# PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^3, z^4 + z^3 + z + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the $(a, b)$-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements $(a, b)$ in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^3, z^4 + z^3 + z + 1)$.

## 1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \ldots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let $a$ and $b$ be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the $(a, b)$-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \ldots) = (a, b, b, a, b, a, a, b, \ldots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

*Date*: 9/04/2020 17:26:06 .

2010 *Mathematics Subject Classification.* 11B85, 11J70, 11B50, 11Y65, 05A15.

*Key words and phrases.* automatic sequence, continued fraction, Thue-Morse sequence.

* This paper was automatically generated from a computer program developed by the authors, with 25.8 minutes CPU time used.

**Theorem 1.1.** *When $(a,b)=(z^3, z^4+z^3+z+1)$, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

*where, for* $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{20}+z^{19}+z^{17}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^7+z^6+z^5+z^3\\
&\quad +1,\\
p_1(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{10}+z^8\\
&\quad +z^6+z^3,\\
p_2(z) &= z^{28}+z^{24}+z^{22}+z^{20}+z^{16}+z^{14}+z^{10}+z^6,\\
p_3(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{10}+z^8\\
&\quad +z^6+z^3,\\
p_4(z) &= z^{22}+z^{21}+z^{19}+z^{17}+z^{16}+z^{15}+z^{13}+z^7+z^5+z^4+z^3,
\end{aligned}$$

*and for* $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{20}+z^{19}+z^{17}+z^{16}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^7+z^6+z^5\\
&\quad +z^4+z^3,\\
p_1(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{10}+z^8\\
&\quad +z^6+z^3,\\
p_2(z) &= z^{28}+z^{24}+z^{22}+z^{20}+z^{16}+z^{14}+z^{10}+z^6,\\
p_3(z) &= z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{20}+z^{18}+z^{17}+z^{16}+z^{15}+z^{10}+z^8\\
&\quad +z^6+z^3,\\
p_4(z) &= z^{22}+z^{21}+z^{19}+z^{17}+z^{15}+z^{13}+z^7+z^5+z^3+1.
\end{aligned}$$

## 2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad \cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times
\end{aligned}$$

$$\cdots \times x^{\deg(\bar{t}_0)} \begin{pmatrix} \bar{t}_0(1/x) & 1 \\ 1 & 0 \end{pmatrix},$$

where $\bar{\mathbf{t}}$ is the $(b,a)$-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n} \tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n} \tilde{Q}_{2^n-1}(x),$$

for some positive integer $d_n$. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four $2 \times 2$ matrices $M^e, M^o, W^e, W^o$ as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$

for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices $M^e, M^o, W^e, W^o$. The first one is about relations between them; the second, about the structure of the each matrix.

**Lemma 2.1.** *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

*Proof.* We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0} M^o_{0,0}$ and $W^o_{0,1} M^o_{1,0}$. We know that

$$P(x, y) = \mathrm{Res}_z \left(\phi(W^o, 0, 0)(x, z),\ z^{12} \cdot \phi(M^o, 0, 0)(x, y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0} M^o_{0,0}$, that will be called $\phi_0(x, y)$. To prove that $\phi_0(x, y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x, y)$ of multiplicity $m$ and that $Q(x, y) := P(x, y)/\phi_0(x, y)^m$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0} M^o_{0,0}$ to order 630 and substitute it for $y$ in $Q(x, y)$. We get a series of valuation less than 630, which proves that $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0}$. We find the minimal polynomial $\phi_1(x, y)$ of $W^o_{0,1} M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e, 0, 0)$ is the minimal polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We know that

$$S(x, y) = \mathrm{Res}_z \left(\phi_0(x, z),\ \phi_1(x, y + z)\right)$$

is an annihilating of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. We verify that $\phi(M^e, 0, 0)$ is an irreducible factor of $S(x, y)$ of multiplicity $\mu$, and that $Q(x, y) := S(x, y)/\phi(M^e, 0, 0)^\mu$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ to order 770 and substitute it for $y$ in $Q(x, y)$. We get a series of valuation less than 770, and therefore $Q(x, y)$ is not an annihilating polynomial of $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e, 0, 0)$, we know that the two series are one and the same. ∎

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Lemma 2.2.** *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[{:}\,14u] = {} & M^e[{:}\,7u] + x^u M^e[{:}\,6u] + x^{3u} M^e[{:}\,4u] + x^{4u} M^e[{:}\,3u] + x^{6u} M^e[{:}\,u] \\ & + x^u M^e[{:}\,7u] + x^{2u} M^e[{:}\,6u] + x^{4u} M^e[{:}\,4u] + x^{5u} M^e[{:}\,3u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{7u}M^e[:u] + x^{2u}M^e[:7u] + x^{3u}M^e[:6u] + x^{5u}M^e[:4u]\\
&+ x^{6u}M^e[:3u] + x^{8u}M^e[:u] + x^{8u}M^e[:6u] + x^{10u}M^e[:4u]\\
&+ x^{12u}M^e[:2u] + (x^{7u} + x^{8u} + x^{9u})R^e,\\
W^e[:14u] = {}& W^e[:7u] + x^{u}W^e[:6u] + x^{3u}W^e[:4u] + x^{4u}W^e[:3u] + x^{6u}W^e[:u]\\
&+ x^{u}W^e[:7u] + x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{5u}W^e[:3u]\\
&+ x^{7u}W^e[:u] + x^{2u}W^e[:7u] + x^{3u}W^e[:6u] + x^{5u}W^e[:4u]\\
&+ x^{6u}W^e[:3u] + x^{8u}W^e[:u] + x^{8u}W^e[:6u] + x^{10u}W^e[:4u]\\
&+ x^{12u}W^e[:2u] + (x^{7u} + x^{8u} + x^{9u})R^e.
\end{aligned}
$$

*For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.*

$$
\begin{aligned}
M^o[:14u] = {}& M^o[:7u] + x^{u}M^o[:6u] + x^{3u}M^o[:4u] + x^{4u}M^o[:3u] + x^{6u}M^o[:u]\\
&+ x^{u}M^o[:7u] + x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{5u}M^o[:3u]\\
&+ x^{7u}M^o[:u] + x^{2u}M^o[:7u] + x^{3u}M^o[:6u] + x^{5u}M^o[:4u]\\
&+ x^{6u}M^o[:3u] + x^{8u}M^o[:u] + x^{8u}M^o[:6u] + x^{10u}M^o[:4u]\\
&+ x^{12u}M^o[:2u] + (x^{7u} + x^{8u} + x^{9u})R^o,\\
W^o[:14u] = {}& W^o[:7u] + x^{u}W^o[:6u] + x^{3u}W^o[:4u] + x^{4u}W^o[:3u] + x^{6u}W^o[:u]\\
&+ x^{u}W^o[:7u] + x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{5u}W^o[:3u]\\
&+ x^{7u}W^o[:u] + x^{2u}W^o[:7u] + x^{3u}W^o[:6u] + x^{5u}W^o[:4u]\\
&+ x^{6u}W^o[:3u] + x^{8u}W^o[:u] + x^{8u}W^o[:6u] + x^{10u}W^o[:4u]\\
&+ x^{12u}W^o[:2u] + (x^{7u} + x^{8u} + x^{9u})R^o.
\end{aligned}
$$

*Proof.* To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many $k$'s into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {}& T[:7u] + x^{u}T[:6u] + x^{3u}T[:4u] + x^{4u}T[:3u] + x^{6u}T[:u] + x^{u}T[:7u]\\
&+ x^{2u}T[:6u] + x^{4u}T[:4u] + x^{5u}T[:3u] + x^{7u}T[:u] + x^{2u}T[:7u]\\
&+ x^{3u}T[:6u] + x^{5u}T[:4u] + x^{6u}T[:3u] + x^{8u}T[:u] + x^{8u}T[:6u]\\
&+ x^{10u}T[:4u] + x^{12u}T[:2u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{3u}T[4u:5u] + x^{u}T[6u:7u]\\
&\quad + T[7u:8u],\\
0 &= x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] + x^{2u}T[6u:7u]\\
&\quad + T[8u:9u],\\
0 &= x^{8u}T[u:2u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{8u}T[2u:3u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{8u}T[3u:4u] + T[11u:12u],
\end{aligned}
$$

$$0 = x^{12u}T[:u] + x^{10u}T[2u:3u] + x^{8u}T[4u:5u] + T[12u:13u],$$
$$0 = x^{12u}T[u:2u] + x^{10u}T[3u:4u] + x^{8u}T[5u:6u] + T[13u:14u],$$

which can be rewritten as

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.9}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[1010w]_2], \tag{2.10}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.11}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.12}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word $w$ of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$
$$0 = T[[10w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$
$$0 = T[[1w]_2] + T[[1001w]_2], \tag{2.16}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[1010w]_2], \tag{2.17}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[1011w]_2], \tag{2.18}$$
$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[1100w]_2], \tag{2.19}$$
$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates $T$ from its minimal polynomial and its first terms. This automaton has 999 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading $w$ from right to left starting from the state $s$, and $\tau$ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.21}$$
$$0 = \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)), \tag{2.22}$$
$$0 = \tau(A(s,1)) + \tau(A(s,1001)), \tag{2.23}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,1010)), \tag{2.24}$$
$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.25}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1100)), \tag{2.26}$$
$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned} 0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,111)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} 0 &= \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$0 = \tau(A(s,1)) + \tau(A(s,1001)), \tag{2.30}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,1010)), \tag{2.31}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,1011)), \tag{2.32}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,1100)), \tag{2.33}$$

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{32}), E_{32}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 16$, which turns out to be true. ☐

In the following lemma, we express $M_{2k}$, $M_{2k+1}$, $W_{2k}$, and $W_{2k+1}$ in terms of $M^e$, $M^o$, $W^e$, and $W^o$.

**Lemma 2.3.** *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{4u} M^e[:3u] + x^{6u} M^e[:u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{4u} W^e[:3u] + x^{6u} W^e[:u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

*For all integer $k \ge 2$, and $u = 2^{2k}$,*

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{4u} M^o[:3u] + x^{6u} M^o[:u] \\ &\quad + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{4u} W^o[:3u] + x^{6u} W^o[:u] \\ &\quad + x^{7u} R^o. \end{aligned}$$

*Proof.* We call the four identities in Lemma 2.3 also by the name $M_{2k}$, $W_{2k}$, $M_{2k+1}$, and $W_{2k+1}$. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity $M_{2k+1}$ is equal to

$$\begin{aligned} W_{2k} M_{2k} &= \big(W^e[:7v] + x^v W^e[:6v] + x^{3v} W^e[:4v] + x^{4v} W^e[:3v] + x^{6v} W^e[:v] \\ &\quad + x^{7u} R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{3v} M^e[:4v] + x^{4v} M^e[:3v] \\ &\quad + x^{6v} M^e[:v] + x^{7u} R^e\big); \end{aligned} \tag{2.35}$$

Call this expression $lhs$. Note that both sides of identity $M_{2k+1}$ have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity $M_{2k+1}$ is congruent, modulo $x^{14v}$, to

$$W^e[:14v]M^e[:14v] + x^{2v}W^e[:12v]M^e[:12v] + x^{6v}W^e[:8v]M^e[:8v] \\ + x^{8v}W^e[:6v]M^e[:6v] + x^{12v}W^e[:2v]M^e[:2v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n\cdot v]$ and $M^e[:n\cdot v]$ in the above expression by the reduction modulo $x^{n\cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call $rhs$. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n\cdot v:(n+1)\cdot v]/X^n, \\ b_n &:= M^e[n\cdot v:(n+1)\cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions $lhs$ (2.35) and $rhs$ as polynomials in $\mathbb{F}_2[a_1,...,a_{14},b_1,...,b_{14},c][X]$. Note that it is not a problem that $a_j$ commutes with $b_k$ while $W^e$ does not commute with $M^e$, because in the expressions concerned, the products of $W^e$-terms and $M^e$-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

*Proof of Theorem 1.1.* We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned} q_0(x) &= x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^9 \\ &\quad + x^8 + x^7, \\ q_1(x) &= x^{25} + x^{22} + x^{20} + x^{18} + x^{13} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^6 + x^5 + x^4 \end{aligned}$$

$$
\begin{aligned}
&+x^{3},\\
q_2(x) &= x^{22}+x^{18}+x^{14}+x^{12}+x^{8}+x^{6}+x^{4}+1,\\
q_3(x) &= x^{25}+x^{22}+x^{20}+x^{18}+x^{13}+x^{12}+x^{11}+x^{10}+x^{8}+x^{7}+x^{6}+x^{5}+x^{4}\\
&\quad +x^{3},\\
q_4(x) &= x^{25}+x^{24}+x^{23}+x^{21}+x^{15}+x^{13}+x^{12}+x^{11}+x^{9}+x^{7}+x^{6}.
\end{aligned}
$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity $m$ and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for $y$ in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b)=f(1/z)$. □

