PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^4, z^3 + z^2 + z + 1)$

GUO-NIU HAN* AND YINING HU*

ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^4, z^3 + z^2 + z + 1)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

Date: 9/04/2020 17:26:06 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

Key words and phrases. automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 10.0 minutes CPU time used.

Theorem 1.1. *When $(a,b)=(z^4, z^3+z^2+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{21}+z^{20}+z^{15}+z^{13}+z^{12}+z^{10}+z^5+1,\\
p_1(z)&=z^{25}+z^{20}+z^{19}+z^{17}+z^{14}+z^{12}+z^9+z^4,\\
p_2(z)&=z^{28}+z^{24}+z^{22}+z^{20}+z^{18}+z^{16}+z^{12}+z^8,\\
p_3(z)&=z^{25}+z^{20}+z^{19}+z^{17}+z^{14}+z^{12}+z^9+z^4,\\
p_4(z)&=z^{22}+z^{21}+z^{20}+z^{18}+z^{16}+z^{15}+z^{13}+z^{12}+z^{10}+z^5+z^4,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{21}+z^{20}+z^{16}+z^{15}+z^{13}+z^{10}+z^8+z^5+z^4,\\
p_1(z)&=z^{25}+z^{20}+z^{19}+z^{17}+z^{14}+z^{12}+z^9+z^4,\\
p_2(z)&=z^{28}+z^{24}+z^{22}+z^{20}+z^{18}+z^{16}+z^{12}+z^8,\\
p_3(z)&=z^{25}+z^{20}+z^{19}+z^{17}+z^{14}+z^{12}+z^9+z^4,\\
p_4(z)&=z^{22}+z^{21}+z^{20}+z^{18}+z^{15}+z^{13}+z^{10}+z^8+z^5+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z \left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{6u} M^e[:u] \\ & + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{4u} M^e[:4u] \\ & + x^{7u} M^e[:u] + x^{4u} M^e[:7u] + x^{5u} M^e[:6u] + x^{6u} M^e[:5u] \\ & + x^{7u} M^e[:4u] + x^{10u} M^e[:u] + x^{5u} M^e[:7u] + x^{6u} M^e[:6u] \\ & + x^{7u} M^e[:5u] + x^{8u} M^e[:4u] + x^{11u} M^e[:u] + x^{6u} M^e[:7u] \\ & + x^{7u} M^e[:6u] + x^{8u} M^e[:5u] + x^{9u} M^e[:4u] + x^{12u} M^e[:u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:7u] + x^{9u}M^e[:5u] + x^{13u}M^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{11u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^uW^e[:6u] + x^{2u}W^e[:5u] + x^{3u}W^e[:4u] + x^{6u}W^e[:u]\\
&+ x^uW^e[:7u] + x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{4u}W^e[:4u]\\
&+ x^{7u}W^e[:u] + x^{4u}W^e[:7u] + x^{5u}W^e[:6u] + x^{6u}W^e[:5u]\\
&+ x^{7u}W^e[:4u] + x^{10u}W^e[:u] + x^{5u}W^e[:7u] + x^{6u}W^e[:6u]\\
&+ x^{7u}W^e[:5u] + x^{8u}W^e[:4u] + x^{11u}W^e[:u] + x^{6u}W^e[:7u]\\
&+ x^{7u}W^e[:6u] + x^{8u}W^e[:5u] + x^{9u}W^e[:4u] + x^{12u}W^e[:u]\\
&+ x^{7u}W^e[:7u] + x^{9u}W^e[:5u] + x^{13u}W^e[:u]\\
&+ (x^{7u} + x^{8u} + x^{11u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{3u}M^o[:4u] + x^{6u}M^o[:u]\\
&+ x^uM^o[:7u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{4u}M^o[:4u]\\
&+ x^{7u}M^o[:u] + x^{4u}M^o[:7u] + x^{5u}M^o[:6u] + x^{6u}M^o[:5u]\\
&+ x^{7u}M^o[:4u] + x^{10u}M^o[:u] + x^{5u}M^o[:7u] + x^{6u}M^o[:6u]\\
&+ x^{7u}M^o[:5u] + x^{8u}M^o[:4u] + x^{11u}M^o[:u] + x^{6u}M^o[:7u]\\
&+ x^{7u}M^o[:6u] + x^{8u}M^o[:5u] + x^{9u}M^o[:4u] + x^{12u}M^o[:u]\\
&+ x^{7u}M^o[:7u] + x^{9u}M^o[:5u] + x^{13u}M^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{11u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{3u}W^o[:4u] + x^{6u}W^o[:u]\\
&+ x^uW^o[:7u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{4u}W^o[:4u]\\
&+ x^{7u}W^o[:u] + x^{4u}W^o[:7u] + x^{5u}W^o[:6u] + x^{6u}W^o[:5u]\\
&+ x^{7u}W^o[:4u] + x^{10u}W^o[:u] + x^{5u}W^o[:7u] + x^{6u}W^o[:6u]\\
&+ x^{7u}W^o[:5u] + x^{8u}W^o[:4u] + x^{11u}W^o[:u] + x^{6u}W^o[:7u]\\
&+ x^{7u}W^o[:6u] + x^{8u}W^o[:5u] + x^{9u}W^o[:4u] + x^{12u}W^o[:u]\\
&+ x^{7u}W^o[:7u] + x^{9u}W^o[:5u] + x^{13u}W^o[:u]\\
&+ (x^{7u} + x^{8u} + x^{11u} + x^{12u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^uT[:6u] + x^{2u}T[:5u] + x^{3u}T[:4u] + x^{6u}T[:u] + x^uT[:7u]\\
&+ x^{2u}T[:6u] + x^{3u}T[:5u] + x^{4u}T[:4u] + x^{7u}T[:u] + x^{4u}T[:7u]\\
&+ x^{5u}T[:6u] + x^{6u}T[:5u] + x^{7u}T[:4u] + x^{10u}T[:u] + x^{5u}T[:7u]\\
&+ x^{6u}T[:6u] + x^{7u}T[:5u] + x^{8u}T[:4u] + x^{11u}T[:u] + x^{6u}T[:7u]\\
&+ x^{7u}T[:6u] + x^{8u}T[:5u] + x^{9u}T[:4u] + x^{12u}T[:u] + x^{7u}T[:7u]
\end{aligned}
$$

