PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^5, z^2 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^5, z^2 + z + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 17:03:12 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 34.5 minutes CPU time used.

Theorem 1.1. *When $(a,b)=(z^5, z^2+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over $\mathbb{F}_2(z)$, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{21}+z^{20}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^6+z^5+z^4+1,\\
p_1(z)&=z^{26}+z^{25}+z^{22}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^9+z^5,\\
p_2(z)&=z^{28}+z^{22}+z^{18}+z^{14}+z^{10},\\
p_3(z)&=z^{26}+z^{25}+z^{22}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^9+z^5,\\
p_4(z)&=z^{24}+z^{22}+z^{21}+z^{20}+z^{17}+z^{12}+z^{11}+z^{10}+z^8+z^6+z^5,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{21}+z^{18}+z^{17}+z^{14}+z^{12}+z^{11}+z^8+z^6+z^5,\\
p_1(z)&=z^{26}+z^{25}+z^{22}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^9+z^5,\\
p_2(z)&=z^{28}+z^{22}+z^{18}+z^{14}+z^{10},\\
p_3(z)&=z^{26}+z^{25}+z^{22}+z^{17}+z^{16}+z^{15}+z^{11}+z^{10}+z^9+z^5,\\
p_4(z)&=z^{24}+z^{22}+z^{21}+z^{17}+z^{12}+z^{11}+z^{10}+z^6+z^5+z^4+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ & + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{5u} M^e[:3u] \\ & + x^{6u} M^e[:2u] + x^{3u} M^e[:7u] + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] \\ & + x^{7u} M^e[:3u] + x^{8u} M^e[:2u] + x^{5u} M^e[:7u] + x^{6u} M^e[:6u] \\ & + x^{7u} M^e[:5u] + x^{9u} M^e[:3u] + x^{10u} M^e[:2u] + x^{7u} M^e[:7u] \\ & + x^{11u} M^e[:3u] + x^{12u} M^e[:2u] + x^{13u} M^e[:u] \end{aligned}$$

$$
\begin{aligned}
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u]\\
&+ x^u W^e[:7u] + x^{2u} W^e[:6u] + x^{3u} W^e[:5u] + x^{5u} W^e[:3u]\\
&+ x^{6u} W^e[:2u] + x^{3u} W^e[:7u] + x^{4u} W^e[:6u] + x^{5u} W^e[:5u]\\
&+ x^{7u} W^e[:3u] + x^{8u} W^e[:2u] + x^{5u} W^e[:7u] + x^{6u} W^e[:6u]\\
&+ x^{7u} W^e[:5u] + x^{9u} W^e[:3u] + x^{10u} W^e[:2u] + x^{7u} W^e[:7u]\\
&+ x^{11u} W^e[:3u] + x^{12u} W^e[:2u] + x^{13u} W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u]\\
&+ x^u M^o[:7u] + x^{2u} M^o[:6u] + x^{3u} M^o[:5u] + x^{5u} M^o[:3u]\\
&+ x^{6u} M^o[:2u] + x^{3u} M^o[:7u] + x^{4u} M^o[:6u] + x^{5u} M^o[:5u]\\
&+ x^{7u} M^o[:3u] + x^{8u} M^o[:2u] + x^{5u} M^o[:7u] + x^{6u} M^o[:6u]\\
&+ x^{7u} M^o[:5u] + x^{9u} M^o[:3u] + x^{10u} M^o[:2u] + x^{7u} M^o[:7u]\\
&+ x^{11u} M^o[:3u] + x^{12u} M^o[:2u] + x^{13u} M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u]\\
&+ x^u W^o[:7u] + x^{2u} W^o[:6u] + x^{3u} W^o[:5u] + x^{5u} W^o[:3u]\\
&+ x^{6u} W^o[:2u] + x^{3u} W^o[:7u] + x^{4u} W^o[:6u] + x^{5u} W^o[:5u]\\
&+ x^{7u} W^o[:3u] + x^{8u} W^o[:2u] + x^{5u} W^o[:7u] + x^{6u} W^o[:6u]\\
&+ x^{7u} W^o[:5u] + x^{9u} W^o[:3u] + x^{10u} W^o[:2u] + x^{7u} W^o[:7u]\\
&+ x^{11u} W^o[:3u] + x^{12u} W^o[:2u] + x^{13u} W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{10u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^u T[:6u] + x^{2u} T[:5u] + x^{4u} T[:3u] + x^{5u} T[:2u] + x^u T[:7u]\\
&+ x^{2u} T[:6u] + x^{3u} T[:5u] + x^{5u} T[:3u] + x^{6u} T[:2u] + x^{3u} T[:7u]\\
&+ x^{4u} T[:6u] + x^{5u} T[:5u] + x^{7u} T[:3u] + x^{8u} T[:2u] + x^{5u} T[:7u]\\
&+ x^{6u} T[:6u] + x^{7u} T[:5u] + x^{9u} T[:3u] + x^{10u} T[:2u] + x^{7u} T[:7u]\\
&+ x^{11u} T[:3u] + x^{12u} T[:2u] + x^{13u} T[:u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 = {}& x^{7u} T[:u] + x^{5u} T[2u:3u] + x^{4u} T[3u:4u] + x^{2u} T[5u:6u]\\
&+ x^u T[6u:7u] + T[7u:8u],
\end{aligned}
$$

$$
\begin{aligned}
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{4u}T[4u:5u] \\
&\quad + x^{3u}T[5u:6u] + T[8u:9u], \\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] \\
&\quad + x^{4u}T[5u:6u] + x^{3u}T[6u:7u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{6u}T[4u:5u] + x^{5u}T[5u:6u] \\
&\quad + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{10u}T[u:2u] + x^{9u}T[2u:3u] + x^{6u}T[5u:6u] \\
&\quad + x^{5u}T[6u:7u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{7u}T[5u:6u] + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{7u}T[6u:7u] \\
&\quad + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.7) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1000w]_2], \qquad (2.8) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] \\
&\quad + T[[110w]_2] + T[[1001w]_2], \qquad (2.9) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \qquad (2.10) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1011w]_2], \qquad (2.11) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[101w]_2] + T[[1100w]_2], \qquad (2.12) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1101w]_2], \qquad (2.13)
\end{aligned}
$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$
\begin{aligned}
0 &= T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.14) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] \\
&\quad + T[[1000w]_2], \qquad (2.15) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[101w]_2] \\
&\quad + T[[110w]_2] + T[[1001w]_2], \qquad (2.16) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1010w]_2], \qquad (2.17) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[1011w]_2], \qquad (2.18) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[101w]_2] + T[[1100w]_2], \qquad (2.19) \\
0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[110w]_2] + T[[1101w]_2], \qquad (2.20)
\end{aligned}
$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1377 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.22}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ + \tau(A(s,1010)), \end{aligned} \tag{2.24}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.25}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1100)), \tag{2.26}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,110)) \\ + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ + \tau(A(s,101)) + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.32}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1100)), \tag{2.33}$$

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,110)) \\ + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{30}), E_{30}) = (A(i, 0^{18}), E_{18})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \leq k \leq 15$, which turns out to be true. □

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \geq 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{5u} M^e[:2u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{5u} W^e[:2u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

For all integer $k \geq 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{5u} M^o[:2u] \\ &\quad + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{5u} W^o[:2u] \\ &\quad + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \geq 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = &\big(W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{4v} W^e[:3v] \\ &+ x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] \\ &+ x^{4v} M^e[:3v] + x^{5v} M^e[:2v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} W^e[:14v] M^e[:14v] + x^{2v} W^e[:12v] M^e[:12v] + x^{4v} W^e[:10v] M^e[:10v] \\ + x^{8v} W^e[:6v] M^e[:6v] + x^{10v} W^e[:4v] M^e[:4v]. \end{aligned}$$

For all $n \leq 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$q_0(x) = x^{28} + x^{24} + x^{23} + x^{22} + x^{17} + x^{16} + x^{14} + x^{11} + x^{10} + x^8 + x^7,$$
$$q_1(x) = x^{23} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12} + x^{11} + x^6 + x^3 + x^2,$$
$$q_2(x) = x^{18} + x^{14} + x^{10} + x^6 + 1,$$
$$q_3(x) = x^{23} + x^{19} + x^{18} + x^{17} + x^{13} + x^{12} + x^{11} + x^6 + x^3 + x^2,$$
$$q_4(x) = x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{11} + x^8 + x^7 + x^6 + x^4.$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$p_0(x) = x^{138} + x^{136} + x^{134} + x^{124} + x^{122} + x^{118} + x^{114} + x^{112} + x^{104} + x^{102}$$

$$
\begin{aligned}
&+ x^{98} + x^{96} + x^{92} + x^{88} + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68}\\
&+ x^{64} + x^{56} + x^{54} + x^{42} + x^{40} + x^{30} + x^{28} + x^{24},\\
p_3(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{104} + x^{100}\\
&+ x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{72} + x^{66} + x^{56} + x^{50}\\
&+ x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{20}\\
&+ x^{18} + x^{16},\\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{110} + x^{106} + x^{102} + x^{98} + x^{96} + x^{94} + x^{90}\\
&+ x^{88} + x^{84} + x^{70} + x^{64} + x^{52} + x^{46} + x^{44} + x^{38} + x^{36} + x^{34} + x^{22}\\
&+ x^{20} + x^{12} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{128} + x^{120} + x^{118} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96}\\
&+ x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68}\\
&+ x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{40} + x^{38}\\
&+ x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{8} + x^{6} + x^{4},\\
p_{12}(x) &= x^{128} + x^{120} + x^{104} + x^{88} + x^{80} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{143} + x^{142} + x^{141} + x^{139} + x^{137} + x^{136} + x^{133} + x^{132}\\
&+ x^{130} + x^{128} + x^{125} + x^{124} + x^{116} + x^{114} + x^{113} + x^{112} + x^{110}\\
&+ x^{109} + x^{108} + x^{107} + x^{106} + x^{96} + x^{95} + x^{92} + x^{88} + x^{87} + x^{82}\\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60}\\
&+ x^{58} + x^{55} + x^{52} + x^{49} + x^{48} + x^{46} + x^{45} + x^{41} + x^{38} + x^{36} + x^{33},\\
p_3(x) &= x^{137} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127} + x^{125} + x^{123} + x^{121}\\
&+ x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} + x^{105} + x^{102}\\
&+ x^{99} + x^{98} + x^{96} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{78} + x^{75}\\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55}\\
&+ x^{53} + x^{51} + x^{49} + x^{46} + x^{43} + x^{42} + x^{40} + x^{33} + x^{32} + x^{30} + x^{27}\\
&+ x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18},\\
p_6(x) &= x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{119}\\
&+ x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{90}\\
&+ x^{89} + x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72}\\
&+ x^{68} + x^{67} + x^{65} + x^{64} + x^{56} + x^{55} + x^{54} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{36} + x^{35} + x^{34} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{16} + x^{15} + x^{13}\\
&+ x^{12},\\
p_9(x) &= x^{127} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{110} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{81} + x^{80} + x^{78} + x^{75}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55}\\
&+ x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{106}\\
&+ x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{90} + x^{89}\\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{26} + x^{25} + x^{24} + x^{22}\\
&+ x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{2} + x\\
&+ 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {}& x^{147} + x^{145} + x^{143} + x^{142} + x^{141} + x^{139} + x^{137} + x^{136} + x^{133} + x^{132}\\
&+ x^{130} + x^{128} + x^{125} + x^{124} + x^{116} + x^{114} + x^{113} + x^{112} + x^{110}\\
&+ x^{109} + x^{108} + x^{107} + x^{106} + x^{96} + x^{95} + x^{92} + x^{88} + x^{87} + x^{82}\\
&+ x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{71} + x^{68} + x^{65} + x^{64} + x^{63} + x^{60}\\
&+ x^{58} + x^{55} + x^{52} + x^{49} + x^{48} + x^{46} + x^{45} + x^{41} + x^{38} + x^{36} + x^{33},\\
p_3(x) = {}& x^{137} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127} + x^{125} + x^{123} + x^{121}\\
&+ x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} + x^{105} + x^{102}\\
&+ x^{99} + x^{98} + x^{96} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{78} + x^{75}\\
&+ x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55}\\
&+ x^{53} + x^{51} + x^{49} + x^{46} + x^{43} + x^{42} + x^{40} + x^{33} + x^{32} + x^{30} + x^{27}\\
&+ x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18},\\
p_6(x) = {}& x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{119}\\
&+ x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{90}\\
&+ x^{89} + x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72}\\
&+ x^{68} + x^{67} + x^{65} + x^{64} + x^{56} + x^{55} + x^{54} + x^{48} + x^{47} + x^{45} + x^{44}\\
&+ x^{36} + x^{35} + x^{34} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{16} + x^{15} + x^{13}\\
&+ x^{12},\\
p_9(x) = {}& x^{127} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113}\\
&+ x^{110} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{81} + x^{80} + x^{78} + x^{75}\\
&+ x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55}\\
&+ x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31}\\
&+ x^{30} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13}\\
&+ x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{106}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{90} + x^{89} \\
&+ x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{26} + x^{25} + x^{24} + x^{22} \\
&+ x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{2} + x \\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0].
For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{156} + x^{154} + x^{152} + x^{146} + x^{140} + x^{138} + x^{136} + x^{130} + x^{126} + x^{124} \\
&+ x^{122} + x^{120} + x^{114} + x^{110} + x^{106} + x^{102} + x^{96} + x^{94} + x^{90} + x^{88} \\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} \\
&+ x^{54} + x^{50} + x^{46} + x^{42}, \\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} \\
&+ x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{80} + x^{72} + x^{68} + x^{66} + x^{64} \\
&+ x^{60} + x^{56} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{28} + x^{20} \\
&+ x^{18} + x^{12}, \\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{96} + x^{92} \\
&+ x^{86} + x^{84} + x^{80} + x^{76} + x^{70} + x^{62} + x^{60} + x^{52} + x^{48} + x^{46} + x^{36} \\
&+ x^{32} + x^{30} + x^{24} + x^{22} + x^{20} + x^{18} + x^{10} + x^{2} + 1, \\
p_9(x) &= x^{116} + x^{114} + x^{112} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} \\
&+ x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{76} + x^{64} + x^{60} + x^{58} + x^{54} + x^{48} \\
&+ x^{46} + x^{44} + x^{42} + x^{40} + x^{26} + x^{24} + x^{16} + x^{8} + x^{4}, \\
p_{12}(x) &= x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96} + x^{84} + x^{82} + x^{80} + x^{20} + x^{18} \\
&+ x^{16} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0].
For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{156} + x^{154} + x^{152} + x^{146} + x^{140} + x^{138} + x^{134} + x^{132} + x^{130} + x^{126} \\
&+ x^{124} + x^{122} + x^{118} + x^{112} + x^{110} + x^{104} + x^{98} + x^{94} + x^{92} + x^{90} \\
&+ x^{86} + x^{84} + x^{72} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{52} + x^{50} + x^{44} \\
&+ x^{36} + x^{34} + x^{26} + x^{24}, \\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{100} \\
&+ x^{98} + x^{94} + x^{92} + x^{90} + x^{84} + x^{80} + x^{78} + x^{76} + x^{68} + x^{56} + x^{54} \\
&+ x^{52} + x^{50} + x^{48} + x^{44} + x^{42} + x^{40} + x^{34} + x^{32} + x^{26} + x^{24} + x^{16}, \\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{110} + x^{108} + x^{96} + x^{94} + x^{90} + x^{88} + x^{86} + x^{74} \\
&+ x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{56} + x^{50} + x^{44} + x^{38} + x^{34} + x^{32} \\
&+ x^{30} + x^{24} + x^{22} + x^{14} + x^{12} + x^{8} + x^{2} + 1, \\
p_9(x) &= x^{116} + x^{114} + x^{112} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94}
\end{aligned}
$$

$$+ x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{76} + x^{64} + x^{60} + x^{58} + x^{54} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{26} + x^{24} + x^{16} + x^{8} + x^{4},$$

$$p_{12}(x) = x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96} + x^{84} + x^{82} + x^{80} + x^{20} + x^{18} + x^{16} + x^{4} + x^{2} + 1,$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{143} + x^{137} + x^{132} + x^{130} + x^{129} + x^{125} + x^{123} + x^{122} \\
& + x^{120} + x^{116} + x^{114} + x^{113} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} \\
& + x^{98} + x^{92} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{78} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} \\
& + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{46} + x^{42} + x^{41} \\
& + x^{36} + x^{34} + x^{33},
\end{aligned}$$

$$\begin{aligned}
p_3(x) = {} & x^{137} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127} + x^{125} + x^{123} + x^{121} \\
& + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} + x^{105} + x^{102} \\
& + x^{99} + x^{98} + x^{96} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{78} + x^{75} \\
& + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} \\
& + x^{53} + x^{51} + x^{49} + x^{46} + x^{43} + x^{42} + x^{40} + x^{33} + x^{32} + x^{30} + x^{27} \\
& + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18},
\end{aligned}$$

$$\begin{aligned}
p_6(x) = {} & x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{119} \\
& + x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{90} \\
& + x^{89} + x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} \\
& + x^{68} + x^{67} + x^{65} + x^{64} + x^{56} + x^{55} + x^{54} + x^{48} + x^{47} + x^{45} + x^{44} \\
& + x^{36} + x^{35} + x^{34} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{16} + x^{15} + x^{13} \\
& + x^{12},
\end{aligned}$$

$$\begin{aligned}
p_9(x) = {} & x^{127} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} \\
& + x^{110} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{81} + x^{80} + x^{78} + x^{75} \\
& + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} \\
& + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31} \\
& + x^{30} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} \\
& + x^{10} + x^{9} + x^{8} + x^{6},
\end{aligned}$$

$$\begin{aligned}
p_{12}(x) = {} & x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{106} \\
& + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{90} + x^{89} \\
& + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{26} + x^{25} + x^{24} + x^{22} \\
& + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{2} + x \\
& + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{143} + x^{137} + x^{132} + x^{130} + x^{129} + x^{125} + x^{123} + x^{122} \\
& + x^{120} + x^{116} + x^{114} + x^{113} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} \\
& + x^{98} + x^{92} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{78} \\
& + x^{76} + x^{75} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} \\
& + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{52} + x^{51} + x^{46} + x^{42} + x^{41} \\
& + x^{36} + x^{34} + x^{33}, \\
p_3(x) = {} & x^{137} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127} + x^{125} + x^{123} + x^{121} \\
& + x^{118} + x^{115} + x^{114} + x^{113} + x^{111} + x^{109} + x^{107} + x^{105} + x^{102} \\
& + x^{99} + x^{98} + x^{96} + x^{89} + x^{88} + x^{86} + x^{83} + x^{82} + x^{81} + x^{78} + x^{75} \\
& + x^{74} + x^{73} + x^{71} + x^{69} + x^{67} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} \\
& + x^{53} + x^{51} + x^{49} + x^{46} + x^{43} + x^{42} + x^{40} + x^{33} + x^{32} + x^{30} + x^{27} \\
& + x^{26} + x^{24} + x^{23} + x^{22} + x^{21} + x^{18}, \\
p_6(x) = {} & x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{124} + x^{123} + x^{121} + x^{119} \\
& + x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{90} \\
& + x^{89} + x^{87} + x^{86} + x^{82} + x^{81} + x^{79} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} \\
& + x^{68} + x^{67} + x^{65} + x^{64} + x^{56} + x^{55} + x^{54} + x^{48} + x^{47} + x^{45} + x^{44} \\
& + x^{36} + x^{35} + x^{34} + x^{28} + x^{27} + x^{25} + x^{23} + x^{22} + x^{16} + x^{15} + x^{13} \\
& + x^{12}, \\
p_9(x) = {} & x^{127} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{116} + x^{115} + x^{114} + x^{113} \\
& + x^{110} + x^{93} + x^{92} + x^{91} + x^{89} + x^{87} + x^{84} + x^{81} + x^{80} + x^{78} + x^{75} \\
& + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{63} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} \\
& + x^{52} + x^{49} + x^{48} + x^{47} + x^{45} + x^{42} + x^{37} + x^{36} + x^{35} + x^{32} + x^{31} \\
& + x^{30} + x^{29} + x^{27} + x^{24} + x^{21} + x^{20} + x^{19} + x^{16} + x^{15} + x^{14} + x^{13} \\
& + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) = {} & x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{116} + x^{114} + x^{113} + x^{112} + x^{106} \\
& + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{96} + x^{90} + x^{89} \\
& + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{80} + x^{26} + x^{25} + x^{24} + x^{22} \\
& + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} + x^{2} + x \\
& + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{138} + x^{136} + x^{134} + x^{128} + x^{122} + x^{120} + x^{116} + x^{114} + x^{112} + x^{106} \\
& + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{78}
\end{aligned}$$

$$\begin{aligned}
&+ x^{72} + x^{70} + x^{66} + x^{64} + x^{58} + x^{56} + x^{54} + x^{50} + x^{48} + x^{42},\\
p_3(x) &= x^{126} + x^{122} + x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{102} + x^{100} + x^{96}\\
&+ x^{90} + x^{86} + x^{82} + x^{76} + x^{68} + x^{66} + x^{64} + x^{60} + x^{54} + x^{52} + x^{50}\\
&+ x^{48} + x^{44} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20}\\
&+ x^{14} + x^{12},\\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{100} + x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{76}\\
&+ x^{70} + x^{68} + x^{64} + x^{60} + x^{54} + x^{46} + x^{42} + x^{38} + x^{36} + x^{32} + x^{28}\\
&+ x^{24} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} + 1,\\
p_9(x) &= x^{128} + x^{120} + x^{118} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96}\\
&+ x^{92} + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68}\\
&+ x^{66} + x^{64} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{40} + x^{38}\\
&+ x^{34} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22} + x^{20} + x^{8} + x^{6} + x^{4},\\
p_{12}(x) &= x^{128} + x^{120} + x^{104} + x^{88} + x^{80} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{143} + x^{142} + x^{141} + x^{139} + x^{136} + x^{135} + x^{132} + x^{129}\\
&+ x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{120} + x^{118} + x^{115}\\
&+ x^{114} + x^{110} + x^{107} + x^{103} + x^{102} + x^{100} + x^{94} + x^{93} + x^{91} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{86} + x^{83} + x^{81} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70}\\
&+ x^{69} + x^{68} + x^{67} + x^{66} + x^{62} + x^{61} + x^{59} + x^{55} + x^{53} + x^{51} + x^{49}\\
&+ x^{47} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} + x^{37} + x^{36} + x^{33} + x^{32}\\
&+ x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24},\\
p_3(x) &= x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128}\\
&+ x^{127} + x^{123} + x^{122} + x^{121} + x^{120} + x^{115} + x^{112} + x^{109} + x^{105}\\
&+ x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{89} + x^{87} + x^{85} + x^{82}\\
&+ x^{75} + x^{72} + x^{69} + x^{68} + x^{67} + x^{62} + x^{61} + x^{58} + x^{55} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34}\\
&+ x^{27} + x^{25} + x^{24} + x^{23} + x^{18} + x^{17} + x^{16},\\
p_6(x) &= x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128}\\
&+ x^{127} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{115} + x^{113} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{103} + x^{99} + x^{96} + x^{95} + x^{91} + x^{89}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70}\\
&+ x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{51} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{6} + x^{5} + x^{4} + x^{2} + x + 1,\\
p_9(x) &= x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{126}\\
&+ x^{125} + x^{124} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108} + x^{102}\\
&+ x^{101} + x^{100} + x^{90} + x^{89} + x^{88} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40}\\
&+ x^{38} + x^{37} + x^{36} + x^{30} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20} + x^{10} + x^{9}\\
&+ x^{8},\\
p_{12}(x) &= x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{126}\\
&+ x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{114} + x^{113} + x^{112} + x^{110}\\
&+ x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{86} + x^{85}\\
&+ x^{84} + x^{82} + x^{81} + x^{80} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38}\\
&+ x^{37} + x^{36} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16}\\
&+ x^{6} + x^{5} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{160} + x^{156} + x^{152} + x^{148} + x^{145} + x^{140} + x^{139} + x^{135} + x^{134} + x^{132}\\
&+ x^{130} + x^{128} + x^{124} + x^{115} + x^{113} + x^{111} + x^{110} + x^{106} + x^{105}\\
&+ x^{102} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83}\\
&+ x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{69} + x^{68} + x^{63} + x^{62} + x^{60} + x^{59}\\
&+ x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{37} + x^{34} + x^{33},\\
p_3(x) &= x^{125} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{109} + x^{105} + x^{104} + x^{103}\\
&+ x^{99} + x^{98} + x^{93} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80}\\
&+ x^{79} + x^{75} + x^{74} + x^{69} + x^{65} + x^{64} + x^{63} + x^{59} + x^{58} + x^{53} + x^{49}\\
&+ x^{48} + x^{47} + x^{43} + x^{42} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{19} + x^{18},\\
p_6(x) &= x^{130} + x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{104} + x^{96} + x^{88}\\
&+ x^{86} + x^{80} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{54} + x^{46} + x^{44} + x^{34}\\
&+ x^{26} + x^{24} + x^{22} + x^{14} + x^{12},\\
p_9(x) &= x^{135} + x^{131} + x^{130} + x^{129} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{115}\\
&+ x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{97}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{39} + x^{35} + x^{34} + x^{33} + x^{28}\\
&+ x^{27} + x^{25} + x^{24} + x^{22} + x^{19} + x^{18} + x^{17} + x^{12} + x^{11} + x^{9} + x^{8} + x^{7}\\
&+ x^{6},\\
p_{12}(x) &= x^{140} + x^{136} + x^{132} + x^{124} + x^{120} + x^{112} + x^{108} + x^{104} + x^{96} + x^{84}\\
&+ x^{80} + x^{44} + x^{40} + x^{36} + x^{28} + x^{24} + x^{16} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0].
For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{138} + x^{136} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{126} + x^{122} + x^{121} \\
&\quad + x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{94} + x^{92} + x^{89} \\
&\quad + x^{86} + x^{83} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} \\
&\quad + x^{65} + x^{63} + x^{62} + x^{60} + x^{57} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{42} \\
&\quad + x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33}, \\
p_3(x) &= x^{133} + x^{128} + x^{127} + x^{124} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} \\
&\quad + x^{114} + x^{112} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} \\
&\quad + x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} \\
&\quad + x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{66} \\
&\quad + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} \\
&\quad + x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{19} + x^{18}, \\
p_6(x) &= x^{138} + x^{130} + x^{128} + x^{126} + x^{122} + x^{116} + x^{114} + x^{112} + x^{108} + x^{100} \\
&\quad + x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64} \\
&\quad + x^{62} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{32} + x^{28} \\
&\quad + x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12}, \\
p_9(x) &= x^{143} + x^{138} + x^{137} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{126} + x^{125} \\
&\quad + x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{104} \\
&\quad + x^{103} + x^{101} + x^{100} + x^{98} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{47} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{34} + x^{31} + x^{30} + x^{29} + x^{24} \\
&\quad + x^{23} + x^{21} + x^{20} + x^{18} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) &= x^{148} + x^{140} + x^{112} + x^{100} + x^{96} + x^{92} + x^{80} + x^{52} + x^{44} + x^{20} + x^{16} \\
&\quad + x^{12} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0].
For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{143} + x^{135} + x^{133} + x^{131} + x^{129} + x^{125} + x^{123} + x^{122} \\
&\quad + x^{121} + x^{120} + x^{119} + x^{117} + x^{109} + x^{106} + x^{104} + x^{103} + x^{98} + x^{97} \\
&\quad + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} \\
&\quad + x^{44} + x^{42}, \\
p_3(x) &= x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{129} + x^{126} + x^{124}
\end{aligned}$$

$$
\begin{aligned}
&+x^{121}+x^{119}+x^{118}+x^{116}+x^{113}+x^{112}+x^{111}+x^{110}+x^{109}\\
&+x^{108}+x^{107}+x^{106}+x^{105}+x^{103}+x^{101}+x^{98}+x^{97}+x^{95}+x^{94}\\
&+x^{89}+x^{87}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{79}+x^{78}+x^{76}+x^{73}\\
&+x^{72}+x^{71}+x^{70}+x^{69}+x^{66}+x^{64}+x^{63}+x^{61}+x^{60}+x^{59}+x^{58}\\
&+x^{57}+x^{54}+x^{52}+x^{49}+x^{47}+x^{45}+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}\\
&+x^{37}+x^{36}+x^{34}+x^{28}+x^{24}+x^{22},\\
p_6(x)=&\,x^{142}+x^{141}+x^{140}+x^{138}+x^{136}+x^{134}+x^{132}+x^{129}+x^{126}+x^{124}\\
&+x^{121}+x^{119}+x^{118}+x^{117}+x^{116}+x^{112}+x^{111}+x^{110}+x^{108}\\
&+x^{107}+x^{106}+x^{103}+x^{99}+x^{98}+x^{97}+x^{95}+x^{94}+x^{93}+x^{92}+x^{91}\\
&+x^{89}+x^{86}+x^{85}+x^{84}+x^{81}+x^{80}+x^{76}+x^{74}+x^{73}+x^{63}+x^{62}\\
&+x^{60}+x^{59}+x^{56}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}\\
&+x^{44}+x^{41}+x^{38}+x^{37}+x^{36}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{26}\\
&+x^{25}+x^{22}+x^{21}+x^{20}+x^{17}+x^{14}+x^{10}+x^{9}+x^{8}+x^{6}+x^{5}+x^{4}\\
&+x^{2}+x+1,\\
p_9(x)=&\,x^{142}+x^{141}+x^{140}+x^{138}+x^{137}+x^{136}+x^{134}+x^{133}+x^{132}+x^{126}\\
&+x^{125}+x^{124}+x^{118}+x^{117}+x^{116}+x^{110}+x^{109}+x^{108}+x^{102}\\
&+x^{101}+x^{100}+x^{90}+x^{89}+x^{88}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}\\
&+x^{38}+x^{37}+x^{36}+x^{30}+x^{29}+x^{28}+x^{22}+x^{21}+x^{20}+x^{10}+x^{9}\\
&+x^{8},\\
p_{12}(x)=&\,x^{142}+x^{141}+x^{140}+x^{138}+x^{137}+x^{136}+x^{134}+x^{133}+x^{132}+x^{126}\\
&+x^{125}+x^{124}+x^{122}+x^{121}+x^{120}+x^{114}+x^{113}+x^{112}+x^{110}\\
&+x^{109}+x^{108}+x^{106}+x^{105}+x^{104}+x^{98}+x^{97}+x^{96}+x^{86}+x^{85}\\
&+x^{84}+x^{82}+x^{81}+x^{80}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{40}+x^{38}\\
&+x^{37}+x^{36}+x^{30}+x^{29}+x^{28}+x^{26}+x^{25}+x^{24}+x^{18}+x^{17}+x^{16}\\
&+x^{6}+x^{5}+x^{4}+x^{2}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{147}+x^{145}+x^{143}+x^{142}+x^{141}+x^{139}+x^{136}+x^{135}+x^{132}+x^{129}\\
&+x^{128}+x^{126}+x^{125}+x^{124}+x^{123}+x^{122}+x^{120}+x^{118}+x^{115}\\
&+x^{114}+x^{110}+x^{107}+x^{103}+x^{102}+x^{100}+x^{94}+x^{93}+x^{91}+x^{90}\\
&+x^{89}+x^{88}+x^{87}+x^{86}+x^{83}+x^{81}+x^{79}+x^{77}+x^{74}+x^{73}+x^{70}\\
&+x^{69}+x^{68}+x^{67}+x^{66}+x^{62}+x^{61}+x^{59}+x^{55}+x^{53}+x^{51}+x^{49}\\
&+x^{47}+x^{45}+x^{44}+x^{43}+x^{41}+x^{39}+x^{38}+x^{37}+x^{36}+x^{33}+x^{32}\\
&+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{25}+x^{24},\\
p_3(x)=&\,x^{142}+x^{141}+x^{140}+x^{138}+x^{136}+x^{134}+x^{131}+x^{130}+x^{129}+x^{128}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{127} + x^{123} + x^{122} + x^{121} + x^{120} + x^{115} + x^{112} + x^{109} + x^{105}\\
&+ x^{104} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{89} + x^{87} + x^{85} + x^{82}\\
&+ x^{75} + x^{72} + x^{69} + x^{68} + x^{67} + x^{62} + x^{61} + x^{58} + x^{55} + x^{51} + x^{50}\\
&+ x^{49} + x^{48} + x^{46} + x^{45} + x^{42} + x^{41} + x^{40} + x^{38} + x^{37} + x^{36} + x^{34}\\
&+ x^{27} + x^{25} + x^{24} + x^{23} + x^{18} + x^{17} + x^{16},\\
p_6(x) = {}& x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{134} + x^{131} + x^{130} + x^{129} + x^{128}\\
&+ x^{127} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} + x^{115} + x^{113} + x^{111}\\
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{103} + x^{99} + x^{96} + x^{95} + x^{91} + x^{89}\\
&+ x^{88} + x^{87} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70}\\
&+ x^{69} + x^{67} + x^{66} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{51} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20}\\
&+ x^{18} + x^{15} + x^{14} + x^{13} + x^{12} + x^{6} + x^{5} + x^{4} + x^{2} + x + 1,\\
p_9(x) = {}& x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{126}\\
&+ x^{125} + x^{124} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108} + x^{102}\\
&+ x^{101} + x^{100} + x^{90} + x^{89} + x^{88} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40}\\
&+ x^{38} + x^{37} + x^{36} + x^{30} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20} + x^{10} + x^{9}\\
&+ x^{8},\\
p_{12}(x) = {}& x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{126}\\
&+ x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{114} + x^{113} + x^{112} + x^{110}\\
&+ x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{86} + x^{85}\\
&+ x^{84} + x^{82} + x^{81} + x^{80} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38}\\
&+ x^{37} + x^{36} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16}\\
&+ x^{6} + x^{5} + x^{4} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{138} + x^{136} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{126} + x^{122} + x^{121}\\
&+ x^{120} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{108} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{98} + x^{94} + x^{92} + x^{89}\\
&+ x^{86} + x^{83} + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66}\\
&+ x^{65} + x^{63} + x^{62} + x^{60} + x^{57} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{37} + x^{35} + x^{33},\\
p_3(x) = {}& x^{133} + x^{128} + x^{127} + x^{124} + x^{122} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115}\\
&+ x^{114} + x^{112} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101}\\
&+ x^{99} + x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{83} + x^{81} + x^{80} + x^{78} + x^{77} + x^{75} + x^{74} + x^{72} + x^{71} + x^{68} + x^{66}\\
&+ x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49}\\
&+ x^{48} + x^{46} + x^{45} + x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33}\\
&+ x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{22} + x^{19} + x^{18},\\
p_6(x) = {}& x^{138} + x^{130} + x^{128} + x^{126} + x^{122} + x^{116} + x^{114} + x^{112} + x^{108} + x^{100}\\
&+ x^{94} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{74} + x^{70} + x^{68} + x^{66} + x^{64}\\
&+ x^{62} + x^{58} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{32} + x^{28}\\
&+ x^{24} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12},\\
p_9(x) = {}& x^{143} + x^{138} + x^{137} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{126} + x^{125}\\
&+ x^{120} + x^{119} + x^{117} + x^{116} + x^{114} + x^{111} + x^{110} + x^{109} + x^{104}\\
&+ x^{103} + x^{101} + x^{100} + x^{98} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86}\\
&+ x^{47} + x^{42} + x^{41} + x^{38} + x^{37} + x^{36} + x^{34} + x^{31} + x^{30} + x^{29} + x^{24}\\
&+ x^{23} + x^{21} + x^{20} + x^{18} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) = {}& x^{148} + x^{140} + x^{112} + x^{100} + x^{96} + x^{92} + x^{80} + x^{52} + x^{44} + x^{20} + x^{16}\\
&+ x^{12} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{160} + x^{156} + x^{152} + x^{148} + x^{145} + x^{140} + x^{139} + x^{135} + x^{134} + x^{132}\\
&+ x^{130} + x^{128} + x^{124} + x^{115} + x^{113} + x^{111} + x^{110} + x^{106} + x^{105}\\
&+ x^{102} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{88} + x^{87} + x^{86} + x^{84} + x^{83}\\
&+ x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{69} + x^{68} + x^{63} + x^{62} + x^{60} + x^{59}\\
&+ x^{58} + x^{56} + x^{55} + x^{54} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{37} + x^{34} + x^{33},\\
p_3(x) = {}& x^{125} + x^{121} + x^{120} + x^{119} + x^{115} + x^{114} + x^{109} + x^{105} + x^{104} + x^{103}\\
&+ x^{99} + x^{98} + x^{93} + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80}\\
&+ x^{79} + x^{75} + x^{74} + x^{69} + x^{65} + x^{64} + x^{63} + x^{59} + x^{58} + x^{53} + x^{49}\\
&+ x^{48} + x^{47} + x^{43} + x^{42} + x^{37} + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{26}\\
&+ x^{25} + x^{24} + x^{23} + x^{19} + x^{18},\\
p_6(x) = {}& x^{130} + x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{104} + x^{96} + x^{88}\\
&+ x^{86} + x^{80} + x^{78} + x^{76} + x^{72} + x^{66} + x^{64} + x^{54} + x^{46} + x^{44} + x^{34}\\
&+ x^{26} + x^{24} + x^{22} + x^{14} + x^{12},\\
p_9(x) = {}& x^{135} + x^{131} + x^{130} + x^{129} + x^{124} + x^{123} + x^{121} + x^{120} + x^{118} + x^{115}\\
&+ x^{114} + x^{113} + x^{108} + x^{107} + x^{105} + x^{104} + x^{102} + x^{99} + x^{98} + x^{97}\\
&+ x^{92} + x^{91} + x^{89} + x^{88} + x^{87} + x^{86} + x^{39} + x^{35} + x^{34} + x^{33} + x^{28}\\
&+ x^{27} + x^{25} + x^{24} + x^{22} + x^{19} + x^{18} + x^{17} + x^{12} + x^{11} + x^{9} + x^{8} + x^{7}
\end{aligned}
$$

$$+ x^{6},$$

$$p_{12}(x) = x^{140} + x^{136} + x^{132} + x^{124} + x^{120} + x^{112} + x^{108} + x^{104} + x^{96} + x^{84} + x^{80} + x^{44} + x^{40} + x^{36} + x^{28} + x^{24} + x^{16} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{143} + x^{135} + x^{133} + x^{131} + x^{129} + x^{125} + x^{123} + x^{122} \\
& + x^{121} + x^{120} + x^{119} + x^{117} + x^{109} + x^{106} + x^{104} + x^{103} + x^{98} + x^{97} \\
& + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{86} + x^{84} + x^{83} + x^{81} + x^{79} + x^{78} \\
& + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{67} + x^{65} + x^{64} + x^{62} + x^{60} \\
& + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} \\
& + x^{44} + x^{42}, \\
p_3(x) = {} & x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{129} + x^{126} + x^{124} \\
& + x^{121} + x^{119} + x^{118} + x^{116} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
& + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{98} + x^{97} + x^{95} + x^{94} \\
& + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{76} + x^{73} \\
& + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{64} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} \\
& + x^{57} + x^{54} + x^{52} + x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} \\
& + x^{37} + x^{36} + x^{34} + x^{28} + x^{24} + x^{22}, \\
p_6(x) = {} & x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{134} + x^{132} + x^{129} + x^{126} + x^{124} \\
& + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{112} + x^{111} + x^{110} + x^{108} \\
& + x^{107} + x^{106} + x^{103} + x^{99} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} \\
& + x^{89} + x^{86} + x^{85} + x^{84} + x^{81} + x^{80} + x^{76} + x^{74} + x^{73} + x^{63} + x^{62} \\
& + x^{60} + x^{59} + x^{56} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} \\
& + x^{44} + x^{41} + x^{38} + x^{37} + x^{36} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{26} \\
& + x^{25} + x^{22} + x^{21} + x^{20} + x^{17} + x^{14} + x^{10} + x^{9} + x^{8} + x^{6} + x^{5} + x^{4} \\
& + x^{2} + x + 1, \\
p_9(x) = {} & x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{126} \\
& + x^{125} + x^{124} + x^{118} + x^{117} + x^{116} + x^{110} + x^{109} + x^{108} + x^{102} \\
& + x^{101} + x^{100} + x^{90} + x^{89} + x^{88} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} \\
& + x^{38} + x^{37} + x^{36} + x^{30} + x^{29} + x^{28} + x^{22} + x^{21} + x^{20} + x^{10} + x^{9} \\
& + x^{8}, \\
p_{12}(x) = {} & x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{126} \\
& + x^{125} + x^{124} + x^{122} + x^{121} + x^{120} + x^{114} + x^{113} + x^{112} + x^{110} \\
& + x^{109} + x^{108} + x^{106} + x^{105} + x^{104} + x^{98} + x^{97} + x^{96} + x^{86} + x^{85} \\
& + x^{84} + x^{82} + x^{81} + x^{80} + x^{46} + x^{45} + x^{44} + x^{42} + x^{41} + x^{40} + x^{38}
\end{aligned}$$

$$+ x^{37} + x^{36} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24} + x^{18} + x^{17} + x^{16}$$
$$+ x^{6} + x^{5} + x^{4} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	345	[12, 153]	690	[894, 158]	1035	[467, 891]
1	[3, 4]	346	[419, 466]	691	[32, 506]	1036	[1197, 200]
2	[5, 6]	347	[513, 389]	692	[895, 896]	1037	[1198, 511]
3	[7, 8]	348	[514, 481]	693	[883, 540]	1038	[1199, 837]
4	[9, 10]	349	[403, 515]	694	[849, 854]	1039	[849, 196]
5	[11, 12]	350	[516, 50]	695	[205, 177]	1040	[430, 163]
6	[13, 14]	351	[517, 503]	696	[897, 53]	1041	[182, 45]
7	[15, 16]	352	[518, 519]	697	[513, 56]	1042	[148, 183]
8	[17, 18]	353	[520, 459]	698	[178, 154]	1043	[57, 889]
9	[19, 20]	354	[521, 167]	699	[443, 163]	1044	[1200, 539]
10	[21, 22]	355	[522, 434]	700	[898, 541]	1045	[431, 883]
11	[23, 24]	356	[523, 524]	701	[899, 50]	1046	[1201, 921]
12	[25, 26]	357	[156, 525]	702	[900, 858]	1047	[534, 820]
13	[27, 28]	358	[190, 61]	703	[855, 495]	1048	[1172, 1202]
14	[29, 30]	359	[526, 215]	704	[434, 145]	1049	[451, 541]
15	[31, 32]	360	[527, 528]	705	[46, 12]	1050	[510, 479]
16	[33, 34]	361	[529, 479]	706	[901, 215]	1051	[430, 873]
17	[35, 36]	362	[182, 424]	707	[160, 533]	1052	[876, 1203]
18	[37, 38]	363	[59, 530]	708	[902, 903]	1053	[883, 522]
19	[39, 34]	364	[531, 39]	709	[500, 56]	1054	[1180, 945]
20	[40, 41]	365	[532, 533]	710	[904, 446]	1055	[386, 172]
21	[42, 43]	366	[529, 211]	711	[875, 515]	1056	[1204, 436]
22	[44, 45]	367	[61, 188]	712	[905, 462]	1057	[517, 945]
23	[46, 47]	368	[534, 489]	713	[853, 196]	1058	[468, 540]
24	[48, 49]	369	[508, 525]	714	[892, 906]	1059	[433, 861]
25	[50, 51]	370	[535, 503]	715	[481, 440]	1060	[1205, 454]
26	[52, 53]	371	[35, 148]	716	[47, 153]	1061	[502, 829]
27	[54, 55]	372	[415, 536]	717	[907, 837]	1062	[394, 453]
28	[56, 57]	373	[126, 201]	718	[908, 483]	1063	[139, 387]
29	[58, 59]	374	[537, 462]	719	[478, 211]	1064	[1206, 37]
30	[60, 61]	375	[538, 539]	720	[473, 149]	1065	[510, 164]
31	[62, 63]	376	[432, 540]	721	[53, 530]	1066	[1207, 57]
32	[64, 65]	377	[505, 442]	722	[909, 450]	1067	[846, 389]
33	[66, 67]	378	[417, 41]	723	[517, 829]	1068	[1208, 915]
34	[68, 69]	379	[430, 450]	724	[910, 544]	1069	[394, 212]
35	[70, 71]	380	[498, 541]	725	[210, 199]	1070	[206, 162]

36	[72, 73]	381	[178, 542]	726	[911, 143]	1071	[185, 533]
37	[74, 75]	382	[525, 543]	727	[145, 439]	1072	[160, 186]
38	[76, 77]	383	[544, 129]	728	[912, 9]	1073	[39, 2]
39	[78, 79]	384	[545, 156]	729	[913, 456]	1074	[1195, 1202]
40	[80, 81]	385	[153, 542]	730	[390, 204]	1075	[845, 58]
41	[82, 83]	386	[546, 19]	731	[914, 915]	1076	[1209, 13]
42	[84, 85]	387	[547, 21]	732	[860, 37]	1077	[500, 389]
43	[86, 87]	388	[548, 549]	733	[916, 917]	1078	[1210, 45]
44	[88, 89]	389	[550, 551]	734	[846, 135]	1079	[512, 176]
45	[90, 91]	390	[552, 553]	735	[171, 387]	1080	[1211, 903]
46	[92, 93]	391	[554, 555]	736	[135, 57]	1081	[172, 37]
47	[94, 95]	392	[556, 557]	737	[918, 399]	1082	[508, 153]
48	[96, 97]	393	[558, 230]	738	[479, 830]	1083	[1176, 193]
49	[98, 99]	394	[559, 560]	739	[868, 393]	1084	[523, 407]
50	[100, 101]	395	[561, 562]	740	[512, 874]	1085	[1212, 152]
51	[102, 103]	396	[563, 561]	741	[851, 9]	1086	[1167, 1213]
52	[104, 105]	397	[564, 565]	742	[484, 198]	1087	[943, 541]
53	[106, 107]	398	[566, 567]	743	[919, 420]	1088	[1214, 130]
54	[108, 109]	399	[568, 569]	744	[898, 191]	1089	[451, 393]
55	[110, 111]	400	[570, 571]	745	[920, 386]	1090	[1215, 43]
56	[112, 113]	401	[572, 573]	746	[36, 921]	1091	[853, 822]
57	[114, 115]	402	[574, 575]	747	[922, 153]	1092	[1216, 132]
58	[116, 117]	403	[576, 577]	748	[134, 56]	1093	[487, 1166]
59	[118, 119]	404	[578, 579]	749	[13, 449]	1094	[1217, 413]
60	[120, 121]	405	[580, 581]	750	[401, 499]	1095	[871, 182]
61	[122, 123]	406	[582, 583]	751	[513, 135]	1096	[1218, 7]
62	[124, 57]	407	[584, 585]	752	[486, 831]	1097	[198, 474]
63	[125, 12]	408	[586, 284]	753	[199, 830]	1098	[1219, 430]
64	[126, 127]	409	[587, 588]	754	[171, 440]	1099	[1220, 865]
65	[128, 129]	410	[589, 590]	755	[393, 13]	1100	[421, 409]
66	[130, 131]	411	[591, 592]	756	[916, 429]	1101	[1221, 921]
67	[132, 133]	412	[593, 594]	757	[845, 52]	1102	[203, 176]
68	[134, 135]	413	[595, 596]	758	[923, 202]	1103	[487, 492]
69	[136, 137]	414	[597, 598]	759	[892, 822]	1104	[538, 1222]
70	[138, 139]	415	[599, 600]	760	[924, 40]	1105	[168, 520]
71	[140, 141]	416	[601, 602]	761	[215, 186]	1106	[1223, 188]
72	[142, 143]	417	[69, 603]	762	[925, 829]	1107	[892, 854]
73	[144, 145]	418	[604, 605]	763	[871, 875]	1108	[467, 212]
74	[146, 147]	419	[606, 607]	764	[926, 389]	1109	[408, 448]
75	[33, 2]	420	[608, 112]	765	[927, 162]	1110	[1224, 402]
76	[128, 133]	421	[609, 368]	766	[825, 492]	1111	[496, 461]
77	[148, 149]	422	[610, 611]	767	[517, 928]	1112	[878, 212]
78	[150, 151]	423	[612, 84]	768	[7, 206]	1113	[926, 135]
79	[152, 10]	424	[613, 614]	769	[929, 212]	1114	[1225, 38]

80	[153, 154]	425	[615, 616]	770	[197, 148]	1115	[833, 858]
81	[37, 10]	426	[617, 618]	771	[930, 931]	1116	[1226, 42]
82	[155, 156]	427	[251, 619]	772	[11, 47]	1117	[502, 928]
83	[53, 157]	428	[620, 621]	773	[932, 45]	1118	[472, 878]
84	[158, 159]	429	[622, 623]	774	[488, 820]	1119	[205, 8]
85	[160, 161]	430	[624, 625]	775	[933, 934]	1120	[12, 525]
86	[162, 163]	431	[626, 627]	776	[878, 453]	1121	[1206, 152]
87	[164, 165]	432	[628, 256]	777	[935, 474]	1122	[872, 182]
88	[166, 167]	433	[629, 630]	778	[406, 209]	1123	[1227, 200]
89	[168, 168]	434	[631, 632]	779	[540, 861]	1124	[943, 393]
90	[169, 170]	435	[297, 633]	780	[936, 824]	1125	[884, 882]
91	[171, 172]	436	[560, 634]	781	[872, 875]	1126	[498, 393]
92	[173, 174]	437	[635, 636]	782	[410, 131]	1127	[850, 156]
93	[175, 176]	438	[637, 638]	783	[863, 209]	1128	[414, 1166]
94	[7, 177]	439	[639, 628]	784	[52, 59]	1129	[891, 436]
95	[47, 178]	440	[245, 640]	785	[937, 153]	1130	[1228, 446]
96	[179, 180]	441	[641, 642]	786	[938, 882]	1131	[199, 880]
97	[181, 182]	442	[602, 643]	787	[452, 409]	1132	[453, 436]
98	[36, 183]	443	[16, 644]	788	[939, 189]	1133	[859, 38]
99	[13, 131]	444	[645, 646]	789	[148, 921]	1134	[1180, 503]
100	[184, 32]	445	[647, 648]	790	[940, 139]	1135	[481, 387]
101	[185, 186]	446	[649, 225]	791	[941, 829]	1136	[518, 1229]
102	[187, 44]	447	[650, 234]	792	[942, 525]	1137	[145, 433]
103	[132, 188]	448	[651, 652]	793	[898, 193]	1138	[848, 507]
104	[189, 190]	449	[339, 653]	794	[413, 163]	1139	[857, 180]
105	[191, 130]	450	[18, 654]	795	[943, 191]	1140	[51, 453]
106	[192, 49]	451	[655, 656]	796	[410, 14]	1141	[832, 888]
107	[193, 194]	452	[657, 658]	797	[944, 822]	1142	[434, 432]
108	[195, 196]	453	[659, 660]	798	[187, 182]	1143	[930, 948]
109	[197, 198]	454	[661, 662]	799	[469, 934]	1144	[943, 193]
110	[199, 165]	455	[663, 664]	800	[134, 389]	1145	[1194, 129]
111	[200, 201]	456	[665, 666]	801	[941, 945]	1146	[169, 461]
112	[202, 190]	457	[667, 668]	802	[189, 60]	1147	[912, 401]
113	[12, 178]	458	[669, 670]	803	[893, 61]	1148	[1216, 61]
114	[203, 204]	459	[671, 672]	804	[946, 880]	1149	[867, 148]
115	[191, 194]	460	[673, 674]	805	[175, 204]	1150	[1220, 1190]
116	[205, 206]	461	[675, 372]	806	[947, 948]	1151	[513, 174]
117	[58, 207]	462	[676, 677]	807	[8, 430]	1152	[56, 200]
118	[208, 209]	463	[678, 679]	808	[883, 439]	1153	[173, 56]
119	[135, 200]	464	[349, 680]	809	[145, 522]	1154	[1189, 865]
120	[210, 211]	465	[681, 682]	810	[462, 177]	1155	[898, 393]
121	[212, 43]	466	[30, 683]	811	[193, 130]	1156	[1230, 205]
122	[213, 214]	467	[684, 685]	812	[399, 136]	1157	[911, 1231]
123	[215, 216]	468	[630, 686]	813	[878, 891]	1158	[520, 435]

124	[217, 218]	469	[687, 688]	814	[447, 128]	1159	[909, 873]
125	[219, 220]	470	[268, 3]	815	[32, 442]	1160	[460, 824]
126	[221, 222]	471	[689, 102]	816	[505, 147]	1161	[448, 430]
127	[223, 224]	472	[690, 691]	817	[949, 393]	1162	[949, 541]
128	[225, 226]	473	[692, 693]	818	[950, 82]	1163	[1232, 276]
129	[121, 227]	474	[694, 695]	819	[951, 952]	1164	[1233, 1234]
130	[228, 229]	475	[696, 697]	820	[953, 954]	1165	[319, 1235]
131	[230, 231]	476	[698, 699]	821	[955, 956]	1166	[1236, 1237]
132	[232, 233]	477	[700, 327]	822	[957, 958]	1167	[1238, 1239]
133	[234, 235]	478	[701, 702]	823	[959, 960]	1168	[1240, 684]
134	[236, 237]	479	[703, 704]	824	[961, 962]	1169	[1241, 1242]
135	[238, 239]	480	[705, 344]	825	[963, 617]	1170	[1243, 17]
136	[240, 241]	481	[706, 707]	826	[685, 964]	1171	[1244, 313]
137	[242, 243]	482	[708, 709]	827	[965, 966]	1172	[1245, 1246]
138	[244, 245]	483	[710, 711]	828	[967, 968]	1173	[1247, 1248]
139	[246, 247]	484	[712, 713]	829	[969, 311]	1174	[1249, 1250]
140	[248, 249]	485	[714, 715]	830	[702, 970]	1175	[1251, 1252]
141	[250, 251]	486	[24, 716]	831	[971, 972]	1176	[1253, 1254]
142	[252, 253]	487	[717, 665]	832	[973, 974]	1177	[1255, 971]
143	[254, 255]	488	[718, 719]	833	[975, 976]	1178	[1256, 5]
144	[256, 257]	489	[720, 721]	834	[977, 978]	1179	[1257, 1258]
145	[258, 259]	490	[722, 723]	835	[979, 980]	1180	[1259, 1260]
146	[260, 261]	491	[724, 725]	836	[981, 570]	1181	[1261, 1262]
147	[63, 262]	492	[726, 727]	837	[982, 983]	1182	[1263, 1264]
148	[263, 264]	493	[728, 304]	838	[984, 985]	1183	[1265, 240]
149	[71, 265]	494	[729, 730]	839	[986, 987]	1184	[1242, 242]
150	[266, 267]	495	[731, 732]	840	[988, 989]	1185	[1266, 1267]
151	[218, 268]	496	[733, 734]	841	[990, 991]	1186	[1268, 1269]
152	[247, 269]	497	[735, 736]	842	[992, 993]	1187	[1270, 1271]
153	[93, 270]	498	[737, 228]	843	[994, 995]	1188	[1272, 1273]
154	[95, 271]	499	[305, 738]	844	[996, 997]	1189	[1274, 1275]
155	[272, 273]	500	[739, 740]	845	[998, 106]	1190	[358, 558]
156	[274, 275]	501	[741, 742]	846	[999, 221]	1191	[1276, 1020]
157	[276, 277]	502	[743, 744]	847	[997, 1000]	1192	[1277, 1278]
158	[278, 279]	503	[745, 746]	848	[1001, 1002]	1193	[1279, 1280]
159	[280, 78]	504	[747, 748]	849	[1003, 1004]	1194	[1281, 1282]
160	[281, 282]	505	[749, 750]	850	[1005, 783]	1195	[1283, 1284]
161	[283, 260]	506	[751, 752]	851	[781, 1006]	1196	[1285, 120]
162	[284, 285]	507	[594, 753]	852	[1007, 283]	1197	[614, 223]
163	[286, 287]	508	[754, 755]	853	[1008, 1009]	1198	[1286, 591]
164	[288, 289]	509	[756, 757]	854	[1010, 953]	1199	[1287, 1288]
165	[290, 291]	510	[758, 759]	855	[1011, 1012]	1200	[1289, 1290]
166	[292, 293]	511	[760, 761]	856	[1013, 1014]	1201	[1291, 1292]
167	[294, 295]	512	[762, 763]	857	[1015, 1016]	1202	[972, 104]

168	[296, 297]	513	[764, 266]	858	[1017, 1018]	1203	[1293, 1294]
169	[298, 299]	514	[765, 766]	859	[233, 1019]	1204	[1019, 1295]
170	[300, 301]	515	[767, 768]	860	[1020, 1021]	1205	[1072, 1296]
171	[302, 303]	516	[769, 770]	861	[259, 629]	1206	[1297, 1298]
172	[304, 305]	517	[771, 772]	862	[1022, 1023]	1207	[579, 1299]
173	[306, 114]	518	[773, 774]	863	[1024, 1025]	1208	[1300, 1301]
174	[307, 308]	519	[775, 776]	864	[1026, 1027]	1209	[1302, 1303]
175	[309, 310]	520	[520, 520]	865	[1028, 1029]	1210	[1304, 1305]
176	[311, 312]	521	[777, 778]	866	[1030, 1031]	1211	[1306, 1307]
177	[313, 286]	522	[632, 779]	867	[1032, 323]	1212	[371, 76]
178	[314, 315]	523	[780, 781]	868	[1033, 1034]	1213	[1308, 1309]
179	[316, 317]	524	[711, 782]	869	[588, 1035]	1214	[1118, 1310]
180	[318, 319]	525	[783, 784]	870	[1036, 563]	1215	[20, 1281]
181	[320, 90]	526	[785, 384]	871	[1037, 578]	1216	[1311, 1312]
182	[321, 322]	527	[786, 787]	872	[1038, 1039]	1217	[109, 1313]
183	[323, 324]	528	[788, 789]	873	[1040, 1041]	1218	[1314, 651]
184	[325, 326]	529	[790, 791]	874	[1042, 1043]	1219	[97, 1040]
185	[327, 328]	530	[569, 622]	875	[1044, 1045]	1220	[1315, 1316]
186	[329, 64]	531	[792, 793]	876	[1046, 1047]	1221	[1317, 1318]
187	[330, 331]	532	[262, 794]	877	[1048, 1049]	1222	[1319, 1320]
188	[332, 333]	533	[795, 796]	878	[1050, 1051]	1223	[378, 1321]
189	[334, 335]	534	[797, 798]	879	[1052, 1053]	1224	[226, 1322]
190	[336, 337]	535	[799, 800]	880	[1054, 1055]	1225	[1323, 1324]
191	[338, 339]	536	[801, 802]	881	[1056, 1057]	1226	[1325, 86]
192	[340, 341]	537	[803, 366]	882	[653, 550]	1227	[1326, 1327]
193	[342, 343]	538	[804, 805]	883	[1058, 1059]	1228	[1328, 336]
194	[344, 345]	539	[806, 807]	884	[1060, 1061]	1229	[1329, 1330]
195	[346, 347]	540	[808, 809]	885	[1062, 1063]	1230	[1331, 1332]
196	[348, 349]	541	[810, 811]	886	[1064, 1065]	1231	[1333, 1334]
197	[350, 351]	542	[812, 813]	887	[1066, 1067]	1232	[7, 448]
198	[352, 353]	543	[275, 814]	888	[1068, 1069]	1233	[405, 12]
199	[354, 355]	544	[815, 816]	889	[239, 1070]	1234	[488, 464]
200	[356, 357]	545	[817, 314]	890	[1071, 1072]	1235	[136, 869]
201	[113, 358]	546	[818, 40]	891	[983, 1073]	1236	[1335, 896]
202	[359, 122]	547	[181, 44]	892	[1074, 1075]	1237	[861, 883]
203	[360, 361]	548	[819, 174]	893	[642, 332]	1238	[1223, 133]
204	[362, 363]	549	[397, 820]	894	[1076, 1077]	1239	[140, 858]
205	[364, 365]	550	[472, 467]	895	[1078, 1079]	1240	[1336, 891]
206	[366, 367]	551	[541, 13]	896	[1080, 1081]	1241	[842, 906]
207	[368, 369]	552	[821, 822]	897	[970, 1082]	1242	[446, 411]
208	[370, 371]	553	[487, 415]	898	[1083, 1084]	1243	[1212, 37]
209	[372, 373]	554	[479, 165]	899	[1085, 559]	1244	[1336, 453]
210	[374, 290]	555	[130, 449]	900	[1086, 1087]	1245	[1337, 454]
211	[375, 376]	556	[823, 824]	901	[1088, 1089]	1246	[857, 141]

212	[377, 378]	557	[825, 455]	902	[1090, 1091]	1247	[1217, 443]
213	[83, 379]	558	[423, 51]	903	[243, 274]	1248	[478, 164]
214	[380, 381]	559	[825, 415]	904	[1092, 1093]	1249	[437, 1203]
215	[382, 383]	560	[152, 38]	905	[1094, 1095]	1250	[861, 468]
216	[384, 385]	561	[200, 127]	906	[1096, 1097]	1251	[1338, 163]
217	[386, 387]	562	[453, 826]	907	[1098, 1050]	1252	[877, 170]
218	[52, 136]	563	[827, 409]	908	[1099, 1100]	1253	[388, 135]
219	[388, 389]	564	[828, 829]	909	[596, 1001]	1254	[392, 1165]
220	[390, 391]	565	[414, 455]	910	[798, 4]	1255	[139, 860]
221	[392, 55]	566	[164, 830]	911	[1101, 1102]	1256	[1339, 13]
222	[58, 136]	567	[525, 154]	912	[1103, 574]	1257	[927, 430]
223	[393, 194]	568	[139, 172]	913	[1104, 1105]	1258	[35, 198]
224	[394, 42]	569	[389, 126]	914	[1106, 1107]	1259	[902, 1213]
225	[395, 159]	570	[421, 52]	915	[1108, 1109]	1260	[827, 58]
226	[213, 396]	571	[525, 831]	916	[1110, 1111]	1261	[838, 492]
227	[211, 165]	572	[532, 216]	917	[1112, 25]	1262	[37, 499]
228	[397, 398]	573	[532, 161]	918	[1113, 118]	1263	[843, 1229]
229	[399, 59]	574	[443, 450]	919	[1114, 1115]	1264	[144, 432]
230	[135, 126]	575	[492, 456]	920	[1116, 547]	1265	[1183, 409]
231	[51, 42]	576	[832, 458]	921	[1117, 1118]	1266	[1340, 450]
232	[185, 161]	577	[433, 431]	922	[1119, 1120]	1267	[900, 141]
233	[395, 400]	578	[833, 141]	923	[1121, 1122]	1268	[1181, 891]
234	[401, 402]	579	[511, 411]	924	[1123, 1124]	1269	[529, 199]
235	[403, 404]	580	[834, 156]	925	[1125, 1126]	1270	[1200, 1222]
236	[405, 47]	581	[192, 427]	926	[1127, 356]	1271	[522, 861]
237	[406, 407]	582	[835, 196]	927	[1128, 1129]	1272	[1341, 188]
238	[408, 206]	583	[210, 479]	928	[1130, 1131]	1273	[496, 483]
239	[56, 150]	584	[198, 149]	929	[310, 1132]	1274	[1341, 133]
240	[390, 176]	585	[153, 543]	930	[1133, 1134]	1275	[941, 503]
241	[409, 136]	586	[836, 158]	931	[814, 1135]	1276	[1342, 544]
242	[193, 410]	587	[837, 467]	932	[1136, 1137]	1277	[1343, 948]
243	[411, 128]	588	[156, 486]	933	[1138, 1139]	1278	[411, 544]
244	[412, 413]	589	[406, 524]	934	[1140, 342]	1279	[1185, 934]
245	[414, 415]	590	[541, 410]	935	[1141, 1142]	1280	[545, 12]
246	[416, 146]	591	[838, 455]	936	[1143, 1144]	1281	[532, 186]
247	[417, 418]	592	[128, 839]	937	[666, 812]	1282	[158, 418]
248	[419, 420]	593	[478, 199]	938	[1145, 1146]	1283	[1344, 129]
249	[421, 399]	594	[132, 129]	939	[1147, 650]	1284	[842, 196]
250	[422, 423]	595	[160, 216]	940	[1148, 1149]	1285	[1226, 212]
251	[44, 424]	596	[840, 396]	941	[1150, 1151]	1286	[1342, 128]
252	[425, 426]	597	[841, 171]	942	[802, 1152]	1287	[910, 128]
253	[54, 427]	598	[842, 822]	943	[1153, 27]	1288	[181, 875]
254	[428, 429]	599	[843, 519]	944	[1154, 1155]	1289	[1345, 404]
255	[206, 430]	600	[540, 431]	945	[1156, 362]	1290	[192, 1165]

256	[431, 432]	601	[844, 130]	946	[1157, 1158]	1291	[166, 1192]
257	[432, 433]	602	[845, 409]	947	[1159, 1160]	1292	[432, 522]
258	[433, 434]	603	[61, 133]	948	[1161, 587]	1293	[1346, 903]
259	[168, 435]	604	[846, 174]	949	[1162, 589]	1294	[51, 212]
260	[207, 137]	605	[194, 847]	950	[1163, 53]	1295	[40, 159]
261	[212, 436]	606	[848, 163]	951	[1164, 52]	1296	[146, 506]
262	[57, 151]	607	[849, 822]	952	[54, 1165]	1297	[529, 164]
263	[437, 438]	608	[206, 413]	953	[1166, 536]	1298	[401, 10]
264	[439, 434]	609	[850, 508]	954	[194, 449]	1299	[47, 525]
265	[139, 440]	610	[851, 401]	955	[1167, 903]	1300	[1215, 436]
266	[441, 398]	611	[414, 492]	956	[11, 156]	1301	[900, 180]
267	[389, 200]	612	[852, 160]	957	[1168, 423]	1302	[481, 860]
268	[387, 9]	613	[853, 854]	958	[211, 1169]	1303	[156, 153]
269	[146, 442]	614	[423, 394]	959	[919, 466]	1304	[895, 1347]
270	[174, 57]	615	[855, 856]	960	[155, 12]	1305	[435, 520]
271	[177, 443]	616	[433, 144]	961	[1170, 205]	1306	[1337, 826]
272	[444, 58]	617	[857, 858]	962	[164, 1169]	1307	[849, 906]
273	[208, 445]	618	[408, 177]	963	[1171, 408]	1308	[177, 430]
274	[446, 447]	619	[178, 543]	964	[473, 183]	1309	[462, 8]
275	[174, 150]	620	[859, 10]	965	[1164, 399]	1310	[389, 57]
276	[409, 59]	621	[169, 483]	966	[175, 391]	1311	[510, 199]
277	[448, 443]	622	[172, 152]	967	[1172, 915]	1312	[42, 826]
278	[410, 449]	623	[386, 860]	968	[125, 156]	1313	[152, 499]
279	[162, 450]	624	[476, 147]	969	[1173, 472]	1314	[1219, 162]
280	[451, 191]	625	[185, 216]	970	[50, 394]	1315	[1204, 43]
281	[194, 131]	626	[540, 434]	971	[541, 194]	1316	[460, 170]
282	[42, 436]	627	[522, 144]	972	[860, 152]	1317	[494, 856]
283	[452, 52]	628	[861, 432]	973	[1174, 404]	1318	[459, 435]
284	[40, 418]	629	[434, 468]	974	[523, 209]	1319	[1348, 466]
285	[158, 400]	630	[435, 435]	975	[1175, 931]	1320	[440, 401]
286	[453, 454]	631	[145, 540]	976	[500, 135]	1321	[840, 34]
287	[455, 456]	632	[459, 520]	977	[1176, 191]	1322	[417, 159]
288	[457, 458]	633	[439, 144]	978	[534, 464]	1323	[32, 477]
289	[459, 459]	634	[415, 426]	979	[1175, 948]	1324	[39, 214]
290	[460, 461]	635	[862, 426]	980	[827, 52]	1325	[210, 164]
291	[462, 206]	636	[863, 524]	981	[1177, 525]	1326	[462, 448]
292	[463, 456]	637	[864, 865]	982	[1178, 39]	1327	[174, 200]
293	[441, 464]	638	[60, 132]	983	[840, 214]	1328	[412, 443]
294	[465, 466]	639	[439, 431]	984	[1179, 511]	1329	[1349, 429]
295	[467, 453]	640	[413, 507]	985	[1180, 829]	1330	[447, 544]
296	[432, 439]	641	[866, 528]	986	[544, 133]	1331	[929, 42]
297	[144, 468]	642	[867, 198]	987	[1166, 426]	1332	[838, 1166]
298	[469, 470]	643	[130, 847]	988	[486, 543]	1333	[1350, 934]
299	[452, 399]	644	[32, 147]	989	[891, 826]	1334	[387, 401]

300	[471, 472]	645	[868, 541]	990	[1181, 453]	1335	[1351, 1352]
301	[473, 474]	646	[48, 427]	991	[187, 875]	1336	[763, 661]
302	[475, 213]	647	[211, 830]	992	[938, 885]	1337	[75, 1353]
303	[476, 477]	648	[207, 869]	993	[134, 174]	1338	[285, 1354]
304	[478, 479]	649	[870, 213]	994	[1182, 1169]	1339	[324, 29]
305	[9, 402]	650	[871, 403]	995	[208, 524]	1340	[1002, 1355]
306	[480, 191]	651	[872, 403]	996	[408, 8]	1341	[1296, 1323]
307	[481, 172]	652	[162, 873]	997	[191, 410]	1342	[1356, 986]
308	[399, 53]	653	[440, 9]	998	[480, 193]	1343	[1357, 1358]
309	[482, 483]	654	[36, 149]	999	[1183, 58]	1344	[1073, 1071]
310	[484, 148]	655	[834, 508]	1000	[8, 162]	1345	[1359, 1360]
311	[455, 485]	656	[203, 874]	1001	[505, 477]	1346	[1361, 1362]
312	[486, 154]	657	[819, 56]	1002	[215, 533]	1347	[1363, 1364]
313	[487, 455]	658	[863, 445]	1003	[490, 917]	1348	[1365, 1366]
314	[488, 489]	659	[215, 161]	1004	[125, 47]	1349	[1367, 1368]
315	[393, 410]	660	[476, 442]	1005	[918, 52]	1350	[1369, 1370]
316	[490, 429]	661	[61, 129]	1006	[212, 826]	1351	[1371, 1169]
317	[451, 193]	662	[875, 404]	1007	[1184, 207]	1352	[392, 427]
318	[491, 7]	663	[876, 438]	1008	[1185, 470]	1353	[33, 214]
319	[492, 485]	664	[435, 168]	1009	[498, 191]	1354	[395, 418]
320	[493, 171]	665	[877, 461]	1010	[1186, 189]	1355	[213, 34]
321	[494, 495]	666	[837, 878]	1011	[1187, 474]	1356	[825, 1166]
322	[468, 439]	667	[879, 880]	1012	[48, 1165]	1357	[1224, 499]
323	[496, 170]	668	[397, 464]	1013	[1188, 882]	1358	[941, 928]
324	[202, 60]	669	[881, 882]	1014	[177, 413]	1359	[142, 1231]
325	[497, 207]	670	[190, 132]	1015	[1189, 1190]	1360	[540, 144]
326	[498, 193]	671	[522, 431]	1016	[155, 508]	1361	[1344, 839]
327	[59, 157]	672	[883, 433]	1017	[1191, 481]	1362	[833, 528]
328	[9, 499]	673	[884, 885]	1018	[479, 880]	1363	[1372, 865]
329	[500, 174]	674	[545, 508]	1019	[476, 506]	1364	[448, 413]
330	[501, 408]	675	[886, 386]	1020	[197, 473]	1365	[1340, 873]
331	[502, 503]	676	[887, 160]	1021	[544, 839]	1366	[853, 906]
332	[413, 450]	677	[146, 477]	1022	[521, 1192]	1367	[1338, 507]
333	[198, 183]	678	[457, 888]	1023	[431, 468]	1368	[502, 945]
334	[504, 146]	679	[468, 433]	1024	[1193, 180]	1369	[1225, 10]
335	[505, 506]	680	[150, 889]	1025	[838, 415]	1370	[842, 854]
336	[484, 36]	681	[890, 499]	1026	[1194, 839]	1371	[1373, 1374]
337	[443, 507]	682	[140, 528]	1027	[1180, 928]	1372	[1375, 1376]
338	[386, 440]	683	[205, 448]	1028	[8, 413]	1373	[1335, 1347]
339	[508, 486]	684	[867, 473]	1029	[171, 860]	1374	[439, 861]
340	[509, 503]	685	[891, 454]	1030	[1195, 915]	1375	[1205, 826]
341	[510, 211]	686	[468, 522]	1031	[846, 56]	1376	[877, 824]
342	[511, 447]	687	[890, 402]	1032	[1196, 202]		
343	[409, 207]	688	[892, 196]	1033	[444, 409]		

344	[512, 204]	689	[893, 132]	1034	[441, 820]	

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	230	0	460	1	690	1	920	0	1150	0
1	0	231	1	461	1	691	0	921	0	1151	0
2	0	232	1	462	1	692	0	922	1	1152	0
3	0	233	1	463	1	693	1	923	1	1153	0
4	1	234	0	464	0	694	1	924	1	1154	1
5	0	235	0	465	1	695	1	925	1	1155	1
6	1	236	1	466	0	696	1	926	0	1156	1
7	0	237	0	467	0	697	0	927	1	1157	0
8	0	238	0	468	1	698	0	928	0	1158	0
9	1	239	0	469	1	699	0	929	1	1159	1
10	0	240	0	470	0	700	1	930	1	1160	1
11	0	241	0	471	0	701	0	931	0	1161	1
12	1	242	1	472	1	702	0	932	1	1162	0
13	1	243	1	473	0	703	1	933	1	1163	0
14	1	244	0	474	1	704	0	934	1	1164	1
15	0	245	1	475	1	705	0	935	0	1165	0
16	0	246	0	476	0	706	1	936	1	1166	1
17	0	247	0	477	1	707	1	937	0	1167	0
18	1	248	1	478	0	708	1	938	0	1168	0
19	1	249	0	479	1	709	1	939	0	1169	0
20	1	250	1	480	0	710	1	940	1	1170	0
21	0	251	1	481	1	711	1	941	0	1171	0
22	1	252	1	482	1	712	1	942	1	1172	0
23	0	253	1	483	1	713	0	943	0	1173	1
24	1	254	1	484	1	714	1	944	1	1174	0
25	1	255	1	485	1	715	1	945	1	1175	0
26	1	256	1	486	1	716	0	946	0	1176	1
27	1	257	0	487	0	717	0	947	1	1177	0
28	0	258	0	488	0	718	0	948	1	1178	0
29	1	259	0	489	0	719	0	949	0	1179	1
30	0	260	1	490	1	720	0	950	0	1180	1
31	0	261	1	491	1	721	0	951	1	1181	1
32	0	262	0	492	0	722	1	952	1	1182	0
33	0	263	0	493	0	723	1	953	1	1183	0
34	1	264	0	494	0	724	1	954	1	1184	0
35	0	265	0	495	0	725	0	955	0	1185	0
36	1	266	1	496	0	726	0	956	0	1186	1
37	1	267	1	497	1	727	0	957	0	1187	1
38	1	268	0	498	0	728	0	958	0	1188	1
39	1	269	1	499	1	729	0	959	0	1189	1

