PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^3, z^4 + z^3 + z)$

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ABSTRACT. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^3, z^4 + z^3 + z)$.

1. INTRODUCTION

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^3, z^4+z^3+z)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z) &= z^{17}+z^{16}+z^{15}+z^{10}+z^5+z^2+1,\\
p_1(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{17}+z^{13}+z^9+z^8+z^5+z^3,\\
p_2(z) &= z^{24}+z^{20}+z^{18}+z^{16}+z^{12}+z^{10}+z^6,\\
p_3(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{17}+z^{13}+z^9+z^8+z^5+z^3,\\
p_4(z) &= z^{18}+z^{17}+z^{15}+z^{12}+z^6+z^5+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z) &= z^{17}+z^{16}+z^{15}+z^{12}+z^{10}+z^5+z^2,\\
p_1(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{17}+z^{13}+z^9+z^8+z^5+z^3,\\
p_2(z) &= z^{24}+z^{20}+z^{18}+z^{16}+z^{12}+z^{10}+z^6,\\
p_3(z) &= z^{21}+z^{20}+z^{19}+z^{18}+z^{17}+z^{13}+z^9+z^8+z^5+z^3,\\
p_4(z) &= z^{18}+z^{17}+z^{15}+z^6+z^5+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z) \\ P_{n-1}(z) & Q_{n-1}(z) \end{pmatrix} := \begin{pmatrix} t_n(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{n-1}(z) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} t_0(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} \in \mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) = {}& x^{\deg(t_{2^n-1})} \begin{pmatrix} t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(t_{2^n-2})} \begin{pmatrix} t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
& \cdots \times x^{\deg(t_0)} \begin{pmatrix} t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},\\
W_n(x) = {}& x^{\deg(\bar t_{2^n-1})} \begin{pmatrix} \bar t_{2^n-1}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times x^{\deg(\bar t_{2^n-2})} \begin{pmatrix} \bar t_{2^n-2}(1/x) & 1 \\ 1 & 0 \end{pmatrix} \times \\
& \cdots \times x^{\deg(\bar t_0)} \begin{pmatrix} \bar t_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x) &= W_n(x)\cdot M_n(x),\\
W_{n+1}(x) &= M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$
$$M^o = W^e \cdot M^e, \tag{2.4}$$
$$W^e = M^o \cdot W^o, \tag{2.5}$$
$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{3u}M^e[:4u] + x^{6u}M^e[:u] + x^u M^e[:7u]\\ & + x^{2u}M^e[:6u] + x^{4u}M^e[:4u] + x^{7u}M^e[:u] + x^{2u}M^e[:7u]\\ & + x^{3u}M^e[:6u] + x^{5u}M^e[:4u] + x^{8u}M^e[:u] + x^{4u}M^e[:7u]\\ & + x^{5u}M^e[:6u] + x^{7u}M^e[:4u] + x^{10u}M^e[:u] + x^{6u}M^e[:7u]\\ & + x^{7u}M^e[:6u] + x^{9u}M^e[:4u] + x^{12u}M^e[:u] + x^{8u}M^e[:6u]\\ & + x^{9u}M^e[:5u] + x^{10u}M^e[:4u] + x^{12u}M^e[:2u]\end{aligned}$$

$$
\begin{aligned}
&\quad + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{13u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^uW^e[:6u] + x^{3u}W^e[:4u] + x^{6u}W^e[:u] + x^uW^e[:7u]\\
&\quad + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{7u}W^e[:u] + x^{2u}W^e[:7u]\\
&\quad + x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{8u}W^e[:u] + x^{4u}W^e[:7u]\\
&\quad + x^{5u}W^e[:6u] + x^{7u}W^e[:4u] + x^{10u}W^e[:u] + x^{6u}W^e[:7u]\\
&\quad + x^{7u}W^e[:6u] + x^{9u}W^e[:4u] + x^{12u}W^e[:u] + x^{8u}W^e[:6u]\\
&\quad + x^{9u}W^e[:5u] + x^{10u}W^e[:4u] + x^{12u}W^e[:2u]\\
&\quad + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^uM^o[:6u] + x^{3u}M^o[:4u] + x^{6u}M^o[:u] + x^uM^o[:7u]\\
&\quad + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{7u}M^o[:u] + x^{2u}M^o[:7u]\\
&\quad + x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{8u}M^o[:u] + x^{4u}M^o[:7u]\\
&\quad + x^{5u}M^o[:6u] + x^{7u}M^o[:4u] + x^{10u}M^o[:u] + x^{6u}M^o[:7u]\\
&\quad + x^{7u}M^o[:6u] + x^{9u}M^o[:4u] + x^{12u}M^o[:u] + x^{8u}M^o[:6u]\\
&\quad + x^{9u}M^o[:5u] + x^{10u}M^o[:4u] + x^{12u}M^o[:2u]\\
&\quad + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{13u})R^o,\\
W^o[:14u] &= W^o[:7u] + x^uW^o[:6u] + x^{3u}W^o[:4u] + x^{6u}W^o[:u] + x^uW^o[:7u]\\
&\quad + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{7u}W^o[:u] + x^{2u}W^o[:7u]\\
&\quad + x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{8u}W^o[:u] + x^{4u}W^o[:7u]\\
&\quad + x^{5u}W^o[:6u] + x^{7u}W^o[:4u] + x^{10u}W^o[:u] + x^{6u}W^o[:7u]\\
&\quad + x^{7u}W^o[:6u] + x^{9u}W^o[:4u] + x^{12u}W^o[:u] + x^{8u}W^o[:6u]\\
&\quad + x^{9u}W^o[:5u] + x^{10u}W^o[:4u] + x^{12u}W^o[:2u]\\
&\quad + (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] &= T[:7u] + x^uT[:6u] + x^{3u}T[:4u] + x^{6u}T[:u] + x^uT[:7u] + x^{2u}T[:6u]\\
&\quad + x^{4u}T[:4u] + x^{7u}T[:u] + x^{2u}T[:7u] + x^{3u}T[:6u] + x^{5u}T[:4u]\\
&\quad + x^{8u}T[:u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{7u}T[:4u] + x^{10u}T[:u]\\
&\quad + x^{6u}T[:7u] + x^{7u}T[:6u] + x^{9u}T[:4u] + x^{12u}T[:u] + x^{8u}T[:6u]\\
&\quad + x^{9u}T[:5u] + x^{10u}T[:4u] + x^{12u}T[:2u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{3u}T[4u:5u] + x^uT[6u:7u] + T[7u:8u],\\
0 &= x^{6u}T[2u:3u] + x^{4u}T[4u:5u] + x^{3u}T[5u:6u] + x^{2u}T[6u:7u]
\end{aligned}
$$

$$\begin{aligned}
&\quad + T[8u\!:\!9u],\\
0 &= x^{8u}T[u\!:\!2u] + x^{6u}T[3u\!:\!4u] + x^{5u}T[4u\!:\!5u] + x^{4u}T[5u\!:\!6u]\\
&\quad + T[9u\!:\!10u],\\
0 &= x^{8u}T[2u\!:\!3u] + x^{6u}T[4u\!:\!5u] + x^{5u}T[5u\!:\!6u] + x^{4u}T[6u\!:\!7u]\\
&\quad + T[10u\!:\!11u],\\
0 &= x^{10u}T[u\!:\!2u] + x^{8u}T[3u\!:\!4u] + x^{7u}T[4u\!:\!5u] + x^{6u}T[5u\!:\!6u]\\
&\quad + T[11u\!:\!12u],\\
0 &= x^{10u}T[2u\!:\!3u] + x^{8u}T[4u\!:\!5u] + x^{7u}T[5u\!:\!6u] + x^{6u}T[6u\!:\!7u]\\
&\quad + T[12u\!:\!13u],\\
0 &= x^{12u}T[u\!:\!2u] + x^{10u}T[3u\!:\!4u] + x^{9u}T[4u\!:\!5u] + x^{8u}T[5u\!:\!6u]\\
&\quad + T[13u\!:\!14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[1w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 898 states; its transition function and output function can be found in the annex. Let $A(s,w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i,w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}
0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,110))\\
&\quad + \tau(A(s,111)),
\end{aligned} \tag{2.21}$$

$$0 = \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110))$$

$$\qquad + \tau(A(s,1000)), \tag{2.22}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1001)), \end{aligned} \tag{2.23}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.24}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.25}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1100)), \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1101)), \end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,100)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1011)), \end{aligned} \tag{2.32}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1100)), \end{aligned} \tag{2.33}$$

$$\begin{aligned} 0 &= \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,1101)), \end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{26}), E_{26}) = (A(i, 0^{14}), E_{14})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 13$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{6u} M^e[:u] + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{6u} W^e[:u] + x^{7u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{6u} M^o[:u] + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{6u} W^o[:u] + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n=2$, the identities can be verified directly. For $n\geq 2$, we claim that

$$\begin{aligned} M_{2k}\wedge W_{2k} &\Rightarrow M_{2k+1}\wedge W_{2k+1},\\ M_{2k+1}\wedge W_{2k+1} &\Rightarrow M_{2k+2}\wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k}\wedge W_{2k}\Rightarrow M_{2k+1},$$

the other ones being similar. Set $u=2^{2k}$ and $v=2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(W^e[:7v]+x^vW^e[:6v]+x^{3v}W^e[:4v]+x^{6v}W^e[:v]+x^{7u}R^e\big)\times\big(\\ M^e[:7v]+x^vM^e[:6v]+x^{3v}M^e[:4v]+x^{6v}M^e[:v]+x^{7u}R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} W^e[:14v]M^e[:14v]+x^{2v}W^e[:12v]M^e[:12v]+x^{6v}W^e[:8v]M^e[:8v]\\ +x^{12v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n\leq 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v,\\ a_n &:= W^e[n\cdot v:(n+1)\cdot v]/X^n,\\ b_n &:= M^e[n\cdot v:(n+1)\cdot v]/X^n,\\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1,...,a_{14},b_1,...,b_{14},c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0\leq j,k\leq 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z=1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b)=\frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{22} + x^{19} + x^{14} + x^9 + x^8 + x^7,\\
q_1(x) &= x^{21} + x^{19} + x^{16} + x^{15} + x^{11} + x^7 + x^6 + x^5 + x^4 + x^3,\\
q_2(x) &= x^{18} + x^{14} + x^{12} + x^8 + x^6 + x^4 + 1,\\
q_3(x) &= x^{21} + x^{19} + x^{16} + x^{15} + x^{11} + x^7 + x^6 + x^5 + x^4 + x^3,\\
q_4(x) &= x^{22} + x^{19} + x^{18} + x^{12} + x^9 + x^7 + x^6.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) = {}& x^{132} + x^{126} + x^{124} + x^{120} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{102}\\
&+ x^{96} + x^{94} + x^{92} + x^{88} + x^{80} + x^{74} + x^{72} + x^{68} + x^{64} + x^{60} + x^{54}\\
&+ x^{52} + x^{50} + x^{46} + x^{40} + x^{38} + x^{36},\\
p_3(x) = {}& x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102}\\
&+ x^{96} + x^{90} + x^{88} + x^{84} + x^{82} + x^{80} + x^{74} + x^{72} + x^{68} + x^{64} + x^{60}\\
&+ x^{58} + x^{54} + x^{50} + x^{44} + x^{42} + x^{40} + x^{38} + x^{34} + x^{30} + x^{26} + x^{20},\\
p_6(x) = {}& x^{122} + x^{120} + x^{118} + x^{114} + x^{106} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92}\\
&+ x^{90} + x^{88} + x^{82} + x^{80} + x^{78} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56}\\
&+ x^{54} + x^{48} + x^{44} + x^{42} + x^{40} + x^{32} + x^{30} + x^{26} + x^{24} + x^{16} + x^{10}\\
&+ x^6 + 1,\\
p_9(x) = {}& x^{120} + x^{114} + x^{110} + x^{108} + x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{86}\\
&+ x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{56} + x^{46} + x^{44} + x^{40}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{16} + x^{14} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{120} + x^{104} + x^{96} + x^{40} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{122} + x^{118} + x^{116} + x^{114} + x^{112} + x^{105} + x^{102} + x^{101}\\
&+ x^{100} + x^{99} + x^{98} + x^{91} + x^{89} + x^{87} + x^{83} + x^{80} + x^{78} + x^{77} + x^{74}\\
&+ x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{39},\\
p_3(x) &= x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106}\\
&+ x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86}\\
&+ x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43}\\
&+ x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{27},\\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{107} + x^{105} + x^{104} + x^{101}\\
&+ x^{99} + x^{98} + x^{97} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84}\\
&+ x^{83} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{45} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24}\\
&+ x^{22} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{114} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{84} + x^{82} + x^{81}\\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{48} + x^{45} + x^{38} + x^{36} + x^{35}\\
&+ x^{34} + x^{33} + x^{31} + x^{26} + x^{23} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12}\\
&+ x^{11} + x^{9},\\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{31}\\
&+ x^{29} + x^{28} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12}\\
&+ x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{122} + x^{118} + x^{116} + x^{114} + x^{112} + x^{105} + x^{102} + x^{101}\\
&+ x^{100} + x^{99} + x^{98} + x^{91} + x^{89} + x^{87} + x^{83} + x^{80} + x^{78} + x^{77} + x^{74}\\
&+ x^{73} + x^{72} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58}\\
&+ x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{39},
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106} \\
&\quad + x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86} \\
&\quad + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} \\
&\quad + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} \\
&\quad + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43} \\
&\quad + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{27}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{107} + x^{105} + x^{104} + x^{101} \\
&\quad + x^{99} + x^{98} + x^{97} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84} \\
&\quad + x^{83} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} \\
&\quad + x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{45} + x^{42} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24} \\
&\quad + x^{22} + x^{20} + x^{19} + x^{18}, \\
p_9(x) &= x^{114} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98} \\
&\quad + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{84} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{48} + x^{45} + x^{38} + x^{36} + x^{35} \\
&\quad + x^{34} + x^{33} + x^{31} + x^{26} + x^{23} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12} \\
&\quad + x^{11} + x^{9}, \\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{31} \\
&\quad + x^{29} + x^{28} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12} \\
&\quad + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{126} + x^{120} + x^{110} + x^{102} + x^{94} + x^{90} + x^{88} + x^{82} + x^{76} + x^{66} + x^{64} \\
&\quad + x^{60} + x^{58} + x^{52} + x^{50} + x^{46} + x^{42}, \\
p_3(x) &= x^{108} + x^{104} + x^{98} + x^{92} + x^{90} + x^{72} + x^{66} + x^{64} + x^{60} + x^{58} + x^{56} \\
&\quad + x^{52} + x^{44} + x^{42} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{18}, \\
p_6(x) &= x^{108} + x^{104} + x^{102} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{88} + x^{86} + x^{84} \\
&\quad + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{52} \\
&\quad + x^{50} + x^{48} + x^{40} + x^{32} + x^{28} + x^{24} + x^{20} + x^{16} + x^{14} + x^{12} + x^{2} + 1, \\
p_9(x) &= x^{102} + x^{98} + x^{88} + x^{86} + x^{78} + x^{74} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} \\
&\quad + x^{50} + x^{48} + x^{46} + x^{40} + x^{34} + x^{30} + x^{28} + x^{24} + x^{20} + x^{18} + x^{16} \\
&\quad + x^{14} + x^{6}, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{22} + x^{18} + x^{16} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{126}+x^{120}+x^{114}+x^{108}+x^{96}+x^{84}+x^{66}+x^{60}+x^{36},\\
p_3(x) &= x^{108}+x^{104}+x^{98}+x^{96}+x^{92}+x^{88}+x^{82}+x^{76}+x^{70}+x^{64}+x^{60}\\
&\quad +x^{58}+x^{52}+x^{50}+x^{44}+x^{42}+x^{40}+x^{36}+x^{34}+x^{30}+x^{22}+x^{20},\\
p_6(x) &= x^{108}+x^{104}+x^{102}+x^{98}+x^{94}+x^{92}+x^{90}+x^{82}+x^{76}+x^{74}+x^{70}\\
&\quad +x^{68}+x^{62}+x^{60}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{36}+x^{30}+x^{28}\\
&\quad +x^{26}+x^{24}+x^{22}+x^{18}+x^{2}+1,\\
p_9(x) &= x^{102}+x^{98}+x^{88}+x^{86}+x^{78}+x^{74}+x^{68}+x^{66}+x^{64}+x^{62}+x^{58}\\
&\quad +x^{50}+x^{48}+x^{46}+x^{40}+x^{34}+x^{30}+x^{28}+x^{24}+x^{20}+x^{18}+x^{16}\\
&\quad +x^{14}+x^{6},\\
p_{12}(x) &= x^{102}+x^{98}+x^{96}+x^{22}+x^{18}+x^{16}+x^{6}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{129}+x^{123}+x^{121}+x^{119}+x^{118}+x^{117}+x^{115}+x^{112}+x^{111}+x^{107}\\
&\quad +x^{105}+x^{104}+x^{103}+x^{102}+x^{101}+x^{98}+x^{97}+x^{95}+x^{94}+x^{93}\\
&\quad +x^{91}+x^{90}+x^{89}+x^{85}+x^{84}+x^{83}+x^{82}+x^{81}+x^{78}+x^{76}+x^{75}\\
&\quad +x^{74}+x^{71}+x^{70}+x^{55}+x^{54}+x^{52}+x^{51}+x^{49}+x^{45}+x^{42}+x^{40}\\
&\quad +x^{39},\\
p_3(x) &= x^{120}+x^{117}+x^{116}+x^{114}+x^{113}+x^{111}+x^{110}+x^{108}+x^{107}+x^{106}\\
&\quad +x^{105}+x^{103}+x^{102}+x^{99}+x^{96}+x^{93}+x^{92}+x^{90}+x^{89}+x^{87}+x^{86}\\
&\quad +x^{84}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{74}\\
&\quad +x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{67}+x^{60}+x^{58}+x^{57}+x^{56}+x^{55}\\
&\quad +x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{43}\\
&\quad +x^{36}+x^{34}+x^{33}+x^{31}+x^{30}+x^{27},\\
p_6(x) &= x^{117}+x^{115}+x^{114}+x^{113}+x^{111}+x^{110}+x^{107}+x^{105}+x^{104}+x^{101}\\
&\quad +x^{99}+x^{98}+x^{97}+x^{94}+x^{92}+x^{91}+x^{90}+x^{89}+x^{86}+x^{85}+x^{84}\\
&\quad +x^{83}+x^{80}+x^{79}+x^{78}+x^{75}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{67}\\
&\quad +x^{64}+x^{63}+x^{62}+x^{61}+x^{58}+x^{56}+x^{54}+x^{53}+x^{52}+x^{45}+x^{42}\\
&\quad +x^{41}+x^{40}+x^{39}+x^{36}+x^{35}+x^{34}+x^{31}+x^{29}+x^{28}+x^{27}+x^{24}\\
&\quad +x^{22}+x^{20}+x^{19}+x^{18},\\
p_9(x) &= x^{114}+x^{111}+x^{108}+x^{106}+x^{105}+x^{104}+x^{103}+x^{101}+x^{100}+x^{98}\\
&\quad +x^{97}+x^{96}+x^{95}+x^{93}+x^{92}+x^{90}+x^{89}+x^{87}+x^{84}+x^{82}+x^{81}\\
&\quad +x^{79}+x^{78}+x^{75}+x^{74}+x^{71}+x^{70}+x^{68}+x^{67}+x^{66}+x^{65}+x^{64}\\
&\quad +x^{63}+x^{62}+x^{61}+x^{60}+x^{59}+x^{57}+x^{48}+x^{45}+x^{38}+x^{36}+x^{35}\\
&\quad +x^{34}+x^{33}+x^{31}+x^{26}+x^{23}+x^{20}+x^{18}+x^{17}+x^{15}+x^{14}+x^{12}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{11} + x^{9},\\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{31}\\
&+ x^{29} + x^{28} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12}\\
&+ x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{115} + x^{112} + x^{111} + x^{107}\\
&+ x^{105} + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&+ x^{91} + x^{90} + x^{89} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{78} + x^{76} + x^{75}\\
&+ x^{74} + x^{71} + x^{70} + x^{55} + x^{54} + x^{52} + x^{51} + x^{49} + x^{45} + x^{42} + x^{40}\\
&+ x^{39},\\
p_3(x) &= x^{120} + x^{117} + x^{116} + x^{114} + x^{113} + x^{111} + x^{110} + x^{108} + x^{107} + x^{106}\\
&+ x^{105} + x^{103} + x^{102} + x^{99} + x^{96} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{86}\\
&+ x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74}\\
&+ x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{43}\\
&+ x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{27},\\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{113} + x^{111} + x^{110} + x^{107} + x^{105} + x^{104} + x^{101}\\
&+ x^{99} + x^{98} + x^{97} + x^{94} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{85} + x^{84}\\
&+ x^{83} + x^{80} + x^{79} + x^{78} + x^{75} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{58} + x^{56} + x^{54} + x^{53} + x^{52} + x^{45} + x^{42}\\
&+ x^{41} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{31} + x^{29} + x^{28} + x^{27} + x^{24}\\
&+ x^{22} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{114} + x^{111} + x^{108} + x^{106} + x^{105} + x^{104} + x^{103} + x^{101} + x^{100} + x^{98}\\
&+ x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{84} + x^{82} + x^{81}\\
&+ x^{79} + x^{78} + x^{75} + x^{74} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{48} + x^{45} + x^{38} + x^{36} + x^{35}\\
&+ x^{34} + x^{33} + x^{31} + x^{26} + x^{23} + x^{20} + x^{18} + x^{17} + x^{15} + x^{14} + x^{12}\\
&+ x^{11} + x^{9},\\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{31}\\
&+ x^{29} + x^{28} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{15} + x^{13} + x^{12}\\
&+ x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0].

For $\phi(W^e, 1, 1)$:

$$
p_0(x) = x^{132} + x^{126} + x^{120} + x^{116} + x^{114} + x^{112} + x^{106} + x^{102} + x^{90} + x^{86}
$$

$$
\begin{aligned}
&+ x^{84} + x^{72} + x^{62} + x^{54} + x^{48} + x^{44} + x^{42},\\
p_3(x) &= x^{122} + x^{120} + x^{118} + x^{116} + x^{108} + x^{106} + x^{102} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{86} + x^{84} + x^{76} + x^{72} + x^{70} + x^{62} + x^{60} + x^{58} + x^{56} + x^{52}\\
&+ x^{50} + x^{48} + x^{40} + x^{36} + x^{32} + x^{30} + x^{18},\\
p_6(x) &= x^{122} + x^{120} + x^{118} + x^{110} + x^{108} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{78} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64}\\
&+ x^{58} + x^{52} + x^{48} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{20} + x^{18}\\
&+ x^{12} + x^{10} + x^{6} + 1,\\
p_9(x) &= x^{120} + x^{114} + x^{110} + x^{108} + x^{102} + x^{100} + x^{98} + x^{92} + x^{90} + x^{86}\\
&+ x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{56} + x^{46} + x^{44} + x^{40}\\
&+ x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{16} + x^{14} + x^{10} + x^{8} + x^{6},\\
p_{12}(x) &= x^{120} + x^{104} + x^{96} + x^{40} + x^{16} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{114} + x^{113} + x^{111} + x^{110}\\
&+ x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{93} + x^{92}\\
&+ x^{91} + x^{89} + x^{86} + x^{85} + x^{81} + x^{78} + x^{75} + x^{72} + x^{69} + x^{66} + x^{65}\\
&+ x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{54} + x^{51} + x^{49} + x^{45} + x^{44}\\
&+ x^{43} + x^{41} + x^{38} + x^{37} + x^{36},\\
p_3(x) &= x^{123} + x^{120} + x^{114} + x^{113} + x^{110} + x^{109} + x^{107} + x^{105} + x^{102} + x^{101}\\
&+ x^{98} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84} + x^{83} + x^{81}\\
&+ x^{79} + x^{77} + x^{75} + x^{70} + x^{69} + x^{67} + x^{66} + x^{59} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{40}\\
&+ x^{38} + x^{37} + x^{36} + x^{34} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24},\\
p_6(x) &= x^{123} + x^{120} + x^{114} + x^{113} + x^{110} + x^{107} + x^{105} + x^{101} + x^{99} + x^{94}\\
&+ x^{92} + x^{89} + x^{88} + x^{85} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70}\\
&+ x^{69} + x^{64} + x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{33} + x^{32} + x^{31} + x^{26} + x^{22} + x^{20} + x^{18} + x^{17} + x^{15}\\
&+ x^{14} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{111} + x^{109} + x^{108} + x^{43} + x^{41} + x^{40} + x^{31}\\
&+ x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{43}\\
&+ x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16}\\
&+ x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{126} + x^{124} + x^{121} + x^{120} + x^{117} + x^{116} + x^{110} + x^{108} \\
&\quad + x^{107} + x^{106} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91} \\
&\quad + x^{88} + x^{87} + x^{85} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} \\
&\quad + x^{69} + x^{62} + x^{60} + x^{57} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{39}, \\
p_3(x) &= x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} \\
&\quad + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{71} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{47} + x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{31} + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{114} + x^{110} + x^{104} + x^{98} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{36} \\
&\quad + x^{34} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18}, \\
p_9(x) &= x^{117} + x^{115} + x^{112} + x^{109} + x^{106} + x^{105} + x^{37} + x^{35} + x^{32} + x^{29} \\
&\quad + x^{26} + x^{25} + x^{21} + x^{19} + x^{16} + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{120} + x^{100} + x^{96} + x^{40} + x^{24} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{126} + x^{125} + x^{121} + x^{120} + x^{117} + x^{112} + x^{110} + x^{109} \\
&\quad + x^{107} + x^{104} + x^{102} + x^{101} + x^{96} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{85} + x^{84} + x^{80} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{69} + x^{66} + x^{64} \\
&\quad + x^{63} + x^{61} + x^{59} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{43} + x^{39}, \\
p_3(x) &= x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{104} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{87} + x^{86} \\
&\quad + x^{83} + x^{81} + x^{80} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{66} + x^{63} \\
&\quad + x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42} \\
&\quad + x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{126} + x^{122} + x^{118} + x^{116} + x^{108} + x^{106} + x^{98} + x^{90} + x^{88} + x^{86} \\
&\quad + x^{84} + x^{82} + x^{78} + x^{74} + x^{70} + x^{68} + x^{60} + x^{48} + x^{44} + x^{42} + x^{40} \\
&\quad + x^{38} + x^{30} + x^{26} + x^{20} + x^{18}, \\
p_9(x) &= x^{129} + x^{127} + x^{124} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110} \\
&\quad + x^{106} + x^{105} + x^{49} + x^{47} + x^{44} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14} \\
&\quad + x^{10} + x^{9}, \\
p_{12}(x) &= x^{132} + x^{124} + x^{120} + x^{112} + x^{108} + x^{104} + x^{96} + x^{52} + x^{44} + x^{40}
\end{aligned}$$

$$+ x^{36} + x^{32} + x^{12} + x^{8} + 1,$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{119} + x^{115} + x^{113} + x^{112} + x^{109} + x^{104} + x^{103} + x^{100} + x^{99} \\
&\quad + x^{97} + x^{93} + x^{92} + x^{91} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{123} + x^{120} + x^{117} + x^{115} + x^{108} + x^{105} + x^{104} + x^{100} + x^{99} + x^{98} \\
&\quad + x^{97} + x^{95} + x^{93} + x^{92} + x^{90} + x^{87} + x^{85} + x^{81} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{59} \\
&\quad + x^{58} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}, \\
p_6(x) &= x^{123} + x^{120} + x^{117} + x^{115} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100} \\
&\quad + x^{98} + x^{97} + x^{91} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{79} + x^{78} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{62} + x^{59} + x^{58} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{34} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} \\
&\quad + x^{19} + x^{17} + x^{14} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{111} + x^{109} + x^{108} + x^{43} + x^{41} + x^{40} + x^{31} \\
&\quad + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{15} + x^{13} + x^{12}, \\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} \\
&\quad + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1].

For $\phi(W^o, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{123} + x^{122} + x^{121} + x^{119} + x^{114} + x^{113} + x^{111} + x^{110} \\
&\quad + x^{108} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101} + x^{98} + x^{96} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{89} + x^{86} + x^{85} + x^{81} + x^{78} + x^{75} + x^{72} + x^{69} + x^{66} + x^{65} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{57} + x^{54} + x^{51} + x^{49} + x^{45} + x^{44} \\
&\quad + x^{43} + x^{41} + x^{38} + x^{37} + x^{36}, \\
p_3(x) &= x^{123} + x^{120} + x^{114} + x^{113} + x^{110} + x^{109} + x^{107} + x^{105} + x^{102} + x^{101} \\
&\quad + x^{98} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{87} + x^{86} + x^{84} + x^{83} + x^{81} \\
&\quad + x^{79} + x^{77} + x^{75} + x^{70} + x^{69} + x^{67} + x^{66} + x^{59} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{40} \\
&\quad + x^{38} + x^{37} + x^{36} + x^{34} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24}, \\
p_6(x) &= x^{123} + x^{120} + x^{114} + x^{113} + x^{110} + x^{107} + x^{105} + x^{101} + x^{99} + x^{94}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{92} + x^{89} + x^{88} + x^{85} + x^{81} + x^{80} + x^{79} + x^{77} + x^{74} + x^{73} + x^{70}\\
&+ x^{69} + x^{64} + x^{60} + x^{59} + x^{56} + x^{55} + x^{54} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{40} + x^{39} + x^{33} + x^{32} + x^{31} + x^{26} + x^{22} + x^{20} + x^{18} + x^{17} + x^{15}\\
&+ x^{14} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,\\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{111} + x^{109} + x^{108} + x^{43} + x^{41} + x^{40} + x^{31}\\
&+ x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{43}\\
&+ x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16}\\
&+ x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{126} + x^{125} + x^{121} + x^{120} + x^{117} + x^{112} + x^{110} + x^{109}\\
&+ x^{107} + x^{104} + x^{102} + x^{101} + x^{96} + x^{92} + x^{91} + x^{88} + x^{87} + x^{86}\\
&+ x^{85} + x^{84} + x^{80} + x^{78} + x^{76} + x^{75} + x^{74} + x^{73} + x^{69} + x^{66} + x^{64}\\
&+ x^{63} + x^{61} + x^{59} + x^{57} + x^{54} + x^{51} + x^{50} + x^{49} + x^{43} + x^{39},\\
p_3(x) &= x^{123} + x^{121} + x^{119} + x^{118} + x^{117} + x^{114} + x^{112} + x^{111} + x^{110} + x^{104}\\
&+ x^{102} + x^{101} + x^{100} + x^{97} + x^{95} + x^{94} + x^{93} + x^{90} + x^{88} + x^{87} + x^{86}\\
&+ x^{83} + x^{81} + x^{80} + x^{79} + x^{76} + x^{74} + x^{73} + x^{72} + x^{69} + x^{66} + x^{63}\\
&+ x^{61} + x^{60} + x^{57} + x^{56} + x^{55} + x^{52} + x^{50} + x^{49} + x^{48} + x^{45} + x^{42}\\
&+ x^{39} + x^{37} + x^{36} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{126} + x^{122} + x^{118} + x^{116} + x^{108} + x^{106} + x^{98} + x^{90} + x^{88} + x^{86}\\
&+ x^{84} + x^{82} + x^{78} + x^{74} + x^{70} + x^{68} + x^{60} + x^{48} + x^{44} + x^{42} + x^{40}\\
&+ x^{38} + x^{30} + x^{26} + x^{20} + x^{18},\\
p_9(x) &= x^{129} + x^{127} + x^{124} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110}\\
&+ x^{106} + x^{105} + x^{49} + x^{47} + x^{44} + x^{39} + x^{38} + x^{36} + x^{35} + x^{32} + x^{31}\\
&+ x^{30} + x^{28} + x^{26} + x^{25} + x^{23} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16} + x^{14}\\
&+ x^{10} + x^{9},\\
p_{12}(x) &= x^{132} + x^{124} + x^{120} + x^{112} + x^{108} + x^{104} + x^{96} + x^{52} + x^{44} + x^{40}\\
&+ x^{36} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{132} + x^{128} + x^{126} + x^{124} + x^{121} + x^{120} + x^{117} + x^{116} + x^{110} + x^{108}\\
&+ x^{107} + x^{106} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{93} + x^{92} + x^{91}\\
&+ x^{88} + x^{87} + x^{85} + x^{80} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71}\\
&+ x^{69} + x^{62} + x^{60} + x^{57} + x^{56} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{39},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{103} + x^{102} + x^{101} + x^{100} + x^{99} \\
&\quad + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{71} \\
&\quad + x^{69} + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{47} + x^{45} + x^{43} + x^{42} + x^{41} + x^{38} + x^{36} + x^{34} + x^{33} \\
&\quad + x^{31} + x^{30} + x^{29} + x^{28} + x^{27}, \\
p_6(x) &= x^{114} + x^{110} + x^{104} + x^{98} + x^{94} + x^{92} + x^{90} + x^{86} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{72} + x^{68} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{42} + x^{40} + x^{36} \\
&\quad + x^{34} + x^{28} + x^{24} + x^{22} + x^{20} + x^{18}, \\
p_9(x) &= x^{117} + x^{115} + x^{112} + x^{109} + x^{106} + x^{105} + x^{37} + x^{35} + x^{32} + x^{29} \\
&\quad + x^{26} + x^{25} + x^{21} + x^{19} + x^{16} + x^{13} + x^{10} + x^{9}, \\
p_{12}(x) &= x^{120} + x^{100} + x^{96} + x^{40} + x^{24} + x^{20} + x^{16} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{119} + x^{115} + x^{113} + x^{112} + x^{109} + x^{104} + x^{103} + x^{100} + x^{99} \\
&\quad + x^{97} + x^{93} + x^{92} + x^{91} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{70} + x^{69} \\
&\quad + x^{68} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{50} + x^{49} + x^{47} + x^{46} + x^{44} + x^{43} + x^{42}, \\
p_3(x) &= x^{123} + x^{120} + x^{117} + x^{115} + x^{108} + x^{105} + x^{104} + x^{100} + x^{99} + x^{98} \\
&\quad + x^{97} + x^{95} + x^{93} + x^{92} + x^{90} + x^{87} + x^{85} + x^{81} + x^{80} + x^{79} + x^{78} \\
&\quad + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{59} \\
&\quad + x^{58} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{44} + x^{43} + x^{42} + x^{41} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}, \\
p_6(x) &= x^{123} + x^{120} + x^{117} + x^{115} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{100} \\
&\quad + x^{98} + x^{97} + x^{91} + x^{90} + x^{88} + x^{86} + x^{84} + x^{80} + x^{79} + x^{78} + x^{76} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{62} + x^{59} + x^{58} + x^{46} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{39} + x^{37} + x^{34} + x^{29} + x^{28} + x^{27} + x^{26} + x^{24} \\
&\quad + x^{19} + x^{17} + x^{14} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{111} + x^{109} + x^{108} + x^{43} + x^{41} + x^{40} + x^{31} \\
&\quad + x^{29} + x^{28} + x^{27} + x^{25} + x^{24} + x^{15} + x^{13} + x^{12}, \\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{27} + x^{25} + x^{24} + x^{23} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} \\
&\quad + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	180	[304, 305]	360	[514, 515]	540	[680, 34]	720	[787, 272]
1	[3, 4]	181	[306, 307]	361	[516, 517]	541	[685, 684]	721	[815, 309]
2	[5, 6]	182	[213, 273]	362	[510, 518]	542	[701, 367]	722	[636, 816]
3	[7, 8]	183	[308, 309]	363	[519, 491]	543	[658, 147]	723	[817, 818]
4	[9, 10]	184	[310, 311]	364	[520, 300]	544	[407, 152]	724	[819, 627]
5	[11, 12]	185	[312, 313]	365	[17, 93]	545	[383, 149]	725	[89, 820]
6	[13, 14]	186	[233, 254]	366	[521, 408]	546	[355, 702]	726	[821, 822]
7	[4, 15]	187	[314, 90]	367	[522, 523]	547	[44, 703]	727	[85, 291]
8	[16, 17]	188	[315, 316]	368	[524, 256]	548	[183, 58]	728	[794, 823]
9	[18, 19]	189	[317, 318]	369	[226, 525]	549	[474, 389]	729	[824, 505]
10	[20, 21]	190	[319, 320]	370	[526, 527]	550	[691, 671]	730	[825, 767]
11	[22, 23]	191	[321, 322]	371	[528, 529]	551	[704, 373]	731	[826, 579]
12	[24, 25]	192	[323, 324]	372	[530, 531]	552	[705, 706]	732	[67, 493]
13	[26, 27]	193	[112, 232]	373	[532, 533]	553	[399, 707]	733	[640, 103]
14	[28, 29]	194	[325, 326]	374	[534, 535]	554	[390, 150]	734	[827, 776]
15	[8, 30]	195	[327, 328]	375	[536, 518]	555	[152, 409]	735	[828, 315]
16	[31, 32]	196	[285, 258]	376	[537, 320]	556	[397, 132]	736	[761, 829]
17	[33, 34]	197	[329, 330]	377	[538, 539]	557	[708, 2]	737	[830, 104]
18	[35, 36]	198	[331, 210]	378	[530, 540]	558	[350, 446]	738	[203, 786]
19	[37, 38]	199	[290, 332]	379	[541, 259]	559	[197, 672]	739	[831, 299]
20	[39, 40]	200	[333, 334]	380	[542, 68]	560	[675, 709]	740	[809, 832]
21	[41, 42]	201	[36, 335]	381	[543, 544]	561	[386, 370]	741	[569, 542]
22	[43, 44]	202	[130, 336]	382	[273, 258]	562	[710, 378]	742	[833, 625]
23	[45, 46]	203	[337, 338]	383	[237, 545]	563	[651, 373]	743	[105, 834]
24	[47, 48]	204	[338, 339]	384	[512, 546]	564	[711, 135]	744	[510, 835]
25	[49, 50]	205	[340, 167]	385	[547, 548]	565	[384, 186]	745	[836, 305]
26	[51, 52]	206	[341, 7]	386	[549, 527]	566	[699, 415]	746	[6, 837]
27	[53, 54]	207	[342, 343]	387	[550, 528]	567	[57, 184]	747	[838, 318]
28	[55, 56]	208	[162, 344]	388	[551, 552]	568	[478, 698]	748	[593, 839]
29	[57, 58]	209	[345, 143]	389	[553, 236]	569	[673, 14]	749	[83, 840]
30	[15, 59]	210	[346, 347]	390	[554, 555]	570	[712, 44]	750	[841, 680]
31	[60, 61]	211	[348, 349]	391	[556, 295]	571	[397, 681]	751	[742, 729]
32	[62, 63]	212	[350, 2]	392	[554, 557]	572	[713, 44]	752	[415, 690]
33	[64, 65]	213	[351, 352]	393	[558, 85]	573	[441, 442]	753	[47, 842]
34	[66, 67]	214	[353, 56]	394	[559, 529]	574	[41, 382]	754	[185, 385]
35	[68, 69]	215	[354, 355]	395	[560, 561]	575	[135, 382]	755	[38, 470]
36	[70, 71]	216	[356, 357]	396	[562, 563]	576	[438, 188]	756	[843, 662]
37	[72, 73]	217	[358, 359]	397	[502, 232]	577	[714, 394]	757	[349, 844]
38	[74, 75]	218	[360, 361]	398	[564, 250]	578	[379, 452]	758	[356, 174]
39	[76, 77]	219	[362, 125]	399	[565, 566]	579	[437, 157]	759	[455, 845]
40	[29, 78]	220	[129, 363]	400	[567, 299]	580	[715, 182]	760	[846, 418]
41	[79, 80]	221	[188, 364]	401	[509, 536]	581	[716, 455]	761	[649, 847]
42	[81, 82]	222	[362, 365]	402	[568, 569]	582	[51, 717]	762	[848, 687]

43	[83, 84]	223	[366, 364]	403	[488, 570]	583	[430, 424]	763	[710, 129]
44	[85, 86]	224	[154, 124]	404	[571, 510]	584	[688, 126]	764	[150, 409]
45	[87, 88]	225	[367, 368]	405	[572, 573]	585	[196, 663]	765	[666, 726]
46	[89, 90]	226	[129, 369]	406	[574, 316]	586	[709, 662]	766	[849, 850]
47	[91, 92]	227	[370, 371]	407	[575, 251]	587	[718, 179]	767	[189, 738]
48	[93, 94]	228	[372, 373]	408	[408, 408]	588	[141, 719]	768	[173, 357]
49	[95, 96]	229	[374, 117]	409	[251, 236]	589	[715, 470]	769	[470, 436]
50	[97, 98]	230	[137, 375]	410	[576, 220]	590	[433, 199]	770	[851, 128]
51	[99, 100]	231	[376, 377]	411	[504, 65]	591	[720, 147]	771	[471, 852]
52	[101, 63]	232	[378, 369]	412	[543, 302]	592	[9, 170]	772	[466, 853]
53	[102, 103]	233	[379, 380]	413	[514, 577]	593	[713, 654]	773	[854, 52]
54	[104, 105]	234	[381, 149]	414	[578, 281]	594	[721, 357]	774	[687, 855]
55	[106, 107]	235	[149, 152]	415	[579, 80]	595	[722, 723]	775	[744, 739]
56	[3, 108]	236	[382, 383]	416	[580, 581]	596	[724, 451]	776	[856, 442]
57	[109, 110]	237	[42, 383]	417	[582, 557]	597	[170, 395]	777	[359, 694]
58	[111, 112]	238	[384, 385]	418	[583, 307]	598	[372, 652]	778	[857, 853]
59	[10, 113]	239	[386, 387]	419	[22, 66]	599	[701, 474]	779	[858, 391]
60	[114, 115]	240	[388, 373]	420	[584, 585]	600	[697, 370]	780	[859, 168]
61	[116, 117]	241	[367, 389]	421	[496, 586]	601	[725, 726]	781	[860, 152]
62	[118, 119]	242	[390, 152]	422	[548, 587]	602	[676, 196]	782	[56, 397]
63	[120, 121]	243	[391, 138]	423	[588, 549]	603	[727, 179]	783	[655, 861]
64	[122, 123]	244	[392, 367]	424	[561, 497]	604	[355, 347]	784	[711, 41]
65	[124, 125]	245	[393, 344]	425	[589, 96]	605	[728, 729]	785	[862, 123]
66	[8, 126]	246	[168, 394]	426	[590, 563]	606	[730, 176]	786	[389, 686]
67	[44, 127]	247	[10, 395]	427	[107, 501]	607	[159, 7]	787	[720, 409]
68	[128, 129]	248	[396, 397]	428	[591, 85]	608	[714, 169]	788	[468, 724]
69	[130, 50]	249	[398, 143]	429	[322, 312]	609	[731, 140]	789	[746, 732]
70	[131, 10]	250	[399, 400]	430	[274, 592]	610	[155, 182]	790	[155, 470]
71	[132, 133]	251	[135, 42]	431	[593, 552]	611	[413, 732]	791	[427, 132]
72	[134, 135]	252	[401, 402]	432	[99, 594]	612	[659, 402]	792	[670, 692]
73	[136, 2]	253	[403, 404]	433	[595, 596]	613	[733, 656]	793	[349, 664]
74	[137, 138]	254	[405, 168]	434	[597, 215]	614	[734, 172]	794	[862, 200]
75	[139, 140]	255	[406, 406]	435	[598, 23]	615	[198, 434]	795	[39, 847]
76	[141, 142]	256	[407, 150]	436	[599, 600]	616	[374, 693]	796	[863, 465]
77	[143, 144]	257	[151, 149]	437	[601, 602]	617	[412, 9]	797	[118, 845]
78	[145, 146]	258	[406, 408]	438	[87, 520]	618	[444, 124]	798	[857, 54]
79	[147, 148]	259	[158, 135]	439	[603, 604]	619	[735, 187]	799	[140, 689]
80	[149, 150]	260	[382, 151]	440	[605, 606]	620	[646, 154]	800	[461, 411]
81	[42, 151]	261	[409, 148]	441	[607, 608]	621	[736, 737]	801	[864, 407]
82	[152, 147]	262	[148, 135]	442	[6, 609]	622	[12, 738]	802	[749, 865]
83	[153, 154]	263	[410, 362]	443	[608, 3]	623	[114, 739]	803	[730, 866]
84	[37, 155]	264	[143, 379]	444	[592, 17]	624	[445, 2]	804	[696, 386]
85	[156, 157]	265	[387, 371]	445	[310, 610]	625	[740, 741]	805	[13, 674]
86	[147, 158]	266	[35, 411]	446	[493, 98]	626	[160, 688]	806	[743, 142]

87	[159, 160]	267	[412, 131]	447	[611, 592]	627	[742, 465]	807	[747, 128]
88	[58, 161]	268	[413, 414]	448	[541, 612]	628	[743, 719]	808	[851, 710]
89	[162, 163]	269	[140, 415]	449	[613, 614]	629	[709, 196]	809	[867, 348]
90	[164, 165]	270	[416, 12]	450	[615, 69]	630	[744, 115]	810	[376, 187]
91	[166, 167]	271	[417, 418]	451	[616, 93]	631	[745, 706]	811	[653, 868]
92	[168, 169]	272	[408, 406]	452	[617, 618]	632	[672, 479]	812	[398, 665]
93	[170, 113]	273	[409, 158]	453	[329, 619]	633	[676, 662]	813	[472, 348]
94	[171, 172]	274	[419, 33]	454	[620, 596]	634	[746, 414]	814	[401, 422]
95	[173, 174]	275	[335, 420]	455	[245, 267]	635	[377, 188]	815	[728, 465]
96	[175, 176]	276	[358, 338]	456	[621, 622]	636	[462, 707]	816	[423, 853]
97	[46, 177]	277	[144, 380]	457	[277, 287]	637	[648, 717]	817	[733, 861]
98	[178, 179]	278	[421, 422]	458	[623, 110]	638	[747, 710]	818	[869, 355]
99	[180, 181]	279	[423, 54]	459	[624, 495]	639	[735, 377]	819	[456, 868]
100	[38, 182]	280	[403, 424]	460	[625, 221]	640	[748, 349]	820	[724, 460]
101	[183, 184]	281	[425, 426]	461	[626, 627]	641	[464, 729]	821	[470, 870]
102	[185, 186]	282	[134, 41]	462	[628, 629]	642	[749, 741]	822	[871, 426]
103	[187, 188]	283	[396, 427]	463	[491, 614]	643	[346, 702]	823	[175, 866]
104	[11, 189]	284	[428, 165]	464	[630, 631]	644	[375, 680]	824	[442, 361]
105	[190, 191]	285	[156, 429]	465	[632, 89]	645	[421, 402]	825	[867, 473]
106	[192, 193]	286	[430, 404]	466	[288, 633]	646	[750, 751]	826	[864, 149]
107	[131, 170]	287	[431, 432]	467	[634, 635]	647	[531, 752]	827	[843, 196]
108	[31, 194]	288	[433, 434]	468	[576, 625]	648	[753, 264]	828	[860, 150]
109	[195, 196]	289	[435, 115]	469	[534, 599]	649	[754, 755]	829	[53, 853]
110	[197, 46]	290	[182, 436]	470	[636, 257]	650	[756, 757]	830	[872, 137]
111	[198, 199]	291	[437, 429]	471	[637, 315]	651	[758, 642]	831	[455, 119]
112	[122, 200]	292	[438, 366]	472	[106, 500]	652	[759, 78]	832	[873, 476]
113	[19, 201]	293	[337, 359]	473	[269, 540]	653	[760, 608]	833	[863, 729]
114	[202, 203]	294	[439, 181]	474	[638, 639]	654	[761, 545]	834	[189, 874]
115	[204, 205]	295	[440, 194]	475	[640, 641]	655	[762, 549]	835	[856, 360]
116	[206, 207]	296	[441, 360]	476	[293, 223]	656	[763, 289]	836	[341, 160]
117	[208, 209]	297	[442, 443]	477	[20, 636]	657	[756, 631]	837	[12, 874]
118	[105, 75]	298	[153, 444]	478	[323, 642]	658	[72, 764]	838	[427, 681]
119	[210, 211]	299	[393, 163]	479	[643, 644]	659	[765, 241]	839	[193, 844]
120	[212, 213]	300	[160, 8]	480	[645, 73]	660	[766, 767]	840	[184, 161]
121	[214, 215]	301	[445, 446]	481	[360, 443]	661	[768, 581]	841	[580, 875]
122	[216, 217]	302	[447, 52]	482	[646, 444]	662	[487, 82]	842	[876, 104]
123	[218, 105]	303	[353, 396]	483	[426, 343]	663	[328, 485]	843	[482, 877]
124	[202, 219]	304	[448, 449]	484	[432, 647]	664	[324, 769]	844	[540, 752]
125	[220, 221]	305	[450, 451]	485	[473, 349]	665	[770, 542]	845	[878, 791]
126	[222, 223]	306	[144, 452]	486	[648, 52]	666	[771, 643]	846	[250, 553]
127	[84, 224]	307	[416, 189]	487	[649, 40]	667	[59, 322]	847	[879, 533]
128	[225, 226]	308	[453, 366]	488	[650, 124]	668	[772, 773]	848	[109, 584]
129	[25, 227]	309	[454, 455]	489	[651, 652]	669	[308, 774]	849	[291, 880]
130	[228, 229]	310	[456, 457]	490	[653, 457]	670	[775, 776]	850	[881, 332]

131	[230, 231]	311	[458, 357]	491	[654, 127]	671	[777, 569]	851	[600, 25]
132	[108, 15]	312	[459, 146]	492	[655, 656]	672	[771, 539]	852	[882, 822]
133	[232, 233]	313	[450, 460]	493	[657, 450]	673	[778, 579]	853	[883, 334]
134	[234, 235]	314	[461, 36]	494	[658, 409]	674	[779, 780]	854	[884, 570]
135	[236, 237]	315	[462, 400]	495	[659, 422]	675	[781, 487]	855	[65, 277]
136	[238, 107]	316	[150, 147]	496	[660, 48]	676	[782, 515]	856	[885, 622]
137	[239, 240]	317	[180, 463]	497	[661, 115]	677	[783, 548]	857	[811, 799]
138	[241, 219]	318	[464, 465]	498	[662, 663]	678	[784, 20]	858	[750, 635]
139	[242, 27]	319	[466, 54]	499	[474, 368]	679	[642, 769]	859	[886, 217]
140	[243, 244]	320	[467, 422]	500	[440, 32]	680	[295, 71]	860	[801, 255]
141	[245, 246]	321	[55, 396]	501	[193, 664]	681	[785, 275]	861	[887, 550]
142	[247, 29]	322	[468, 450]	502	[665, 379]	682	[219, 786]	862	[888, 573]
143	[248, 249]	323	[469, 470]	503	[666, 667]	683	[787, 213]	863	[16, 616]
144	[250, 251]	324	[378, 363]	504	[665, 144]	684	[788, 89]	864	[784, 260]
145	[252, 250]	325	[359, 339]	505	[668, 167]	685	[789, 790]	865	[889, 249]
146	[253, 254]	326	[471, 191]	506	[669, 335]	686	[523, 791]	866	[890, 231]
147	[255, 256]	327	[472, 473]	507	[165, 44]	687	[792, 793]	867	[506, 891]
148	[21, 257]	328	[392, 474]	508	[195, 662]	688	[794, 265]	868	[892, 818]
149	[258, 259]	329	[475, 386]	509	[439, 463]	689	[795, 237]	869	[893, 606]
150	[260, 21]	330	[172, 476]	510	[670, 671]	690	[244, 517]	870	[267, 894]
151	[261, 86]	331	[477, 121]	511	[672, 177]	691	[796, 293]	871	[895, 229]
152	[86, 262]	332	[478, 432]	512	[673, 674]	692	[797, 587]	872	[263, 241]
153	[263, 202]	333	[46, 479]	513	[675, 676]	693	[326, 244]	873	[896, 240]
154	[264, 265]	334	[480, 121]	514	[145, 677]	694	[577, 556]	874	[23, 67]
155	[266, 267]	335	[69, 226]	515	[678, 131]	695	[798, 524]	875	[335, 855]
156	[268, 269]	336	[481, 6]	516	[432, 679]	696	[762, 526]	876	[182, 870]
157	[270, 271]	337	[482, 483]	517	[138, 680]	697	[799, 528]	877	[345, 665]
158	[272, 273]	338	[312, 284]	518	[681, 133]	698	[753, 794]	878	[58, 897]
159	[274, 16]	339	[484, 485]	519	[33, 682]	699	[800, 209]	879	[431, 698]
160	[77, 275]	340	[486, 487]	520	[683, 155]	700	[234, 773]	880	[467, 402]
161	[276, 277]	341	[488, 489]	521	[351, 684]	701	[801, 524]	881	[872, 391]
162	[278, 279]	342	[490, 108]	522	[56, 427]	702	[802, 233]	882	[858, 137]
163	[280, 281]	343	[229, 491]	523	[683, 38]	703	[513, 763]	883	[657, 724]
164	[282, 279]	344	[492, 493]	524	[685, 352]	704	[25, 803]	884	[725, 667]
165	[283, 284]	345	[494, 495]	525	[368, 686]	705	[804, 604]	885	[475, 697]
166	[213, 285]	346	[496, 497]	526	[49, 336]	706	[805, 639]	886	[469, 182]
167	[286, 287]	347	[61, 498]	527	[687, 420]	707	[287, 205]	887	[7, 688]
168	[288, 289]	348	[499, 227]	528	[688, 30]	708	[806, 507]	888	[704, 652]
169	[217, 290]	349	[500, 501]	529	[689, 690]	709	[807, 609]	889	[459, 677]
170	[291, 262]	350	[111, 502]	530	[691, 692]	710	[566, 793]	890	[190, 852]
171	[292, 293]	351	[503, 504]	531	[116, 693]	711	[781, 81]	891	[388, 652]
172	[294, 295]	352	[307, 505]	532	[338, 694]	712	[808, 513]	892	[654, 703]
173	[295, 296]	353	[506, 507]	533	[695, 418]	713	[809, 810]	893	[453, 188]
174	[297, 271]	354	[508, 61]	534	[696, 697]	714	[811, 550]	894	[411, 335]