$$+ x^{9u}T[:5u] + x^{13u}T[:u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{3u}T[4u:5u] + x^{2u}T[5u:6u] + x^{u}T[6u:7u] \\
&\quad + T[7u:8u], \\
0 &= x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + T[8u:9u], \\
0 &= x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u], \\
0 &= x^{10u}T[:u] + x^{6u}T[4u:5u] + x^{4u}T[6u:7u] + T[10u:11u], \\
0 &= x^{11u}T[:u] + x^{7u}T[4u:5u] + x^{5u}T[6u:7u] + T[11u:12u], \\
0 &= x^{12u}T[:u] + x^{8u}T[4u:5u] + x^{6u}T[6u:7u] + T[12u:13u], \\
0 &= x^{13u}T[:u] + x^{9u}T[4u:5u] + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.7) \\
0 &= T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \qquad (2.8) \\
0 &= T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \qquad (2.9) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \qquad (2.10) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1011w]_2], \qquad (2.11) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.12) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1101w]_2], \qquad (2.13)
\end{aligned}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$\begin{aligned}
0 &= T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\
&\quad + T[[111w]_2], \qquad (2.14) \\
0 &= T[[10w]_2] + T[[100w]_2] + T[[1000w]_2], \qquad (2.15) \\
0 &= T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \qquad (2.16) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \qquad (2.17) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1011w]_2], \qquad (2.18) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1100w]_2], \qquad (2.19) \\
0 &= T[[w]_2] + T[[100w]_2] + T[[110w]_2] + T[[1101w]_2], \qquad (2.20)
\end{aligned}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 305 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s, \epsilon)) + \tau(A(s, 1)) + \tau(A(s, 100)) + \tau(A(s, 101))$$

$$+ \tau(A(s,110)) + \tau(A(s,111)), \tag{2.21}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1000)), \tag{2.22}$$
$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.23}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1010)), \tag{2.24}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1011)), \tag{2.25}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,101)) \\ + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.28}$$
$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1000)), \tag{2.29}$$
$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.30}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1010)), \tag{2.31}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1011)), \tag{2.32}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{32}), E_{32}) = (A(i,0^{20}), E_{20})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 16$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{3u} M^e[:4u] + x^{6u} M^e[:u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{3u} W^e[:4u] + x^{6u} W^e[:u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{3u} M^o[:4u] + x^{6u} M^o[:u] \\ &\quad + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{3u} W^o[:4u] + x^{6u} W^o[:u] \\ &\quad + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:7v] + x^v W^e[:6v] + x^{2v} W^e[:5v] + x^{3v} W^e[:4v] + x^{6v} W^e[:v] \\ &+ x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{2v} M^e[:5v] + x^{3v} M^e[:4v] \\ &+ x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} W^e[:14v]M^e[:14v] + x^{2v} W^e[:12v]M^e[:12v] + x^{4v} W^e[:10v]M^e[:10v] \\ + x^{6v} W^e[:8v]M^e[:8v] + x^{12v} W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{28} + x^{23} + x^{18} + x^{16} + x^{15} + x^{13} + x^8 + x^7,\\
q_1(x) &= x^{24} + x^{19} + x^{16} + x^{14} + x^{11} + x^9 + x^8 + x^3,\\
q_2(x) &= x^{20} + x^{16} + x^{12} + x^{10} + x^8 + x^6 + x^4 + 1,\\
q_3(x) &= x^{24} + x^{19} + x^{16} + x^{14} + x^{11} + x^9 + x^8 + x^3,\\
q_4(x) &= x^{24} + x^{23} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12} + x^{10} + x^8 + x^7 + x^6.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) = {} & x^{144} + x^{142} + x^{140} + x^{138} + x^{132} + x^{126} + x^{124} + x^{112} + x^{102} + x^{86}\\
& + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{56} + x^{54} + x^{52} + x^{50}\\
& + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36},\\
p_3(x) = {} & x^{134} + x^{128} + x^{120} + x^{118} + x^{112} + x^{108} + x^{104} + x^{102} + x^{100} + x^{92}\\
& + x^{88} + x^{84} + x^{78} + x^{76} + x^{70} + x^{64} + x^{62} + x^{56} + x^{54} + x^{46} + x^{40}\\
& + x^{36} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20},\\
p_6(x) = {} & x^{134} + x^{126} + x^{120} + x^{118} + x^{116} + x^{108} + x^{106} + x^{104} + x^{100} + x^{94}\\
& + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{70} + x^{66} + x^{62} + x^{60}\\
& + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{42} + x^{38} + x^{26} + x^{24} + x^{22} + x^{14}\\
& + x^{12} + x^8 + x^6 + 1,\\
p_9(x) = {} & x^{136} + x^{120} + x^{118} + x^{112} + x^{108} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84}\\
& + x^{82} + x^{80} + x^{78} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{58} + x^{54} + x^{52}\\
& + x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{34} + x^{30} + x^{24} + x^{20} + x^{18} + x^{16}\\
& + x^{12} + x^6,\\
p_{12}(x) = {} & x^{136} + x^{128} + x^{72} + x^{64} + x^{56} + x^{48} + x^8 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0].
For $\phi(M^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{142} + x^{141} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} \\
&\quad + x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{120} + x^{114} + x^{112} + x^{111} \\
&\quad + x^{108} + x^{107} + x^{103} + x^{102} + x^{99} + x^{97} + x^{95} + x^{94} + x^{91} + x^{90} \\
&\quad + x^{88} + x^{87} + x^{86} + x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{70} + x^{69} + x^{65} + x^{59} + x^{54} + x^{53} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41} \\
&\quad + x^{40} + x^{39}, \\
p_3(x) &= x^{139} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} \\
&\quad + x^{122} + x^{121} + x^{115} + x^{114} + x^{111} + x^{109} + x^{107} + x^{104} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{99} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} \\
&\quad + x^{51} + x^{50} + x^{49} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{27}, \\
p_6(x) &= x^{135} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} \\
&\quad + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{81} + x^{80} + x^{79} + x^{78} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{131} + x^{130} + x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{110} + x^{108} + x^{104} + x^{103} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} \\
&\quad + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{64} + x^{60} + x^{59} + x^{51} + x^{50} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{24} + x^{23} \\
&\quad + x^{22} + x^{21} + x^{19} + x^{18} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} \\
&\quad + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
&\quad + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} \\
&\quad + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{64} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} \\
&\quad + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}
\end{aligned}$$

$$+ x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1].