## 3. Annexe

All 16 polynomials are of the form

$$p_0(x)+p_3(x)y^3+p_6(x)y^6+p_9(x)y^9+p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$
\begin{aligned}
p_0(x) &= x^{150}+x^{148}+x^{144}+x^{142}+x^{140}+x^{132}+x^{126}+x^{124}+x^{122}+x^{120}\\
&\quad +x^{118}+x^{116}+x^{112}+x^{108}+x^{106}+x^{104}+x^{102}+x^{100}+x^{98}+x^{94}\\
&\quad +x^{92}+x^{90}+x^{86}+x^{82}+x^{78}+x^{72}+x^{70}+x^{68}+x^{66}+x^{64}+x^{62}\\
&\quad +x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{40}+x^{38}+x^{36},\\
p_3(x) &= x^{142}+x^{140}+x^{134}+x^{132}+x^{124}+x^{122}+x^{118}+x^{116}+x^{114}+x^{108}\\
&\quad +x^{104}+x^{98}+x^{96}+x^{94}+x^{88}+x^{82}+x^{78}+x^{76}+x^{66}+x^{64}+x^{62}\\
&\quad +x^{60}+x^{58}+x^{56}+x^{48}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}\\
&\quad +x^{26}+x^{20},\\
p_6(x) &= x^{142}+x^{140}+x^{138}+x^{136}+x^{132}+x^{130}+x^{126}+x^{124}+x^{120}+x^{118}\\
&\quad +x^{114}+x^{112}+x^{110}+x^{108}+x^{106}+x^{104}+x^{100}+x^{98}+x^{96}+x^{90}\\
&\quad +x^{88}+x^{84}+x^{74}+x^{64}+x^{62}+x^{56}+x^{54}+x^{52}+x^{48}+x^{44}+x^{42}\\
&\quad +x^{40}+x^{38}+x^{36}+x^{32}+x^{30}+x^{26}+x^{22}+x^{20}+x^{10}+x^{6}+1,\\
p_9(x) &= x^{144}+x^{138}+x^{136}+x^{122}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}+x^{104}\\
&\quad +x^{102}+x^{100}+x^{98}+x^{96}+x^{92}+x^{88}+x^{86}+x^{84}+x^{82}+x^{72}+x^{68}\\
&\quad +x^{60}+x^{38}+x^{36}+x^{34}+x^{28}+x^{24}+x^{20}+x^{14}+x^{10}+x^{8}+x^{6},\\
p_{12}(x) &= x^{144}+x^{136}+x^{128}+x^{112}+x^{104}+x^{88}+x^{72}+x^{64}+x^{48}+x^{40}+x^{32}\\
&\quad +x^{16}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136} \\
& + x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{122} + x^{121} + x^{120} + x^{119} \\
& + x^{115} + x^{111} + x^{110} + x^{108} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} + x^{90} \\
& + x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76} \\
& + x^{73} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{57} \\
& + x^{56} + x^{54} + x^{53} + x^{52} + x^{47} + x^{42} + x^{40} + x^{39}, \\
p_3(x) = {} & x^{141} + x^{140} + x^{136} + x^{132} + x^{131} + x^{130} + x^{129} + x^{126} + x^{122} + x^{121} \\
& + x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107} \\
& + x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84} \\
& + x^{83} + x^{80} + x^{76} + x^{72} + x^{69} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} \\
& + x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} \\
& + x^{30} + x^{27}, \\
p_6(x) = {} & x^{138} + x^{137} + x^{135} + x^{133} + x^{132} + x^{128} + x^{127} + x^{125} + x^{124} + x^{118} \\
& + x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} \\
& + x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} \\
& + x^{81} + x^{80} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} \\
& + x^{61} + x^{60} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35} \\
& + x^{33} + x^{32} + x^{28} + x^{27} + x^{26} + x^{20} + x^{19} + x^{18}, \\
p_9(x) = {} & x^{135} + x^{134} + x^{130} + x^{129} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{120} \\
& + x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} \\
& + x^{104} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} \\
& + x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{71} + x^{70} + x^{69} \\
& + x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{45} + x^{44} \\
& + x^{43} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} \\
& + x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} \\
& + x^{9}, \\
p_{12}(x) = {} & x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116} \\
& + x^{112} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} \\
& + x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} \\
& + x^{64} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} \\
& + x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0].

For $\phi(M^e, 1, 0)$:

$$
p_0(x) = x^{147} + x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{137} + x^{136}
$$

$$
\begin{aligned}
&+ x^{135} + x^{134} + x^{131} + x^{130} + x^{127} + x^{122} + x^{121} + x^{120} + x^{119}\\
&+ x^{115} + x^{111} + x^{110} + x^{108} + x^{100} + x^{98} + x^{96} + x^{95} + x^{94} + x^{90}\\
&+ x^{89} + x^{88} + x^{87} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{78} + x^{77} + x^{76}\\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{53} + x^{52} + x^{47} + x^{42} + x^{40} + x^{39},\\
p_3(x) = {}& x^{141} + x^{140} + x^{136} + x^{132} + x^{131} + x^{130} + x^{129} + x^{126} + x^{122} + x^{121}\\
&+ x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84}\\
&+ x^{83} + x^{80} + x^{76} + x^{72} + x^{69} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53}\\
&+ x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33}\\
&+ x^{30} + x^{27},\\
p_6(x) = {}& x^{138} + x^{137} + x^{135} + x^{133} + x^{132} + x^{128} + x^{127} + x^{125} + x^{124} + x^{118}\\
&+ x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100}\\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82}\\
&+ x^{81} + x^{80} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63}\\
&+ x^{61} + x^{60} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35}\\
&+ x^{33} + x^{32} + x^{28} + x^{27} + x^{26} + x^{20} + x^{19} + x^{18},\\
p_9(x) = {}& x^{135} + x^{134} + x^{130} + x^{129} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{120}\\
&+ x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107}\\
&+ x^{104} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{45} + x^{44}\\
&+ x^{43} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{9},\\
p_{12}(x) = {}& x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96}\\
&+ x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65}\\
&+ x^{64} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20}\\
&+ x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{144} + x^{142} + x^{136} + x^{128} + x^{120} + x^{114} + x^{108} + x^{104} + x^{94} + x^{90}\\
&+ x^{88} + x^{86} + x^{84} + x^{78} + x^{76} + x^{72} + x^{64} + x^{52} + x^{50} + x^{46} + x^{42},\\
p_3(x) = {}& x^{126} + x^{124} + x^{110} + x^{106} + x^{102} + x^{100} + x^{96} + x^{86} + x^{84} + x^{82}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{78} + x^{76} + x^{74} + x^{70} + x^{68} + x^{64} + x^{62} + x^{56} + x^{52} + x^{50} + x^{48}\\
&+ x^{46} + x^{44} + x^{38} + x^{28} + x^{24} + x^{20} + x^{18},\\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{112} + x^{110} + x^{108} + x^{102} + x^{98} + x^{88}\\
&+ x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{68} + x^{66} + x^{60} + x^{52} + x^{50}\\
&+ x^{48} + x^{42} + x^{40} + x^{36} + x^{30} + x^{28} + x^{26} + x^{20} + x^{16} + x^{12} + x^{8}\\
&+ x^{2} + 1,\\
p_9(x) &= x^{120} + x^{118} + x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{82}\\
&+ x^{80} + x^{72} + x^{68} + x^{64} + x^{60} + x^{52} + x^{48} + x^{46} + x^{44} + x^{36} + x^{34}\\
&+ x^{32} + x^{30} + x^{26} + x^{18} + x^{16} + x^{6},\\
p_{12}(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{104} + x^{102} + x^{98} + x^{96} + x^{72} + x^{70}\\
&+ x^{66} + x^{64} + x^{24} + x^{22} + x^{18} + x^{16} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{142} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124} + x^{118} + x^{116}\\
&+ x^{112} + x^{110} + x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{86} + x^{82}\\
&+ x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{60} + x^{58} + x^{44} + x^{36},\\
p_3(x) &= x^{126} + x^{124} + x^{114} + x^{112} + x^{110} + x^{96} + x^{92} + x^{86} + x^{82} + x^{76} + x^{74}\\
&+ x^{70} + x^{68} + x^{66} + x^{60} + x^{48} + x^{46} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32}\\
&+ x^{22} + x^{20},\\
p_6(x) &= x^{126} + x^{124} + x^{120} + x^{118} + x^{114} + x^{110} + x^{102} + x^{98} + x^{96} + x^{94}\\
&+ x^{92} + x^{90} + x^{88} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{68} + x^{64} + x^{62}\\
&+ x^{60} + x^{58} + x^{48} + x^{42} + x^{38} + x^{36} + x^{32} + x^{30} + x^{26} + x^{24} + x^{22}\\
&+ x^{20} + x^{18} + x^{14} + x^{8} + x^{2} + 1,\\
p_9(x) &= x^{120} + x^{118} + x^{108} + x^{104} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{82}\\
&+ x^{80} + x^{72} + x^{68} + x^{64} + x^{60} + x^{52} + x^{48} + x^{46} + x^{44} + x^{36} + x^{34}\\
&+ x^{32} + x^{30} + x^{26} + x^{18} + x^{16} + x^{6},\\
p_{12}(x) &= x^{120} + x^{118} + x^{114} + x^{112} + x^{104} + x^{102} + x^{98} + x^{96} + x^{72} + x^{70}\\
&+ x^{66} + x^{64} + x^{24} + x^{22} + x^{18} + x^{16} + x^{8} + x^{6} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{139} + x^{138} + x^{137} + x^{134} + x^{133} + x^{132} + x^{129} + x^{128}\\
&+ x^{127} + x^{122} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{108} + x^{107}\\
&+ x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&+ x^{92} + x^{90} + x^{87} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39},\\
p_3(x) &= x^{141} + x^{140} + x^{136} + x^{132} + x^{131} + x^{130} + x^{129} + x^{126} + x^{122} + x^{121}\\
&+ x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107}\\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84}\\
&+ x^{83} + x^{80} + x^{76} + x^{72} + x^{69} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53}\\
&+ x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33}\\
&+ x^{30} + x^{27},\\
p_6(x) &= x^{138} + x^{137} + x^{135} + x^{133} + x^{132} + x^{128} + x^{127} + x^{125} + x^{124} + x^{118}\\
&+ x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100}\\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82}\\
&+ x^{81} + x^{80} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63}\\
&+ x^{61} + x^{60} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35}\\
&+ x^{33} + x^{32} + x^{28} + x^{27} + x^{26} + x^{20} + x^{19} + x^{18},\\
p_9(x) &= x^{135} + x^{134} + x^{130} + x^{129} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{120}\\
&+ x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107}\\
&+ x^{104} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90}\\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{71} + x^{70} + x^{69}\\
&+ x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{45} + x^{44}\\
&+ x^{43} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27}\\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11}\\
&+ x^{9},\\
p_{12}(x) &= x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96}\\
&+ x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65}\\
&+ x^{64} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20}\\
&+ x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{139} + x^{138} + x^{137} + x^{134} + x^{133} + x^{132} + x^{129} + x^{128}\\
&+ x^{127} + x^{122} + x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{108} + x^{107}\\
&+ x^{105} + x^{104} + x^{102} + x^{101} + x^{100} + x^{98} + x^{97} + x^{95} + x^{94} + x^{93}\\
&+ x^{92} + x^{90} + x^{87} + x^{84} + x^{82} + x^{80} + x^{79} + x^{77} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{64} + x^{62} + x^{61} + x^{60} + x^{54} + x^{53} + x^{52} + x^{50} + x^{49} + x^{47}\\
&+ x^{46} + x^{45} + x^{44} + x^{42} + x^{40} + x^{39},\\
p_3(x) &= x^{141} + x^{140} + x^{136} + x^{132} + x^{131} + x^{130} + x^{129} + x^{126} + x^{122} + x^{121}
\end{aligned}
$$

