}_2[x] \setminus \mathbb{F}_2$ . In this paper we prove the conjecture for  $(a, b) = (z^5, z^2)$ .

## 1. INTRODUCTION

Let  $\mathbb{F}_2$  be the finite field of cardinality 2. Given a sequence of polynomials  $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$  of elements from  $\mathbb{F}_2[z] \setminus \mathbb{F}_2$ , we define the infinite continued fraction

$$(1.1) \quad \text{CF}(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\ddots}}}}$$

as the limit of the finite continued fractions

$$(1.2) \quad \text{CF}_n(\mathbf{a}(z)) = \frac{1}{a_0(z) + \frac{1}{a_1(z) + \frac{1}{\ddots + \frac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)).$$

Conversely, each power series in  $\mathbb{F}_2[[1/z]]$  without constant term

$$(1.3) \quad c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \dots$$

admits a continued fraction expansion of form (1.1).

Let  $a$  and  $b$  be two distinct elements from  $\mathbb{F}_2[z] \setminus \mathbb{F}_2$ . We consider the continued fraction (1.1) associated with the  $(a, b)$ -Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$(1.4) \quad \text{CF}(a, b) := \text{CF}(\mathbf{t}).$$

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**Theorem 1.1.** *When  $(a, b) = (z^5, z^2)$ , the two power series  $\text{CF}(a, b)$  and  $\text{CF}(b, a)$  are algebraic over  $\mathbb{F}_2(z)$ , with minimal polynomials of the form*

$$p_4(z)y^4 + p_3(z)y^3 + p_2(z)y^2 + p_1(z)y + p_0(z) = 0$$

where, for  $\text{CF}(a, b)$ ,

$$\begin{aligned} p_0(z) &= z^{13} + z^9 + z^6 + z^3 + 1, \\ p_1(z) &= z^{18} + z^{15} + z^{14} + z^{11} + z^8 + z^5, \\ p_2(z) &= z^{20} + z^{16} + z^{10}, \\ p_3(z) &= z^{18} + z^{15} + z^{14} + z^{11} + z^8 + z^5, \\ p_4(z) &= z^{16} + z^{13} + z^{12} + z^{10} + z^9 + z^6 + z^3, \end{aligned}$$

and for  $\text{CF}(b, a)$ ,

$$\begin{aligned} p_0(z) &= z^{13} + z^{12} + z^9 + z^6 + z^3, \\ p_1(z) &= z^{18} + z^{15} + z^{14} + z^{11} + z^8 + z^5, \\ p_2(z) &= z^{20} + z^{16} + z^{10}, \\ p_3(z) &= z^{18} + z^{15} + z^{14} + z^{11} + z^8 + z^5, \\ p_4(z) &= z^{16} + z^{13} + z^{10} + z^9 + z^6 + z^3 + 1. \end{aligned}$$

## 2. PROOF

Define the sequences of polynomials  $(P_n(z))_{n \geq 0}$  and  $(Q_n(z))_{n \geq 0}$  by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$(2.1) \quad \text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)).$$

As formula (2.1) is not efficient for calculating  $\text{CF}_n(a, b)$ , a fast method is to be derived and described as follows. Define

$$\begin{aligned} M_n(x) &= x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \\ W_n(x) &= x^{\deg(\bar{t}_{2^n-1})} \begin{pmatrix} \bar{t}_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar{t}_{2^n-2})} \begin{pmatrix} \bar{t}_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\ &\quad \cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where  $\bar{t}$  is the  $(b, a)$ -Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for  $n \geq 0$

$$\begin{aligned} M_{n+1}(x) &= W_n(x) \cdot M_n(x), \\ W_{n+1}(x) &= M_n(x) \cdot W_n(x). \end{aligned}$$

Let  $x := 1/z$ . For each non-zero polynomial  $P(z)$ , define  $\tilde{P}(x)$  to be  $P(1/x)$ . Then,

$$\text{CF}_n(a, b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of  $M_n(x)$  with the definition of  $P_n(x)$  and  $Q_n(x)$ , we see that

$$\begin{aligned} M_n(x)_{0,1} &= x^{d_n} \tilde{P}_{2^n-1}(x), \\ M_n(x)_{0,0} &= x^{d_n} \tilde{Q}_{2^n-1}(x), \end{aligned}$$

for some positive integer  $d_n$ . Hence,

$$(2.2) \quad \text{CF}_{2^{2^n}-1}(a, b) = \frac{\tilde{P}_{2^{2^n}-1}(x)}{\tilde{Q}_{2^{2^n}-1}(x)} = \frac{M_{2^n}(x)_{0,1}}{M_{2^n}(x)_{0,0}}.$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of  $\text{CF}(a, b)$  will be established if it is shown that both  $M_{2^n}(x)_{0,1}$  and  $M_{2^n}(x)_{0,0}$  converge to algebraic series in  $\mathbb{F}_2[[x]]$ .