For $\phi(M^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{142} + x^{141} + x^{135} + x^{133} + x^{132} + x^{131} + x^{130} + x^{128} \\
&+ x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{120} + x^{114} + x^{112} + x^{111} \\
&+ x^{108} + x^{107} + x^{103} + x^{102} + x^{99} + x^{97} + x^{95} + x^{94} + x^{91} + x^{90} \\
&+ x^{88} + x^{87} + x^{86} + x^{82} + x^{81} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} \\
&+ x^{70} + x^{69} + x^{65} + x^{59} + x^{54} + x^{53} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41} \\
&+ x^{40} + x^{39}, \\
p_3(x) &= x^{139} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} \\
&+ x^{122} + x^{121} + x^{115} + x^{114} + x^{111} + x^{109} + x^{107} + x^{104} + x^{102} \\
&+ x^{101} + x^{100} + x^{99} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} \\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} \\
&+ x^{51} + x^{50} + x^{49} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} \\
&+ x^{28} + x^{27}, \\
p_6(x) &= x^{135} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} \\
&+ x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{99} + x^{98} + x^{97} + x^{96} \\
&+ x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{81} + x^{80} + x^{79} + x^{78} + x^{67} \\
&+ x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{51} + x^{50} \\
&+ x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} \\
&+ x^{36} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{131} + x^{130} + x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{113} + x^{112} \\
&+ x^{111} + x^{110} + x^{108} + x^{104} + x^{103} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} \\
&+ x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{64} + x^{60} + x^{59} + x^{51} + x^{50} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} \\
&+ x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{24} + x^{23} \\
&+ x^{22} + x^{21} + x^{19} + x^{18} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} \\
&+ x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
&+ x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} \\
&+ x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
&+ x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} \\
&+ x^{66} + x^{65} + x^{64} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40}
\end{aligned}$$

$$+ x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}$$
$$+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}$$
$$+ x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6}$$
$$+ x^{5} + x^{4} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1].

For $\phi(M^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{150} + x^{148} + x^{146} + x^{144} + x^{142} + x^{134} + x^{132} + x^{130} + x^{128} + x^{118}\\
&\quad + x^{116} + x^{114} + x^{112} + x^{110} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90}\\
&\quad + x^{86} + x^{76} + x^{72} + x^{70} + x^{64} + x^{56} + x^{54} + x^{46} + x^{42},\\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{102} + x^{100}\\
&\quad + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74}\\
&\quad + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{52} + x^{48} + x^{46} + x^{42} + x^{38} + x^{36}\\
&\quad + x^{34} + x^{32} + x^{22} + x^{18},\\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{110} + x^{102} + x^{100} + x^{92} + x^{90} + x^{82}\\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{62} + x^{58} + x^{52} + x^{42} + x^{38}\\
&\quad + x^{36} + x^{30} + x^{28} + x^{26} + x^{22} + x^{20} + x^{16} + x^{12} + x^{10} + x^{8} + x^{4} + x^{2}\\
&\quad + 1,\\
p_9(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{94}\\
&\quad + x^{92} + x^{84} + x^{80} + x^{78} + x^{76} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{44}\\
&\quad + x^{42} + x^{38} + x^{36} + x^{30} + x^{26} + x^{24} + x^{20} + x^{16} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{102} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{80} + x^{70} + x^{68} + x^{66} + x^{64} + x^{38} + x^{36} + x^{34} + x^{32} + x^{22}\\
&\quad + x^{20} + x^{18} + x^{16} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{150} + x^{148} + x^{146} + x^{144} + x^{142} + x^{118} + x^{116} + x^{114} + x^{112} + x^{106}\\
&\quad + x^{94} + x^{86} + x^{74} + x^{64} + x^{58} + x^{54} + x^{52} + x^{48} + x^{46} + x^{42} + x^{36},\\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108}\\
&\quad + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{84} + x^{70} + x^{62} + x^{58}\\
&\quad + x^{56} + x^{54} + x^{46} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{26} + x^{24} + x^{20},\\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{108} + x^{106} + x^{104} + x^{102} + x^{98} + x^{96}\\
&\quad + x^{92} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} + x^{66} + x^{64}\\
&\quad + x^{62} + x^{60} + x^{58} + x^{54} + x^{50} + x^{46} + x^{44} + x^{36} + x^{34} + x^{32} + x^{30}\\
&\quad + x^{28} + x^{24} + x^{14} + x^{10} + x^{8} + x^{4} + x^{2} + 1,\\
p_9(x) &= x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} + x^{94}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{92} + x^{84} + x^{80} + x^{78} + x^{76} + x^{62} + x^{58} + x^{56} + x^{52} + x^{50} + x^{44} \\
&+ x^{42} + x^{38} + x^{36} + x^{30} + x^{26} + x^{24} + x^{20} + x^{16} + x^{10} + x^{8} + x^{6}, \\
p_{12}(x) = &\, x^{118} + x^{116} + x^{114} + x^{112} + x^{102} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84} \\
&+ x^{82} + x^{80} + x^{70} + x^{68} + x^{66} + x^{64} + x^{38} + x^{36} + x^{34} + x^{32} + x^{22} \\
&+ x^{20} + x^{18} + x^{16} + x^{6} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) = &\, x^{147} + x^{145} + x^{143} + x^{141} + x^{139} + x^{135} + x^{134} + x^{133} + x^{132} + x^{127} \\
&+ x^{126} + x^{125} + x^{124} + x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&+ x^{113} + x^{110} + x^{109} + x^{105} + x^{104} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} \\
&+ x^{84} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} \\
&+ x^{59} + x^{58} + x^{55} + x^{51} + x^{50} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39}, \\
p_3(x) = &\, x^{139} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} \\
&+ x^{122} + x^{121} + x^{115} + x^{114} + x^{111} + x^{109} + x^{107} + x^{104} + x^{102} \\
&+ x^{101} + x^{100} + x^{99} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} \\
&+ x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} \\
&+ x^{51} + x^{50} + x^{49} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} \\
&+ x^{28} + x^{27}, \\
p_6(x) = &\, x^{135} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} \\
&+ x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{99} + x^{98} + x^{97} + x^{96} \\
&+ x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{81} + x^{80} + x^{79} + x^{78} + x^{67} \\
&+ x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{51} + x^{50} \\
&+ x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} \\
&+ x^{36} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) = &\, x^{131} + x^{130} + x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{113} + x^{112} \\
&+ x^{111} + x^{110} + x^{108} + x^{104} + x^{103} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} \\
&+ x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} \\
&+ x^{66} + x^{64} + x^{60} + x^{59} + x^{51} + x^{50} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} \\
&+ x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{24} + x^{23} \\
&+ x^{22} + x^{21} + x^{19} + x^{18} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) = &\, x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} \\
&+ x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
&+ x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} \\
&+ x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78}
\end{aligned}
$$