$$\begin{aligned}
&+ x^{120} + x^{118} + x^{117} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109} + x^{107} \\
&+ x^{99} + x^{98} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{89} + x^{88} + x^{86} + x^{84} \\
&+ x^{83} + x^{80} + x^{76} + x^{72} + x^{69} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{53} \\
&+ x^{50} + x^{49} + x^{48} + x^{46} + x^{45} + x^{43} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} \\
&+ x^{30} + x^{27}, \\
p_6(x) = {}& x^{138} + x^{137} + x^{135} + x^{133} + x^{132} + x^{128} + x^{127} + x^{125} + x^{124} + x^{118} \\
&+ x^{117} + x^{115} + x^{114} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{100} \\
&+ x^{96} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} \\
&+ x^{81} + x^{80} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} \\
&+ x^{61} + x^{60} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{36} + x^{35} \\
&+ x^{33} + x^{32} + x^{28} + x^{27} + x^{26} + x^{20} + x^{19} + x^{18}, \\
p_9(x) = {}& x^{135} + x^{134} + x^{130} + x^{129} + x^{128} + x^{127} + x^{124} + x^{123} + x^{122} + x^{120} \\
&+ x^{118} + x^{117} + x^{115} + x^{113} + x^{112} + x^{111} + x^{110} + x^{108} + x^{107} \\
&+ x^{104} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{94} + x^{93} + x^{92} + x^{90} \\
&+ x^{89} + x^{87} + x^{85} + x^{84} + x^{83} + x^{82} + x^{81} + x^{79} + x^{71} + x^{70} + x^{69} \\
&+ x^{68} + x^{67} + x^{64} + x^{63} + x^{62} + x^{60} + x^{58} + x^{57} + x^{55} + x^{45} + x^{44} \\
&+ x^{43} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{33} + x^{31} + x^{29} + x^{28} + x^{27} \\
&+ x^{26} + x^{25} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{13} + x^{12} + x^{11} \\
&+ x^{9}, \\
p_{12}(x) = {}& x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{116} \\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{104} + x^{103} + x^{101} + x^{99} + x^{97} + x^{96} \\
&+ x^{84} + x^{83} + x^{81} + x^{79} + x^{77} + x^{76} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} \\
&+ x^{64} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} \\
&+ x^{16} + x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) = {}& x^{150} + x^{148} + x^{142} + x^{138} + x^{136} + x^{134} + x^{130} + x^{126} + x^{120} + x^{118} \\
&+ x^{116} + x^{114} + x^{108} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{90} + x^{84} \\
&+ x^{80} + x^{74} + x^{72} + x^{70} + x^{54} + x^{50} + x^{48} + x^{44} + x^{42}, \\
p_3(x) = {}& x^{142} + x^{140} + x^{138} + x^{134} + x^{128} + x^{126} + x^{120} + x^{118} + x^{116} + x^{112} \\
&+ x^{110} + x^{108} + x^{98} + x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{74} \\
&+ x^{72} + x^{70} + x^{68} + x^{66} + x^{64} + x^{56} + x^{54} + x^{52} + x^{50} + x^{44} + x^{36} \\
&+ x^{30} + x^{18}, \\
p_6(x) = {}& x^{142} + x^{140} + x^{136} + x^{132} + x^{130} + x^{128} + x^{124} + x^{122} + x^{120} + x^{118} \\
&+ x^{106} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{76} + x^{70} + x^{64} + x^{60} + x^{58}
\end{aligned}$$

$$
\begin{aligned}
&+x^{50}+x^{44}+x^{42}+x^{40}+x^{36}+x^{34}+x^{30}+x^{28}+x^{22}+x^{18}+x^{16}\\
&+x^{12}+x^{10}+x^{6}+1,\\
p_9(x) &= x^{144}+x^{138}+x^{136}+x^{122}+x^{116}+x^{114}+x^{110}+x^{108}+x^{106}+x^{104}\\
&+x^{102}+x^{100}+x^{98}+x^{96}+x^{92}+x^{88}+x^{86}+x^{84}+x^{82}+x^{72}+x^{68}\\
&+x^{60}+x^{38}+x^{36}+x^{34}+x^{28}+x^{24}+x^{20}+x^{14}+x^{10}+x^{8}+x^{6},\\
p_{12}(x) &= x^{144}+x^{136}+x^{128}+x^{112}+x^{104}+x^{88}+x^{72}+x^{64}+x^{48}+x^{40}+x^{32}\\
&+x^{16}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147}+x^{145}+x^{144}+x^{143}+x^{142}+x^{140}+x^{139}+x^{138}+x^{137}+x^{134}\\
&+x^{132}+x^{129}+x^{128}+x^{126}+x^{123}+x^{119}+x^{118}+x^{116}+x^{115}\\
&+x^{113}+x^{112}+x^{111}+x^{108}+x^{107}+x^{106}+x^{104}+x^{103}+x^{102}\\
&+x^{101}+x^{98}+x^{97}+x^{93}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{79}+x^{78}+x^{76}+x^{73}+x^{69}+x^{67}+x^{64}+x^{61}+x^{57}\\
&+x^{53}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{38}\\
&+x^{37}+x^{36},\\
p_3(x) &= x^{144}+x^{143}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{129}+x^{128}+x^{127}\\
&+x^{123}+x^{122}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{111}+x^{108}\\
&+x^{107}+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{94}\\
&+x^{91}+x^{89}+x^{87}+x^{85}+x^{84}+x^{83}+x^{79}+x^{78}+x^{77}+x^{73}+x^{72}\\
&+x^{71}+x^{70}+x^{68}+x^{67}+x^{66}+x^{62}+x^{61}+x^{58}+x^{57}+x^{56}+x^{53}\\
&+x^{52}+x^{51}+x^{49}+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{39}+x^{36}\\
&+x^{35}+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{25}+x^{24},\\
p_6(x) &= x^{144}+x^{143}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{128}+x^{127}+x^{126}\\
&+x^{122}+x^{117}+x^{113}+x^{105}+x^{104}+x^{100}+x^{99}+x^{98}+x^{97}+x^{93}\\
&+x^{92}+x^{91}+x^{89}+x^{87}+x^{86}+x^{85}+x^{81}+x^{80}+x^{79}+x^{77}+x^{76}\\
&+x^{75}+x^{74}+x^{72}+x^{67}+x^{64}+x^{63}+x^{61}+x^{59}+x^{56}+x^{55}+x^{54}\\
&+x^{52}+x^{51}+x^{50}+x^{48}+x^{47}+x^{45}+x^{44}+x^{43}+x^{40}+x^{39}+x^{37}\\
&+x^{33}+x^{32}+x^{31}+x^{30}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{19}+x^{16}\\
&+x^{15}+x^{14}+x^{8}+x^{7}+x^{5}+x^{3}+x+1,\\
p_9(x) &= x^{144}+x^{143}+x^{141}+x^{139}+x^{137}+x^{136}+x^{124}+x^{123}+x^{121}+x^{120}\\
&+x^{112}+x^{111}+x^{109}+x^{108}+x^{96}+x^{95}+x^{93}+x^{91}+x^{89}+x^{88}\\
&+x^{80}+x^{79}+x^{77}+x^{76}+x^{48}+x^{47}+x^{45}+x^{43}+x^{41}+x^{40}+x^{28}\\
&+x^{27}+x^{25}+x^{24}+x^{16}+x^{15}+x^{13}+x^{12},\\
p_{12}(x) &= x^{144}+x^{143}+x^{141}+x^{139}+x^{137}+x^{136}+x^{132}+x^{131}+x^{129}+x^{127}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{125}+x^{123}+x^{121}+x^{119}+x^{117}+x^{116}+x^{104}+x^{103}+x^{101}\\
&+x^{99}+x^{97}+x^{95}+x^{93}+x^{91}+x^{89}+x^{88}+x^{84}+x^{83}+x^{81}+x^{80}\\
&+x^{72}+x^{71}+x^{69}+x^{67}+x^{65}+x^{64}+x^{48}+x^{47}+x^{45}+x^{43}+x^{41}\\
&+x^{40}+x^{36}+x^{35}+x^{33}+x^{31}+x^{29}+x^{27}+x^{25}+x^{23}+x^{21}+x^{20}\\
&+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}&x^{152}+x^{148}+x^{144}+x^{143}+x^{141}+x^{140}+x^{139}+x^{137}+x^{136}+x^{134}\\
&+x^{131}+x^{130}+x^{128}+x^{125}+x^{122}+x^{118}+x^{117}+x^{116}+x^{114}\\
&+x^{113}+x^{111}+x^{110}+x^{107}+x^{106}+x^{104}+x^{103}+x^{101}+x^{97}+x^{96}\\
&+x^{94}+x^{93}+x^{91}+x^{90}+x^{89}+x^{87}+x^{86}+x^{84}+x^{79}+x^{78}+x^{77}\\
&+x^{75}+x^{74}+x^{72}+x^{69}+x^{67}+x^{66}+x^{65}+x^{64}+x^{63}+x^{60}+x^{59}\\
&+x^{57}+x^{56}+x^{55}+x^{53}+x^{48}+x^{46}+x^{44}+x^{43}+x^{39},\\
p_3(x) = {}&x^{131}+x^{128}+x^{126}+x^{124}+x^{123}+x^{121}+x^{120}+x^{119}+x^{118}+x^{117}\\
&+x^{116}+x^{112}+x^{110}+x^{108}+x^{105}+x^{103}+x^{101}+x^{97}+x^{96}+x^{95}\\
&+x^{94}+x^{93}+x^{92}+x^{88}+x^{86}+x^{84}+x^{81}+x^{79}+x^{77}+x^{75}+x^{73}\\
&+x^{71}+x^{69}+x^{67}+x^{65}+x^{63}+x^{61}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}\\
&+x^{52}+x^{48}+x^{46}+x^{44}+x^{41}+x^{39}+x^{37}+x^{33}+x^{32}+x^{31}+x^{30}\\
&+x^{29}+x^{28}+x^{27},\\
p_6(x) = {}&x^{134}+x^{130}+x^{128}+x^{126}+x^{122}+x^{120}+x^{118}+x^{116}+x^{112}+x^{110}\\
&+x^{108}+x^{106}+x^{104}+x^{98}+x^{96}+x^{94}+x^{88}+x^{82}+x^{80}+x^{70}+x^{68}\\
&+x^{64}+x^{62}+x^{58}+x^{56}+x^{54}+x^{52}+x^{48}+x^{44}+x^{32}+x^{24}+x^{22}\\
&+x^{20}+x^{18},\\
p_9(x) = {}&x^{137}+x^{134}+x^{132}+x^{131}+x^{130}+x^{128}+x^{126}+x^{125}+x^{122}+x^{118}\\
&+x^{116}+x^{115}+x^{114}+x^{112}+x^{110}+x^{109}+x^{106}+x^{105}+x^{89}+x^{86}\\
&+x^{84}+x^{83}+x^{82}+x^{80}+x^{78}+x^{77}+x^{74}+x^{73}+x^{41}+x^{38}+x^{36}\\
&+x^{35}+x^{34}+x^{32}+x^{30}+x^{29}+x^{26}+x^{22}+x^{20}+x^{19}+x^{18}+x^{16}\\
&+x^{14}+x^{13}+x^{10}+x^{9},\\
p_{12}(x) = {}&x^{140}+x^{136}+x^{128}+x^{124}+x^{120}+x^{116}+x^{100}+x^{96}+x^{92}+x^{88}\\
&+x^{80}+x^{68}+x^{64}+x^{44}+x^{40}+x^{32}+x^{28}+x^{24}+x^{20}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}&x^{150}+x^{148}+x^{147}+x^{146}+x^{145}+x^{143}+x^{142}+x^{136}+x^{134}+x^{133}\\
&+x^{128}+x^{121}+x^{118}+x^{115}+x^{114}+x^{113}+x^{112}+x^{110}+x^{109}\\
&+x^{108}+x^{105}+x^{104}+x^{103}+x^{102}+x^{100}+x^{99}+x^{98}+x^{93}+x^{92}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{91} + x^{90} + x^{88} + x^{86} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{72} + x^{71}\\
&+ x^{69} + x^{66} + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{49}\\
&+ x^{47} + x^{39},\\
p_3(x) &= x^{147} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{132}\\
&+ x^{129} + x^{127} + x^{125} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100}\\
&+ x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{75}\\
&+ x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{150} + x^{144} + x^{140} + x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124}\\
&+ x^{116} + x^{112} + x^{108} + x^{102} + x^{98} + x^{88} + x^{80} + x^{78} + x^{76} + x^{74}\\
&+ x^{70} + x^{64} + x^{60} + x^{56} + x^{54} + x^{48} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28}\\
&+ x^{26} + x^{20} + x^{18},\\
p_9(x) &= x^{153} + x^{150} + x^{149} + x^{148} + x^{147} + x^{143} + x^{140} + x^{138} + x^{136} + x^{135}\\
&+ x^{132} + x^{129} + x^{128} + x^{127} + x^{124} + x^{120} + x^{119} + x^{118} + x^{117}\\
&+ x^{115} + x^{113} + x^{112} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{92}\\
&+ x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{81} + x^{80} + x^{74} + x^{73}\\
&+ x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{44} + x^{42} + x^{40} + x^{39} + x^{36}\\
&+ x^{33} + x^{32} + x^{31} + x^{28} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16}\\
&+ x^{10} + x^{9},\\
p_{12}(x) &= x^{156} + x^{148} + x^{136} + x^{132} + x^{128} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88}\\
&+ x^{80} + x^{76} + x^{72} + x^{64} + x^{60} + x^{52} + x^{40} + x^{36} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{141} + x^{132} + x^{129} + x^{128} + x^{125} + x^{119} + x^{117} + x^{115}\\
&+ x^{113} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101}\\
&+ x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85}\\
&+ x^{84} + x^{83} + x^{82} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65}\\
&+ x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42},\\
p_3(x) &= x^{144} + x^{143} + x^{139} + x^{136} + x^{133} + x^{131} + x^{127} + x^{126} + x^{125} + x^{122}\\
&+ x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110}\\
&+ x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{91}\\
&+ x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{75} + x^{72} + x^{70} + x^{69}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{68}+x^{62}+x^{61}+x^{58}+x^{57}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}\\
&+x^{45}+x^{44}+x^{40}+x^{39}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{28}\\
&+x^{27}+x^{26},\\
p_6(x)=&\,x^{144}+x^{143}+x^{139}+x^{136}+x^{133}+x^{131}+x^{129}+x^{127}+x^{126}+x^{122}\\
&+x^{118}+x^{116}+x^{114}+x^{111}+x^{109}+x^{107}+x^{106}+x^{105}+x^{104}\\
&+x^{102}+x^{101}+x^{99}+x^{98}+x^{97}+x^{94}+x^{92}+x^{85}+x^{82}+x^{80}+x^{76}\\
&+x^{71}+x^{70}+x^{68}+x^{66}+x^{60}+x^{59}+x^{56}+x^{55}+x^{53}+x^{51}+x^{50}\\
&+x^{48}+x^{47}+x^{45}+x^{44}+x^{43}+x^{36}+x^{33}+x^{30}+x^{29}+x^{27}+x^{25}\\
&+x^{24}+x^{23}+x^{22}+x^{20}+x^{18}+x^{14}+x^{13}+x^{12}+x^{8}+x^{7}+x^{5}+x^{3}\\
&+x+1,\\
p_9(x)=&\,x^{144}+x^{143}+x^{141}+x^{139}+x^{137}+x^{136}+x^{124}+x^{123}+x^{121}+x^{120}\\
&+x^{112}+x^{111}+x^{109}+x^{108}+x^{96}+x^{95}+x^{93}+x^{91}+x^{89}+x^{88}\\
&+x^{80}+x^{79}+x^{77}+x^{76}+x^{48}+x^{47}+x^{45}+x^{43}+x^{41}+x^{40}+x^{28}\\
&+x^{27}+x^{25}+x^{24}+x^{16}+x^{15}+x^{13}+x^{12},\\
p_{12}(x)=&\,x^{144}+x^{143}+x^{141}+x^{139}+x^{137}+x^{136}+x^{132}+x^{131}+x^{129}+x^{127}\\
&+x^{125}+x^{123}+x^{121}+x^{119}+x^{117}+x^{116}+x^{104}+x^{103}+x^{101}\\
&+x^{99}+x^{97}+x^{95}+x^{93}+x^{91}+x^{89}+x^{88}+x^{84}+x^{83}+x^{81}+x^{80}\\
&+x^{72}+x^{71}+x^{69}+x^{67}+x^{65}+x^{64}+x^{48}+x^{47}+x^{45}+x^{43}+x^{41}\\
&+x^{40}+x^{36}+x^{35}+x^{33}+x^{31}+x^{29}+x^{27}+x^{25}+x^{23}+x^{21}+x^{20}\\
&+x^{8}+x^{7}+x^{5}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1].