Actually, we will prove that for all  $0 \leq i, j \leq 1$  the four sequences  $(M_{2^n}(x)_{i,j})_n$ ,  $(M_{2^{n+1}}(x)_{i,j})_n$ ,  $(W_{2^n}(x)_{i,j})_n$ , and  $(W_{2^{n+1}}(x)_{i,j})_n$  converge to algebraic series in  $\mathbb{F}_2[[x]]$ . For this purpose, we define four  $2 \times 2$  matrices  $M^e, M^o, W^e, W^o$  as follows: For each  $T \in \{M^e, M^o, W^e, W^o\}$  and  $0 \leq i, j \leq 1$ ,  $T_{i,j}$  is defined to be the unique solution in  $\mathbb{F}_2[[x]]$  of the polynomial  $\phi(T, i, j)$  under certain initial conditions; the polynomials  $\phi(T, i, j)$  and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of  $(M_{2^n}(x))_n$ ,  $(M_{2^{n+1}}(x))_n$ ,  $(W_{2^n}(x))_n$ , and  $(W_{2^{n+1}}(x))_n$ .

Let us explain how the polynomials  $\phi(T, i, j)$  and initial conditions are found, and why the solutions exist and are unique. For  $0 \leq i, j \leq 1$ , the coefficients of the polynomial  $\phi(M^e, i, j)$  (resp.  $\phi(M^o, i, j)$ ,  $\phi(W^e, i, j)$ , and  $\phi(W^o, i, j)$ ) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where  $T = M_{12,i,j}$  (resp.  $M_{11,i,j}$ ,  $W_{12,i,j}$ , and  $W_{11,i,j}$ ), found by the Derksen algorithm. We take the first 8 terms of  $M_{12,i,j}$  (resp.  $M_{11,i,j}$ ,  $W_{12,i,j}$ , and  $W_{11,i,j}$ ) as initial conditions for  $\phi(M^e, i, j)$  (resp.  $\phi(M^o, i, j)$ ,  $\phi(W^e, i, j)$ , and  $\phi(W^o, i, j)$ ). The following fact will be used to ensure that the solution exists and is unique: let  $P(x, y) \in \mathbb{F}_2[x, y]$  and for each series  $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$  denote the partial sum  $\sum_{j=0}^{n-1} a_j x^j$  by  $f_n(x)$  for  $n \geq 0$ . If for some  $n \geq 0$  and  $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$   $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$  and  $Q(x, y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$  can be written as  $x^m \sum_{j=0}^{\infty} q_j(x) y^j$  where  $q_j(x)$  are polynomials for  $j \geq 0$ ,  $q_1(0) = 1$ , and  $q_j(0) = 0$  for  $j > 1$ , then there exists a unique solution  $f(x) \in \mathbb{F}_2[[x]]$  of  $P(x, f(x)) = 0$  that satisfies the initial condition  $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$ .

We state two lemmas concerning the four matrices  $M^e, M^o, W^e, W^o$ . The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$(2.3) \quad M^e = W^o \cdot M^o,$$

$$(2.4) \quad M^o = W^e \cdot M^e,$$

$$(2.5) \quad W^e = M^o \cdot W^o,$$

$$(2.6) \quad W^o = M^e \cdot W^e.$$

*Proof.* We give the proof of the identity

$$M_{0,0}^e = W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o,$$

the proofs of the others being similar.

First, we compute the minimal polynomials of  $W_{0,0}^o M_{0,0}^o$  and  $W_{0,1}^o M_{1,0}^o$ . We know that

$$P(x, y) = \text{Res}_z (\phi(W^o, 0, 0)(x, z), z^{12} \cdot \phi(M^o, 0, 0)(x, y/z))$$

is an annihilating polynomial of  $W_{0,0}^o M_{0,0}^o$ . We use Padé-Hermite approximation to find a candidate for the minimal polynomial of  $W_{0,0}^o M_{0,0}^o$ , that will be called  $\phi_0(x, y)$ . To prove that  $\phi_0(x, y)$  is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of  $P(x, y)$  of multiplicity  $m$  and that  $Q(x, y) := P(x, y)/\phi_0(x, y)^m$  is not an annihilating polynomial of  $W_{0,0}^o M_{0,0}^o$ . We verify the first point directly. For the second point, we truncate  $W_{0,0}^o M_{0,0}^o$  to order 630 and substitute it for  $y$  in  $Q(x, y)$ . We get a series of valuation less than 630, which proves that  $Q(x, y)$  is not an annihilating polynomial of  $W_{0,0}^o M_{0,0}^o$ . We find the minimal polynomial  $\phi_1(x, y)$  of  $W_{0,1}^o M_{1,0}^o$  in a similar way.