$$+ x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}$$
$$+ x^{66} + x^{65} + x^{64} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40}$$
$$+ x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}$$
$$+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}$$
$$+ x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6}$$
$$+ x^{5} + x^{4} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{143} + x^{141} + x^{139} + x^{135} + x^{134} + x^{133} + x^{132} + x^{127} \\
&\quad + x^{126} + x^{125} + x^{124} + x^{123} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{113} + x^{110} + x^{109} + x^{105} + x^{104} + x^{93} + x^{92} + x^{91} + x^{86} + x^{85} \\
&\quad + x^{84} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} \\
&\quad + x^{59} + x^{58} + x^{55} + x^{51} + x^{50} + x^{46} + x^{44} + x^{43} + x^{41} + x^{40} + x^{39}, \\
p_3(x) &= x^{139} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} + x^{130} + x^{127} + x^{125} \\
&\quad + x^{122} + x^{121} + x^{115} + x^{114} + x^{111} + x^{109} + x^{107} + x^{104} + x^{102} \\
&\quad + x^{101} + x^{100} + x^{99} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} \\
&\quad + x^{51} + x^{50} + x^{49} + x^{43} + x^{42} + x^{39} + x^{37} + x^{35} + x^{32} + x^{30} + x^{29} \\
&\quad + x^{28} + x^{27}, \\
p_6(x) &= x^{135} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{125} + x^{124} \\
&\quad + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{92} + x^{89} + x^{88} + x^{81} + x^{80} + x^{79} + x^{78} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{51} + x^{50} \\
&\quad + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{39} + x^{38} + x^{37} \\
&\quad + x^{36} + x^{29} + x^{28} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{131} + x^{130} + x^{129} + x^{128} + x^{126} + x^{125} + x^{124} + x^{123} + x^{113} + x^{112} \\
&\quad + x^{111} + x^{110} + x^{108} + x^{104} + x^{103} + x^{99} + x^{98} + x^{95} + x^{92} + x^{90} \\
&\quad + x^{87} + x^{84} + x^{82} + x^{81} + x^{80} + x^{79} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} \\
&\quad + x^{66} + x^{64} + x^{60} + x^{59} + x^{51} + x^{50} + x^{46} + x^{45} + x^{41} + x^{40} + x^{39} \\
&\quad + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{24} + x^{23} \\
&\quad + x^{22} + x^{21} + x^{19} + x^{18} + x^{14} + x^{12} + x^{11} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{127} + x^{126} + x^{125} + x^{124} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} \\
&\quad + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} \\
&\quad + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78}\\
&+ x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{64} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29}\\
&+ x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18}\\
&+ x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6}\\
&+ x^{5} + x^{4} + x^{3} + x^{2} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{140} + x^{138} + x^{136} + x^{120} + x^{108} + x^{106} + x^{104} + x^{102}\\
&\quad + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{68} + x^{66} + x^{56}\\
&\quad + x^{54} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42},\\
p_3(x) &= x^{134} + x^{120} + x^{104} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84} + x^{82} + x^{80} + x^{72}\\
&\quad + x^{68} + x^{66} + x^{60} + x^{58} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42}\\
&\quad + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20} + x^{18},\\
p_6(x) &= x^{134} + x^{128} + x^{126} + x^{116} + x^{106} + x^{102} + x^{98} + x^{94} + x^{90} + x^{88}\\
&\quad + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{64} + x^{56} + x^{44} + x^{40} + x^{34} + x^{32}\\
&\quad + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{14} + x^{8} + x^{6} + 1,\\
p_9(x) &= x^{136} + x^{120} + x^{118} + x^{112} + x^{108} + x^{100} + x^{98} + x^{96} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{80} + x^{78} + x^{72} + x^{70} + x^{66} + x^{64} + x^{60} + x^{58} + x^{54} + x^{52}\\
&\quad + x^{50} + x^{48} + x^{44} + x^{42} + x^{38} + x^{34} + x^{30} + x^{24} + x^{20} + x^{18} + x^{16}\\
&\quad + x^{12} + x^{6},\\
p_{12}(x) &= x^{136} + x^{128} + x^{72} + x^{64} + x^{56} + x^{48} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{142} + x^{141} + x^{139} + x^{137} + x^{133} + x^{130} + x^{128} + x^{126}\\
&\quad + x^{124} + x^{123} + x^{118} + x^{117} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110}\\
&\quad + x^{109} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{96}\\
&\quad + x^{95} + x^{92} + x^{91} + x^{87} + x^{86} + x^{82} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71}\\
&\quad + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55}\\
&\quad + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{43} + x^{37} + x^{36},\\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{131} + x^{129}\\
&\quad + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117}\\
&\quad + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{107} + x^{103} + x^{101} + x^{100}\\
&\quad + x^{97} + x^{96} + x^{91} + x^{90} + x^{87} + x^{86} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{64}+x^{60}+x^{57}+x^{52}+x^{51}+x^{50}\\
&+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{40}+x^{39}+x^{36}+x^{34}+x^{31}+x^{29}\\
&+x^{28}+x^{25}+x^{24},\\
p_6(x)=&\,x^{143}+x^{142}+x^{141}+x^{140}+x^{138}+x^{137}+x^{136}+x^{135}+x^{131}+x^{129}\\
&+x^{126}+x^{125}+x^{122}+x^{121}+x^{120}+x^{119}+x^{109}+x^{108}+x^{102}\\
&+x^{99}+x^{97}+x^{96}+x^{94}+x^{93}+x^{91}+x^{90}+x^{88}+x^{83}+x^{80}+x^{77}\\
&+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{67}+x^{66}+x^{65}+x^{63}+x^{59}+x^{57}\\
&+x^{56}+x^{55}+x^{53}+x^{52}+x^{50}+x^{47}+x^{46}+x^{45}+x^{42}+x^{41}+x^{38}\\
&+x^{36}+x^{35}+x^{34}+x^{29}+x^{26}+x^{24}+x^{23}+x^{21}+x^{20}+x^{18}+x^{15}\\
&+x^{14}+x^{11}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+1,\\
p_9(x)=&\,x^{143}+x^{142}+x^{141}+x^{140}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{134}\\
&+x^{133}+x^{132}+x^{131}+x^{130}+x^{129}+x^{128}+x^{123}+x^{122}+x^{121}\\
&+x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}\\
&+x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}\\
&+x^{97}+x^{96}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{63}+x^{62}+x^{61}+x^{60}\\
&+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}\\
&+x^{48}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}\\
&+x^{33}+x^{32}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}\\
&+x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12},\\
p_{12}(x)=&\,x^{143}+x^{142}+x^{141}+x^{140}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{134}\\
&+x^{133}+x^{132}+x^{131}+x^{130}+x^{129}+x^{128}+x^{127}+x^{126}+x^{125}\\
&+x^{124}+x^{111}+x^{110}+x^{109}+x^{108}+x^{95}+x^{94}+x^{93}+x^{92}+x^{75}\\
&+x^{74}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}\\
&+x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{31}+x^{30}\\
&+x^{29}+x^{28}+x^{11}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x\\
&+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{156}+x^{152}+x^{148}+x^{146}+x^{143}+x^{142}+x^{140}+x^{136}+x^{126}+x^{125}\\
&+x^{124}+x^{122}+x^{120}+x^{117}+x^{116}+x^{115}+x^{114}+x^{107}+x^{106}\\
&+x^{105}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{93}+x^{90}+x^{88}+x^{84}\\
&+x^{83}+x^{78}+x^{75}+x^{72}+x^{71}+x^{67}+x^{66}+x^{65}+x^{63}+x^{62}+x^{60}\\
&+x^{57}+x^{56}+x^{55}+x^{54}+x^{48}+x^{47}+x^{46}+x^{44}+x^{42}+x^{39},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{128} + x^{123} + x^{120} + x^{115} + x^{112} + x^{107} + x^{104} + x^{99} + x^{64} + x^{59} \\
&\quad + x^{48} + x^{43} + x^{40} + x^{35} + x^{32} + x^{27}, \\
p_6(x) &= x^{132} + x^{128} + x^{122} + x^{118} + x^{96} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} \\
&\quad + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{48} + x^{44} + x^{40} + x^{38} \\
&\quad + x^{34} + x^{32} + x^{30} + x^{28} + x^{22} + x^{18}, \\
p_9(x) &= x^{136} + x^{131} + x^{126} + x^{121} + x^{120} + x^{115} + x^{110} + x^{105} + x^{104} + x^{99} \\
&\quad + x^{94} + x^{89} + x^{88} + x^{83} + x^{78} + x^{73} + x^{56} + x^{51} + x^{46} + x^{41} + x^{40} \\
&\quad + x^{35} + x^{30} + x^{25} + x^{24} + x^{19} + x^{14} + x^{9}, \\
p_{12}(x) &= x^{140} + x^{136} + x^{132} + x^{128} + x^{124} + x^{108} + x^{92} + x^{72} + x^{68} + x^{64} \\
&\quad + x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{28} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{139} + x^{138} + x^{136} + x^{134} + x^{132} + x^{131} + x^{129} + x^{128} \\
&\quad + x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{120} + x^{110} + x^{109} + x^{108} \\
&\quad + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{93} + x^{90} + x^{88} \\
&\quad + x^{85} + x^{82} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} + x^{73} + x^{67} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{49} + x^{43} \\
&\quad + x^{42} + x^{39}, \\
p_3(x) &= x^{140} + x^{136} + x^{135} + x^{131} + x^{108} + x^{104} + x^{103} + x^{99} + x^{76} + x^{72} \\
&\quad + x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56} + x^{55} + x^{51} + x^{36} \\
&\quad + x^{32} + x^{31} + x^{27}, \\
p_6(x) &= x^{144} + x^{134} + x^{128} + x^{118} + x^{108} + x^{98} + x^{96} + x^{92} + x^{88} + x^{86} + x^{82} \\
&\quad + x^{78} + x^{76} + x^{68} + x^{66} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42} \\
&\quad + x^{40} + x^{38} + x^{34} + x^{30} + x^{28} + x^{18}, \\
p_9(x) &= x^{148} + x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{134} + x^{133} \\
&\quad + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} \\
&\quad + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} \\
&\quad + x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} \\
&\quad + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} \\
&\quad + x^{83} + x^{82} + x^{81} + x^{78} + x^{77} + x^{73} + x^{68} + x^{64} + x^{63} + x^{60} + x^{59} \\
&\quad + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} \\
&\quad + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} \\
&\quad + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} \\
&\quad + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{9},
\end{aligned}
$$