For $\phi(W^o,0,0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{147}+x^{145}+x^{144}+x^{143}+x^{142}+x^{140}+x^{139}+x^{138}+x^{137}+x^{134}\\
&+x^{132}+x^{129}+x^{128}+x^{126}+x^{123}+x^{119}+x^{118}+x^{116}+x^{115}\\
&+x^{113}+x^{112}+x^{111}+x^{108}+x^{107}+x^{106}+x^{104}+x^{103}+x^{102}\\
&+x^{101}+x^{98}+x^{97}+x^{93}+x^{89}+x^{88}+x^{87}+x^{86}+x^{85}+x^{84}+x^{83}\\
&+x^{82}+x^{81}+x^{79}+x^{78}+x^{76}+x^{73}+x^{69}+x^{67}+x^{64}+x^{61}+x^{57}\\
&+x^{53}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{38}\\
&+x^{37}+x^{36},\\
p_3(x)=&\,x^{144}+x^{143}+x^{139}+x^{138}+x^{137}+x^{136}+x^{135}+x^{129}+x^{128}+x^{127}\\
&+x^{123}+x^{122}+x^{117}+x^{116}+x^{115}+x^{114}+x^{112}+x^{111}+x^{108}\\
&+x^{107}+x^{106}+x^{104}+x^{103}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{94}\\
&+x^{91}+x^{89}+x^{87}+x^{85}+x^{84}+x^{83}+x^{79}+x^{78}+x^{77}+x^{73}+x^{72}\\
&+x^{71}+x^{70}+x^{68}+x^{67}+x^{66}+x^{62}+x^{61}+x^{58}+x^{57}+x^{56}+x^{53}\\
&+x^{52}+x^{51}+x^{49}+x^{47}+x^{46}+x^{45}+x^{44}+x^{42}+x^{41}+x^{39}+x^{36}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{35} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{25} + x^{24},\\
p_6(x) &= x^{144} + x^{143} + x^{139} + x^{138} + x^{137} + x^{136} + x^{135} + x^{128} + x^{127} + x^{126}\\
&+ x^{122} + x^{117} + x^{113} + x^{105} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{93}\\
&+ x^{92} + x^{91} + x^{89} + x^{87} + x^{86} + x^{85} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76}\\
&+ x^{75} + x^{74} + x^{72} + x^{67} + x^{64} + x^{63} + x^{61} + x^{59} + x^{56} + x^{55} + x^{54}\\
&+ x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{40} + x^{39} + x^{37}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{19} + x^{16}\\
&+ x^{15} + x^{14} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) &= x^{144} + x^{143} + x^{141} + x^{139} + x^{137} + x^{136} + x^{124} + x^{123} + x^{121} + x^{120}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88}\\
&+ x^{80} + x^{79} + x^{77} + x^{76} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41} + x^{40} + x^{28}\\
&+ x^{27} + x^{25} + x^{24} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{144} + x^{143} + x^{141} + x^{139} + x^{137} + x^{136} + x^{132} + x^{131} + x^{129} + x^{127}\\
&+ x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{104} + x^{103} + x^{101}\\
&+ x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} + x^{83} + x^{81} + x^{80}\\
&+ x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41}\\
&+ x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20}\\
&+ x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{150} + x^{148} + x^{147} + x^{146} + x^{145} + x^{143} + x^{142} + x^{136} + x^{134} + x^{133}\\
&+ x^{128} + x^{121} + x^{118} + x^{115} + x^{114} + x^{113} + x^{112} + x^{110} + x^{109}\\
&+ x^{108} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{93} + x^{92}\\
&+ x^{91} + x^{90} + x^{88} + x^{86} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{72} + x^{71}\\
&+ x^{69} + x^{66} + x^{64} + x^{63} + x^{60} + x^{58} + x^{57} + x^{56} + x^{55} + x^{51} + x^{49}\\
&+ x^{47} + x^{39},\\
p_3(x) &= x^{147} + x^{144} + x^{143} + x^{142} + x^{139} + x^{138} + x^{137} + x^{135} + x^{134} + x^{132}\\
&+ x^{129} + x^{127} + x^{125} + x^{121} + x^{120} + x^{118} + x^{117} + x^{115} + x^{114}\\
&+ x^{112} + x^{111} + x^{109} + x^{108} + x^{106} + x^{105} + x^{103} + x^{102} + x^{100}\\
&+ x^{99} + x^{97} + x^{96} + x^{94} + x^{93} + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{75}\\
&+ x^{72} + x^{71} + x^{70} + x^{67} + x^{66} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57}\\
&+ x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42}\\
&+ x^{41} + x^{40} + x^{36} + x^{35} + x^{34} + x^{30} + x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{150} + x^{144} + x^{140} + x^{138} + x^{136} + x^{132} + x^{130} + x^{128} + x^{126} + x^{124}\\
&+ x^{116} + x^{112} + x^{108} + x^{102} + x^{98} + x^{88} + x^{80} + x^{78} + x^{76} + x^{74}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{70} + x^{64} + x^{60} + x^{56} + x^{54} + x^{48} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28}\\
&+ x^{26} + x^{20} + x^{18},\\
p_9(x) &= x^{153} + x^{150} + x^{149} + x^{148} + x^{147} + x^{143} + x^{140} + x^{138} + x^{136} + x^{135}\\
&+ x^{132} + x^{129} + x^{128} + x^{127} + x^{124} + x^{120} + x^{119} + x^{118} + x^{117}\\
&+ x^{115} + x^{113} + x^{112} + x^{106} + x^{102} + x^{101} + x^{100} + x^{99} + x^{95} + x^{92}\\
&+ x^{90} + x^{89} + x^{88} + x^{87} + x^{86} + x^{85} + x^{83} + x^{81} + x^{80} + x^{74} + x^{73}\\
&+ x^{57} + x^{54} + x^{53} + x^{52} + x^{51} + x^{47} + x^{44} + x^{42} + x^{40} + x^{39} + x^{36}\\
&+ x^{33} + x^{32} + x^{31} + x^{28} + x^{24} + x^{23} + x^{22} + x^{21} + x^{19} + x^{17} + x^{16}\\
&+ x^{10} + x^{9},\\
p_{12}(x) &= x^{156} + x^{148} + x^{136} + x^{132} + x^{128} + x^{104} + x^{100} + x^{96} + x^{92} + x^{88}\\
&+ x^{80} + x^{76} + x^{72} + x^{64} + x^{60} + x^{52} + x^{40} + x^{36} + x^{32} + x^{12} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{152} + x^{148} + x^{144} + x^{143} + x^{141} + x^{140} + x^{139} + x^{137} + x^{136} + x^{134}\\
&+ x^{131} + x^{130} + x^{128} + x^{125} + x^{122} + x^{118} + x^{117} + x^{116} + x^{114}\\
&+ x^{113} + x^{111} + x^{110} + x^{107} + x^{106} + x^{104} + x^{103} + x^{101} + x^{97} + x^{96}\\
&+ x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{87} + x^{86} + x^{84} + x^{79} + x^{78} + x^{77}\\
&+ x^{75} + x^{74} + x^{72} + x^{69} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{60} + x^{59}\\
&+ x^{57} + x^{56} + x^{55} + x^{53} + x^{48} + x^{46} + x^{44} + x^{43} + x^{39},\\
p_3(x) &= x^{131} + x^{128} + x^{126} + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{117}\\
&+ x^{116} + x^{112} + x^{110} + x^{108} + x^{105} + x^{103} + x^{101} + x^{97} + x^{96} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{88} + x^{86} + x^{84} + x^{81} + x^{79} + x^{77} + x^{75} + x^{73}\\
&+ x^{71} + x^{69} + x^{67} + x^{65} + x^{63} + x^{61} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53}\\
&+ x^{52} + x^{48} + x^{46} + x^{44} + x^{41} + x^{39} + x^{37} + x^{33} + x^{32} + x^{31} + x^{30}\\
&+ x^{29} + x^{28} + x^{27},\\
p_6(x) &= x^{134} + x^{130} + x^{128} + x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{110}\\
&+ x^{108} + x^{106} + x^{104} + x^{98} + x^{96} + x^{94} + x^{88} + x^{82} + x^{80} + x^{70} + x^{68}\\
&+ x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{52} + x^{48} + x^{44} + x^{32} + x^{24} + x^{22}\\
&+ x^{20} + x^{18},\\
p_9(x) &= x^{137} + x^{134} + x^{132} + x^{131} + x^{130} + x^{128} + x^{126} + x^{125} + x^{122} + x^{118}\\
&+ x^{116} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{106} + x^{105} + x^{89} + x^{86}\\
&+ x^{84} + x^{83} + x^{82} + x^{80} + x^{78} + x^{77} + x^{74} + x^{73} + x^{41} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{32} + x^{30} + x^{29} + x^{26} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16}\\
&+ x^{14} + x^{13} + x^{10} + x^{9},\\
p_{12}(x) &= x^{140} + x^{136} + x^{128} + x^{124} + x^{120} + x^{116} + x^{100} + x^{96} + x^{92} + x^{88}
\end{aligned}
$$

$$+ x^{80} + x^{68} + x^{64} + x^{44} + x^{40} + x^{32} + x^{28} + x^{24} + x^{20} + x^{4} + 1,$$