175	[298, 92]	355	[509, 510]	535	[698, 647]	715	[206, 812]	895	[43, 654]
176	[299, 19]	356	[64, 276]	536	[699, 689]	716	[813, 210]	896	[660, 842]
177	[88, 300]	357	[511, 312]	537	[381, 407]	717	[332, 326]	897	[110, 585]
178	[301, 302]	358	[327, 484]	538	[391, 375]	718	[814, 20]		
179	[303, 94]	359	[512, 513]	539	[700, 676]	719	[769, 780]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	129	0	258	1	387	1	516	0	645	0	774	0
1	0	130	1	259	0	388	0	517	0	646	1	775	0
2	0	131	1	260	0	389	1	518	0	647	1	776	1
3	0	132	1	261	0	390	1	519	1	648	1	777	1
4	0	133	1	262	1	391	0	520	0	649	0	778	1
5	0	134	1	263	0	392	1	521	1	650	1	779	1
6	0	135	0	264	1	393	1	522	0	651	1	780	0
7	0	136	1	265	1	394	1	523	0	652	1	781	0
8	1	137	1	266	0	395	0	524	0	653	1	782	0
9	0	138	0	267	1	396	1	525	0	654	0	783	0
10	1	139	1	268	1	397	0	526	0	655	0	784	0
11	0	140	0	269	0	398	0	527	0	656	1	785	0
12	1	141	1	270	1	399	1	528	0	657	1	786	1
13	0	142	1	271	0	400	0	529	1	658	1	787	0
14	0	143	1	272	0	401	1	530	1	659	1	788	0
15	1	144	1	273	0	402	0	531	1	660	0	789	0
16	1	145	1	274	0	403	1	532	0	661	0	790	0
17	1	146	1	275	1	404	0	533	1	662	0	791	1
18	0	147	1	276	0	405	1	534	0	663	1	792	0
19	1	148	1	277	1	406	1	535	1	664	1	793	0
20	1	149	1	278	0	407	0	536	1	665	0	794	0
21	1	150	0	279	1	408	0	537	1	666	1	795	1
22	0	151	0	280	1	409	0	538	0	667	1	796	1
23	0	152	1	281	1	410	0	539	1	668	0	797	0
24	1	153	0	282	1	411	0	540	0	669	1	798	1
25	0	154	1	283	1	412	1	541	0	670	0	799	0
26	0	155	0	284	0	413	1	542	0	671	1	800	1
27	0	156	1	285	1	414	0	543	1	672	1	801	0
28	0	157	1	286	0	415	0	544	0	673	1	802	0
29	0	158	0	287	1	416	1	545	1	674	1	803	1
30	1	159	0	288	0	417	0	546	1	675	0	804	0
31	1	160	1	289	1	418	0	547	1	676	0	805	0
32	0	161	0	290	1	419	0	548	1	677	0	806	0
33	1	162	0	291	0	420	0	549	1	678	0	807	1
34	1	163	1	292	0	421	0	550	1	679	0	808	0
35	0	164	1	293	1	422	1	551	0	680	0	809	1

36	1	165	1	294	1	423	1	552	0	681	0	810	1
37	1	166	1	295	0	424	1	553	1	682	0	811	1
38	1	167	0	296	0	425	1	554	1	683	0	812	0
39	1	168	0	297	0	426	0	555	1	684	0	813	0
40	0	169	0	298	0	427	1	556	0	685	0	814	1
41	1	170	0	299	1	428	0	557	0	686	0	815	0
42	1	171	0	300	1	429	0	558	1	687	0	816	1
43	0	172	1	301	0	430	0	559	1	688	0	817	1
44	1	173	0	302	1	431	1	560	0	689	1	818	1
45	0	174	0	303	1	432	0	561	1	690	1	819	0
46	0	175	0	304	0	433	0	562	1	691	1	820	0
47	1	176	1	305	1	434	0	563	1	692	0	821	0
48	0	177	1	306	1	435	1	564	0	693	1	822	0
49	0	178	0	307	1	436	0	565	1	694	1	823	0
50	0	179	1	308	1	437	0	566	1	695	1	824	0
51	0	180	0	309	1	438	0	567	0	696	0	825	1
52	1	181	1	310	0	439	1	568	0	697	0	826	0
53	0	182	1	311	1	440	0	569	1	698	1	827	1
54	0	183	1	312	0	441	0	570	0	699	1	828	0
55	0	184	0	313	1	442	0	571	0	700	1	829	0
56	0	185	0	314	1	443	1	572	1	701	0	830	0
57	0	186	0	315	0	444	0	573	0	702	0	831	1
58	1	187	1	316	0	445	0	574	1	703	0	832	0
59	1	188	0	317	0	446	1	575	0	704	0	833	1
60	1	189	0	318	0	447	1	576	0	705	0	834	0
61	1	190	0	319	0	448	0	577	1	706	0	835	1
62	0	191	0	320	0	449	1	578	0	707	1	836	1
63	1	192	0	321	0	450	1	579	0	708	0	837	1
64	1	193	1	322	0	451	0	580	1	709	1	838	1
65	1	194	1	323	0	452	1	581	0	710	1	839	1
66	1	195	0	324	1	453	1	582	0	711	0	840	0
67	1	196	1	325	1	454	1	583	0	712	0	841	1
68	0	197	1	326	1	455	1	584	0	713	1	842	1
69	1	198	1	327	0	456	0	585	1	714	1	843	1
70	1	199	1	328	1	457	1	586	1	715	1	844	0
71	1	200	0	329	1	458	1	587	1	716	0	845	1
72	1	201	1	330	1	459	0	588	1	717	0	846	1
73	1	202	1	331	1	460	1	589	1	718	1	847	1
74	1	203	1	332	0	461	1	590	0	719	0	848	0
75	1	204	0	333	0	462	0	591	0	720	0	849	0
76	1	205	1	334	0	463	0	592	0	721	0	850	0
77	1	206	1	335	1	464	0	593	1	722	0	851	0
78	1	207	1	336	1	465	1	594	0	723	1	852	1
79	1	208	0	337	1	466	0	595	0	724	0	853	1

80	1	209	1	338	0	467	0	596	0	725	0	854	0
81	1	210	0	339	0	468	0	597	0	726	0	855	1
82	1	211	1	340	1	469	0	598	1	727	1	856	1
83	0	212	1	341	1	470	0	599	0	728	0	857	1
84	1	213	1	342	1	471	1	600	0	729	0	858	1
85	1	214	1	343	0	472	0	601	0	730	1	859	0
86	1	215	0	344	0	473	0	602	0	731	0	860	0
87	0	216	1	345	1	474	1	603	1	732	1	861	0
88	1	217	0	346	0	475	1	604	1	733	1	862	0
89	0	218	1	347	1	476	1	605	0	734	1	863	1
90	1	219	0	348	1	477	1	606	1	735	0	864	0
91	1	220	0	349	0	478	0	607	0	736	0	865	0
92	0	221	0	350	1	479	0	608	1	737	0	866	0
93	0	222	0	351	1	480	0	609	0	738	1	867	1
94	0	223	1	352	1	481	1	610	0	739	1	868	0
95	0	224	1	353	1	482	1	611	1	740	1	869	1
96	0	225	0	354	0	483	0	612	1	741	1	870	1
97	0	226	0	355	1	484	0	613	1	742	1	871	0
98	0	227	0	356	1	485	0	614	1	743	0	872	0
99	0	228	1	357	1	486	1	615	1	744	0	873	0
100	1	229	0	358	0	487	0	616	0	745	1	874	0
101	1	230	1	359	1	488	1	617	1	746	0	875	1
102	0	231	1	360	1	489	1	618	0	747	1	876	1
103	1	232	1	361	0	490	1	619	0	748	1	877	1
104	0	233	0	362	0	491	0	620	1	749	0	878	1
105	0	234	1	363	1	492	0	621	0	750	1	879	1
106	0	235	1	364	0	493	1	622	1	751	1	880	0
107	1	236	0	365	1	494	1	623	1	752	0	881	0
108	1	237	1	366	1	495	1	624	0	753	1	882	1
109	0	238	1	367	0	496	0	625	1	754	0	883	1
110	1	239	1	368	0	497	0	626	1	755	1	884	0
111	1	240	0	369	0	498	0	627	1	756	1	885	1
112	1	241	0	370	0	499	1	628	0	757	0	886	0
113	1	242	1	371	0	500	0	629	1	758	1	887	0
114	1	243	0	372	1	501	1	630	0	759	1	888	0
115	0	244	1	373	0	502	0	631	1	760	1	889	0
116	1	245	1	374	0	503	1	632	1	761	0	890	0
117	0	246	0	375	1	504	0	633	0	762	0	891	0
118	0	247	1	376	1	505	0	634	0	763	1	892	0
119	0	248	1	377	0	506	1	635	0	764	0	893	1
120	1	249	0	378	1	507	1	636	0	765	1	894	0
121	1	250	1	379	0	508	0	637	1	766	0	895	0
122	1	251	0	380	0	509	1	638	1	767	0	896	0
123	1	252	1	381	1	510	0	639	0	768	0	897	1

124	1	253	1	382	0	511	1	640	1	769	0		
125	0	254	1	383	1	512	1	641	0	770	0		
126	0	255	1	384	1	513	0	642	0	771	1		
127	1	256	0	385	1	514	1	643	0	772	0		
128	0	257	0	386	1	515	0	644	1	773	0		

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