$$\begin{aligned}p_{12}(x) = {} & x^{152}+x^{144}+x^{132}+x^{124}+x^{120}+x^{116}+x^{112}+x^{108}+x^{104}+x^{100}\\ & +x^{96}+x^{92}+x^{84}+x^{76}+x^{52}+x^{44}+x^{40}+x^{36}+x^{32}+x^{28}+x^{20}\\ & +x^{12}+x^{8}+1,\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}p_0(x) = {} & x^{147}+x^{145}+x^{143}+x^{141}+x^{137}+x^{135}+x^{133}+x^{127}+x^{126}+x^{125}\\ & +x^{124}+x^{115}+x^{113}+x^{111}+x^{110}+x^{109}+x^{108}+x^{107}+x^{106}\\ & +x^{105}+x^{104}+x^{103}+x^{97}+x^{94}+x^{92}+x^{89}+x^{87}+x^{85}+x^{84}+x^{83}\\ & +x^{82}+x^{81}+x^{79}+x^{78}+x^{77}+x^{75}+x^{74}+x^{70}+x^{68}+x^{67}+x^{66}\\ & +x^{64}+x^{63}+x^{59}+x^{57}+x^{52}+x^{51}+x^{50}+x^{46}+x^{43}+x^{42},\\ p_3(x) = {} & x^{143}+x^{142}+x^{141}+x^{140}+x^{138}+x^{137}+x^{136}+x^{134}+x^{133}+x^{132}\\ & +x^{130}+x^{128}+x^{127}+x^{125}+x^{123}+x^{122}+x^{121}+x^{120}+x^{117}\\ & +x^{116}+x^{114}+x^{112}+x^{111}+x^{108}+x^{107}+x^{103}+x^{99}+x^{98}+x^{91}\\ & +x^{90}+x^{87}+x^{86}+x^{79}+x^{78}+x^{77}+x^{76}+x^{74}+x^{73}+x^{72}+x^{70}\\ & +x^{67}+x^{66}+x^{65}+x^{61}+x^{59}+x^{58}+x^{56}+x^{55}+x^{54}+x^{53}+x^{49}\\ & +x^{48}+x^{45}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{31}+x^{27}+x^{26},\\ p_6(x) = {} & x^{143}+x^{142}+x^{141}+x^{140}+x^{138}+x^{137}+x^{136}+x^{134}+x^{133}+x^{132}\\ & +x^{130}+x^{128}+x^{127}+x^{125}+x^{122}+x^{121}+x^{120}+x^{119}+x^{117}\\ & +x^{116}+x^{115}+x^{114}+x^{113}+x^{112}+x^{109}+x^{108}+x^{102}+x^{98}+x^{95}\\ & +x^{92}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{82}+x^{80}+x^{79}+x^{78}+x^{77}\\ & +x^{75}+x^{73}+x^{71}+x^{70}+x^{68}+x^{64}+x^{62}+x^{61}+x^{60}+x^{59}+x^{54}\\ & +x^{51}+x^{49}+x^{48}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{35}+x^{34}+x^{31}\\ & +x^{30}+x^{29}+x^{26}+x^{25}+x^{23}+x^{21}+x^{20}+x^{18}+x^{17}+x^{16}+x^{13}\\ & +x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+1,\\ p_9(x) = {} & x^{143}+x^{142}+x^{141}+x^{140}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{134}\\ & +x^{133}+x^{132}+x^{131}+x^{130}+x^{129}+x^{128}+x^{123}+x^{122}+x^{121}\\ & +x^{120}+x^{119}+x^{118}+x^{117}+x^{116}+x^{115}+x^{114}+x^{113}+x^{112}\\ & +x^{107}+x^{106}+x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{98}\\ & +x^{97}+x^{96}+x^{91}+x^{90}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\ & +x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{63}+x^{62}+x^{61}+x^{60}\\ & +x^{59}+x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}\\ & +x^{48}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}\\ & +x^{33}+x^{32}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20}+x^{19}\\ & +x^{18}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12},\\ p_{12}(x) = {} & x^{143}+x^{142}+x^{141}+x^{140}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{134}\end{aligned}$$