and the initial terms are [0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{145} + x^{141} + x^{132} + x^{129} + x^{128} + x^{125} + x^{119} + x^{117} + x^{115}\\
&\quad + x^{113} + x^{112} + x^{111} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102} + x^{101}\\
&\quad + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{87} + x^{86} + x^{85}\\
&\quad + x^{84} + x^{83} + x^{82} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{65}\\
&\quad + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51}\\
&\quad + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{43} + x^{42},\\
p_3(x) &= x^{144} + x^{143} + x^{139} + x^{136} + x^{133} + x^{131} + x^{127} + x^{126} + x^{125} + x^{122}\\
&\quad + x^{121} + x^{119} + x^{118} + x^{117} + x^{116} + x^{115} + x^{113} + x^{112} + x^{110}\\
&\quad + x^{108} + x^{107} + x^{105} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{91}\\
&\quad + x^{90} + x^{86} + x^{85} + x^{84} + x^{83} + x^{80} + x^{78} + x^{75} + x^{72} + x^{70} + x^{69}\\
&\quad + x^{68} + x^{62} + x^{61} + x^{58} + x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50}\\
&\quad + x^{45} + x^{44} + x^{40} + x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{28}\\
&\quad + x^{27} + x^{26},\\
p_6(x) &= x^{144} + x^{143} + x^{139} + x^{136} + x^{133} + x^{131} + x^{129} + x^{127} + x^{126} + x^{122}\\
&\quad + x^{118} + x^{116} + x^{114} + x^{111} + x^{109} + x^{107} + x^{106} + x^{105} + x^{104}\\
&\quad + x^{102} + x^{101} + x^{99} + x^{98} + x^{97} + x^{94} + x^{92} + x^{85} + x^{82} + x^{80} + x^{76}\\
&\quad + x^{71} + x^{70} + x^{68} + x^{66} + x^{60} + x^{59} + x^{56} + x^{55} + x^{53} + x^{51} + x^{50}\\
&\quad + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{36} + x^{33} + x^{30} + x^{29} + x^{27} + x^{25}\\
&\quad + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{14} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3}\\
&\quad + x + 1,\\
p_9(x) &= x^{144} + x^{143} + x^{141} + x^{139} + x^{137} + x^{136} + x^{124} + x^{123} + x^{121} + x^{120}\\
&\quad + x^{112} + x^{111} + x^{109} + x^{108} + x^{96} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88}\\
&\quad + x^{80} + x^{79} + x^{77} + x^{76} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41} + x^{40} + x^{28}\\
&\quad + x^{27} + x^{25} + x^{24} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_{12}(x) &= x^{144} + x^{143} + x^{141} + x^{139} + x^{137} + x^{136} + x^{132} + x^{131} + x^{129} + x^{127}\\
&\quad + x^{125} + x^{123} + x^{121} + x^{119} + x^{117} + x^{116} + x^{104} + x^{103} + x^{101}\\
&\quad + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{84} + x^{83} + x^{81} + x^{80}\\
&\quad + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} + x^{48} + x^{47} + x^{45} + x^{43} + x^{41}\\
&\quad + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{23} + x^{21} + x^{20}\\
&\quad + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ ($\Lambda(n) := [\delta(n, 0), \delta(n, 1)]$):

| $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ | $n$ | $\Lambda(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | [1, 2] | 200 | [355, 356] | 400 | [553, 237] | 600 | [744, 762] | 800 | [916, 917] |
| 1 | [3, 4] | 201 | [357, 358] | 401 | [554, 555] | 601 | [13, 763] | 801 | [918, 376] |
| 2 | [5, 6] | 202 | [359, 360] | 402 | [556, 557] | 602 | [421, 58] | 802 | [711, 406] |
| 3 | [7, 8] | 203 | [361, 362] | 403 | [558, 559] | 603 | [728, 160] | 803 | [473, 170] |
| 4 | [9, 10] | 204 | [363, 364] | 404 | [560, 561] | 604 | [751, 494] | 804 | [169, 159] |
| 5 | [11, 12] | 205 | [365, 366] | 405 | [562, 563] | 605 | [393, 136] | 805 | [474, 56] |
| 6 | [13, 14] | 206 | [367, 203] | 406 | [265, 78] | 606 | [719, 397] | 806 | [397, 787] |
| 7 | [15, 16] | 207 | [368, 40] | 407 | [564, 93] | 607 | [121, 726] | 807 | [919, 758] |
| 8 | [17, 18] | 208 | [369, 370] | 408 | [565, 566] | 608 | [9, 469] | 808 | [451, 370] |
| 9 | [19, 20] | 209 | [371, 372] | 409 | [567, 568] | 609 | [759, 423] | 809 | [920, 463] |
| 10 | [21, 22] | 210 | [373, 168] | 410 | [569, 298] | 610 | [764, 471] | 810 | [921, 149] |
| 11 | [23, 24] | 211 | [122, 374] | 411 | [570, 518] | 611 | [376, 186] | 811 | [425, 145] |
| 12 | [25, 26] | 212 | [375, 183] | 412 | [571, 572] | 612 | [765, 405] | 812 | [408, 922] |
| 13 | [27, 28] | 213 | [376, 120] | 413 | [573, 574] | 613 | [177, 762] | 813 | [472, 469] |
| 14 | [29, 30] | 214 | [377, 127] | 414 | [575, 576] | 614 | [415, 172] | 814 | [782, 383] |
| 15 | [31, 32] | 215 | [117, 378] | 415 | [577, 578] | 615 | [48, 766] | 815 | [368, 473] |
| 16 | [33, 34] | 216 | [9, 379] | 416 | [579, 580] | 616 | [713, 721] | 816 | [486, 200] |
| 17 | [35, 36] | 217 | [380, 381] | 417 | [581, 582] | 617 | [131, 170] | 817 | [722, 172] |
| 18 | [37, 38] | 218 | [135, 382] | 418 | [583, 366] | 618 | [767, 14] | 818 | [428, 130] |
| 19 | [39, 40] | 219 | [165, 383] | 419 | [584, 510] | 619 | [163, 442] | 819 | [923, 429] |
| 20 | [41, 42] | 220 | [373, 384] | 420 | [585, 586] | 620 | [183, 176] | 820 | [924, 729] |
| 21 | [43, 44] | 221 | [385, 123] | 421 | [587, 588] | 621 | [768, 395] | 821 | [743, 383] |
| 22 | [45, 33] | 222 | [386, 379] | 422 | [589, 590] | 622 | [438, 395] | 822 | [724, 413] |
| 23 | [46, 33] | 223 | [387, 146] | 423 | [591, 592] | 623 | [731, 769] | 823 | [465, 413] |
| 24 | [47, 48] | 224 | [388, 389] | 424 | [593, 594] | 624 | [473, 132] | 824 | [721, 922] |
| 25 | [49, 50] | 225 | [390, 191] | 425 | [595, 596] | 625 | [770, 404] | 825 | [421, 721] |
| 26 | [51, 52] | 226 | [376, 378] | 426 | [597, 598] | 626 | [722, 440] | 826 | [925, 406] |
| 27 | [53, 54] | 227 | [391, 392] | 427 | [599, 77] | 627 | [771, 164] | 827 | [159, 146] |
| 28 | [55, 56] | 228 | [393, 185] | 428 | [600, 584] | 628 | [390, 431] | 828 | [784, 419] |
| 29 | [57, 58] | 229 | [394, 179] | 429 | [601, 602] | 629 | [417, 192] | 829 | [926, 484] |
| 30 | [43, 59] | 230 | [199, 395] | 430 | [603, 267] | 630 | [47, 772] | 830 | [798, 762] |
| 31 | [60, 61] | 231 | [377, 396] | 431 | [604, 605] | 631 | [133, 471] | 831 | [927, 463] |
| 32 | [62, 63] | 232 | [397, 398] | 432 | [606, 607] | 632 | [49, 752] | 832 | [928, 444] |
| 33 | [64, 65] | 233 | [177, 399] | 433 | [270, 608] | 633 | [773, 401] | 833 | [489, 772] |
| 34 | [66, 67] | 234 | [400, 401] | 434 | [609, 610] | 634 | [774, 204] | 834 | [129, 429] |
| 35 | [68, 69] | 235 | [402, 403] | 435 | [611, 591] | 635 | [146, 473] | 835 | [723, 202] |
| 36 | [70, 71] | 236 | [193, 404] | 436 | [612, 288] | 636 | [775, 492] | 836 | [929, 433] |
| 37 | [72, 73] | 237 | [405, 148] | 437 | [613, 614] | 637 | [469, 492] | 837 | [383, 128] |
| 38 | [74, 75] | 238 | [406, 147] | 438 | [615, 616] | 638 | [776, 748] | 838 | [372, 34] |
| 39 | [76, 77] | 239 | [40, 132] | 439 | [617, 557] | 639 | [760, 494] | 839 | [930, 195] |
| 40 | [78, 79] | 240 | [148, 148] | 440 | [618, 619] | 640 | [116, 152] | 840 | [402, 931] |

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|---|---|---|---|---|---|---|---|---|---|
| 41 | [80, 81] | 241 | [407, 54] | 441 | [620, 323] | 641 | [776, 736] | 841 | [713, 408] |
| 42 | [82, 83] | 242 | [57, 408] | 442 | [621, 622] | 642 | [764, 134] | 842 | [745, 134] |
| 43 | [84, 85] | 243 | [190, 409] | 443 | [623, 624] | 643 | [777, 458] | 843 | [932, 130] |
| 44 | [86, 87] | 244 | [410, 176] | 444 | [625, 626] | 644 | [443, 202] | 844 | [497, 167] |
| 45 | [88, 66] | 245 | [135, 185] | 445 | [627, 628] | 645 | [764, 403] | 845 | [781, 396] |
| 46 | [89, 90] | 246 | [411, 412] | 446 | [629, 630] | 646 | [47, 194] | 846 | [142, 176] |
| 47 | [91, 92] | 247 | [139, 413] | 447 | [631, 632] | 647 | [778, 501] | 847 | [933, 446] |
| 48 | [93, 94] | 248 | [414, 189] | 448 | [633, 634] | 648 | [738, 395] | 848 | [934, 935] |
| 49 | [95, 96] | 249 | [415, 416] | 449 | [224, 635] | 649 | [55, 2] | 849 | [733, 416] |
| 50 | [97, 98] | 250 | [417, 38] | 450 | [636, 216] | 650 | [464, 120] | 850 | [782, 374] |
| 51 | [99, 100] | 251 | [418, 10] | 451 | [637, 349] | 651 | [716, 750] | 851 | [49, 726] |
| 52 | [101, 102] | 252 | [419, 420] | 452 | [638, 639] | 652 | [387, 425] | 852 | [453, 34] |
| 53 | [103, 104] | 253 | [421, 125] | 453 | [61, 640] | 653 | [779, 769] | 853 | [372, 737] |
| 54 | [105, 106] | 254 | [422, 423] | 454 | [641, 642] | 654 | [496, 156] | 854 | [40, 170] |
| 55 | [107, 24] | 255 | [424, 14] | 455 | [643, 644] | 655 | [780, 186] | 855 | [936, 54] |
| 56 | [108, 109] | 256 | [425, 40] | 456 | [645, 646] | 656 | [781, 127] | 856 | [768, 477] |
| 57 | [110, 111] | 257 | [180, 426] | 457 | [647, 92] | 657 | [48, 195] | 857 | [491, 179] |
| 58 | [112, 113] | 258 | [427, 44] | 458 | [648, 649] | 658 | [405, 754] | 858 | [460, 458] |
| 59 | [114, 103] | 259 | [181, 378] | 459 | [650, 228] | 659 | [410, 466] | 859 | [937, 145] |
| 60 | [115, 116] | 260 | [428, 429] | 460 | [651, 591] | 660 | [462, 44] | 860 | [464, 186] |
| 61 | [117, 118] | 261 | [430, 148] | 461 | [652, 346] | 661 | [782, 123] | 861 | [393, 458] |
| 62 | [119, 120] | 262 | [145, 132] | 462 | [653, 596] | 662 | [743, 123] | 862 | [926, 191] |
| 63 | [121, 50] | 263 | [145, 406] | 463 | [654, 613] | 663 | [783, 123] | 863 | [155, 38] |
| 64 | [122, 123] | 264 | [170, 147] | 464 | [655, 656] | 664 | [193, 183] | 864 | [713, 58] |
| 65 | [124, 125] | 265 | [149, 148] | 465 | [657, 590] | 665 | [784, 10] | 865 | [938, 123] |
| 66 | [126, 127] | 266 | [405, 149] | 466 | [288, 658] | 666 | [143, 36] | 866 | [757, 796] |
| 67 | [51, 128] | 267 | [40, 406] | 467 | [659, 25] | 667 | [765, 754] | 867 | [939, 189] |
| 68 | [129, 130] | 268 | [160, 131] | 468 | [660, 661] | 668 | [404, 151] | 868 | [411, 2] |
| 69 | [131, 132] | 269 | [190, 431] | 469 | [628, 662] | 669 | [785, 431] | 869 | [180, 412] |
| 70 | [133, 134] | 270 | [432, 433] | 470 | [663, 664] | 670 | [774, 33] | 870 | [740, 772] |
| 71 | [135, 136] | 271 | [199, 434] | 471 | [665, 666] | 671 | [786, 492] | 871 | [400, 791] |
| 72 | [137, 138] | 272 | [170, 425] | 472 | [667, 585] | 672 | [11, 749] | 872 | [919, 796] |
| 73 | [139, 140] | 273 | [203, 33] | 473 | [358, 595] | 673 | [155, 482] | 873 | [925, 132] |
| 74 | [141, 142] | 274 | [435, 433] | 474 | [668, 664] | 674 | [167, 787] | 874 | [405, 405] |
| 75 | [143, 59] | 275 | [436, 150] | 475 | [669, 670] | 675 | [788, 431] | 875 | [940, 796] |
| 76 | [144, 145] | 276 | [167, 398] | 476 | [671, 549] | 676 | [465, 42] | 876 | [941, 151] |
| 77 | [146, 145] | 277 | [437, 54] | 477 | [672, 673] | 677 | [754, 405] | 877 | [764, 931] |
| 78 | [147, 145] | 278 | [438, 434] | 478 | [515, 674] | 678 | [789, 458] | 878 | [418, 419] |
| 79 | [148, 149] | 279 | [439, 408] | 479 | [675, 215] | 679 | [790, 10] | 879 | [152, 381] |
| 80 | [150, 151] | 280 | [415, 440] | 480 | [676, 677] | 680 | [742, 791] | 880 | [742, 202] |
| 81 | [47, 152] | 281 | [441, 442] | 481 | [678, 258] | 681 | [767, 730] | 881 | [376, 118] |
| 82 | [153, 154] | 282 | [443, 401] | 482 | [679, 680] | 682 | [377, 762] | 882 | [942, 935] |
| 83 | [155, 156] | 283 | [13, 444] | 483 | [677, 582] | 683 | [792, 147] | 883 | [725, 453] |
| 84 | [157, 158] | 284 | [436, 142] | 484 | [681, 682] | 684 | [150, 712] | 884 | [460, 382] |