Now we prove that  $\phi(M^e, 0, 0)$  is the minimal polynomial of  $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ . We know that

$$S(x, y) = \text{Res}_z (\phi_0(x, z), \phi_1(x, y + z))$$

is an annihilating of  $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ . We verify that  $\phi(M^e, 0, 0)$  is an irreducible factor of  $S(x, y)$  of multiplicity  $\mu$ , and that  $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$  is not an annihilating polynomial of  $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ . To see the last point, we truncate  $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$  to order 770 and substitute it for  $y$  in  $Q(x, y)$ . We get a series of valuation less than 770, and therefore  $Q(x, y)$  is not an annihilating polynomial of  $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$ .

Finally, the first 16 terms of  $M_{0,0}^e$  and  $W_{0,0}^o M_{0,0}^o + W_{0,1}^o M_{1,0}^o$  coincide. As these first terms determine a unique solution of  $\phi(M^e, 0, 0)$ , we know that the two series are one and the same.  $\square$

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers  $k \geq 2$  and  $u = 2^{2k-1}$ , the following identities hold.*

$$\begin{aligned} M^e[:14u] &= M^e[:7u] + x^{4u} M^e[:3u] + x^{4u} M^e[:7u] + x^{8u} M^e[:3u] \\ &\quad + x^{10u} M^e[:4u] + (x^{7u} + x^{11u}) R^e, \\ W^e[:14u] &= W^e[:7u] + x^{4u} W^e[:3u] + x^{4u} W^e[:7u] + x^{8u} W^e[:3u] \\ &\quad + x^{10u} W^e[:4u] + (x^{7u} + x^{11u}) R^e. \end{aligned}$$

*For all integers  $k \geq 2$  and  $u = 2^{2k}$ , the following identities hold.*

$$M^o[:14u] = M^o[:7u] + x^{4u} M^o[:3u] + x^{4u} M^o[:7u] + x^{8u} M^o[:3u]$$

$$\begin{aligned}
& + x^{10u} M^o[:4u] + (x^{7u} + x^{11u}) R^o, \\
W^o[:14u] &= W^o[:7u] + x^{4u} W^o[:3u] + x^{4u} W^o[:7u] + x^{8u} W^o[:3u] \\
& + x^{10u} W^o[:4u] + (x^{7u} + x^{11u}) R^o.
\end{aligned}$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many  $k$ 's into finitely many conditions on the states of the automaton. In the following, we will prove that for  $T = M_{0,0}^o$ , for all integer  $k \geq 2$  and  $u = 2^{2k}$ ,

$$T[:14u] = T[:7u] + x^{4u} T[:3u] + x^{4u} T[:7u] + x^{8u} T[:3u] + x^{10u} T[:4u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{4u} T[3u:4u] + T[7u:8u], \\
0 &= x^{8u} T[:u] + x^{4u} T[4u:5u] + T[8u:9u], \\
0 &= x^{8u} T[u:2u] + x^{4u} T[5u:6u] + T[9u:10u], \\
0 &= x^{10u} T[:u] + x^{8u} T[2u:3u] + x^{4u} T[6u:7u] + T[10u:11u], \\
0 &= x^{10u} T[u:2u] + T[11u:12u], \\
0 &= x^{10u} T[2u:3u] + T[12u:13u], \\
0 &= x^{10u} T[3u:4u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
(2.7) \quad 0 &= T[[11w]_2] + T[[111w]_2], \\
(2.8) \quad 0 &= T[[w]_2] + T[[100w]_2] + T[[1000w]_2], \\
(2.9) \quad 0 &= T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \\
(2.10) \quad 0 &= T[[w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1010w]_2], \\
(2.11) \quad 0 &= T[[1w]_2] + T[[1011w]_2], \\
(2.12) \quad 0 &= T[[10w]_2] + T[[1100w]_2], \\
(2.13) \quad 0 &= T[[11w]_2] + T[[1101w]_2],
\end{aligned}$$

for all binary word  $w$  of length  $2k$  and  $w \neq 0^{2k}$ , and

$$\begin{aligned}
(2.14) \quad 0 &= T[[11w]_2] + T[[111w]_2], \\
(2.15) \quad 0 &= T[[w]_2] + T[[100w]_2] + T[[1000w]_2], \\
(2.16) \quad 0 &= T[[1w]_2] + T[[101w]_2] + T[[1001w]_2], \\
(2.17) \quad 0 &= T[[w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1010w]_2], \\
(2.18) \quad 0 &= T[[1w]_2] + T[[1011w]_2], \\
(2.19) \quad 0 &= T[[10w]_2] + T[[1100w]_2], \\
(2.20) \quad 0 &= T[[11w]_2] + T[[1101w]_2],
\end{aligned}$$

for  $w = 0^{2k}$ .

First we calculate an 2-automaton that generates  $T$  from its minimal polynomial and its first terms. This automaton has 65 states; its transition function and output function can be found in the annex. Let  $A(s, w)$  denote the state reached after

reading  $w$  from right to left starting from the state  $s$ , and  $\tau$  the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$(2.21) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 111)),$$

$$(2.22) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 1000)),$$

$$(2.23) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1001)),$$

$$(2.24) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1010)),$$

$$(2.25) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 1011)),$$

$$(2.26) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 1100)),$$

$$(2.27) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 1101)),$$

for all  $s \in E_{2k}$ , and

$$(2.28) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 111)),$$

$$(2.29) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 100)) + \tau(A(s, 1000)),$$

$$(2.30) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 101)) + \tau(A(s, 1001)),$$

$$(2.31) \quad 0 = \tau(A(s, \epsilon)) + \tau(A(s, 10)) + \tau(A(s, 110)) + \tau(A(s, 1010)),$$

$$(2.32) \quad 0 = \tau(A(s, 1)) + \tau(A(s, 1011)),$$

$$(2.33) \quad 0 = \tau(A(s, 10)) + \tau(A(s, 1100)),$$

$$(2.34) \quad 0 = \tau(A(s, 11)) + \tau(A(s, 1101)),$$

for  $s = A(i, 0^{2k})$ . We find that  $(A(i, 0^{12}), E_{12}) = (A(i, 0^{10}), E_{10})$ , so that we only have to verify that identities (2.21) through (2.34) hold for  $2 \leq k \leq 6$ , which turns out to be true.  $\square$

In the following lemma, we express  $M_{2k}$ ,  $M_{2k+1}$ ,  $W_{2k}$ , and  $W_{2k+1}$  in terms of  $M^e$ ,  $M^o$ ,  $W^e$ , and  $W^o$ .

**Lemma 2.3.** *For all integer  $k \geq 2$ , and  $u = 2^{2k-1}$ ,*

$$M_{2k} = M^e[:7u] + x^{4u}M^e[:3u] + x^{7u}R^e,$$

$$W_{2k} = W^e[:7u] + x^{4u}W^e[:3u] + x^{7u}R^e.$$

*For all integer  $k \geq 2$ , and  $u = 2^{2k}$ ,*

$$M_{2k+1} = M^o[:7u] + x^{4u}M^o[:3u] + x^{7u}R^o,$$

$$W_{2k+1} = W^o[:7u] + x^{4u}W^o[:3u] + x^{7u}R^o.$$

*Proof.* We call the four identities in Lemma 2.3 also by the name  $M_{2k}$ ,  $W_{2k}$ ,  $M_{2k+1}$ , and  $W_{2k+1}$ . For  $n = 2$ , the identities can be verified directly. For  $n \geq 2$ , we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set  $u = 2^{2k}$  and  $v = 2^{2k-1}$ . By definition and induction hypothesis, the left side of identity  $M_{2k+1}$  is equal to

$$(2.35) \quad W_{2k}M_{2k} = (W^e[:7v] + x^{4v}W^e[:3v] + x^{7u}R^e) \times (M^e[:7v] + x^{4v}M^e[:3v] + x^{7u}R^e);$$

Call this expression *lhs*. Note that both sides of identity  $M_{2k+1}$  have the same term of highest degree  $x^{14v}R^o$ . Therefore we only need to prove that their difference is  $O(x^{14v})$ . Using Lemma 2.1 it can be seen that the right side of identity  $M_{2k+1}$  is congruent, modulo  $x^{14v}$ , to

$$W^e[:14v]M^e[:14v] + x^{8v}W^e[:6v]M^e[:6v].$$

For all  $n \leq 14$ , replace the occurrences of  $W^e[:n \cdot v]$  and  $M^e[:n \cdot v]$  in the above expression by the reduction modulo  $x^{n \cdot v}$  of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call *rhs*. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v] / X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v] / X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions *lhs* (2.35) and *rhs* as polynomials in  $\mathbb{F}_2[a_1, \dots, a_{14}, b_1, \dots, b_{14}, c][X]$ . Note that it is not a problem that  $a_j$  commutes with  $b_k$  while  $W^e$  does not commute with  $M^e$ , because in the expressions concerned, the products of  $W^e$ -terms and  $M^e$ -terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed  $O(X^{14})$ , which completes the proof.  $\square$

*Proof of Theorem 1.1.* We prove the theorem for  $\text{CF}(a, b)$ ; for  $\text{CF}(b, a)$ , the proof is similar. By Lemma 2.3, we have For all  $0 \leq j, k \leq 1$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} M_{2n,j,k} &= M_{j,k}^e, \\ \lim_{n \rightarrow \infty} M_{2n+1,j,k} &= M_{j,k}^o, \\ \lim_{n \rightarrow \infty} W_{2n,j,k} &= W_{j,k}^e, \\ \lim_{n \rightarrow \infty} W_{2n+1,j,k} &= W_{j,k}^o. \end{aligned}$$

Let  $z = 1/x$ . By the convergence theorem and identity (2.2),

$$\text{CF}(a, b) = \frac{M_{0,1}^e(x)}{M_{0,0}^e(x)}.$$

By definition,  $\phi(M^e, 0, 1)$  and  $\phi(M^e, 0, 0)$  are minimal polynomials of  $M_{0,1}^e$  and  $M_{0,0}^e$ . Therefore

$$P(x, y) = \text{Res}_t (\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y))$$

is an annihilating polynomial of  $f(x) = M_{0,1}^e/M_{0,0}^e$ .

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{20} + x^{17} + x^{14} + x^{11} + x^7,$$

$$\begin{aligned}
q_1(x) &= x^{15} + x^{12} + x^9 + x^6 + x^5 + x^2, \\
q_2(x) &= x^{10} + x^4 + 1, \\
q_3(x) &= x^{15} + x^{12} + x^9 + x^6 + x^5 + x^2, \\
q_4(x) &= x^{17} + x^{14} + x^{11} + x^{10} + x^8 + x^7 + x^4.
\end{aligned}$$

The polynomial  $Q(x, y)$  is the candidate for the minimal polynomial of  $f(x)$  found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of  $f(x)$ , we only need to prove that it is an irreducible factor of  $P(x, y)$  of multiplicity  $m$  and  $R(x, y) := P(x, y)/Q(x, y)^m$  is not an annihilating polynomial of  $f(x)$ . We verify the first point directly. For the second point, we find that when we truncate  $f(x)$  to order 216, and substitute it for  $y$  in  $R(z, y)$ , we get a series with valuation smaller than 216, which proves that  $R(z, y)$  is not an annihilating polynomial of  $f(x)$ . Finally,  $z^{12}Q(1/z, y)$  is the minimal polynomial of  $\text{CF}(a, b) = f(1/z)$ .  $\square$

### 3. ANNEXE

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients  $p_j(x)$  and the 16 initial terms to determine the solutions uniquely are given below.

For  $\phi(M^e, 0, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{102} + x^{96} + x^{88} + x^{82} + x^{74} + x^{70} + x^{68} + x^{60} + x^{58} + x^{48} + x^{46} \\
&\quad + x^{42} + x^{40} + x^{38} + x^{32} + x^{30} + x^{24}, \\
p_3(x) &= x^{84} + x^{80} + x^{64} + x^{58} + x^{44} + x^{36} + x^{34} + x^{32} + x^{24} + x^{20} + x^{18} \\
&\quad + x^{16}, \\
p_6(x) &= x^{84} + x^{80} + x^{74} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{48} \\
&\quad + x^{44} + x^{40} + x^{38} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{10} + x^8 + x^4 + 1, \\
p_9(x) &= x^{80} + x^{70} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18} + x^{16} + x^{12} + x^{10} + x^8 + x^4, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0]$ .

For  $\phi(M^e, 0, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{104} + x^{97} + x^{90} + x^{88} + x^{78} + x^{77} + x^{76} + x^{74} + x^{66} \\
&\quad + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} + x^{36} + x^{33}, \\
p_3(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{85} + x^{82} + x^{55} + x^{52} + x^{49} + x^{47} + x^{46} \\
&\quad + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{22} \\
&\quad + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40}
\end{aligned}$$



$$\begin{aligned}
& + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{85} + x^{82} + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{57} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} \\
& + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{23} \\
& + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^6, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0]$ .

For  $\phi(M^e, 1, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{104} + x^{97} + x^{90} + x^{88} + x^{78} + x^{77} + x^{76} + x^{74} + x^{66} \\
& + x^{63} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
& + x^{47} + x^{46} + x^{42} + x^{41} + x^{40} + x^{36} + x^{33}, \\
p_3(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{85} + x^{82} + x^{55} + x^{52} + x^{49} + x^{47} + x^{46} \\
& + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{22} \\
& + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
& + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{85} + x^{82} + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{57} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} \\
& + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{23} \\
& + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^6, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0]$ .

For  $\phi(M^e, 1, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{94} + x^{82} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{56} \\
& + x^{50} + x^{42}, \\
p_3(x) &= x^{90} + x^{80} + x^{70} + x^{52} + x^{50} + x^{46} + x^{42} + x^{40} + x^{32} + x^{30} + x^{28} \\
& + x^{26} + x^{16} + x^{12}, \\
p_6(x) &= x^{90} + x^{74} + x^{70} + x^{68} + x^{62} + x^{56} + x^{54} + x^{52} + x^{50} + x^{46} + x^{42} \\
& + x^{40} + x^{38} + x^{34} + x^{32} + x^{22} + x^{20} + x^{14} + x^{12} + x^{10} + x^4 + 1, \\
p_9(x) &= x^{80} + x^{70} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{36} \\
& + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18} + x^{16} + x^{12} + x^{10} + x^8 + x^4, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0]$ .

For  $\phi(W^e, 0, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{100} + x^{94} + x^{82} + x^{80} + x^{74} + x^{72} + x^{70} + x^{66} + x^{60} \\
& + x^{58} + x^{54} + x^{52} + x^{44} + x^{42} + x^{38} + x^{30} + x^{24},
\end{aligned}$$

$$\begin{aligned}
p_3(x) &= x^{90} + x^{80} + x^{64} + x^{58} + x^{50} + x^{42} + x^{34} + x^{32} + x^{26} + x^{24} + x^{18} \\
&\quad + x^{16}, \\
p_6(x) &= x^{90} + x^{74} + x^{68} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} \\
&\quad + x^{42} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{22} + x^{18} + x^{14} + x^{10} + x^8 \\
&\quad + x^4 + 1, \\
p_9(x) &= x^{80} + x^{70} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{36} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18} + x^{16} + x^{12} + x^{10} + x^8 + x^4, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0]$ .

For  $\phi(W^e, 0, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{98} + x^{91} + x^{88} + x^{76} + x^{72} + x^{71} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{60} + x^{58} + x^{57} + x^{53} + x^{51} + x^{50} + x^{43} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{35} + x^{34} + x^{33}, \\
p_3(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{85} + x^{82} + x^{55} + x^{52} + x^{49} + x^{47} + x^{46} \\
&\quad + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{22} \\
&\quad + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{85} + x^{82} + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{57} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{23} \\
&\quad + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^6, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0]$ .

For  $\phi(W^e, 1, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{98} + x^{91} + x^{88} + x^{76} + x^{72} + x^{71} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{62} + x^{60} + x^{58} + x^{57} + x^{53} + x^{51} + x^{50} + x^{43} + x^{42} + x^{40} + x^{38} \\
&\quad + x^{35} + x^{34} + x^{33}, \\
p_3(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{85} + x^{82} + x^{55} + x^{52} + x^{49} + x^{47} + x^{46} \\
&\quad + x^{45} + x^{44} + x^{42} + x^{41} + x^{38} + x^{37} + x^{34} + x^{31} + x^{28} + x^{25} + x^{22} \\
&\quad + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{85} + x^{82} + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{57} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{41} + x^{39}
\end{aligned}$$

$$\begin{aligned}
& + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{28} + x^{23} \\
& + x^{21} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + x^9 + x^6, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0]$ .

For  $\phi(W^e, 1, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{102} + x^{96} + x^{82} + x^{70} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{46} \\
& + x^{44} + x^{42}, \\
p_3(x) &= x^{84} + x^{80} + x^{70} + x^{52} + x^{46} + x^{44} + x^{40} + x^{36} + x^{32} + x^{30} + x^{28} \\
& + x^{20} + x^{16} + x^{12}, \\
p_6(x) &= x^{84} + x^{80} + x^{74} + x^{70} + x^{68} + x^{62} + x^{56} + x^{54} + x^{52} + x^{46} + x^{44} \\
& + x^{38} + x^{36} + x^{34} + x^{26} + x^{22} + x^{16} + x^{14} + x^{12} + x^{10} + x^4 + 1, \\
p_9(x) &= x^{80} + x^{70} + x^{64} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{36} \\
& + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{18} + x^{16} + x^{12} + x^{10} + x^8 + x^4, \\
p_{12}(x) &= x^{80} + x^{32} + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0]$ .

For  $\phi(M^o, 0, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{104} + x^{101} + x^{98} + x^{97} + x^{94} + x^{90} + x^{88} + x^{87} + x^{85} \\
& + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
& + x^{68} + x^{66} + x^{63} + x^{61} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46} \\
& + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} \\
& + x^{30} + x^{28} + x^{24}, \\
p_3(x) &= x^{100} + x^{97} + x^{91} + x^{90} + x^{88} + x^{87} + x^{84} + x^{80} + x^{74} + x^{62} + x^{60} \\
& + x^{57} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} \\
& + x^{34} + x^{33} + x^{32} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{16}, \\
p_6(x) &= x^{100} + x^{97} + x^{91} + x^{90} + x^{88} + x^{87} + x^{84} + x^{80} + x^{77} + x^{74} + x^{71} \\
& + x^{70} + x^{67} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} \\
& + x^{51} + x^{49} + x^{46} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} \\
& + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15} \\
& + x^{14} + x^{12} + x^8 + 1, \\
p_9(x) &= x^{100} + x^{88} + x^{52} + x^{40} + x^{20} + x^8, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0]$ .

For  $\phi(M^o, 0, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{110} + x^{107} + x^{100} + x^{97} + x^{94} + x^{91} + x^{90} + x^{87} + x^{85} \\
& + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} \\
& + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52}
\end{aligned}$$

$$\begin{aligned}
& + x^{50} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{33}, \\
p_3(x) &= x^{85} + x^{82} + x^{45} + x^{42} + x^{37} + x^{34} + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
& + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{47} + x^{44} + x^{41} + x^{38} + x^{15} + x^{12} + x^9 + x^6, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1]$ .