$$+ x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{111} + x^{110} + x^{109} + x^{108} + x^{95} + x^{94} + x^{93} + x^{92} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{31} + x^{30} + x^{29} + x^{28} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$p_0(x) = x^{147} + x^{145} + x^{142} + x^{141} + x^{139} + x^{137} + x^{133} + x^{130} + x^{128} + x^{126} + x^{124} + x^{123} + x^{118} + x^{117} + x^{116} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{87} + x^{86} + x^{82} + x^{77} + x^{76} + x^{74} + x^{73} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47} + x^{46} + x^{43} + x^{37} + x^{36},$$

$$p_3(x) = x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{131} + x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{107} + x^{103} + x^{101} + x^{100} + x^{97} + x^{96} + x^{91} + x^{90} + x^{87} + x^{86} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{64} + x^{60} + x^{57} + x^{52} + x^{51} + x^{50} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39} + x^{36} + x^{34} + x^{31} + x^{29} + x^{28} + x^{25} + x^{24},$$

$$p_6(x) = x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{135} + x^{131} + x^{129} + x^{126} + x^{125} + x^{122} + x^{121} + x^{120} + x^{119} + x^{109} + x^{108} + x^{102} + x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{91} + x^{90} + x^{88} + x^{83} + x^{80} + x^{77} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{63} + x^{59} + x^{57} + x^{56} + x^{55} + x^{53} + x^{52} + x^{50} + x^{47} + x^{46} + x^{45} + x^{42} + x^{41} + x^{38} + x^{36} + x^{35} + x^{34} + x^{29} + x^{26} + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{15} + x^{14} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1,$$

$$p_9(x) = x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{63} + x^{62} + x^{61} + x^{60}$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49}\\
&+ x^{48} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19}\\
&+ x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12},\\
p_{12}(x) = {}& x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134}\\
&+ x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125}\\
&+ x^{124} + x^{111} + x^{110} + x^{109} + x^{108} + x^{95} + x^{94} + x^{93} + x^{92} + x^{75}\\
&+ x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x\\
&+ 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{144} + x^{142} + x^{139} + x^{138} + x^{136} + x^{134} + x^{132} + x^{131} + x^{129} + x^{128}\\
&+ x^{127} + x^{126} + x^{125} + x^{122} + x^{121} + x^{120} + x^{110} + x^{109} + x^{108}\\
&+ x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{93} + x^{90} + x^{88}\\
&+ x^{85} + x^{82} + x^{80} + x^{79} + x^{77} + x^{75} + x^{74} + x^{73} + x^{67} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{49} + x^{43}\\
&+ x^{42} + x^{39},\\
p_3(x) = {}& x^{140} + x^{136} + x^{135} + x^{131} + x^{108} + x^{104} + x^{103} + x^{99} + x^{76} + x^{72}\\
&+ x^{71} + x^{68} + x^{67} + x^{64} + x^{63} + x^{60} + x^{59} + x^{56} + x^{55} + x^{51} + x^{36}\\
&+ x^{32} + x^{31} + x^{27},\\
p_6(x) = {}& x^{144} + x^{134} + x^{128} + x^{118} + x^{108} + x^{98} + x^{96} + x^{92} + x^{88} + x^{86} + x^{82}\\
&+ x^{78} + x^{76} + x^{68} + x^{66} + x^{60} + x^{58} + x^{52} + x^{50} + x^{48} + x^{44} + x^{42}\\
&+ x^{40} + x^{38} + x^{34} + x^{30} + x^{28} + x^{18},\\
p_9(x) = {}& x^{148} + x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{136} + x^{135} + x^{134} + x^{133}\\
&+ x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124}\\
&+ x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115}\\
&+ x^{114} + x^{113} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96}\\
&+ x^{95} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{83} + x^{82} + x^{81} + x^{78} + x^{77} + x^{73} + x^{68} + x^{64} + x^{63} + x^{60} + x^{59}\\
&+ x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47}\\
&+ x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} \\
&+ x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{18} + x^{17} + x^{14} + x^{13} + x^{9}, \\
p_{12}(x) &= x^{152} + x^{144} + x^{132} + x^{124} + x^{120} + x^{116} + x^{112} + x^{108} + x^{104} + x^{100} \\
&+ x^{96} + x^{92} + x^{84} + x^{76} + x^{52} + x^{44} + x^{40} + x^{36} + x^{32} + x^{28} + x^{20} \\
&+ x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{156} + x^{152} + x^{148} + x^{146} + x^{143} + x^{142} + x^{140} + x^{136} + x^{126} + x^{125} \\
&+ x^{124} + x^{122} + x^{120} + x^{117} + x^{116} + x^{115} + x^{114} + x^{107} + x^{106} \\
&+ x^{105} + x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{93} + x^{90} + x^{88} + x^{84} \\
&+ x^{83} + x^{78} + x^{75} + x^{72} + x^{71} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} \\
&+ x^{57} + x^{56} + x^{55} + x^{54} + x^{48} + x^{47} + x^{46} + x^{44} + x^{42} + x^{39}, \\
p_3(x) &= x^{128} + x^{123} + x^{120} + x^{115} + x^{112} + x^{107} + x^{104} + x^{99} + x^{64} + x^{59} \\
&+ x^{48} + x^{43} + x^{40} + x^{35} + x^{32} + x^{27}, \\
p_6(x) &= x^{132} + x^{128} + x^{122} + x^{118} + x^{96} + x^{92} + x^{86} + x^{84} + x^{82} + x^{80} + x^{76} \\
&+ x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{58} + x^{48} + x^{44} + x^{40} + x^{38} \\
&+ x^{34} + x^{32} + x^{30} + x^{28} + x^{22} + x^{18}, \\
p_9(x) &= x^{136} + x^{131} + x^{126} + x^{121} + x^{120} + x^{115} + x^{110} + x^{105} + x^{104} + x^{99} \\
&+ x^{94} + x^{89} + x^{88} + x^{83} + x^{78} + x^{73} + x^{56} + x^{51} + x^{46} + x^{41} + x^{40} \\
&+ x^{35} + x^{30} + x^{25} + x^{24} + x^{19} + x^{14} + x^{9}, \\
p_{12}(x) &= x^{140} + x^{136} + x^{132} + x^{128} + x^{124} + x^{108} + x^{92} + x^{72} + x^{68} + x^{64} \\
&+ x^{60} + x^{56} + x^{52} + x^{48} + x^{44} + x^{28} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{143} + x^{141} + x^{137} + x^{135} + x^{133} + x^{127} + x^{126} + x^{125} \\
&+ x^{124} + x^{115} + x^{113} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} \\
&+ x^{105} + x^{104} + x^{103} + x^{97} + x^{94} + x^{92} + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} \\
&+ x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{70} + x^{68} + x^{67} + x^{66} \\
&+ x^{64} + x^{63} + x^{59} + x^{57} + x^{52} + x^{51} + x^{50} + x^{46} + x^{43} + x^{42}, \\
p_3(x) &= x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} \\
&+ x^{130} + x^{128} + x^{127} + x^{125} + x^{123} + x^{122} + x^{121} + x^{120} + x^{117} \\
&+ x^{116} + x^{114} + x^{112} + x^{111} + x^{108} + x^{107} + x^{103} + x^{99} + x^{98} + x^{91} \\
&+ x^{90} + x^{87} + x^{86} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{73} + x^{72} + x^{70} \\
&+ x^{67} + x^{66} + x^{65} + x^{61} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{49} \\
&+ x^{48} + x^{45} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{31} + x^{27} + x^{26},
\end{aligned}
$$