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|---|---|---|---|---|---|---|---|---|---|
| 85 | [159, 160] | 285 | [132, 160] | 485 | [683, 304] | 685 | [418, 379] | 885 | [788, 191] |
| 86 | [161, 162] | 286 | [406, 160] | 486 | [684, 685] | 686 | [408, 729] | 886 | [943, 378] |
| 87 | [163, 164] | 287 | [445, 446] | 487 | [686, 687] | 687 | [489, 48] | 887 | [10, 392] |
| 88 | [165, 51] | 288 | [388, 370] | 488 | [688, 689] | 688 | [774, 453] | 888 | [939, 752] |
| 89 | [166, 167] | 289 | [439, 58] | 489 | [690, 691] | 689 | [467, 168] | 889 | [414, 752] |
| 90 | [163, 168] | 290 | [400, 447] | 490 | [692, 693] | 690 | [755, 754] | 890 | [400, 202] |
| 91 | [169, 170] | 291 | [448, 449] | 491 | [694, 695] | 691 | [727, 426] | 891 | [173, 397] |
| 92 | [41, 140] | 292 | [393, 382] | 492 | [28, 696] | 692 | [793, 794] | 892 | [380, 797] |
| 93 | [171, 172] | 293 | [450, 412] | 493 | [697, 313] | 693 | [421, 408] | 893 | [119, 186] |
| 94 | [173, 174] | 294 | [451, 452] | 494 | [698, 699] | 694 | [194, 766] | 894 | [743, 51] |
| 95 | [175, 176] | 295 | [166, 397] | 495 | [700, 345] | 695 | [418, 469] | 895 | [148, 405] |
| 96 | [177, 127] | 296 | [371, 453] | 496 | [701, 208] | 696 | [146, 40] | 896 | [438, 200] |
| 97 | [178, 179] | 297 | [126, 396] | 497 | [702, 329] | 697 | [53, 490] | 897 | [944, 739] |
| 98 | [180, 2] | 298 | [32, 204] | 498 | [703, 689] | 698 | [790, 419] | 898 | [773, 202] |
| 99 | [181, 118] | 299 | [375, 150] | 499 | [704, 705] | 699 | [722, 741] | 899 | [166, 174] |
| 100 | [182, 183] | 300 | [178, 452] | 500 | [706, 707] | 700 | [771, 442] | 900 | [383, 52] |
| 101 | [184, 185] | 301 | [139, 454] | 501 | [708, 709] | 701 | [795, 796] | 901 | [945, 769] |
| 102 | [186, 187] | 302 | [430, 149] | 502 | [710, 124] | 702 | [774, 372] | 902 | [58, 449] |
| 103 | [188, 189] | 303 | [455, 456] | 503 | [711, 132] | 703 | [475, 797] | 903 | [406, 146] |
| 104 | [190, 191] | 304 | [425, 131] | 504 | [404, 712] | 704 | [780, 120] | 904 | [205, 730] |
| 105 | [155, 192] | 305 | [457, 156] | 505 | [713, 125] | 705 | [498, 384] | 905 | [937, 131] |
| 106 | [193, 142] | 306 | [135, 458] | 506 | [422, 714] | 706 | [798, 399] | 906 | [946, 501] |
| 107 | [194, 195] | 307 | [459, 51] | 507 | [715, 482] | 707 | [799, 431] | 907 | [483, 772] |
| 108 | [196, 197] | 308 | [460, 185] | 508 | [45, 372] | 708 | [760, 800] | 908 | [43, 36] |
| 109 | [198, 134] | 309 | [385, 374] | 509 | [716, 392] | 709 | [489, 194] | 909 | [441, 164] |
| 110 | [160, 40] | 310 | [461, 152] | 510 | [432, 174] | 710 | [801, 225] | 910 | [947, 172] |
| 111 | [199, 200] | 311 | [462, 463] | 511 | [116, 48] | 711 | [802, 578] | 911 | [793, 800] |
| 112 | [201, 202] | 312 | [464, 378] | 512 | [717, 146] | 712 | [325, 803] | 912 | [436, 404] |
| 113 | [203, 204] | 313 | [465, 140] | 513 | [718, 140] | 713 | [804, 805] | 913 | [473, 159] |
| 114 | [205, 14] | 314 | [175, 466] | 514 | [719, 167] | 714 | [607, 806] | 914 | [772, 381] |
| 115 | [206, 104] | 315 | [467, 384] | 515 | [184, 382] | 715 | [807, 808] | 915 | [495, 935] |
| 116 | [207, 208] | 316 | [468, 195] | 516 | [479, 449] | 716 | [809, 524] | 916 | [141, 404] |
| 117 | [209, 210] | 317 | [386, 469] | 517 | [499, 433] | 717 | [810, 811] | 917 | [926, 409] |
| 118 | [211, 212] | 318 | [470, 12] | 518 | [720, 721] | 718 | [812, 813] | 918 | [948, 517] |
| 119 | [213, 214] | 319 | [198, 471] | 519 | [722, 416] | 719 | [814, 245] | 919 | [360, 817] |
| 120 | [215, 216] | 320 | [469, 420] | 520 | [723, 401] | 720 | [815, 816] | 920 | [949, 285] |
| 121 | [217, 218] | 321 | [472, 379] | 521 | [170, 146] | 721 | [817, 211] | 921 | [950, 951] |
| 122 | [219, 220] | 322 | [459, 383] | 522 | [394, 452] | 722 | [818, 264] | 922 | [111, 952] |
| 123 | [221, 222] | 323 | [414, 50] | 523 | [451, 179] | 723 | [819, 78] | 923 | [580, 21] |
| 124 | [223, 224] | 324 | [39, 473] | 524 | [32, 33] | 724 | [820, 608] | 924 | [953, 517] |
| 125 | [225, 226] | 325 | [474, 426] | 525 | [124, 721] | 725 | [821, 96] | 925 | [954, 266] |
| 126 | [227, 228] | 326 | [171, 416] | 526 | [724, 454] | 726 | [822, 823] | 926 | [955, 951] |
| 127 | [229, 230] | 327 | [45, 204] | 527 | [55, 412] | 727 | [824, 825] | 927 | [956, 957] |
| 128 | [231, 232] | 328 | [475, 381] | 528 | [725, 372] | 728 | [826, 237] | 928 | [958, 896] |

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|---|---|---|---|---|---|---|---|---|---|
| 129 | [233, 234] | 329 | [177, 396] | 529 | [414, 726] | 729 | [805, 827] | 929 | [959, 543] |
| 130 | [235, 236] | 330 | [476, 454] | 530 | [717, 425] | 730 | [828, 829] | 930 | [960, 363] |
| 131 | [237, 238] | 331 | [438, 477] | 531 | [727, 56] | 731 | [830, 831] | 931 | [961, 962] |
| 132 | [239, 240] | 332 | [461, 48] | 532 | [728, 147] | 732 | [832, 833] | 932 | [963, 964] |
| 133 | [241, 224] | 333 | [120, 478] | 533 | [721, 729] | 733 | [834, 238] | 933 | [965, 966] |
| 134 | [242, 243] | 334 | [479, 480] | 534 | [153, 197] | 734 | [835, 309] | 934 | [967, 880] |
| 135 | [244, 245] | 335 | [450, 2] | 535 | [13, 730] | 735 | [836, 687] | 935 | [968, 969] |
| 136 | [246, 247] | 336 | [199, 477] | 536 | [731, 732] | 736 | [220, 837] | 936 | [970, 971] |
| 137 | [248, 249] | 337 | [481, 158] | 537 | [159, 147] | 737 | [218, 838] | 937 | [972, 957] |
| 138 | [250, 251] | 338 | [146, 131] | 538 | [733, 172] | 738 | [839, 525] | 938 | [973, 851] |
| 139 | [252, 253] | 339 | [457, 482] | 539 | [464, 118] | 739 | [840, 841] | 939 | [974, 63] |
| 140 | [254, 255] | 340 | [49, 189] | 540 | [451, 389] | 740 | [827, 278] | 940 | [975, 225] |
| 141 | [256, 257] | 341 | [483, 194] | 541 | [159, 425] | 741 | [842, 233] | 941 | [976, 977] |
| 142 | [258, 259] | 342 | [190, 484] | 542 | [448, 480] | 742 | [843, 811] | 942 | [978, 979] |
| 143 | [260, 239] | 343 | [485, 159] | 543 | [188, 726] | 743 | [844, 845] | 943 | [980, 25] |
| 144 | [261, 239] | 344 | [486, 434] | 544 | [734, 151] | 744 | [846, 220] | 944 | [605, 981] |
| 145 | [262, 263] | 345 | [427, 463] | 545 | [735, 736] | 745 | [847, 588] | 945 | [982, 565] |
| 146 | [264, 265] | 346 | [487, 434] | 546 | [424, 730] | 746 | [848, 595] | 946 | [983, 269] |
| 147 | [266, 69] | 347 | [425, 473] | 547 | [476, 413] | 747 | [849, 850] | 947 | [984, 266] |
| 148 | [267, 268] | 348 | [488, 378] | 548 | [491, 452] | 748 | [851, 852] | 948 | [985, 720] |
| 149 | [149, 149] | 349 | [489, 152] | 549 | [720, 125] | 749 | [210, 853] | 949 | [986, 739] |
| 150 | [269, 270] | 350 | [120, 187] | 550 | [453, 737] | 750 | [20, 854] | 950 | [921, 148] |
| 151 | [271, 272] | 351 | [437, 490] | 551 | [738, 477] | 751 | [855, 513] | 951 | [754, 149] |
| 152 | [104, 273] | 352 | [491, 389] | 552 | [465, 454] | 752 | [856, 857] | 952 | [40, 159] |
| 153 | [274, 275] | 353 | [419, 492] | 553 | [481, 739] | 753 | [858, 519] | 953 | [920, 44] |
| 154 | [90, 276] | 354 | [436, 183] | 554 | [161, 494] | 754 | [338, 562] | 954 | [144, 131] |
| 155 | [277, 278] | 355 | [196, 154] | 555 | [377, 399] | 755 | [859, 537] | 955 | [932, 429] |
| 156 | [279, 280] | 356 | [493, 494] | 556 | [137, 501] | 756 | [860, 576] | 956 | [455, 935] |
| 157 | [281, 282] | 357 | [495, 456] | 557 | [180, 56] | 757 | [861, 862] | 957 | [754, 148] |
| 158 | [283, 284] | 358 | [170, 160] | 558 | [740, 194] | 758 | [863, 864] | 958 | [987, 446] |
| 159 | [285, 286] | 359 | [496, 482] | 559 | [415, 741] | 759 | [865, 81] | 959 | [173, 167] |
| 160 | [263, 285] | 360 | [467, 442] | 560 | [742, 401] | 760 | [866, 257] | 960 | [746, 416] |
| 161 | [287, 288] | 361 | [497, 397] | 561 | [743, 374] | 761 | [867, 112] | 961 | [770, 142] |
| 162 | [289, 290] | 362 | [498, 168] | 562 | [132, 147] | 762 | [868, 869] | 962 | [742, 447] |
| 163 | [291, 292] | 363 | [499, 174] | 563 | [131, 406] | 763 | [870, 871] | 963 | [988, 189] |
| 164 | [293, 294] | 364 | [182, 150] | 564 | [744, 399] | 764 | [872, 111] | 964 | [927, 44] |
| 165 | [295, 231] | 365 | [500, 501] | 565 | [43, 463] | 765 | [873, 338] | 965 | [777, 136] |
| 166 | [296, 297] | 366 | [369, 389] | 566 | [173, 433] | 766 | [582, 874] | 966 | [947, 416] |
| 167 | [298, 299] | 367 | [502, 64] | 567 | [745, 471] | 767 | [875, 20] | 967 | [460, 136] |
| 168 | [300, 301] | 368 | [503, 85] | 568 | [467, 164] | 768 | [876, 222] | 968 | [928, 763] |
| 169 | [302, 264] | 369 | [504, 505] | 569 | [746, 172] | 769 | [877, 878] | 969 | [472, 10] |
| 170 | [77, 262] | 370 | [506, 507] | 570 | [747, 729] | 770 | [803, 271] | 970 | [988, 752] |
| 171 | [303, 304] | 371 | [508, 218] | 571 | [735, 748] | 771 | [879, 210] | 971 | [799, 191] |
| 172 | [305, 306] | 372 | [273, 317] | 572 | [715, 156] | 772 | [880, 881] | 972 | [792, 160] |