For  $\phi(M^o, 1, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{102} + x^{96} + x^{95} + x^{92} + x^{88} + x^{85} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} \\
& + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} \\
& + x^{57} + x^{56} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{39} + x^{37} + x^{35} \\
& + x^{34} + x^{33}, \\
p_3(x) &= x^{85} + x^{82} + x^{45} + x^{42} + x^{37} + x^{34} + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
& + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{47} + x^{44} + x^{41} + x^{38} + x^{15} + x^{12} + x^9 + x^6, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0]$ .

For  $\phi(M^o, 1, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{101} + x^{91} + x^{88} + x^{85} + x^{81} + x^{79} + x^{76} + x^{73} + x^{69} \\
& + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{52} + x^{49} + x^{47} + x^{45} \\
& + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{100} + x^{97} + x^{91} + x^{88} + x^{87} + x^{74} + x^{62} + x^{60} + x^{57} + x^{52} + x^{51} \\
& + x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} \\
& + x^{27} + x^{24} + x^{23} + x^{22}, \\
p_6(x) &= x^{100} + x^{97} + x^{91} + x^{88} + x^{87} + x^{77} + x^{74} + x^{71} + x^{67} + x^{65} + x^{62} \\
& + x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
& + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30} + x^{25} + x^{24} \\
& + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + 1, \\
p_9(x) &= x^{100} + x^{88} + x^{52} + x^{40} + x^{20} + x^8, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0]$ .

For  $\phi(W^o, 0, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{104} + x^{101} + x^{98} + x^{97} + x^{94} + x^{90} + x^{88} + x^{87} + x^{85} \\
& + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
& + x^{68} + x^{66} + x^{63} + x^{61} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{48} + x^{46}
\end{aligned}$$

$$\begin{aligned}
& + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} \\
& + x^{30} + x^{28} + x^{24}, \\
p_3(x) &= x^{100} + x^{97} + x^{91} + x^{90} + x^{88} + x^{87} + x^{84} + x^{80} + x^{74} + x^{62} + x^{60} \\
& + x^{57} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} \\
& + x^{34} + x^{33} + x^{32} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{16}, \\
p_6(x) &= x^{100} + x^{97} + x^{91} + x^{90} + x^{88} + x^{87} + x^{84} + x^{80} + x^{77} + x^{74} + x^{71} \\
& + x^{70} + x^{67} + x^{65} + x^{64} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} \\
& + x^{51} + x^{49} + x^{46} + x^{44} + x^{42} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} \\
& + x^{33} + x^{32} + x^{28} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18} + x^{15} \\
& + x^{14} + x^{12} + x^8 + 1, \\
p_9(x) &= x^{100} + x^{88} + x^{52} + x^{40} + x^{20} + x^8, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0]$ .

For  $\phi(W^o, 0, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{102} + x^{96} + x^{95} + x^{92} + x^{88} + x^{85} + x^{82} + x^{81} + x^{79} + x^{78} + x^{75} \\
& + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} \\
& + x^{57} + x^{56} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{39} + x^{37} + x^{35} \\
& + x^{34} + x^{33}, \\
p_3(x) &= x^{85} + x^{82} + x^{45} + x^{42} + x^{37} + x^{34} + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
& + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{47} + x^{44} + x^{41} + x^{38} + x^{15} + x^{12} + x^9 + x^6, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0]$ .

For  $\phi(W^o, 1, 0)$ :

$$\begin{aligned}
p_0(x) &= x^{120} + x^{114} + x^{110} + x^{107} + x^{100} + x^{97} + x^{94} + x^{91} + x^{90} + x^{87} + x^{85} \\
& + x^{84} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{73} + x^{70} + x^{69} + x^{68} \\
& + x^{67} + x^{65} + x^{64} + x^{63} + x^{60} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{52} \\
& + x^{50} + x^{46} + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} + x^{33}, \\
p_3(x) &= x^{85} + x^{82} + x^{45} + x^{42} + x^{37} + x^{34} + x^{21} + x^{18}, \\
p_6(x) &= x^{90} + x^{84} + x^{70} + x^{64} + x^{58} + x^{52} + x^{50} + x^{46} + x^{44} + x^{42} + x^{40} \\
& + x^{36} + x^{34} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18} + x^{12}, \\
p_9(x) &= x^{95} + x^{92} + x^{89} + x^{86} + x^{47} + x^{44} + x^{41} + x^{38} + x^{15} + x^{12} + x^9 + x^6, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1]$ .

For  $\phi(W^o, 1, 1)$ :

$$\begin{aligned}
p_0(x) &= x^{111} + x^{105} + x^{101} + x^{91} + x^{88} + x^{85} + x^{81} + x^{79} + x^{76} + x^{73} + x^{69} \\
&\quad + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{59} + x^{52} + x^{49} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{100} + x^{97} + x^{91} + x^{88} + x^{87} + x^{74} + x^{62} + x^{60} + x^{57} + x^{52} + x^{51} \\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{40} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{27} + x^{24} + x^{23} + x^{22}, \\
p_6(x) &= x^{100} + x^{97} + x^{91} + x^{88} + x^{87} + x^{77} + x^{74} + x^{71} + x^{67} + x^{65} + x^{62} \\
&\quad + x^{60} + x^{59} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{30} + x^{25} + x^{24} \\
&\quad + x^{23} + x^{22} + x^{19} + x^{16} + x^{15} + x^{14} + 1, \\
p_9(x) &= x^{100} + x^{88} + x^{52} + x^{40} + x^{20} + x^8, \\
p_{12}(x) &= x^{100} + x^{88} + x^{80} + x^{52} + x^{40} + x^{32} + x^{20} + x^8 + 1,
\end{aligned}$$

and the initial terms are  $[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0]$ .

Below is the transition function and output function of an 2-automaton that generates  $T = M_{0,0}^o$ :

Transition function  $(n, j) \mapsto \delta(n, j)$  ( $\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$ ):

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|-----|--------------|-----|--------------|-----|--------------|-----|--------------|-----|--------------|
| 0   | [1, 2]       | 14  | [8, 24]      | 28  | [38, 39]     | 42  | [49, 10]     | 56  | [13, 29]     |
| 1   | [3, 4]       | 15  | [13, 8]      | 29  | [40, 41]     | 43  | [12, 2]      | 57  | [48, 24]     |
| 2   | [5, 6]       | 16  | [24, 24]     | 30  | [42, 43]     | 44  | [50, 51]     | 58  | [49, 45]     |
| 3   | [7, 8]       | 17  | [25, 10]     | 31  | [8, 13]      | 45  | [52, 53]     | 59  | [61, 62]     |
| 4   | [9, 10]      | 18  | [26, 27]     | 32  | [44, 2]      | 46  | [54, 55]     | 60  | [63, 64]     |
| 5   | [11, 12]     | 19  | [28, 9]      | 33  | [26, 2]      | 47  | [56, 19]     | 61  | [25, 45]     |
| 6   | [7, 13]      | 20  | [29, 13]     | 34  | [30, 27]     | 48  | [41, 57]     | 62  | [59, 45]     |
| 7   | [14, 15]     | 21  | [13, 7]      | 35  | [9, 45]      | 49  | [58, 4]      | 63  | [48, 8]      |
| 8   | [15, 16]     | 22  | [30, 2]      | 36  | [11, 26]     | 50  | [44, 27]     | 64  | [48, 48]     |
| 9   | [17, 18]     | 23  | [12, 27]     | 37  | [46, 24]     | 51  | [59, 10]     |     |              |
| 10  | [19, 20]     | 24  | [16, 31]     | 38  | [24, 46]     | 52  | [28, 59]     |     |              |
| 11  | [21, 5]      | 25  | [32, 33]     | 39  | [47, 12]     | 53  | [60, 24]     |     |              |
| 12  | [22, 23]     | 26  | [34, 35]     | 40  | [8, 48]      | 54  | [48, 13]     |     |              |
| 13  | [13, 13]     | 27  | [36, 37]     | 41  | [24, 48]     | 55  | [8, 8]       |     |              |

Output function  $n \mapsto \tau(n)$ :

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|-----|-----------|-----|-----------|-----|-----------|-----|-----------|-----|-----------|
| 0   | 0         | 10  | 1         | 20  | 0         | 30  | 0         | 40  | 0         |
| 1   | 0         | 11  | 0         | 21  | 0         | 31  | 0         | 41  | 1         |
| 2   | 0         | 12  | 0         | 22  | 0         | 32  | 1         | 42  | 0         |
| 3   | 0         | 13  | 0         | 23  | 0         | 33  | 0         | 43  | 0         |
| 4   | 1         | 14  | 0         | 24  | 1         | 34  | 0         | 44  | 1         |

|   |   |    |   |    |   |    |   |    |   |    |   |  |  |
|---|---|----|---|----|---|----|---|----|---|----|---|--|--|
| 5 | 0 | 15 | 0 | 25 | 1 | 35 | 1 | 45 | 1 | 55 | 0 |  |  |
| 6 | 0 | 16 | 1 | 26 | 0 | 36 | 0 | 46 | 1 | 56 | 0 |  |  |
| 7 | 0 | 17 | 1 | 27 | 0 | 37 | 1 | 47 | 0 | 57 | 1 |  |  |
| 8 | 0 | 18 | 0 | 28 | 1 | 38 | 1 | 48 | 1 | 58 | 0 |  |  |
| 9 | 1 | 19 | 1 | 29 | 0 | 39 | 0 | 49 | 0 | 59 | 1 |  |  |

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