$$
\begin{aligned}
p_6(x) = {} & x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} + x^{134} + x^{133} + x^{132} \\
& + x^{130} + x^{128} + x^{127} + x^{125} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} \\
& + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{109} + x^{108} + x^{102} + x^{98} + x^{95} \\
& + x^{92} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{80} + x^{79} + x^{78} + x^{77} \\
& + x^{75} + x^{73} + x^{71} + x^{70} + x^{68} + x^{64} + x^{62} + x^{61} + x^{60} + x^{59} + x^{54} \\
& + x^{51} + x^{49} + x^{48} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{35} + x^{34} + x^{31} \\
& + x^{30} + x^{29} + x^{26} + x^{25} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{13} \\
& + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x + 1, \\
p_9(x) = {} & x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} \\
& + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{123} + x^{122} + x^{121} \\
& + x^{120} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} \\
& + x^{107} + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} + x^{98} \\
& + x^{97} + x^{96} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{84} + x^{83} \\
& + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{63} + x^{62} + x^{61} + x^{60} \\
& + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} \\
& + x^{48} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} \\
& + x^{33} + x^{32} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} \\
& + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12}, \\
p_{12}(x) = {} & x^{143} + x^{142} + x^{141} + x^{140} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{134} \\
& + x^{133} + x^{132} + x^{131} + x^{130} + x^{129} + x^{128} + x^{127} + x^{126} + x^{125} \\
& + x^{124} + x^{111} + x^{110} + x^{109} + x^{108} + x^{95} + x^{94} + x^{93} + x^{92} + x^{75} \\
& + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} \\
& + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} \\
& + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{31} + x^{30} \\
& + x^{29} + x^{28} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x \\
& + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	62	[87, 87]	124	[166, 95]	186	[219, 78]	248	[91, 12]
1	[3, 4]	63	[91, 92]	125	[7, 167]	187	[220, 2]	249	[266, 11]
2	[5, 6]	64	[93, 80]	126	[168, 169]	188	[78, 2]	250	[267, 84]
3	[7, 8]	65	[81, 36]	127	[85, 32]	189	[149, 29]	251	[152, 36]
4	[9, 10]	66	[94, 31]	128	[33, 87]	190	[166, 31]	252	[26, 148]
5	[10, 10]	67	[94, 95]	129	[29, 170]	191	[162, 88]	253	[268, 70]

6	[11, 12]	68	[96, 97]	130	[35, 10]	192	[221, 162]	254	[269, 88]
7	[13, 14]	69	[98, 99]	131	[80, 169]	193	[222, 70]	255	[37, 2]
8	[15, 16]	70	[100, 101]	132	[171, 165]	194	[68, 144]	256	[270, 78]
9	[17, 18]	71	[102, 103]	133	[172, 167]	195	[223, 72]	257	[271, 94]
10	[19, 20]	72	[104, 105]	134	[168, 11]	196	[73, 145]	258	[225, 95]
11	[14, 21]	73	[106, 96]	135	[158, 27]	197	[224, 94]	259	[39, 162]
12	[22, 23]	74	[107, 108]	136	[25, 82]	198	[225, 31]	260	[172, 80]
13	[24, 25]	75	[109, 110]	137	[29, 162]	199	[226, 146]	261	[272, 208]
14	[26, 27]	76	[99, 111]	138	[173, 25]	200	[147, 27]	262	[273, 116]
15	[28, 29]	77	[112, 113]	139	[158, 148]	201	[8, 151]	263	[274, 121]
16	[30, 31]	78	[114, 62]	140	[174, 90]	202	[37, 31]	264	[275, 276]
17	[10, 32]	79	[115, 116]	141	[87, 32]	203	[227, 82]	265	[277, 16]
18	[33, 32]	80	[117, 118]	142	[91, 146]	204	[80, 11]	266	[278, 124]
19	[34, 35]	81	[119, 120]	143	[175, 176]	205	[228, 90]	267	[279, 22]
20	[35, 35]	82	[121, 122]	144	[176, 177]	206	[229, 39]	268	[280, 51]
21	[25, 36]	83	[123, 124]	145	[97, 107]	207	[230, 39]	269	[281, 258]
22	[37, 38]	84	[125, 126]	146	[178, 179]	208	[214, 2]	270	[282, 5]
23	[39, 40]	85	[127, 128]	147	[180, 121]	209	[30, 2]	271	[283, 141]
24	[41, 42]	86	[124, 129]	148	[45, 181]	210	[231, 158]	272	[284, 88]
25	[43, 42]	87	[20, 130]	149	[182, 67]	211	[232, 40]	273	[227, 27]
26	[44, 45]	88	[51, 131]	150	[183, 99]	212	[170, 165]	274	[285, 162]
27	[46, 47]	89	[132, 118]	151	[21, 135]	213	[233, 234]	275	[81, 27]
28	[48, 49]	90	[133, 134]	152	[184, 185]	214	[235, 236]	276	[159, 82]
29	[50, 51]	91	[135, 136]	153	[186, 187]	215	[237, 203]	277	[40, 165]
30	[52, 5]	92	[118, 137]	154	[188, 188]	216	[238, 236]	278	[153, 29]
31	[53, 54]	93	[138, 139]	155	[189, 190]	217	[239, 135]	279	[154, 88]
32	[32, 32]	94	[140, 141]	156	[191, 178]	218	[240, 241]	280	[286, 12]
33	[55, 56]	95	[61, 142]	157	[192, 45]	219	[242, 62]	281	[214, 31]
34	[57, 19]	96	[143, 144]	158	[193, 194]	220	[243, 244]	282	[287, 11]
35	[18, 58]	97	[145, 145]	159	[195, 196]	221	[245, 203]	283	[288, 91]
36	[42, 59]	98	[11, 146]	160	[197, 198]	222	[246, 247]	284	[289, 290]
37	[60, 61]	99	[147, 148]	161	[199, 200]	223	[248, 65]	285	[291, 103]
38	[62, 63]	100	[149, 150]	162	[198, 66]	224	[249, 53]	286	[292, 14]
39	[64, 65]	101	[150, 40]	163	[103, 201]	225	[250, 53]	287	[293, 22]
40	[66, 67]	102	[151, 146]	164	[202, 198]	226	[251, 252]	288	[294, 255]
41	[29, 40]	103	[152, 148]	165	[203, 204]	227	[253, 46]	289	[166, 2]
42	[68, 68]	104	[153, 150]	166	[205, 113]	228	[254, 255]	290	[220, 38]
43	[69, 70]	105	[150, 154]	167	[206, 178]	229	[256, 188]	291	[295, 8]
44	[71, 72]	106	[68, 145]	168	[207, 208]	230	[257, 258]	292	[147, 36]
45	[73, 74]	107	[144, 144]	169	[116, 139]	231	[259, 120]	293	[296, 39]
46	[75, 68]	108	[145, 74]	170	[187, 49]	232	[260, 247]	294	[297, 39]
47	[40, 76]	109	[75, 145]	171	[209, 187]	233	[74, 144]	295	[298, 21]
48	[77, 78]	110	[143, 68]	172	[210, 136]	234	[213, 74]	296	[299, 66]
49	[78, 38]	111	[8, 91]	173	[211, 185]	235	[261, 84]	297	[300, 290]