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|---|---|---|---|---|---|---|---|---|---|
| 173 | [307, 308] | 373 | [509, 214] | 573 | [470, 749] | 773 | [882, 286] | 973 | [45, 453] |
| 174 | [309, 310] | 374 | [510, 511] | 574 | [493, 162] | 774 | [883, 884] | 974 | [379, 392] |
| 175 | [311, 312] | 375 | [512, 513] | 575 | [391, 750] | 775 | [885, 221] | 975 | [789, 136] |
| 176 | [313, 256] | 376 | [514, 515] | 576 | [188, 50] | 776 | [886, 354] | 976 | [785, 191] |
| 177 | [314, 315] | 377 | [516, 231] | 577 | [157, 739] | 777 | [887, 705] | 977 | [166, 433] |
| 178 | [316, 317] | 378 | [517, 518] | 578 | [473, 406] | 778 | [888, 326] | 978 | [498, 164] |
| 179 | [318, 319] | 379 | [519, 99] | 579 | [751, 162] | 779 | [889, 890] | 979 | [143, 463] |
| 180 | [320, 321] | 380 | [520, 61] | 580 | [188, 752] | 780 | [891, 292] | 980 | [782, 51] |
| 181 | [322, 323] | 381 | [92, 521] | 581 | [753, 446] | 781 | [892, 851] | 981 | [143, 44] |
| 182 | [324, 325] | 382 | [522, 523] | 582 | [55, 426] | 782 | [893, 101] | 982 | [781, 399] |
| 183 | [326, 327] | 383 | [524, 525] | 583 | [485, 170] | 783 | [894, 315] | 983 | [781, 762] |
| 184 | [328, 329] | 384 | [526, 527] | 584 | [201, 401] | 784 | [895, 896] | 984 | [761, 769] |
| 185 | [330, 331] | 385 | [528, 529] | 585 | [487, 200] | 785 | [897, 268] | 985 | [989, 817] |
| 186 | [113, 332] | 386 | [530, 531] | 586 | [754, 754] | 786 | [898, 899] | 986 | [568, 560] |
| 187 | [214, 333] | 387 | [532, 358] | 587 | [755, 405] | 787 | [245, 900] | 987 | [990, 991] |
| 188 | [334, 90] | 388 | [533, 275] | 588 | [718, 42] | 788 | [901, 562] | 988 | [992, 362] |
| 189 | [335, 336] | 389 | [534, 535] | 589 | [756, 33] | 789 | [902, 607] | 989 | [993, 122] |
| 190 | [337, 338] | 390 | [536, 537] | 590 | [193, 150] | 790 | [903, 28] | 990 | [498, 442] |
| 191 | [339, 340] | 391 | [538, 539] | 591 | [163, 384] | 791 | [904, 861] | 991 | [926, 431] |
| 192 | [341, 342] | 392 | [540, 541] | 592 | [186, 478] | 792 | [905, 624] | 992 | [125, 449] |
| 193 | [343, 344] | 393 | [542, 543] | 593 | [757, 758] | 793 | [906, 271] | 993 | [994, 510] |
| 194 | [345, 64] | 394 | [544, 65] | 594 | [139, 42] | 794 | [907, 908] | 994 | [995, 116] |
| 195 | [346, 347] | 395 | [545, 546] | 595 | [147, 131] | 795 | [909, 910] | 995 | [996, 93] |
| 196 | [348, 349] | 396 | [547, 548] | 596 | [160, 145] | 796 | [911, 912] | 996 | [997, 173] |
| 197 | [63, 350] | 397 | [259, 549] | 597 | [759, 714] | 797 | [808, 913] | 997 | [998, 309] |
| 198 | [351, 352] | 398 | [297, 550] | 598 | [760, 162] | 798 | [914, 529] | 998 | [115, 461] |
| 199 | [353, 354] | 399 | [551, 552] | 599 | [761, 732] | 799 | [915, 267] | | |

Output function $n \mapsto \tau(n)$:

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 143 | 0 | 286 | 0 | 429 | 0 | 572 | 0 | 715 | 0 | 858 | 0 |
| 1 | 0 | 144 | 0 | 287 | 0 | 430 | 0 | 573 | 1 | 716 | 0 | 859 | 1 |
| 2 | 0 | 145 | 0 | 288 | 0 | 431 | 1 | 574 | 0 | 717 | 1 | 860 | 1 |
| 3 | 0 | 146 | 0 | 289 | 1 | 432 | 0 | 575 | 1 | 718 | 1 | 861 | 1 |
| 4 | 0 | 147 | 1 | 290 | 0 | 433 | 0 | 576 | 0 | 719 | 0 | 862 | 1 |
| 5 | 0 | 148 | 1 | 291 | 1 | 434 | 0 | 577 | 1 | 720 | 1 | 863 | 1 |
| 6 | 0 | 149 | 0 | 292 | 1 | 435 | 0 | 578 | 0 | 721 | 0 | 864 | 0 |
| 7 | 0 | 150 | 0 | 293 | 0 | 436 | 0 | 579 | 1 | 722 | 0 | 865 | 0 |
| 8 | 1 | 151 | 0 | 294 | 1 | 437 | 1 | 580 | 0 | 723 | 0 | 866 | 1 |
| 9 | 0 | 152 | 0 | 295 | 0 | 438 | 1 | 581 | 0 | 724 | 0 | 867 | 0 |
| 10 | 1 | 153 | 0 | 296 | 0 | 439 | 1 | 582 | 0 | 725 | 1 | 868 | 1 |
| 11 | 0 | 154 | 1 | 297 | 1 | 440 | 0 | 583 | 1 | 726 | 0 | 869 | 1 |
| 12 | 1 | 155 | 1 | 298 | 0 | 441 | 1 | 584 | 0 | 727 | 0 | 870 | 1 |
| 13 | 0 | 156 | 1 | 299 | 1 | 442 | 1 | 585 | 1 | 728 | 0 | 871 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | 0 | 157 | 1 | 300 | 1 | 443 | 1 | 586 | 0 | 729 | 1 | 872 | 0 |
| 15 | 0 | 158 | 0 | 301 | 0 | 444 | 0 | 587 | 1 | 730 | 1 | 873 | 0 |
| 16 | 0 | 159 | 1 | 302 | 0 | 445 | 0 | 588 | 1 | 731 | 1 | 874 | 1 |
| 17 | 1 | 160 | 0 | 303 | 1 | 446 | 0 | 589 | 1 | 732 | 1 | 875 | 0 |
| 18 | 1 | 161 | 0 | 304 | 1 | 447 | 0 | 590 | 1 | 733 | 1 | 876 | 1 |
| 19 | 0 | 162 | 1 | 305 | 0 | 448 | 1 | 591 | 1 | 734 | 0 | 877 | 0 |
| 20 | 0 | 163 | 1 | 306 | 0 | 449 | 0 | 592 | 1 | 735 | 1 | 878 | 1 |
| 21 | 1 | 164 | 0 | 307 | 1 | 450 | 0 | 593 | 1 | 736 | 0 | 879 | 0 |
| 22 | 0 | 165 | 0 | 308 | 0 | 451 | 1 | 594 | 0 | 737 | 0 | 880 | 1 |
| 23 | 0 | 166 | 0 | 309 | 1 | 452 | 0 | 595 | 1 | 738 | 0 | 881 | 0 |
| 24 | 0 | 167 | 0 | 310 | 0 | 453 | 0 | 596 | 0 | 739 | 1 | 882 | 1 |
| 25 | 1 | 168 | 1 | 311 | 1 | 454 | 0 | 597 | 0 | 740 | 1 | 883 | 1 |
| 26 | 1 | 169 | 0 | 312 | 1 | 455 | 1 | 598 | 1 | 741 | 1 | 884 | 0 |
| 27 | 0 | 170 | 0 | 313 | 1 | 456 | 0 | 599 | 0 | 742 | 1 | 885 | 0 |
| 28 | 0 | 171 | 1 | 314 | 1 | 457 | 0 | 600 | 0 | 743 | 1 | 886 | 0 |
| 29 | 0 | 172 | 0 | 315 | 0 | 458 | 0 | 601 | 0 | 744 | 0 | 887 | 1 |
| 30 | 1 | 173 | 1 | 316 | 1 | 459 | 1 | 602 | 1 | 745 | 1 | 888 | 0 |
| 31 | 0 | 174 | 1 | 317 | 1 | 460 | 0 | 603 | 0 | 746 | 0 | 889 | 1 |
| 32 | 0 | 175 | 1 | 318 | 1 | 461 | 0 | 604 | 1 | 747 | 1 | 890 | 0 |
| 33 | 0 | 176 | 1 | 319 | 1 | 462 | 1 | 605 | 1 | 748 | 1 | 891 | 1 |
| 34 | 1 | 177 | 1 | 320 | 1 | 463 | 1 | 606 | 0 | 749 | 0 | 892 | 0 |
| 35 | 1 | 178 | 1 | 321 | 0 | 464 | 1 | 607 | 0 | 750 | 0 | 893 | 0 |
| 36 | 0 | 179 | 1 | 322 | 1 | 465 | 1 | 608 | 0 | 751 | 1 | 894 | 1 |
| 37 | 1 | 180 | 1 | 323 | 1 | 466 | 0 | 609 | 0 | 752 | 1 | 895 | 1 |
| 38 | 1 | 181 | 1 | 324 | 0 | 467 | 0 | 610 | 0 | 753 | 0 | 896 | 1 |
| 39 | 0 | 182 | 0 | 325 | 1 | 468 | 1 | 611 | 0 | 754 | 0 | 897 | 1 |
| 40 | 1 | 183 | 1 | 326 | 1 | 469 | 1 | 612 | 0 | 755 | 1 | 898 | 1 |
| 41 | 0 | 184 | 1 | 327 | 0 | 470 | 1 | 613 | 1 | 756 | 1 | 899 | 0 |
| 42 | 0 | 185 | 1 | 328 | 1 | 471 | 1 | 614 | 1 | 757 | 1 | 900 | 0 |
| 43 | 1 | 186 | 1 | 329 | 1 | 472 | 0 | 615 | 1 | 758 | 1 | 901 | 0 |
| 44 | 0 | 187 | 0 | 330 | 1 | 473 | 0 | 616 | 0 | 759 | 0 | 902 | 0 |
| 45 | 0 | 188 | 0 | 331 | 1 | 474 | 1 | 617 | 1 | 760 | 1 | 903 | 0 |
| 46 | 0 | 189 | 0 | 332 | 0 | 475 | 1 | 618 | 0 | 761 | 0 | 904 | 1 |
| 47 | 0 | 190 | 0 | 333 | 0 | 476 | 1 | 619 | 1 | 762 | 1 | 905 | 1 |
| 48 | 1 | 191 | 0 | 334 | 0 | 477 | 0 | 620 | 1 | 763 | 1 | 906 | 0 |
| 49 | 1 | 192 | 0 | 335 | 0 | 478 | 1 | 621 | 1 | 764 | 0 | 907 | 0 |
| 50 | 1 | 193 | 1 | 336 | 0 | 479 | 0 | 622 | 1 | 765 | 0 | 908 | 1 |
| 51 | 1 | 194 | 0 | 337 | 0 | 480 | 1 | 623 | 1 | 766 | 0 | 909 | 1 |
| 52 | 1 | 195 | 1 | 338 | 0 | 481 | 0 | 624 | 0 | 767 | 0 | 910 | 0 |
| 53 | 0 | 196 | 1 | 339 | 0 | 482 | 0 | 625 | 0 | 768 | 1 | 911 | 0 |
| 54 | 1 | 197 | 0 | 340 | 1 | 483 | 0 | 626 | 0 | 769 | 0 | 912 | 0 |
| 55 | 0 | 198 | 1 | 341 | 0 | 484 | 0 | 627 | 0 | 770 | 0 | 913 | 0 |
| 56 | 1 | 199 | 0 | 342 | 0 | 485 | 1 | 628 | 1 | 771 | 0 | 914 | 1 |
| 57 | 0 | 200 | 1 | 343 | 1 | 486 | 0 | 629 | 0 | 772 | 1 | 915 | 0 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 58 | 0 | 201 | 0 | 344 | 0 | 487 | 1 | 630 | 0 | 773 | 1 | 916 | 1 |
| 59 | 1 | 202 | 1 | 345 | 0 | 488 | 1 | 631 | 0 | 774 | 1 | 917 | 1 |
| 60 | 0 | 203 | 1 | 346 | 1 | 489 | 1 | 632 | 1 | 775 | 0 | 918 | 0 |
| 61 | 0 | 204 | 1 | 347 | 1 | 490 | 0 | 633 | 1 | 776 | 0 | 919 | 0 |
| 62 | 0 | 205 | 1 | 348 | 1 | 491 | 0 | 634 | 1 | 777 | 1 | 920 | 0 |
| 63 | 0 | 206 | 0 | 349 | 1 | 492 | 0 | 635 | 0 | 778 | 0 | 921 | 1 |
| 64 | 0 | 207 | 1 | 350 | 0 | 493 | 0 | 636 | 0 | 779 | 1 | 922 | 0 |
| 65 | 0 | 208 | 1 | 351 | 1 | 494 | 0 | 637 | 1 | 780 | 1 | 923 | 0 |
| 66 | 1 | 209 | 0 | 352 | 0 | 495 | 0 | 638 | 0 | 781 | 0 | 924 | 0 |
| 67 | 1 | 210 | 0 | 353 | 0 | 496 | 1 | 639 | 1 | 782 | 0 | 925 | 0 |
| 68 | 1 | 211 | 0 | 354 | 0 | 497 | 1 | 640 | 1 | 783 | 1 | 926 | 1 |
| 69 | 1 | 212 | 1 | 355 | 1 | 498 | 1 | 641 | 0 | 784 | 1 | 927 | 1 |
| 70 | 0 | 213 | 0 | 356 | 0 | 499 | 1 | 642 | 0 | 785 | 1 | 928 | 1 |
| 71 | 0 | 214 | 0 | 357 | 0 | 500 | 1 | 643 | 1 | 786 | 1 | 929 | 1 |
| 72 | 1 | 215 | 0 | 358 | 0 | 501 | 1 | 644 | 1 | 787 | 0 | 930 | 0 |
| 73 | 0 | 216 | 0 | 359 | 1 | 502 | 0 | 645 | 0 | 788 | 0 | 931 | 0 |
| 74 | 1 | 217 | 0 | 360 | 0 | 503 | 1 | 646 | 0 | 789 | 0 | 932 | 1 |
| 75 | 0 | 218 | 0 | 361 | 1 | 504 | 1 | 647 | 0 | 790 | 0 | 933 | 1 |
| 76 | 0 | 219 | 0 | 362 | 1 | 505 | 0 | 648 | 0 | 791 | 1 | 934 | 0 |
| 77 | 0 | 220 | 0 | 363 | 1 | 506 | 1 | 649 | 0 | 792 | 1 | 935 | 1 |
| 78 | 1 | 221 | 1 | 364 | 0 | 507 | 0 | 650 | 1 | 793 | 0 | 936 | 1 |
| 79 | 1 | 222 | 1 | 365 | 1 | 508 | 0 | 651 | 0 | 794 | 0 | 937 | 1 |
| 80 | 0 | 223 | 0 | 366 | 1 | 509 | 0 | 652 | 0 | 795 | 1 | 938 | 0 |
| 81 | 0 | 224 | 0 | 367 | 0 | 510 | 0 | 653 | 1 | 796 | 0 | 939 | 0 |
| 82 | 0 | 225 | 1 | 368 | 1 | 511 | 1 | 654 | 1 | 797 | 1 | 940 | 0 |
| 83 | 1 | 226 | 0 | 369 | 1 | 512 | 1 | 655 | 1 | 798 | 1 | 941 | 1 |
| 84 | 1 | 227 | 1 | 370 | 1 | 513 | 1 | 656 | 0 | 799 | 0 | 942 | 1 |
| 85 | 1 | 228 | 1 | 371 | 0 | 514 | 0 | 657 | 1 | 800 | 1 | 943 | 0 |
| 86 | 0 | 229 | 0 | 372 | 1 | 515 | 1 | 658 | 1 | 801 | 0 | 944 | 1 |
| 87 | 1 | 230 | 0 | 373 | 0 | 516 | 0 | 659 | 0 | 802 | 1 | 945 | 0 |
| 88 | 0 | 231 | 0 | 374 | 0 | 517 | 1 | 660 | 1 | 803 | 0 | 946 | 0 |
| 89 | 0 | 232 | 1 | 375 | 1 | 518 | 1 | 661 | 0 | 804 | 0 | 947 | 0 |
| 90 | 1 | 233 | 1 | 376 | 0 | 519 | 0 | 662 | 1 | 805 | 1 | 948 | 0 |
| 91 | 0 | 234 | 0 | 377 | 0 | 520 | 0 | 663 | 1 | 806 | 1 | 949 | 0 |
| 92 | 0 | 235 | 1 | 378 | 1 | 521 | 0 | 664 | 1 | 807 | 0 | 950 | 1 |
| 93 | 1 | 236 | 1 | 379 | 0 | 522 | 0 | 665 | 1 | 808 | 1 | 951 | 0 |
| 94 | 1 | 237 | 1 | 380 | 0 | 523 | 1 | 666 | 0 | 809 | 0 | 952 | 1 |
| 95 | 1 | 238 | 0 | 381 | 0 | 524 | 0 | 667 | 0 | 810 | 1 | 953 | 0 |
| 96 | 1 | 239 | 1 | 382 | 0 | 525 | 0 | 668 | 1 | 811 | 1 | 954 | 0 |
| 97 | 1 | 240 | 1 | 383 | 0 | 526 | 0 | 669 | 1 | 812 | 1 | 955 | 1 |
| 98 | 1 | 241 | 0 | 384 | 0 | 527 | 0 | 670 | 1 | 813 | 0 | 956 | 1 |
| 99 | 1 | 242 | 0 | 385 | 1 | 528 | 1 | 671 | 1 | 814 | 0 | 957 | 0 |
| 100 | 0 | 243 | 0 | 386 | 1 | 529 | 1 | 672 | 0 | 815 | 1 | 958 | 1 |
| 101 | 1 | 244 | 0 | 387 | 0 | 530 | 1 | 673 | 1 | 816 | 0 | 959 | 1 |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 102 | 1 | 245 | 0 | 388 | 0 | 531 | 0 | 674 | 0 | 817 | 0 | 960 | 0 |
| 103 | 0 | 246 | 1 | 389 | 0 | 532 | 0 | 675 | 0 | 818 | 0 | 961 | 0 |
| 104 | 0 | 247 | 0 | 390 | 1 | 533 | 0 | 676 | 1 | 819 | 0 | 962 | 1 |
| 105 | 1 | 248 | 1 | 391 | 1 | 534 | 0 | 677 | 0 | 820 | 0 | 963 | 1 |
| 106 | 1 | 249 | 1 | 392 | 1 | 535 | 0 | 678 | 0 | 821 | 1 | 964 | 1 |
| 107 | 0 | 250 | 0 | 393 | 1 | 536 | 1 | 679 | 0 | 822 | 0 | 965 | 1 |
| 108 | 1 | 251 | 1 | 394 | 0 | 537 | 1 | 680 | 1 | 823 | 1 | 966 | 0 |
| 109 | 1 | 252 | 0 | 395 | 1 | 538 | 1 | 681 | 0 | 824 | 0 | 967 | 0 |
| 110 | 0 | 253 | 1 | 396 | 1 | 539 | 1 | 682 | 0 | 825 | 1 | 968 | 1 |
| 111 | 0 | 254 | 1 | 397 | 1 | 540 | 1 | 683 | 1 | 826 | 0 | 969 | 0 |
| 112 | 0 | 255 | 1 | 398 | 1 | 541 | 1 | 684 | 0 | 827 | 1 | 970 | 1 |
| 113 | 1 | 256 | 1 | 399 | 0 | 542 | 1 | 685 | 1 | 828 | 1 | 971 | 0 |
| 114 | 1 | 257 | 1 | 400 | 0 | 543 | 0 | 686 | 1 | 829 | 1 | 972 | 1 |
| 115 | 0 | 258 | 0 | 401 | 0 | 544 | 0 | 687 | 1 | 830 | 1 | 973 | 0 |
| 116 | 1 | 259 | 1 | 402 | 1 | 545 | 1 | 688 | 1 | 831 | 1 | 974 | 0 |
| 117 | 0 | 260 | 0 | 403 | 1 | 546 | 1 | 689 | 0 | 832 | 1 | 975 | 0 |
| 118 | 0 | 261 | 0 | 404 | 1 | 547 | 1 | 690 | 1 | 833 | 1 | 976 | 1 |
| 119 | 0 | 262 | 0 | 405 | 1 | 548 | 0 | 691 | 0 | 834 | 1 | 977 | 0 |
| 120 | 0 | 263 | 0 | 406 | 0 | 549 | 1 | 692 | 0 | 835 | 0 | 978 | 1 |
| 121 | 0 | 264 | 0 | 407 | 0 | 550 | 0 | 693 | 1 | 836 | 1 | 979 | 0 |
| 122 | 0 | 265 | 0 | 408 | 1 | 551 | 0 | 694 | 0 | 837 | 0 | 980 | 0 |
| 123 | 1 | 266 | 1 | 409 | 1 | 552 | 1 | 695 | 1 | 838 | 1 | 981 | 0 |
| 124 | 0 | 267 | 1 | 410 | 0 | 553 | 0 | 696 | 0 | 839 | 0 | 982 | 0 |
| 125 | 1 | 268 | 0 | 411 | 1 | 554 | 0 | 697 | 0 | 840 | 1 | 983 | 0 |
| 126 | 1 | 269 | 0 | 412 | 1 | 555 | 0 | 698 | 0 | 841 | 0 | 984 | 0 |
| 127 | 0 | 270 | 0 | 413 | 1 | 556 | 1 | 699 | 0 | 842 | 1 | 985 | 0 |
| 128 | 0 | 271 | 0 | 414 | 1 | 557 | 1 | 700 | 0 | 843 | 1 | 986 | 0 |
| 129 | 1 | 272 | 0 | 415 | 1 | 558 | 1 | 701 | 1 | 844 | 1 | 987 | 1 |
| 130 | 1 | 273 | 1 | 416 | 1 | 559 | 1 | 702 | 1 | 845 | 0 | 988 | 1 |
| 131 | 1 | 274 | 0 | 417 | 0 | 560 | 1 | 703 | 1 | 846 | 0 | 989 | 0 |
| 132 | 1 | 275 | 0 | 418 | 1 | 561 | 1 | 704 | 1 | 847 | 1 | 990 | 1 |
| 133 | 0 | 276 | 0 | 419 | 0 | 562 | 1 | 705 | 1 | 848 | 0 | 991 | 1 |
| 134 | 0 | 277 | 1 | 420 | 1 | 563 | 1 | 706 | 1 | 849 | 1 | 992 | 1 |
| 135 | 0 | 278 | 1 | 421 | 1 | 564 | 0 | 707 | 0 | 850 | 0 | 993 | 0 |
| 136 | 1 | 279 | 1 | 422 | 1 | 565 | 1 | 708 | 1 | 851 | 1 | 994 | 0 |
| 137 | 1 | 280 | 1 | 423 | 1 | 566 | 1 | 709 | 1 | 852 | 0 | 995 | 0 |
| 138 | 0 | 281 | 1 | 424 | 1 | 567 | 1 | 710 | 0 | 853 | 1 | 996 | 0 |
| 139 | 0 | 282 | 1 | 425 | 1 | 568 | 0 | 711 | 1 | 854 | 1 | 997 | 0 |
| 140 | 1 | 283 | 0 | 426 | 0 | 569 | 0 | 712 | 1 | 855 | 1 | 998 | 0 |
| 141 | 1 | 284 | 0 | 427 | 0 | 570 | 1 | 713 | 0 | 856 | 1 | | |
| 142 | 0 | 285 | 1 | 428 | 0 | 571 | 1 | 714 | 0 | 857 | 0 | | |

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