40	1	270	1	500	1	730	0	960	1	1190	1
41	1	271	0	501	1	731	0	961	0	1191	0
42	0	272	1	502	0	732	1	962	1	1192	0
43	1	273	1	503	0	733	0	963	0	1193	0
44	1	274	0	504	1	734	0	964	0	1194	0
45	1	275	1	505	1	735	1	965	1	1195	1
46	0	276	0	506	0	736	0	966	1	1196	0
47	0	277	1	507	1	737	0	967	0	1197	0
48	1	278	0	508	1	738	1	968	1	1198	0
49	1	279	1	509	0	739	1	969	1	1199	1
50	1	280	1	510	1	740	1	970	1	1200	1
51	1	281	1	511	1	741	1	971	1	1201	1
52	1	282	0	512	1	742	1	972	1	1202	1
53	0	283	1	513	0	743	0	973	0	1203	1
54	1	284	1	514	1	744	1	974	1	1204	0
55	0	285	0	515	1	745	0	975	0	1205	1
56	0	286	0	516	1	746	1	976	1	1206	1
57	0	287	1	517	1	747	1	977	1	1207	1
58	1	288	1	518	1	748	1	978	1	1208	1
59	1	289	1	519	1	749	1	979	0	1209	1
60	0	290	1	520	0	750	0	980	1	1210	0
61	0	291	1	521	0	751	0	981	0	1211	0
62	0	292	1	522	1	752	1	982	0	1212	0
63	1	293	1	523	1	753	0	983	1	1213	0
64	0	294	1	524	1	754	1	984	1	1214	1
65	1	295	0	525	0	755	0	985	1	1215	1
66	0	296	0	526	0	756	0	986	0	1216	1
67	1	297	1	527	0	757	0	987	1	1217	1
68	1	298	1	528	0	758	1	988	1	1218	0
69	0	299	1	529	1	759	1	989	1	1219	0
70	0	300	0	530	1	760	1	990	1	1220	0
71	1	301	0	531	1	761	0	991	1	1221	0
72	1	302	1	532	0	762	1	992	0	1222	0
73	1	303	0	533	0	763	0	993	1	1223	0
74	1	304	0	534	1	764	0	994	0	1224	0
75	0	305	1	535	1	765	1	995	1	1225	0
76	1	306	0	536	0	766	0	996	0	1226	0
77	0	307	1	537	0	767	1	997	0	1227	1
78	1	308	0	538	0	768	0	998	0	1228	0
79	0	309	1	539	1	769	1	999	0	1229	0
80	1	310	1	540	1	770	1	1000	0	1230	1
81	1	311	1	541	1	771	1	1001	1	1231	0
82	1	312	1	542	0	772	0	1002	0	1232	0
83	0	313	0	543	1	773	1	1003	1	1233	1

84	0	314	0	544	0	774	0	1004	1	1234	0
85	1	315	0	545	0	775	1	1005	0	1235	0
86	1	316	1	546	0	776	1	1006	1	1236	1
87	1	317	1	547	0	777	0	1007	0	1237	0
88	1	318	1	548	1	778	0	1008	0	1238	0
89	0	319	0	549	0	779	1	1009	0	1239	1
90	1	320	0	550	1	780	1	1010	1	1240	0
91	1	321	0	551	1	781	1	1011	1	1241	0
92	0	322	1	552	0	782	0	1012	1	1242	0
93	1	323	0	553	0	783	0	1013	1	1243	0
94	0	324	0	554	1	784	1	1014	0	1244	0
95	0	325	1	555	0	785	0	1015	1	1245	0
96	1	326	0	556	0	786	0	1016	1	1246	1
97	0	327	1	557	0	787	1	1017	0	1247	1
98	1	328	1	558	0	788	0	1018	1	1248	0
99	1	329	1	559	0	789	0	1019	0	1249	0
100	1	330	1	560	0	790	1	1020	1	1250	0
101	1	331	0	561	1	791	0	1021	0	1251	0
102	1	332	1	562	0	792	1	1022	0	1252	0
103	1	333	1	563	1	793	1	1023	1	1253	1
104	1	334	1	564	0	794	1	1024	0	1254	0
105	0	335	1	565	1	795	0	1025	1	1255	0
106	0	336	1	566	1	796	0	1026	0	1256	0
107	1	337	0	567	0	797	1	1027	1	1257	1
108	1	338	0	568	0	798	1	1028	0	1258	0
109	1	339	1	569	1	799	1	1029	1	1259	1
110	0	340	0	570	0	800	1	1030	1	1260	1
111	1	341	1	571	0	801	0	1031	0	1261	1
112	0	342	1	572	0	802	1	1032	0	1262	1
113	1	343	0	573	0	803	0	1033	1	1263	0
114	0	344	1	574	0	804	0	1034	1	1264	1
115	0	345	1	575	0	805	1	1035	0	1265	0
116	1	346	1	576	0	806	1	1036	0	1266	0
117	1	347	0	577	0	807	0	1037	0	1267	0
118	1	348	1	578	0	808	1	1038	1	1268	1
119	0	349	0	579	1	809	0	1039	1	1269	1
120	0	350	1	580	1	810	1	1040	0	1270	1
121	1	351	1	581	0	811	1	1041	0	1271	1
122	0	352	1	582	0	812	0	1042	0	1272	1
123	0	353	0	583	0	813	1	1043	0	1273	0
124	0	354	0	584	1	814	0	1044	1	1274	1
125	1	355	1	585	1	815	0	1045	1	1275	0
126	0	356	1	586	0	816	1	1046	1	1276	0
127	0	357	0	587	0	817	0	1047	1	1277	0

128	1	358	1	588	0	818	0	1048	0	1278	1
129	1	359	0	589	0	819	1	1049	1	1279	0
130	0	360	0	590	1	820	1	1050	1	1280	0
131	0	361	1	591	1	821	0	1051	0	1281	0
132	1	362	0	592	1	822	0	1052	1	1282	0
133	0	363	1	593	0	823	0	1053	1	1283	1
134	1	364	1	594	1	824	0	1054	1	1284	0
135	0	365	0	595	1	825	0	1055	0	1285	0
136	0	366	1	596	1	826	1	1056	0	1286	0
137	1	367	0	597	1	827	1	1057	1	1287	1
138	0	368	1	598	0	828	0	1058	1	1288	0
139	0	369	1	599	0	829	1	1059	0	1289	1
140	1	370	1	600	1	830	0	1060	1	1290	0
141	1	371	0	601	0	831	1	1061	0	1291	1
142	1	372	0	602	0	832	0	1062	0	1292	0
143	1	373	0	603	0	833	0	1063	0	1293	1
144	1	374	0	604	0	834	1	1064	1	1294	1
145	0	375	0	605	1	835	0	1065	1	1295	1
146	1	376	0	606	1	836	0	1066	1	1296	1
147	1	377	1	607	1	837	0	1067	0	1297	1
148	0	378	0	608	1	838	1	1068	1	1298	0
149	1	379	0	609	0	839	0	1069	0	1299	0
150	1	380	0	610	1	840	1	1070	1	1300	1
151	1	381	0	611	1	841	1	1071	1	1301	0
152	0	382	0	612	0	842	0	1072	1	1302	1
153	1	383	0	613	0	843	0	1073	1	1303	0
154	0	384	0	614	0	844	0	1074	1	1304	0
155	1	385	1	615	1	845	0	1075	0	1305	1
156	0	386	0	616	0	846	0	1076	1	1306	0
157	0	387	0	617	1	847	0	1077	1	1307	1
158	0	388	1	618	0	848	1	1078	0	1308	0
159	1	389	1	619	0	849	1	1079	1	1309	1
160	1	390	0	620	1	850	0	1080	0	1310	1
161	1	391	1	621	1	851	1	1081	0	1311	1
162	1	392	0	622	0	852	0	1082	1	1312	0
163	0	393	0	623	0	853	0	1083	1	1313	0
164	1	394	0	624	0	854	1	1084	1	1314	0
165	1	395	1	625	1	855	1	1085	0	1315	0
166	1	396	1	626	1	856	1	1086	0	1316	1
167	1	397	0	627	1	857	1	1087	0	1317	0
168	0	398	1	628	0	858	0	1088	1	1318	1
169	1	399	0	629	0	859	1	1089	1	1319	0
170	0	400	0	630	1	860	1	1090	1	1320	1
171	1	401	0	631	0	861	0	1091	0	1321	1

172	0	402	0	632	1	862	0	1092	1	1322	0
173	0	403	0	633	0	863	0	1093	0	1323	0
174	1	404	0	634	0	864	0	1094	1	1324	1
175	1	405	1	635	0	865	0	1095	0	1325	0
176	1	406	0	636	0	866	1	1096	0	1326	1
177	0	407	1	637	0	867	0	1097	1	1327	1
178	0	408	0	638	0	868	1	1098	0	1328	0
179	1	409	0	639	0	869	0	1099	0	1329	0
180	1	410	0	640	1	870	0	1100	0	1330	0
181	0	411	1	641	1	871	0	1101	0	1331	1
182	0	412	0	642	0	872	1	1102	0	1332	1
183	0	413	1	643	0	873	0	1103	0	1333	0
184	1	414	1	644	0	874	0	1104	0	1334	0
185	1	415	0	645	1	875	1	1105	0	1335	1
186	1	416	0	646	1	876	1	1106	0	1336	0
187	1	417	0	647	0	877	0	1107	1	1337	0
188	1	418	0	648	1	878	1	1108	0	1338	0
189	1	419	1	649	0	879	1	1109	0	1339	0
190	1	420	1	650	0	880	1	1110	0	1340	0
191	0	421	0	651	1	881	0	1111	0	1341	1
192	0	422	1	652	1	882	1	1112	1	1342	0
193	1	423	0	653	1	883	1	1113	0	1343	0
194	1	424	0	654	1	884	1	1114	0	1344	1
195	1	425	1	655	1	885	0	1115	0	1345	1
196	1	426	1	656	0	886	1	1116	0	1346	1
197	1	427	1	657	1	887	1	1117	0	1347	1
198	1	428	1	658	0	888	1	1118	1	1348	0
199	0	429	0	659	0	889	0	1119	1	1349	0
200	1	430	0	660	0	890	1	1120	1	1350	0
201	1	431	1	661	0	891	1	1121	1	1351	1
202	0	432	0	662	1	892	1	1122	1	1352	0
203	0	433	0	663	1	893	0	1123	1	1353	0
204	0	434	0	664	1	894	1	1124	0	1354	1
205	1	435	1	665	0	895	0	1125	1	1355	0
206	1	436	0	666	0	896	0	1126	0	1356	0
207	1	437	0	667	1	897	1	1127	0	1357	0
208	1	438	0	668	0	898	1	1128	1	1358	0
209	0	439	0	669	0	899	0	1129	1	1359	1
210	0	440	1	670	1	900	0	1130	0	1360	1
211	0	441	1	671	1	901	1	1131	0	1361	1
212	1	442	0	672	1	902	1	1132	0	1362	0
213	0	443	0	673	1	903	1	1133	1	1363	1
214	0	444	1	674	0	904	1	1134	1	1364	1
215	0	445	0	675	1	905	1	1135	1	1365	0

216	0	446	0	676	1	906	0	1136	1	1366	0
217	0	447	0	677	1	907	0	1137	0	1367	0
218	1	448	1	678	1	908	0	1138	1	1368	0
219	1	449	1	679	1	909	1	1139	1	1369	0
220	0	450	1	680	1	910	1	1140	1	1370	0
221	0	451	1	681	1	911	0	1141	0	1371	1
222	1	452	1	682	1	912	0	1142	0	1372	1
223	0	453	0	683	1	913	0	1143	1	1373	1
224	0	454	0	684	0	914	0	1144	0	1374	0
225	1	455	1	685	1	915	0	1145	0	1375	1
226	0	456	0	686	1	916	0	1146	1	1376	0
227	0	457	1	687	1	917	1	1147	0		
228	0	458	0	688	1	918	0	1148	1		
229	0	459	1	689	0	919	0	1149	0		

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