50	[79, 80]	112	[155, 91]	174	[212, 190]	236	[10, 35]	298	[301, 158]
51	[81, 82]	113	[9, 32]	175	[73, 144]	237	[262, 12]	299	[302, 94]
52	[83, 84]	114	[156, 90]	176	[213, 145]	238	[8, 11]	300	[303, 78]
53	[85, 10]	115	[157, 158]	177	[144, 68]	239	[263, 25]	301	[101, 46]
54	[11, 86]	116	[159, 36]	178	[214, 38]	240	[264, 146]	302	[304, 61]
55	[87, 35]	117	[160, 29]	179	[39, 154]	241	[152, 27]	303	[134, 244]
56	[34, 10]	118	[30, 95]	180	[215, 72]	242	[80, 91]	304	[167, 91]
57	[9, 35]	119	[161, 72]	181	[162, 165]	243	[265, 84]		
58	[32, 87]	120	[74, 145]	182	[216, 94]	244	[85, 35]		
59	[40, 88]	121	[74, 74]	183	[217, 8]	245	[7, 80]		
60	[89, 90]	122	[162, 163]	184	[218, 70]	246	[169, 12]		
61	[9, 87]	123	[164, 165]	185	[75, 144]	247	[227, 36]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	44	1	88	0	132	1	176	1	220	0	264	0
1	0	45	0	89	1	133	1	177	1	221	0	265	0
2	0	46	1	90	1	134	1	178	0	222	0	266	0
3	0	47	0	91	0	135	0	179	1	223	0	267	1
4	0	48	1	92	1	136	1	180	1	224	0	268	1
5	0	49	1	93	1	137	0	181	1	225	1	269	0
6	1	50	0	94	0	138	1	182	1	226	0	270	0
7	0	51	0	95	0	139	0	183	0	227	1	271	1
8	1	52	1	96	0	140	0	184	0	228	0	272	0
9	0	53	1	97	0	141	1	185	1	229	0	273	1
10	0	54	1	98	1	142	0	186	0	230	1	274	0
11	1	55	1	99	1	143	0	187	0	231	1	275	0
12	1	56	0	100	1	144	1	188	1	232	1	276	0
13	0	57	0	101	0	145	0	189	1	233	1	277	0
14	1	58	0	102	1	146	0	190	0	234	1	278	0
15	1	59	0	103	0	147	1	191	1	235	0	279	1
16	1	60	1	104	0	148	0	192	0	236	0	280	1
17	0	61	0	105	0	149	1	193	0	237	1	281	0
18	1	62	1	106	0	150	0	194	0	238	1	282	0
19	0	63	0	107	1	151	1	195	0	239	0	283	1
20	1	64	1	108	0	152	0	196	0	240	0	284	0
21	1	65	0	109	1	153	0	197	0	241	0	285	0
22	1	66	0	110	0	154	1	198	1	242	0	286	1
23	1	67	0	111	1	155	1	199	0	243	0	287	0
24	0	68	0	112	1	156	1	200	1	244	1	288	1
25	1	69	1	113	0	157	0	201	1	245	0	289	0
26	1	70	1	114	1	158	0	202	1	246	0	290	0
27	1	71	1	115	0	159	0	203	1	247	1	291	0
28	1	72	0	116	0	160	0	204	0	248	0	292	1

29	0	73	0	117	0	161	0	205	0	249	0	293	0
30	1	74	1	118	1	162	1	206	0	250	1	294	1
31	1	75	1	119	0	163	0	207	1	251	0	295	0
32	0	76	1	120	1	164	1	208	0	252	1	296	0
33	1	77	1	121	1	165	1	209	1	253	1	297	1
34	0	78	1	122	1	166	0	210	1	254	0	298	0
35	1	79	0	123	1	167	0	211	1	255	1	299	0
36	0	80	0	124	0	168	1	212	0	256	0	300	1
37	1	81	0	125	0	169	0	213	1	257	1	301	0
38	1	82	1	126	1	170	0	214	0	258	1	302	0
39	1	83	1	127	1	171	1	215	1	259	1	303	1
40	0	84	0	128	1	172	1	216	1	260	1	304	0
41	0	85	1	129	0	173	1	217	0	261	0		
42	0	86	0	130	1	174	0	218	0	262	1		
43	1	87	1	131	0	175	0	219	0	263	0		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn