PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^6+z^5+z^3,z)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^6+z^5+z^3,z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^6+z^5+z^3,z)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z)&=z^{17}+z^{16}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^2+1,\\
p_1(z)&=z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^7+z^6+z^3,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^6,\\
p_3(z)&=z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^7+z^6+z^3,\\
p_4(z)&=z^{22}+z^{20}+z^{18}+z^{17}+z^{14}+z^{13}+z^{11}+z^{10}+z^9+z^5+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z)&=z^{20}+z^{17}+z^{14}+z^{13}+z^{12}+z^{11}+z^{10}+z^9+z^6+z^5+z^2,\\
p_1(z)&=z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^7+z^6+z^3,\\
p_2(z)&=z^{24}+z^{22}+z^{20}+z^{18}+z^6,\\
p_3(z)&=z^{23}+z^{21}+z^{19}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{10}+z^7+z^6+z^3,\\
p_4(z)&=z^{22}+z^{18}+z^{17}+z^{16}+z^{14}+z^{13}+z^{11}+z^{10}+z^9+z^8+z^5+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)&=x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)&=x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\ge 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z \left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] = {}& M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{6u} M^e[:u] \\
& + x^u M^e[:7u] + x^{2u} M^e[:6u] + x^{3u} M^e[:5u] + x^{5u} M^e[:3u] \\
& + x^{7u} M^e[:u] + x^{3u} M^e[:7u] + x^{4u} M^e[:6u] + x^{5u} M^e[:5u] \\
& + x^{7u} M^e[:3u] + x^{9u} M^e[:u] + x^{6u} M^e[:7u] + x^{7u} M^e[:6u] \\
& + x^{8u} M^e[:5u] + x^{10u} M^e[:3u] + x^{12u} M^e[:u] + x^{7u} M^e[:7u] \\
& + x^{8u} M^e[:6u] + x^{12u} M^e[:2u] + (x^{7u} + x^{8u} + x^{10u} + x^{13u}) R^e,
\end{aligned}$$

$$\begin{aligned}W^e[:14u] &= W^e[:7u] + x^uW^e[:6u] + x^{2u}W^e[:5u] + x^{4u}W^e[:3u] + x^{6u}W^e[:u]\\ &+ x^uW^e[:7u] + x^{2u}W^e[:6u] + x^{3u}W^e[:5u] + x^{5u}W^e[:3u]\\ &+ x^{7u}W^e[:u] + x^{3u}W^e[:7u] + x^{4u}W^e[:6u] + x^{5u}W^e[:5u]\\ &+ x^{7u}W^e[:3u] + x^{9u}W^e[:u] + x^{6u}W^e[:7u] + x^{7u}W^e[:6u]\\ &+ x^{8u}W^e[:5u] + x^{10u}W^e[:3u] + x^{12u}W^e[:u] + x^{7u}W^e[:7u]\\ &+ x^{8u}W^e[:6u] + x^{12u}W^e[:2u] + (x^{7u} + x^{8u} + x^{10u} + x^{13u})R^e.\end{aligned}$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$\begin{aligned}M^o[:14u] &= M^o[:7u] + x^uM^o[:6u] + x^{2u}M^o[:5u] + x^{4u}M^o[:3u] + x^{6u}M^o[:u]\\ &+ x^uM^o[:7u] + x^{2u}M^o[:6u] + x^{3u}M^o[:5u] + x^{5u}M^o[:3u]\\ &+ x^{7u}M^o[:u] + x^{3u}M^o[:7u] + x^{4u}M^o[:6u] + x^{5u}M^o[:5u]\\ &+ x^{7u}M^o[:3u] + x^{9u}M^o[:u] + x^{6u}M^o[:7u] + x^{7u}M^o[:6u]\\ &+ x^{8u}M^o[:5u] + x^{10u}M^o[:3u] + x^{12u}M^o[:u] + x^{7u}M^o[:7u]\\ &+ x^{8u}M^o[:6u] + x^{12u}M^o[:2u] + (x^{7u} + x^{8u} + x^{10u} + x^{13u})R^o,\\ W^o[:14u] &= W^o[:7u] + x^uW^o[:6u] + x^{2u}W^o[:5u] + x^{4u}W^o[:3u] + x^{6u}W^o[:u]\\ &+ x^uW^o[:7u] + x^{2u}W^o[:6u] + x^{3u}W^o[:5u] + x^{5u}W^o[:3u]\\ &+ x^{7u}W^o[:u] + x^{3u}W^o[:7u] + x^{4u}W^o[:6u] + x^{5u}W^o[:5u]\\ &+ x^{7u}W^o[:3u] + x^{9u}W^o[:u] + x^{6u}W^o[:7u] + x^{7u}W^o[:6u]\\ &+ x^{8u}W^o[:5u] + x^{10u}W^o[:3u] + x^{12u}W^o[:u] + x^{7u}W^o[:7u]\\ &+ x^{8u}W^o[:6u] + x^{12u}W^o[:2u] + (x^{7u} + x^{8u} + x^{10u} + x^{13u})R^o.\end{aligned}$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$\begin{aligned}T[:14u] &= T[:7u] + x^uT[:6u] + x^{2u}T[:5u] + x^{4u}T[:3u] + x^{6u}T[:u] + x^uT[:7u]\\ &+ x^{2u}T[:6u] + x^{3u}T[:5u] + x^{5u}T[:3u] + x^{7u}T[:u] + x^{3u}T[:7u]\\ &+ x^{4u}T[:6u] + x^{5u}T[:5u] + x^{7u}T[:3u] + x^{9u}T[:u] + x^{6u}T[:7u]\\ &+ x^{7u}T[:6u] + x^{8u}T[:5u] + x^{10u}T[:3u] + x^{12u}T[:u] + x^{7u}T[:7u]\\ &+ x^{8u}T[:6u] + x^{12u}T[:2u].\end{aligned}$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}0 &= x^{6u}T[u:2u] + x^{4u}T[3u:4u] + x^{2u}T[5u:6u] + x^uT[6u:7u]\\ &\quad + T[7u:8u],\\ 0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u]\\ &\quad + x^{3u}T[5u:6u] + T[8u:9u],\\ 0 &= x^{9u}T[:u] + x^{7u}T[2u:3u] + x^{6u}T[3u:4u] + x^{5u}T[4u:5u]\\ &\quad + x^{4u}T[5u:6u] + x^{3u}T[6u:7u] + T[9u:10u],\end{aligned}$$

$$\begin{aligned}
0 &= x^{10u}T[:u] + x^{6u}T[4u:5u] + T[10u:11u],\\
0 &= x^{10u}T[u:2u] + x^{6u}T[5u:6u] + T[11u:12u],\\
0 &= x^{10u}T[2u:3u] + x^{6u}T[6u:7u] + T[12u:13u],\\
0 &= x^{12u}T[u:2u] + x^{8u}T[5u:6u] + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[10w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[w]_2] + T[[100w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[1011w]_2], \tag{2.18}$$

$$0 = T[[10w]_2] + T[[110w]_2] + T[[1100w]_2], \tag{2.19}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1066 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.21}$$

$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.22}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.23}$$

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.24}$$
$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.25}$$
$$0 = \tau(A(s,10)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.26}$$
$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,111)), \tag{2.28}$$
$$0 = \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,1000)), \tag{2.29}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1001)), \tag{2.30}$$
$$0 = \tau(A(s,\epsilon)) + \tau(A(s,100)) + \tau(A(s,1010)), \tag{2.31}$$
$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,1011)), \tag{2.32}$$
$$0 = \tau(A(s,10)) + \tau(A(s,110)) + \tau(A(s,1100)), \tag{2.33}$$
$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i,0^{20}), E_{20}) = (A(i,0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 10$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^u M^e[:6u] + x^{2u} M^e[:5u] + x^{4u} M^e[:3u] + x^{6u} M^e[:u] \\ &\quad + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^u W^e[:6u] + x^{2u} W^e[:5u] + x^{4u} W^e[:3u] + x^{6u} W^e[:u] \\ &\quad + x^{7u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^u M^o[:6u] + x^{2u} M^o[:5u] + x^{4u} M^o[:3u] + x^{6u} M^o[:u] \\ &\quad + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^u W^o[:6u] + x^{2u} W^o[:5u] + x^{4u} W^o[:3u] + x^{6u} W^o[:u] \\ &\quad + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k}M_{2k} = \big(&W^e[:7v] + x^v W^e[:6v] + x^{2v}W^e[:5v] + x^{4v}W^e[:3v] + x^{6v}W^e[:v] \\ &+ x^{7u}R^e\big) \times \big(M^e[:7v] + x^v M^e[:6v] + x^{2v}M^e[:5v] + x^{4v}M^e[:3v] \\ &+ x^{6v}M^e[:v] + x^{7u}R^e\big); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v}R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$\begin{aligned} W^e[:14v]M^e[:14v] + x^{2v}W^e[:12v]M^e[:12v] + x^{4v}W^e[:10v]M^e[:10v] \\ + x^{8v}W^e[:6v]M^e[:6v] + x^{12v}W^e[:2v]M^e[:2v]. \end{aligned}$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned} X &:= x^v, \\ a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n, \\ b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n, \\ c &:= R^e. \end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned} \lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k}, \\ \lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k}, \\ \lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k}, \\ \lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}. \end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{22} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^8 \\
&\quad + x^7, \\
q_1(x) &= x^{21} + x^{18} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^6 + x^5 + x^3 + x, \\
q_2(x) &= x^{18} + x^6 + x^4 + x^2 + 1, \\
q_3(x) &= x^{21} + x^{18} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^6 + x^5 + x^3 + x, \\
q_4(x) &= x^{22} + x^{19} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^7 + x^6 + x^4 + x^2.
\end{aligned}$$

The polynomial $Q(x, y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x, y)$ of multiplicity m and $R(x, y) := P(x, y)/Q(x, y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z, y)$, we get a series with valuation smaller than 216, which proves that $R(z, y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z, y)$ is the minimal polynomial of $\mathrm{CF}(a, b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{104} + x^{100} + x^{98} + x^{94} + x^{88} + x^{86} + x^{84} + x^{80} + x^{74} \\
&\quad + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{54} + x^{50} + x^{46} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} \\
&\quad + x^{14} + x^{12}, \\
p_3(x) &= x^{104} + x^{102} + x^{100} + x^{96} + x^{92} + x^{86} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} \\
&\quad + x^{68} + x^{60} + x^{58} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{42} + x^{40} + x^{32} \\
&\quad + x^{26} + x^{24} + x^{22} + x^{20} + x^{16} + x^{14} + x^{12} + x^8, \\
p_6(x) &= x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} + x^{88} + x^{84} + x^{76} + x^{66} \\
&\quad + x^{62} + x^{56} + x^{54} + x^{48} + x^{46} + x^{40} + x^{38} + x^{30} + x^{28} + x^{26} + x^{24} \\
&\quad + x^{22} + x^{18} + x^{14} + 1, \\
p_9(x) &= x^{102} + x^{98} + x^{92} + x^{90} + x^{88} + x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{66} + x^{62} + x^{56} + x^{52} + x^{48} + x^{46} + x^{44} + x^{42} + x^{38} + x^{36} + x^{34} \\
&\quad + x^{26} + x^{24} + x^{22} + x^{18} + x^{14} + x^{10} + x^8 + x^2, \\
p_{12}(x) &= x^{102} + x^{98} + x^{96} + x^{70} + x^{66} + x^{64} + x^{54} + x^{50} + x^{48} + x^{38} + x^{34} \\
&\quad + x^{32} + x^6 + x^2 + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {} & x^{129} + x^{123} + x^{122} + x^{119} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108} \\
& + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{95} + x^{90} + x^{89} + x^{86} \\
& + x^{81} + x^{80} + x^{79} + x^{75} + x^{73} + x^{70} + x^{68} + x^{65} + x^{63} + x^{60} + x^{57} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{46} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} \\
& + x^{33} + x^{32} + x^{31} + x^{29} + x^{27}, \\
p_3(x) = {} & x^{120} + x^{118} + x^{116} + x^{115} + x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} \\
& + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{86} + x^{83} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65} \\
& + x^{60} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{39} + x^{37} \\
& + x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} \\
& + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_6(x) = {} & x^{117} + x^{115} + x^{114} + x^{111} + x^{109} + x^{108} + x^{101} + x^{98} + x^{97} + x^{96} \\
& + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{71} \\
& + x^{68} + x^{67} + x^{66} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\
& + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} \\
& + x^{27} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{9} + x^{7} + x^{6}, \\
p_9(x) = {} & x^{114} + x^{112} + x^{110} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} \\
& + x^{96} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{80} + x^{77} + x^{73} + x^{71} \\
& + x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{54} + x^{52} + x^{49} + x^{48} \\
& + x^{46} + x^{45} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{25} + x^{23} \\
& + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{10} + x^{5} + x^{4} + x^{3}, \\
p_{12}(x) = {} & x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79} \\
& + x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60} \\
& + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{39} + x^{37} \\
& + x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {} & x^{129} + x^{123} + x^{122} + x^{119} + x^{115} + x^{114} + x^{112} + x^{110} + x^{109} + x^{108} \\
& + x^{106} + x^{105} + x^{104} + x^{103} + x^{102} + x^{100} + x^{95} + x^{90} + x^{89} + x^{86} \\
& + x^{81} + x^{80} + x^{79} + x^{75} + x^{73} + x^{70} + x^{68} + x^{65} + x^{63} + x^{60} + x^{57} \\
& + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{46} + x^{40} + x^{39} + x^{37} + x^{35} + x^{34} \\
& + x^{33} + x^{32} + x^{31} + x^{29} + x^{27}, \\
p_3(x) = {} & x^{120} + x^{118} + x^{116} + x^{115} + x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} \\
& + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{86} + x^{83} + x^{80} + x^{79}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65}\\
&+ x^{60} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{39} + x^{37}\\
&+ x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{9},\\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{111} + x^{109} + x^{108} + x^{101} + x^{98} + x^{97} + x^{96}\\
&+ x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{71}\\
&+ x^{68} + x^{67} + x^{66} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28}\\
&+ x^{27} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{9} + x^{7} + x^{6},\\
p_9(x) &= x^{114} + x^{112} + x^{110} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97}\\
&+ x^{96} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{80} + x^{77} + x^{73} + x^{71}\\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{54} + x^{52} + x^{49} + x^{48}\\
&+ x^{46} + x^{45} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{25} + x^{23}\\
&+ x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{10} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) &= x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79}\\
&+ x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{136} + x^{134} + x^{128} + x^{124} + x^{122} + x^{116} + x^{110} + x^{108}\\
&+ x^{106} + x^{104} + x^{100} + x^{98} + x^{94} + x^{92} + x^{88} + x^{86} + x^{80} + x^{78} + x^{70}\\
&+ x^{68} + x^{64} + x^{62} + x^{58} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44} + x^{42},\\
p_3(x) &= x^{126} + x^{122} + x^{118} + x^{114} + x^{112} + x^{104} + x^{98} + x^{96} + x^{90} + x^{88}\\
&+ x^{86} + x^{84} + x^{78} + x^{76} + x^{74} + x^{70} + x^{66} + x^{62} + x^{56} + x^{52} + x^{46}\\
&+ x^{34} + x^{32} + x^{30} + x^{28} + x^{24} + x^{22} + x^{20} + x^{8} + x^{6},\\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{114} + x^{112} + x^{110} + x^{108} + x^{100}\\
&+ x^{92} + x^{90} + x^{88} + x^{86} + x^{82} + x^{80} + x^{78} + x^{76} + x^{70} + x^{68} + x^{66}\\
&+ x^{64} + x^{56} + x^{54} + x^{50} + x^{48} + x^{42} + x^{38} + x^{34} + x^{30} + x^{26} + x^{22}\\
&+ x^{20} + x^{12} + x^{8} + x^{6} + x^{2} + 1,\\
p_9(x) &= x^{120} + x^{114} + x^{106} + x^{104} + x^{100} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80}\\
&+ x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{52} + x^{46} + x^{42} + x^{38} + x^{34}\\
&+ x^{32} + x^{28} + x^{26} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} + x^{2},\\
p_{12}(x) &= x^{120} + x^{104} + x^{96} + x^{88} + x^{64} + x^{48} + x^{40} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{136} + x^{134} + x^{132} + x^{128} + x^{122} + x^{108} + x^{106} + x^{102} \\
&\quad + x^{96} + x^{92} + x^{88} + x^{86} + x^{80} + x^{74} + x^{72} + x^{68} + x^{62} + x^{60} + x^{58} \\
&\quad + x^{48} + x^{46} + x^{42} + x^{40} + x^{36} + x^{34} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20} \\
&\quad + x^{12}, \\
p_3(x) &= x^{126} + x^{122} + x^{118} + x^{112} + x^{108} + x^{106} + x^{104} + x^{102} + x^{100} + x^{98} \\
&\quad + x^{96} + x^{92} + x^{84} + x^{76} + x^{74} + x^{70} + x^{62} + x^{60} + x^{56} + x^{54} + x^{52} \\
&\quad + x^{44} + x^{42} + x^{40} + x^{36} + x^{34} + x^{28} + x^{26} + x^{22} + x^{18} + x^{16} + x^{8}, \\
p_6(x) &= x^{126} + x^{122} + x^{120} + x^{118} + x^{116} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} \\
&\quad + x^{96} + x^{84} + x^{82} + x^{80} + x^{70} + x^{68} + x^{64} + x^{56} + x^{52} + x^{50} + x^{46} \\
&\quad + x^{38} + x^{36} + x^{34} + x^{22} + x^{18} + x^{16} + x^{12} + x^{8} + x^{4} + x^{2} + 1, \\
p_9(x) &= x^{120} + x^{114} + x^{106} + x^{104} + x^{100} + x^{94} + x^{92} + x^{90} + x^{88} + x^{82} + x^{80} \\
&\quad + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{52} + x^{46} + x^{42} + x^{38} + x^{34} \\
&\quad + x^{32} + x^{28} + x^{26} + x^{18} + x^{14} + x^{12} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{120} + x^{104} + x^{96} + x^{88} + x^{64} + x^{48} + x^{40} + x^{32} + x^{24} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{118} + x^{115} + x^{114} + x^{109} + x^{105} + x^{102} + x^{100} + x^{93} \\
&\quad + x^{92} + x^{90} + x^{88} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70} \\
&\quad + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49} \\
&\quad + x^{48} + x^{47} + x^{46} + x^{42} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29} \\
&\quad + x^{28} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18}, \\
p_3(x) &= x^{120} + x^{118} + x^{116} + x^{115} + x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99} \\
&\quad + x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{86} + x^{83} + x^{80} + x^{79} \\
&\quad + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65} \\
&\quad + x^{60} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{39} + x^{37} \\
&\quad + x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} \\
&\quad + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{9}, \\
p_6(x) &= x^{117} + x^{115} + x^{114} + x^{111} + x^{109} + x^{108} + x^{101} + x^{98} + x^{97} + x^{96} \\
&\quad + x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{71} \\
&\quad + x^{68} + x^{67} + x^{66} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} \\
&\quad + x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28} \\
&\quad + x^{27} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{9} + x^{7} + x^{6}, \\
p_9(x) &= x^{114} + x^{112} + x^{110} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97} \\
&\quad + x^{96} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{80} + x^{77} + x^{73} + x^{71}
\end{aligned}$$

$$\begin{aligned}
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{54} + x^{52} + x^{49} + x^{48}\\
&+ x^{46} + x^{45} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{25} + x^{23}\\
&+ x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{10} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) = {}& x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79}\\
&+ x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) = {}& x^{129} + x^{125} + x^{118} + x^{115} + x^{114} + x^{109} + x^{105} + x^{102} + x^{100} + x^{93}\\
&+ x^{92} + x^{90} + x^{88} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{71} + x^{70}\\
&+ x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{56} + x^{54} + x^{52} + x^{51} + x^{50} + x^{49}\\
&+ x^{48} + x^{47} + x^{46} + x^{42} + x^{38} + x^{37} + x^{34} + x^{33} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{24} + x^{23} + x^{22} + x^{19} + x^{18},\\
p_3(x) = {}& x^{120} + x^{118} + x^{116} + x^{115} + x^{108} + x^{104} + x^{103} + x^{102} + x^{101} + x^{99}\\
&+ x^{98} + x^{97} + x^{96} + x^{94} + x^{92} + x^{91} + x^{88} + x^{86} + x^{83} + x^{80} + x^{79}\\
&+ x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{71} + x^{69} + x^{68} + x^{66} + x^{65}\\
&+ x^{60} + x^{55} + x^{53} + x^{52} + x^{50} + x^{49} + x^{48} + x^{46} + x^{43} + x^{39} + x^{37}\\
&+ x^{34} + x^{33} + x^{31} + x^{29} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20}\\
&+ x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^{9},\\
p_6(x) = {}& x^{117} + x^{115} + x^{114} + x^{111} + x^{109} + x^{108} + x^{101} + x^{98} + x^{97} + x^{96}\\
&+ x^{93} + x^{91} + x^{90} + x^{85} + x^{82} + x^{81} + x^{80} + x^{79} + x^{77} + x^{76} + x^{71}\\
&+ x^{68} + x^{67} + x^{66} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48}\\
&+ x^{47} + x^{45} + x^{44} + x^{41} + x^{39} + x^{38} + x^{35} + x^{32} + x^{30} + x^{29} + x^{28}\\
&+ x^{27} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{16} + x^{9} + x^{7} + x^{6},\\
p_9(x) = {}& x^{114} + x^{112} + x^{110} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{98} + x^{97}\\
&+ x^{96} + x^{91} + x^{89} + x^{88} + x^{86} + x^{85} + x^{82} + x^{80} + x^{77} + x^{73} + x^{71}\\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{60} + x^{58} + x^{57} + x^{54} + x^{52} + x^{49} + x^{48}\\
&+ x^{46} + x^{45} + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{31} + x^{30} + x^{25} + x^{23}\\
&+ x^{22} + x^{20} + x^{19} + x^{18} + x^{16} + x^{14} + x^{13} + x^{10} + x^{5} + x^{4} + x^{3},\\
p_{12}(x) = {}& x^{111} + x^{109} + x^{108} + x^{103} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{79}\\
&+ x^{77} + x^{76} + x^{71} + x^{69} + x^{68} + x^{67} + x^{65} + x^{64} + x^{63} + x^{61} + x^{60}\\
&+ x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{45} + x^{44} + x^{39} + x^{37}\\
&+ x^{36} + x^{35} + x^{33} + x^{32} + x^{15} + x^{13} + x^{12} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0].

For $\phi(W^e,1,1)$:

$$\begin{aligned}
p_0(x) &= x^{114}+x^{110}+x^{106}+x^{104}+x^{102}+x^{82}+x^{80}+x^{74}+x^{60}+x^{58}+x^{50}\\
&\quad +x^{44}+x^{38}+x^{36}+x^{34}+x^{26}+x^{24},\\
p_3(x) &= x^{104}+x^{102}+x^{100}+x^{88}+x^{80}+x^{76}+x^{70}+x^{64}+x^{60}+x^{54}+x^{38}\\
&\quad +x^{36}+x^{32}+x^{24}+x^{16}+x^{6},\\
p_6(x) &= x^{104}+x^{102}+x^{100}+x^{98}+x^{94}+x^{90}+x^{84}+x^{78}+x^{76}+x^{74}+x^{66}\\
&\quad +x^{64}+x^{52}+x^{50}+x^{48}+x^{44}+x^{42}+x^{40}+x^{38}+x^{28}+x^{22}+x^{14}\\
&\quad +x^{10}+x^{8}+x^{4}+1,\\
p_9(x) &= x^{102}+x^{98}+x^{92}+x^{90}+x^{88}+x^{84}+x^{80}+x^{76}+x^{74}+x^{72}+x^{70}\\
&\quad +x^{66}+x^{62}+x^{56}+x^{52}+x^{48}+x^{46}+x^{44}+x^{42}+x^{38}+x^{36}+x^{34}\\
&\quad +x^{26}+x^{24}+x^{22}+x^{18}+x^{14}+x^{10}+x^{8}+x^{2},\\
p_{12}(x) &= x^{102}+x^{98}+x^{96}+x^{70}+x^{66}+x^{64}+x^{54}+x^{50}+x^{48}+x^{38}+x^{34}\\
&\quad +x^{32}+x^{6}+x^{2}+1,
\end{aligned}$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{129}+x^{122}+x^{121}+x^{118}+x^{117}+x^{116}+x^{115}+x^{112}+x^{111}+x^{106}\\
&\quad +x^{102}+x^{101}+x^{99}+x^{94}+x^{93}+x^{92}+x^{91}+x^{90}+x^{89}+x^{86}+x^{84}\\
&\quad +x^{82}+x^{79}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{71}+x^{68}+x^{67}+x^{64}\\
&\quad +x^{62}+x^{61}+x^{59}+x^{56}+x^{52}+x^{50}+x^{49}+x^{48}+x^{47}+x^{44}+x^{41}\\
&\quad +x^{36}+x^{34}+x^{33}+x^{31}+x^{28}+x^{25}+x^{24}+x^{23}+x^{20}+x^{19}+x^{17}\\
&\quad +x^{14}+x^{13}+x^{12},\\
p_3(x) &= x^{123}+x^{120}+x^{117}+x^{114}+x^{113}+x^{112}+x^{109}+x^{107}+x^{103}+x^{102}\\
&\quad +x^{101}+x^{100}+x^{97}+x^{95}+x^{92}+x^{91}+x^{90}+x^{86}+x^{84}+x^{83}+x^{79}\\
&\quad +x^{78}+x^{77}+x^{76}+x^{74}+x^{69}+x^{68}+x^{65}+x^{63}+x^{62}+x^{58}+x^{52}\\
&\quad +x^{50}+x^{48}+x^{47}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{37}+x^{36}\\
&\quad +x^{35}+x^{34}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{23}+x^{22}+x^{20}\\
&\quad +x^{19}+x^{14}+x^{13}+x^{10}+x^{9}+x^{8},\\
p_6(x) &= x^{123}+x^{120}+x^{117}+x^{114}+x^{113}+x^{112}+x^{107}+x^{105}+x^{101}+x^{100}\\
&\quad +x^{97}+x^{95}+x^{93}+x^{90}+x^{89}+x^{87}+x^{85}+x^{84}+x^{77}+x^{76}+x^{75}\\
&\quad +x^{73}+x^{72}+x^{70}+x^{69}+x^{66}+x^{65}+x^{63}+x^{62}+x^{61}+x^{60}+x^{58}\\
&\quad +x^{57}+x^{56}+x^{55}+x^{53}+x^{50}+x^{49}+x^{48}+x^{45}+x^{41}+x^{38}+x^{35}\\
&\quad +x^{34}+x^{29}+x^{28}+x^{27}+x^{26}+x^{23}+x^{20}+x^{18}+x^{16}+x^{14}+x^{12}\\
&\quad +x^{8}+x^{7}+x^{6}+x^{3}+x+1,\\
p_9(x) &= x^{123}+x^{121}+x^{120}+x^{115}+x^{113}+x^{112}+x^{107}+x^{105}+x^{104}+x^{103}\\
&\quad +x^{101}+x^{100}+x^{91}+x^{89}+x^{88}+x^{83}+x^{81}+x^{80}+x^{71}+x^{69}+x^{68}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{67} + x^{65} + x^{64} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41}\\
&+ x^{40} + x^{39} + x^{37} + x^{36} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16} + x^{11}\\
&+ x^{9} + x^{8} + x^{7} + x^{5} + x^{4},\\
p_{12}(x) = {}& x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{107}\\
&+ x^{105} + x^{104} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80}\\
&+ x^{79} + x^{77} + x^{76} + x^{63} + x^{61} + x^{60} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41}\\
&+ x^{40} + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16} + x^{15}\\
&+ x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0].

For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{144} + x^{140} + x^{138} + x^{134} + x^{133} + x^{132} + x^{130} + x^{129} + x^{128} + x^{123}\\
&+ x^{122} + x^{121} + x^{119} + x^{117} + x^{115} + x^{110} + x^{108} + x^{107} + x^{103}\\
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{88} + x^{83}\\
&+ x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62}\\
&+ x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44}\\
&+ x^{43} + x^{37} + x^{35} + x^{34} + x^{29} + x^{28} + x^{27},\\
p_3(x) = {}& x^{123} + x^{120} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{109} + x^{106} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{99} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89}\\
&+ x^{87} + x^{85} + x^{84} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{70} + x^{68} + x^{66}\\
&+ x^{64} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{45}\\
&+ x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29}\\
&+ x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{9},\\
p_6(x) = {}& x^{126} + x^{120} + x^{118} + x^{114} + x^{112} + x^{110} + x^{100} + x^{98} + x^{96} + x^{92}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64}\\
&+ x^{58} + x^{56} + x^{54} + x^{50} + x^{40} + x^{38} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22}\\
&+ x^{18} + x^{16} + x^{10} + x^{6},\\
p_9(x) = {}& x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{113} + x^{109}\\
&+ x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{99} + x^{97} + x^{94} + x^{93} + x^{91}\\
&+ x^{90} + x^{89} + x^{85} + x^{84} + x^{78} + x^{76} + x^{72} + x^{69} + x^{68} + x^{67} + x^{62}\\
&+ x^{60} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{35} + x^{33} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{21} + x^{20} + x^{17}\\
&+ x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{3},\\
p_{12}(x) = {}& x^{132} + x^{108} + x^{104} + x^{96} + x^{84} + x^{76} + x^{72} + x^{64} + x^{60} + x^{56} + x^{52}\\
&+ x^{48} + x^{44} + x^{40} + x^{32} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(M^o,1,0)$:

$$
\begin{aligned}
p_0(x) &= x^{120}+x^{116}+x^{114}+x^{113}+x^{110}+x^{109}+x^{108}+x^{105}+x^{104}+x^{103}\\
&\quad +x^{102}+x^{101}+x^{98}+x^{95}+x^{93}+x^{92}+x^{90}+x^{89}+x^{83}+x^{81}+x^{78}\\
&\quad +x^{77}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{63}+x^{61}\\
&\quad +x^{60}+x^{59}+x^{57}+x^{54}+x^{53}+x^{52}+x^{48}+x^{46}+x^{43}+x^{39}+x^{38}\\
&\quad +x^{37}+x^{35}+x^{32}+x^{31}+x^{30}+x^{29}+x^{22}+x^{21}+x^{20}+x^{18},\\
p_3(x) &= x^{111}+x^{108}+x^{107}+x^{104}+x^{103}+x^{102}+x^{101}+x^{97}+x^{87}+x^{84}\\
&\quad +x^{83}+x^{80}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}\\
&\quad +x^{65}+x^{63}+x^{60}+x^{59}+x^{56}+x^{55}+x^{54}+x^{53}+x^{49}+x^{47}+x^{44}\\
&\quad +x^{43}+x^{40}+x^{38}+x^{37}+x^{36}+x^{35}+x^{33}+x^{32}+x^{30}+x^{29}+x^{28}\\
&\quad +x^{27}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}\\
&\quad +x^{13}+x^{9},\\
p_6(x) &= x^{114}+x^{108}+x^{98}+x^{96}+x^{90}+x^{82}+x^{80}+x^{76}+x^{68}+x^{66}+x^{56}\\
&\quad +x^{54}+x^{52}+x^{48}+x^{44}+x^{38}+x^{32}+x^{30}+x^{28}+x^{24}+x^{22}+x^{20}\\
&\quad +x^{16}+x^{6},\\
p_9(x) &= x^{117}+x^{114}+x^{113}+x^{111}+x^{110}+x^{106}+x^{105}+x^{104}+x^{103}+x^{99}\\
&\quad +x^{85}+x^{82}+x^{81}+x^{79}+x^{78}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{67}\\
&\quad +x^{66}+x^{65}+x^{63}+x^{62}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{50}\\
&\quad +x^{49}+x^{47}+x^{46}+x^{42}+x^{41}+x^{40}+x^{39}+x^{35}+x^{21}+x^{18}+x^{17}\\
&\quad +x^{15}+x^{14}+x^{10}+x^{9}+x^{8}+x^{7}+x^{3},\\
p_{12}(x) &= x^{120}+x^{112}+x^{108}+x^{104}+x^{96}+x^{88}+x^{80}+x^{76}+x^{60}+x^{44}+x^{40}\\
&\quad +x^{32}+x^{24}+x^{16}+x^{12}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(M^o,1,1)$:

$$
\begin{aligned}
p_0(x) &= x^{129}+x^{125}+x^{123}+x^{121}+x^{115}+x^{112}+x^{111}+x^{109}+x^{101}+x^{96}\\
&\quad +x^{95}+x^{93}+x^{88}+x^{83}+x^{80}+x^{78}+x^{75}+x^{70}+x^{69}+x^{68}+x^{67}\\
&\quad +x^{66}+x^{65}+x^{62}+x^{61}+x^{60}+x^{57}+x^{52}+x^{50}+x^{49}+x^{46}+x^{44}\\
&\quad +x^{43}+x^{42}+x^{40}+x^{39}+x^{37}+x^{36}+x^{33},\\
p_3(x) &= x^{123}+x^{120}+x^{115}+x^{113}+x^{112}+x^{109}+x^{107}+x^{105}+x^{103}+x^{101}\\
&\quad +x^{98}+x^{96}+x^{95}+x^{93}+x^{92}+x^{86}+x^{85}+x^{84}+x^{82}+x^{81}+x^{79}\\
&\quad +x^{77}+x^{72}+x^{70}+x^{67}+x^{64}+x^{63}+x^{62}+x^{58}+x^{57}+x^{54}+x^{53}\\
&\quad +x^{51}+x^{50}+x^{49}+x^{47}+x^{44}+x^{40}+x^{38}+x^{34}+x^{31}+x^{27}+x^{26}\\
&\quad +x^{24}+x^{23}+x^{19}+x^{18}+x^{16}+x^{13}+x^{12},\\
p_6(x) &= x^{123}+x^{120}+x^{115}+x^{113}+x^{112}+x^{107}+x^{105}+x^{103}+x^{101}+x^{98}\\
&\quad +x^{96}+x^{95}+x^{88}+x^{85}+x^{83}+x^{82}+x^{77}+x^{76}+x^{70}+x^{64}+x^{62}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{59} + x^{58} + x^{57} + x^{54} + x^{53} + x^{51} + x^{50} + x^{45} + x^{39} + x^{34} + x^{32} \\
&+ x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20} + x^{18} + x^{16} + x^{14} \\
&+ x^{12} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{3} + x + 1, \\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{107} + x^{105} + x^{104} + x^{103} \\
&+ x^{101} + x^{100} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} + x^{71} + x^{69} + x^{68} \\
&+ x^{67} + x^{65} + x^{64} + x^{55} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{43} + x^{41} \\
&+ x^{40} + x^{39} + x^{37} + x^{36} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16} + x^{11} \\
&+ x^{9} + x^{8} + x^{7} + x^{5} + x^{4}, \\
p_{12}(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{111} + x^{109} + x^{108} + x^{107} \\
&+ x^{105} + x^{104} + x^{99} + x^{97} + x^{96} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} \\
&+ x^{79} + x^{77} + x^{76} + x^{63} + x^{61} + x^{60} + x^{47} + x^{45} + x^{44} + x^{43} + x^{41} \\
&+ x^{40} + x^{35} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{19} + x^{17} + x^{16} + x^{15} \\
&+ x^{13} + x^{12} + x^{11} + x^{9} + x^{8} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{122} + x^{121} + x^{118} + x^{117} + x^{116} + x^{115} + x^{112} + x^{111} + x^{106} \\
&+ x^{102} + x^{101} + x^{99} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} \\
&+ x^{82} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{71} + x^{68} + x^{67} + x^{64} \\
&+ x^{62} + x^{61} + x^{59} + x^{56} + x^{52} + x^{50} + x^{49} + x^{48} + x^{47} + x^{44} + x^{41} \\
&+ x^{36} + x^{34} + x^{33} + x^{31} + x^{28} + x^{25} + x^{24} + x^{23} + x^{20} + x^{19} + x^{17} \\
&+ x^{14} + x^{13} + x^{12}, \\
p_3(x) &= x^{123} + x^{120} + x^{117} + x^{114} + x^{113} + x^{112} + x^{109} + x^{107} + x^{103} + x^{102} \\
&+ x^{101} + x^{100} + x^{97} + x^{95} + x^{92} + x^{91} + x^{90} + x^{86} + x^{84} + x^{83} + x^{79} \\
&+ x^{78} + x^{77} + x^{76} + x^{74} + x^{69} + x^{68} + x^{65} + x^{63} + x^{62} + x^{58} + x^{52} \\
&+ x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{37} + x^{36} \\
&+ x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{26} + x^{23} + x^{22} + x^{20} \\
&+ x^{19} + x^{14} + x^{13} + x^{10} + x^{9} + x^{8}, \\
p_6(x) &= x^{123} + x^{120} + x^{117} + x^{114} + x^{113} + x^{112} + x^{107} + x^{105} + x^{101} + x^{100} \\
&+ x^{97} + x^{95} + x^{93} + x^{90} + x^{89} + x^{87} + x^{85} + x^{84} + x^{77} + x^{76} + x^{75} \\
&+ x^{73} + x^{72} + x^{70} + x^{69} + x^{66} + x^{65} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} \\
&+ x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{45} + x^{41} + x^{38} + x^{35} \\
&+ x^{34} + x^{29} + x^{28} + x^{27} + x^{26} + x^{23} + x^{20} + x^{18} + x^{16} + x^{14} + x^{12} \\
&+ x^{8} + x^{7} + x^{6} + x^{3} + x + 1, \\
p_9(x) &= x^{123} + x^{121} + x^{120} + x^{115} + x^{113} + x^{112} + x^{107} + x^{105} + x^{104} + x^{103} \\
&+ x^{101} + x^{100} + x^{91} + x^{89} + x^{88} + x^{83} + x^{81} + x^{80} + x^{71} + x^{69} + x^{68}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{67}+x^{65}+x^{64}+x^{55}+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{43}+x^{41}\\
&+x^{40}+x^{39}+x^{37}+x^{36}+x^{27}+x^{25}+x^{24}+x^{19}+x^{17}+x^{16}+x^{11}\\
&+x^{9}+x^{8}+x^{7}+x^{5}+x^{4},\\
p_{12}(x)=&\,x^{123}+x^{121}+x^{120}+x^{115}+x^{113}+x^{112}+x^{111}+x^{109}+x^{108}+x^{107}\\
&+x^{105}+x^{104}+x^{99}+x^{97}+x^{96}+x^{91}+x^{89}+x^{88}+x^{83}+x^{81}+x^{80}\\
&+x^{79}+x^{77}+x^{76}+x^{63}+x^{61}+x^{60}+x^{47}+x^{45}+x^{44}+x^{43}+x^{41}\\
&+x^{40}+x^{35}+x^{33}+x^{32}+x^{27}+x^{25}+x^{24}+x^{19}+x^{17}+x^{16}+x^{15}\\
&+x^{13}+x^{12}+x^{11}+x^{9}+x^{8}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x)=&\,x^{120}+x^{116}+x^{114}+x^{113}+x^{110}+x^{109}+x^{108}+x^{105}+x^{104}+x^{103}\\
&+x^{102}+x^{101}+x^{98}+x^{95}+x^{93}+x^{92}+x^{90}+x^{89}+x^{83}+x^{81}+x^{78}\\
&+x^{77}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{63}+x^{61}\\
&+x^{60}+x^{59}+x^{57}+x^{54}+x^{53}+x^{52}+x^{48}+x^{46}+x^{43}+x^{39}+x^{38}\\
&+x^{37}+x^{35}+x^{32}+x^{31}+x^{30}+x^{29}+x^{22}+x^{21}+x^{20}+x^{18},\\
p_3(x)=&\,x^{111}+x^{108}+x^{107}+x^{104}+x^{103}+x^{102}+x^{101}+x^{97}+x^{87}+x^{84}\\
&+x^{83}+x^{80}+x^{78}+x^{77}+x^{76}+x^{75}+x^{73}+x^{72}+x^{71}+x^{70}+x^{69}\\
&+x^{65}+x^{63}+x^{60}+x^{59}+x^{56}+x^{55}+x^{54}+x^{53}+x^{49}+x^{47}+x^{44}\\
&+x^{43}+x^{40}+x^{38}+x^{37}+x^{36}+x^{35}+x^{33}+x^{32}+x^{30}+x^{29}+x^{28}\\
&+x^{27}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{17}+x^{16}+x^{15}+x^{14}\\
&+x^{13}+x^{9},\\
p_6(x)=&\,x^{114}+x^{108}+x^{98}+x^{96}+x^{90}+x^{82}+x^{80}+x^{76}+x^{68}+x^{66}+x^{56}\\
&+x^{54}+x^{52}+x^{48}+x^{44}+x^{38}+x^{32}+x^{30}+x^{28}+x^{24}+x^{22}+x^{20}\\
&+x^{16}+x^{6},\\
p_9(x)=&\,x^{117}+x^{114}+x^{113}+x^{111}+x^{110}+x^{106}+x^{105}+x^{104}+x^{103}+x^{99}\\
&+x^{85}+x^{82}+x^{81}+x^{79}+x^{78}+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{67}\\
&+x^{66}+x^{65}+x^{63}+x^{62}+x^{58}+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{50}\\
&+x^{49}+x^{47}+x^{46}+x^{42}+x^{41}+x^{40}+x^{39}+x^{35}+x^{21}+x^{18}+x^{17}\\
&+x^{15}+x^{14}+x^{10}+x^{9}+x^{8}+x^{7}+x^{3},\\
p_{12}(x)=&\,x^{120}+x^{112}+x^{108}+x^{104}+x^{96}+x^{88}+x^{80}+x^{76}+x^{60}+x^{44}+x^{40}\\
&+x^{32}+x^{24}+x^{16}+x^{12}+x^{8}+1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{144}+x^{140}+x^{138}+x^{134}+x^{133}+x^{132}+x^{130}+x^{129}+x^{128}+x^{123}\\
&+x^{122}+x^{121}+x^{119}+x^{117}+x^{115}+x^{110}+x^{108}+x^{107}+x^{103}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{97} + x^{96} + x^{94} + x^{92} + x^{89} + x^{88} + x^{83}\\
&+ x^{81} + x^{79} + x^{77} + x^{75} + x^{74} + x^{70} + x^{67} + x^{66} + x^{65} + x^{64} + x^{62}\\
&+ x^{57} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{44}\\
&+ x^{43} + x^{37} + x^{35} + x^{34} + x^{29} + x^{28} + x^{27},\\
p_3(x) &= x^{123} + x^{120} + x^{119} + x^{116} + x^{114} + x^{113} + x^{112} + x^{109} + x^{106} + x^{105}\\
&+ x^{104} + x^{103} + x^{102} + x^{99} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{90} + x^{89}\\
&+ x^{87} + x^{85} + x^{84} + x^{79} + x^{78} + x^{77} + x^{75} + x^{74} + x^{70} + x^{68} + x^{66}\\
&+ x^{64} + x^{61} + x^{59} + x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{50} + x^{47} + x^{45}\\
&+ x^{44} + x^{43} + x^{40} + x^{38} + x^{37} + x^{36} + x^{35} + x^{33} + x^{32} + x^{30} + x^{29}\\
&+ x^{28} + x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{16} + x^{15} + x^{14} + x^{9},\\
p_6(x) &= x^{126} + x^{120} + x^{118} + x^{114} + x^{112} + x^{110} + x^{100} + x^{98} + x^{96} + x^{92}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64}\\
&+ x^{58} + x^{56} + x^{54} + x^{50} + x^{40} + x^{38} + x^{32} + x^{30} + x^{28} + x^{26} + x^{22}\\
&+ x^{18} + x^{16} + x^{10} + x^{6},\\
p_9(x) &= x^{129} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{113} + x^{109}\\
&+ x^{108} + x^{107} + x^{106} + x^{105} + x^{104} + x^{99} + x^{97} + x^{94} + x^{93} + x^{91}\\
&+ x^{90} + x^{89} + x^{85} + x^{84} + x^{78} + x^{76} + x^{72} + x^{69} + x^{68} + x^{67} + x^{62}\\
&+ x^{60} + x^{56} + x^{53} + x^{52} + x^{51} + x^{49} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41}\\
&+ x^{40} + x^{35} + x^{33} + x^{30} + x^{29} + x^{27} + x^{26} + x^{25} + x^{21} + x^{20} + x^{17}\\
&+ x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{3},\\
p_{12}(x) &= x^{132} + x^{108} + x^{104} + x^{96} + x^{84} + x^{76} + x^{72} + x^{64} + x^{60} + x^{56} + x^{52}\\
&+ x^{48} + x^{44} + x^{40} + x^{32} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0].

For $\phi(W^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{123} + x^{121} + x^{115} + x^{112} + x^{111} + x^{109} + x^{101} + x^{96}\\
&+ x^{95} + x^{93} + x^{88} + x^{83} + x^{80} + x^{78} + x^{75} + x^{70} + x^{69} + x^{68} + x^{67}\\
&+ x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{57} + x^{52} + x^{50} + x^{49} + x^{46} + x^{44}\\
&+ x^{43} + x^{42} + x^{40} + x^{39} + x^{37} + x^{36} + x^{33},\\
p_3(x) &= x^{123} + x^{120} + x^{115} + x^{113} + x^{112} + x^{109} + x^{107} + x^{105} + x^{103} + x^{101}\\
&+ x^{98} + x^{96} + x^{95} + x^{93} + x^{92} + x^{86} + x^{85} + x^{84} + x^{82} + x^{81} + x^{79}\\
&+ x^{77} + x^{72} + x^{70} + x^{67} + x^{64} + x^{63} + x^{62} + x^{58} + x^{57} + x^{54} + x^{53}\\
&+ x^{51} + x^{50} + x^{49} + x^{47} + x^{44} + x^{40} + x^{38} + x^{34} + x^{31} + x^{27} + x^{26}\\
&+ x^{24} + x^{23} + x^{19} + x^{18} + x^{16} + x^{13} + x^{12},\\
p_6(x) &= x^{123} + x^{120} + x^{115} + x^{113} + x^{112} + x^{107} + x^{105} + x^{103} + x^{101} + x^{98}\\
&+ x^{96} + x^{95} + x^{88} + x^{85} + x^{83} + x^{82} + x^{77} + x^{76} + x^{70} + x^{64} + x^{62}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{59}+x^{58}+x^{57}+x^{54}+x^{53}+x^{51}+x^{50}+x^{45}+x^{39}+x^{34}+x^{32}\\
&+x^{31}+x^{30}+x^{29}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20}+x^{18}+x^{16}+x^{14}\\
&+x^{12}+x^{9}+x^{8}+x^{7}+x^{5}+x^{4}+x^{3}+x+1,\\
p_9(x)={}&x^{123}+x^{121}+x^{120}+x^{115}+x^{113}+x^{112}+x^{107}+x^{105}+x^{104}+x^{103}\\
&+x^{101}+x^{100}+x^{91}+x^{89}+x^{88}+x^{83}+x^{81}+x^{80}+x^{71}+x^{69}+x^{68}\\
&+x^{67}+x^{65}+x^{64}+x^{55}+x^{53}+x^{52}+x^{51}+x^{49}+x^{48}+x^{43}+x^{41}\\
&+x^{40}+x^{39}+x^{37}+x^{36}+x^{27}+x^{25}+x^{24}+x^{19}+x^{17}+x^{16}+x^{11}\\
&+x^{9}+x^{8}+x^{7}+x^{5}+x^{4},\\
p_{12}(x)={}&x^{123}+x^{121}+x^{120}+x^{115}+x^{113}+x^{112}+x^{111}+x^{109}+x^{108}+x^{107}\\
&+x^{105}+x^{104}+x^{99}+x^{97}+x^{96}+x^{91}+x^{89}+x^{88}+x^{83}+x^{81}+x^{80}\\
&+x^{79}+x^{77}+x^{76}+x^{63}+x^{61}+x^{60}+x^{47}+x^{45}+x^{44}+x^{43}+x^{41}\\
&+x^{40}+x^{35}+x^{33}+x^{32}+x^{27}+x^{25}+x^{24}+x^{19}+x^{17}+x^{16}+x^{15}\\
&+x^{13}+x^{12}+x^{11}+x^{9}+x^{8}+x^{3}+x+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	267	[427, 428]	534	[741, 742]	801	[940, 941]
1	[3, 4]	268	[159, 31]	535	[743, 373]	802	[942, 522]
2	[5, 6]	269	[45, 344]	536	[744, 427]	803	[943, 944]
3	[7, 8]	270	[429, 430]	537	[745, 497]	804	[651, 106]
4	[9, 10]	271	[431, 432]	538	[746, 400]	805	[678, 945]
5	[11, 12]	272	[433, 350]	539	[747, 451]	806	[252, 946]
6	[13, 14]	273	[396, 434]	540	[162, 456]	807	[941, 947]
7	[15, 16]	274	[417, 435]	541	[444, 745]	808	[595, 587]
8	[17, 18]	275	[436, 437]	542	[426, 358]	809	[948, 949]
9	[4, 19]	276	[157, 150]	543	[748, 393]	810	[950, 555]
10	[20, 21]	277	[438, 358]	544	[749, 480]	811	[951, 660]
11	[22, 23]	278	[439, 440]	545	[750, 10]	812	[952, 238]
12	[24, 25]	279	[441, 111]	546	[751, 483]	813	[669, 608]
13	[26, 27]	280	[159, 112]	547	[54, 152]	814	[953, 565]
14	[28, 29]	281	[113, 441]	548	[123, 168]	815	[954, 955]
15	[30, 31]	282	[442, 443]	549	[460, 190]	816	[613, 956]
16	[32, 33]	283	[444, 396]	550	[375, 465]	817	[200, 524]
17	[34, 35]	284	[445, 172]	551	[144, 140]	818	[645, 338]
18	[36, 37]	285	[446, 447]	552	[494, 392]	819	[957, 700]
19	[8, 38]	286	[448, 449]	553	[121, 145]	820	[958, 305]
20	[39, 40]	287	[163, 153]	554	[735, 364]	821	[959, 212]
21	[41, 42]	288	[450, 451]	555	[478, 119]	822	[960, 307]

22	[43, 44]	289	[452, 453]	556	[744, 729]	823	[961, 571]
23	[45, 46]	290	[454, 56]	557	[51, 498]	824	[962, 245]
24	[36, 47]	291	[52, 179]	558	[752, 753]	825	[963, 964]
25	[48, 49]	292	[455, 456]	559	[194, 426]	826	[965, 99]
26	[50, 51]	293	[457, 38]	560	[9, 754]	827	[966, 647]
27	[52, 53]	294	[118, 458]	561	[393, 11]	828	[551, 104]
28	[54, 55]	295	[459, 460]	562	[499, 755]	829	[915, 840]
29	[39, 56]	296	[406, 461]	563	[404, 745]	830	[967, 968]
30	[57, 58]	297	[391, 174]	564	[387, 726]	831	[686, 904]
31	[59, 60]	298	[462, 427]	565	[487, 364]	832	[932, 969]
32	[61, 62]	299	[463, 183]	566	[756, 381]	833	[970, 209]
33	[63, 64]	300	[411, 147]	567	[475, 757]	834	[19, 971]
34	[65, 66]	301	[464, 465]	568	[758, 759]	835	[972, 620]
35	[67, 68]	302	[453, 466]	569	[446, 760]	836	[973, 206]
36	[69, 70]	303	[467, 468]	570	[736, 11]	837	[341, 342]
37	[71, 72]	304	[469, 470]	571	[761, 762]	838	[974, 430]
38	[16, 73]	305	[463, 416]	572	[481, 381]	839	[454, 40]
39	[74, 75]	306	[471, 37]	573	[763, 764]	840	[359, 975]
40	[76, 77]	307	[472, 132]	574	[390, 152]	841	[740, 181]
41	[78, 79]	308	[468, 473]	575	[505, 128]	842	[384, 496]
42	[80, 81]	309	[456, 397]	576	[489, 449]	843	[381, 458]
43	[82, 83]	310	[474, 435]	577	[181, 386]	844	[976, 806]
44	[84, 85]	311	[475, 188]	578	[378, 765]	845	[977, 978]
45	[86, 87]	312	[422, 194]	579	[766, 54]	846	[778, 419]
46	[88, 89]	313	[476, 164]	580	[457, 728]	847	[441, 45]
47	[90, 91]	314	[477, 351]	581	[767, 768]	848	[472, 130]
48	[92, 93]	315	[153, 171]	582	[769, 755]	849	[769, 14]
49	[94, 95]	316	[414, 478]	583	[502, 762]	850	[354, 402]
50	[96, 97]	317	[479, 480]	584	[770, 150]	851	[979, 419]
51	[98, 99]	318	[481, 118]	585	[771, 473]	852	[980, 820]
52	[100, 101]	319	[482, 483]	586	[772, 44]	853	[981, 420]
53	[102, 103]	320	[183, 484]	587	[773, 774]	854	[801, 982]
54	[104, 105]	321	[479, 485]	588	[469, 136]	855	[727, 831]
55	[106, 107]	322	[184, 486]	589	[504, 41]	856	[470, 349]
56	[108, 109]	323	[487, 488]	590	[775, 114]	857	[182, 416]
57	[110, 111]	324	[489, 490]	591	[131, 130]	858	[765, 983]
58	[112, 113]	325	[403, 491]	592	[745, 434]	859	[177, 392]
59	[31, 114]	326	[492, 493]	593	[776, 777]	860	[984, 975]
60	[112, 114]	327	[494, 174]	594	[778, 437]	861	[985, 366]
61	[115, 116]	328	[431, 372]	595	[158, 342]	862	[788, 736]
62	[117, 118]	329	[495, 496]	596	[386, 362]	863	[438, 51]
63	[119, 120]	330	[396, 497]	597	[451, 490]	864	[986, 447]
64	[121, 35]	331	[41, 352]	598	[401, 355]	865	[987, 171]
65	[122, 10]	332	[358, 498]	599	[772, 33]	866	[815, 978]

66	[123, 124]	333	[499, 14]	600	[779, 139]	867	[410, 465]
67	[125, 126]	334	[470, 433]	601	[780, 158]	868	[471, 47]
68	[127, 128]	335	[500, 501]	602	[781, 782]	869	[829, 416]
69	[129, 130]	336	[502, 503]	603	[492, 783]	870	[835, 31]
70	[131, 132]	337	[477, 41]	604	[365, 784]	871	[773, 988]
71	[133, 134]	338	[140, 189]	605	[496, 425]	872	[799, 138]
72	[135, 136]	339	[504, 351]	606	[464, 376]	873	[982, 466]
73	[31, 113]	340	[505, 506]	607	[784, 405]	874	[170, 154]
74	[137, 138]	341	[507, 198]	608	[145, 346]	875	[737, 488]
75	[139, 140]	342	[276, 201]	609	[785, 786]	876	[751, 796]
76	[141, 142]	343	[508, 88]	610	[766, 390]	877	[989, 983]
77	[143, 144]	344	[73, 280]	611	[176, 787]	878	[379, 805]
78	[34, 145]	345	[509, 237]	612	[367, 48]	879	[755, 380]
79	[146, 147]	346	[510, 511]	613	[133, 421]	880	[164, 486]
80	[148, 149]	347	[512, 513]	614	[788, 393]	881	[749, 485]
81	[150, 150]	348	[514, 515]	615	[789, 736]	882	[990, 460]
82	[151, 152]	349	[516, 517]	616	[790, 380]	883	[985, 784]
83	[153, 154]	350	[260, 518]	617	[381, 348]	884	[991, 992]
84	[155, 12]	351	[6, 519]	618	[353, 785]	885	[358, 794]
85	[156, 48]	352	[70, 520]	619	[791, 420]	886	[191, 468]
86	[150, 157]	353	[521, 522]	620	[343, 45]	887	[818, 149]
87	[158, 159]	354	[523, 524]	621	[792, 793]	888	[781, 993]
88	[46, 158]	355	[294, 525]	622	[371, 432]	889	[362, 994]
89	[160, 159]	356	[526, 527]	623	[51, 794]	890	[193, 423]
90	[154, 161]	357	[311, 528]	624	[795, 186]	891	[474, 186]
91	[162, 163]	358	[529, 530]	625	[482, 796]	892	[995, 742]
92	[164, 116]	359	[531, 532]	626	[156, 368]	893	[797, 490]
93	[165, 166]	360	[533, 534]	627	[491, 381]	894	[822, 832]
94	[167, 168]	361	[535, 536]	628	[172, 382]	895	[812, 133]
95	[154, 169]	362	[203, 537]	629	[450, 797]	896	[996, 425]
96	[170, 171]	363	[538, 245]	630	[779, 144]	897	[409, 460]
97	[172, 119]	364	[539, 540]	631	[115, 486]	898	[382, 120]
98	[173, 174]	365	[541, 542]	632	[729, 798]	899	[741, 356]
99	[175, 176]	366	[543, 544]	633	[495, 125]	900	[122, 754]
100	[177, 174]	367	[545, 546]	634	[361, 387]	901	[187, 757]
101	[178, 179]	368	[547, 297]	635	[363, 488]	902	[110, 45]
102	[180, 181]	369	[548, 549]	636	[114, 441]	903	[989, 806]
103	[182, 183]	370	[550, 551]	637	[171, 161]	904	[742, 430]
104	[184, 116]	371	[552, 510]	638	[775, 113]	905	[997, 486]
105	[185, 186]	372	[228, 553]	639	[799, 800]	906	[807, 373]
106	[187, 188]	373	[554, 555]	640	[366, 405]	907	[119, 383]
107	[189, 190]	374	[556, 557]	641	[141, 765]	908	[998, 46]
108	[191, 192]	375	[558, 79]	642	[801, 453]	909	[979, 437]
109	[193, 194]	376	[559, 221]	643	[802, 473]	910	[999, 993]

110	[195, 73]	377	[560, 561]	644	[803, 734]	911	[452, 982]
111	[196, 197]	378	[562, 309]	645	[48, 759]	912	[151, 55]
112	[198, 199]	379	[563, 203]	646	[804, 54]	913	[140, 460]
113	[58, 200]	380	[328, 564]	647	[181, 805]	914	[1000, 393]
114	[201, 86]	381	[565, 238]	648	[451, 449]	915	[807, 442]
115	[202, 203]	382	[524, 508]	649	[761, 503]	916	[1001, 342]
116	[204, 205]	383	[300, 320]	650	[395, 745]	917	[1002, 152]
117	[206, 207]	384	[566, 567]	651	[409, 189]	918	[785, 787]
118	[208, 209]	385	[568, 569]	652	[445, 478]	919	[783, 800]
119	[210, 211]	386	[230, 570]	653	[765, 806]	920	[146, 412]
120	[212, 213]	387	[323, 514]	654	[453, 807]	921	[398, 41]
121	[214, 215]	388	[571, 547]	655	[808, 172]	922	[791, 149]
122	[216, 217]	389	[572, 573]	656	[809, 396]	923	[160, 342]
123	[218, 106]	390	[574, 575]	657	[127, 506]	924	[742, 357]
124	[219, 220]	391	[576, 577]	658	[13, 755]	925	[770, 157]
125	[221, 222]	392	[578, 292]	659	[810, 362]	926	[974, 357]
126	[223, 224]	393	[579, 571]	660	[32, 44]	927	[385, 484]
127	[225, 226]	394	[580, 90]	661	[811, 777]	928	[466, 373]
128	[227, 228]	395	[87, 581]	662	[812, 350]	929	[991, 346]
129	[229, 230]	396	[582, 583]	663	[724, 440]	930	[986, 760]
130	[231, 232]	397	[62, 293]	664	[802, 813]	931	[754, 1003]
131	[233, 234]	398	[584, 80]	665	[814, 491]	932	[387, 994]
132	[235, 236]	399	[585, 583]	666	[356, 430]	933	[468, 813]
133	[237, 217]	400	[226, 92]	667	[377, 51]	934	[1004, 164]
134	[238, 239]	401	[586, 587]	668	[408, 145]	935	[1005, 465]
135	[240, 241]	402	[564, 588]	669	[423, 426]	936	[977, 816]
136	[242, 237]	403	[589, 569]	670	[493, 138]	937	[368, 49]
137	[243, 244]	404	[590, 591]	671	[815, 816]	938	[976, 983]
138	[245, 246]	405	[542, 592]	672	[817, 172]	939	[368, 759]
139	[247, 248]	406	[553, 593]	673	[748, 736]	940	[817, 478]
140	[249, 250]	407	[594, 595]	674	[165, 415]	941	[405, 754]
141	[251, 252]	408	[596, 510]	675	[818, 420]	942	[1006, 1007]
142	[253, 254]	409	[256, 331]	676	[819, 820]	943	[1008, 130]
143	[255, 256]	410	[597, 598]	677	[424, 126]	944	[1009, 768]
144	[257, 258]	411	[599, 581]	678	[433, 133]	945	[175, 785]
145	[259, 260]	412	[528, 600]	679	[125, 425]	946	[35, 346]
146	[261, 262]	413	[199, 70]	680	[821, 753]	947	[478, 382]
147	[263, 264]	414	[601, 602]	681	[795, 435]	948	[780, 160]
148	[265, 266]	415	[603, 604]	682	[763, 739]	949	[819, 1010]
149	[267, 17]	416	[605, 208]	683	[756, 118]	950	[459, 189]
150	[268, 269]	417	[606, 607]	684	[822, 732]	951	[1011, 44]
151	[270, 258]	418	[608, 542]	685	[823, 501]	952	[990, 189]
152	[271, 71]	419	[609, 610]	686	[379, 386]	953	[789, 393]
153	[272, 273]	420	[611, 612]	687	[824, 493]	954	[1008, 132]

154	[274, 97]	421	[517, 613]	688	[825, 814]	955	[1012, 774]
155	[275, 276]	422	[614, 560]	689	[826, 442]	956	[825, 403]
156	[277, 278]	423	[615, 529]	690	[827, 40]	957	[1013, 369]
157	[157, 157]	424	[616, 529]	691	[439, 725]	958	[427, 798]
158	[81, 279]	425	[567, 617]	692	[350, 134]	959	[500, 831]
159	[280, 281]	426	[266, 618]	693	[747, 797]	960	[1011, 33]
160	[197, 59]	427	[619, 620]	694	[442, 374]	961	[399, 1007]
161	[21, 282]	428	[244, 574]	695	[137, 800]	962	[804, 390]
162	[283, 272]	429	[621, 544]	696	[423, 377]	963	[1000, 736]
163	[284, 274]	430	[622, 623]	697	[824, 783]	964	[805, 362]
164	[246, 285]	431	[624, 625]	698	[135, 470]	965	[810, 387]
165	[286, 287]	432	[264, 626]	699	[752, 828]	966	[746, 1007]
166	[288, 289]	433	[627, 628]	700	[491, 118]	967	[830, 501]
167	[290, 291]	434	[262, 198]	701	[730, 407]	968	[803, 1014]
168	[292, 78]	435	[629, 630]	702	[829, 183]	969	[836, 403]
169	[293, 294]	436	[631, 4]	703	[784, 9]	970	[826, 373]
170	[295, 296]	437	[632, 633]	704	[830, 831]	971	[10, 834]
171	[297, 298]	438	[634, 298]	705	[731, 832]	972	[1015, 343]
172	[299, 300]	439	[635, 548]	706	[821, 828]	973	[413, 351]
173	[301, 302]	440	[109, 521]	707	[833, 133]	974	[1016, 20]
174	[303, 304]	441	[60, 636]	708	[178, 53]	975	[890, 658]
175	[305, 306]	442	[29, 536]	709	[496, 126]	976	[1017, 593]
176	[307, 268]	443	[530, 637]	710	[114, 343]	977	[1018, 846]
177	[308, 309]	444	[520, 262]	711	[754, 834]	978	[95, 841]
178	[310, 311]	445	[638, 210]	712	[783, 138]	979	[924, 1019]
179	[312, 313]	446	[639, 640]	713	[144, 409]	980	[1020, 1021]
180	[314, 315]	447	[641, 642]	714	[129, 132]	981	[1022, 560]
181	[316, 317]	448	[643, 644]	715	[143, 139]	982	[1023, 930]
182	[318, 319]	449	[75, 645]	716	[835, 112]	983	[577, 605]
183	[320, 321]	450	[646, 647]	717	[767, 2]	984	[1024, 223]
184	[322, 323]	451	[648, 649]	718	[448, 490]	985	[1025, 1019]
185	[324, 325]	452	[650, 651]	719	[136, 349]	986	[1026, 665]
186	[326, 312]	453	[652, 554]	720	[467, 192]	987	[1027, 21]
187	[327, 328]	454	[653, 654]	721	[836, 814]	988	[1028, 1029]
188	[91, 329]	455	[655, 596]	722	[837, 196]	989	[1030, 850]
189	[248, 330]	456	[656, 657]	723	[838, 274]	990	[1031, 580]
190	[331, 332]	457	[658, 644]	724	[839, 840]	991	[1032, 674]
191	[333, 334]	458	[207, 263]	725	[841, 842]	992	[66, 1033]
192	[335, 336]	459	[659, 330]	726	[570, 843]	993	[1034, 1035]
193	[337, 338]	460	[105, 580]	727	[844, 845]	994	[536, 1036]
194	[339, 340]	461	[660, 196]	728	[846, 847]	995	[1037, 1038]
195	[341, 159]	462	[103, 661]	729	[848, 58]	996	[1039, 340]
196	[342, 31]	463	[662, 663]	730	[849, 850]	997	[1040, 317]
197	[342, 112]	464	[664, 519]	731	[851, 852]	998	[1041, 81]

198	[46, 160]	465	[665, 666]	732	[205, 102]	999	[1042, 1043]
199	[343, 111]	466	[667, 556]	733	[853, 619]	1000	[281, 888]
200	[111, 344]	467	[668, 669]	734	[843, 854]	1001	[1044, 710]
201	[344, 158]	468	[670, 671]	735	[855, 856]	1002	[1045, 248]
202	[345, 346]	469	[672, 627]	736	[857, 845]	1003	[217, 889]
203	[347, 40]	470	[673, 674]	737	[858, 703]	1004	[1046, 1047]
204	[167, 124]	471	[675, 676]	738	[859, 860]	1005	[1048, 1049]
205	[118, 348]	472	[677, 678]	739	[861, 582]	1006	[1050, 866]
206	[349, 350]	473	[325, 679]	740	[862, 230]	1007	[1051, 229]
207	[351, 352]	474	[680, 669]	741	[863, 622]	1008	[25, 315]
208	[347, 56]	475	[681, 682]	742	[679, 864]	1009	[1052, 249]
209	[353, 176]	476	[683, 204]	743	[865, 530]	1010	[1053, 1054]
210	[43, 33]	477	[211, 676]	744	[610, 244]	1011	[1055, 272]
211	[113, 343]	478	[236, 684]	745	[64, 866]	1012	[1056, 667]
212	[354, 355]	479	[685, 686]	746	[867, 868]	1013	[1057, 565]
213	[356, 357]	480	[687, 688]	747	[869, 654]	1014	[511, 861]
214	[50, 358]	481	[689, 537]	748	[870, 871]	1015	[1058, 595]
215	[359, 360]	482	[690, 691]	749	[872, 873]	1016	[723, 759]
216	[361, 362]	483	[525, 299]	750	[874, 518]	1017	[1059, 753]
217	[351, 42]	484	[663, 692]	751	[875, 876]	1018	[981, 149]
218	[363, 364]	485	[693, 533]	752	[877, 300]	1019	[466, 442]
219	[365, 366]	486	[691, 694]	753	[878, 879]	1020	[996, 126]
220	[367, 368]	487	[695, 696]	754	[527, 296]	1021	[397, 154]
221	[183, 369]	488	[697, 698]	755	[880, 881]	1022	[14, 793]
222	[370, 166]	489	[699, 18]	756	[882, 514]	1023	[809, 745]
223	[371, 372]	490	[647, 28]	757	[883, 579]	1024	[827, 56]
224	[373, 374]	491	[700, 701]	758	[884, 657]	1025	[808, 478]
225	[375, 376]	492	[702, 703]	759	[278, 885]	1026	[823, 831]
226	[194, 377]	493	[704, 705]	760	[886, 883]	1027	[743, 442]
227	[378, 142]	494	[706, 252]	761	[887, 888]	1028	[155, 388]
228	[180, 379]	495	[707, 708]	762	[889, 890]	1029	[476, 184]
229	[14, 380]	496	[709, 98]	763	[891, 94]	1030	[771, 813]
230	[117, 381]	497	[591, 710]	764	[540, 892]	1031	[987, 154]
231	[382, 383]	498	[561, 711]	765	[893, 894]	1032	[792, 380]
232	[384, 125]	499	[315, 682]	766	[895, 271]	1033	[10, 1003]
233	[385, 369]	500	[712, 24]	767	[896, 897]	1034	[729, 428]
234	[386, 387]	501	[713, 259]	768	[898, 899]	1035	[995, 356]
235	[11, 388]	502	[714, 22]	769	[900, 901]	1036	[373, 443]
236	[389, 390]	503	[617, 715]	770	[902, 268]	1037	[750, 754]
237	[391, 392]	504	[716, 717]	771	[903, 306]	1038	[738, 764]
238	[393, 155]	505	[718, 719]	772	[904, 596]	1039	[758, 49]
239	[189, 394]	506	[720, 721]	773	[905, 906]	1040	[429, 357]
240	[395, 396]	507	[722, 343]	774	[907, 857]	1041	[722, 441]
241	[397, 171]	508	[344, 160]	775	[908, 508]	1042	[997, 116]

242	[398, 351]	509	[723, 49]	776	[909, 910]	1043	[805, 387]
243	[399, 400]	510	[724, 725]	777	[637, 911]	1044	[1060, 111]
244	[401, 402]	511	[362, 726]	778	[912, 913]	1045	[345, 992]
245	[403, 117]	512	[727, 501]	779	[914, 915]	1046	[833, 350]
246	[390, 55]	513	[456, 153]	780	[916, 23]	1047	[984, 360]
247	[404, 396]	514	[8, 728]	781	[917, 516]	1048	[1005, 376]
248	[173, 392]	515	[460, 394]	782	[918, 849]	1049	[811, 1061]
249	[405, 10]	516	[349, 133]	783	[919, 920]	1050	[1059, 828]
250	[406, 407]	517	[462, 729]	784	[921, 20]	1051	[814, 117]
251	[408, 35]	518	[730, 461]	785	[922, 923]	1052	[1013, 484]
252	[139, 409]	519	[731, 732]	786	[336, 924]	1053	[176, 786]
253	[410, 376]	520	[111, 46]	787	[306, 526]	1054	[1004, 184]
254	[411, 412]	521	[389, 54]	788	[620, 22]	1055	[145, 992]
255	[413, 41]	522	[733, 734]	789	[925, 275]	1056	[1002, 55]
256	[377, 358]	523	[418, 437]	790	[926, 554]	1057	[790, 793]
257	[414, 172]	524	[436, 419]	791	[927, 928]	1058	[1062, 344]
258	[370, 415]	525	[455, 163]	792	[929, 930]	1059	[1063, 684]
259	[416, 369]	526	[416, 484]	793	[901, 931]	1060	[1064, 847]
260	[417, 186]	527	[735, 488]	794	[298, 932]	1061	[692, 219]
261	[418, 419]	528	[171, 169]	795	[933, 334]	1062	[1065, 520]
262	[148, 420]	529	[185, 435]	796	[604, 934]	1063	[1006, 400]
263	[350, 421]	530	[736, 155]	797	[935, 936]	1064	[1062, 46]
264	[422, 423]	531	[737, 364]	798	[920, 937]	1065	[1060, 45]
265	[424, 425]	532	[738, 739]	799	[938, 661]		
266	[426, 51]	533	[740, 379]	800	[696, 939]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	178	0	356	0	534	0	712	0	890	1
1	0	179	0	357	1	535	0	713	1	891	0
2	1	180	1	358	0	536	0	714	0	892	1
3	0	181	1	359	1	537	1	715	0	893	1
4	0	182	1	360	0	538	0	716	1	894	0
5	1	183	1	361	0	539	1	717	1	895	0
6	1	184	0	362	1	540	0	718	1	896	1
7	0	185	0	363	0	541	0	719	0	897	0
8	0	186	1	364	1	542	1	720	0	898	0
9	0	187	0	365	0	543	1	721	0	899	0
10	0	188	0	366	1	544	1	722	0	900	0
11	1	189	1	367	1	545	1	723	0	901	0
12	0	190	0	368	0	546	1	724	1	902	0
13	1	191	1	369	0	547	0	725	0	903	0
14	0	192	0	370	0	548	0	726	1	904	1
15	0	193	1	371	0	549	0	727	1	905	1

16	0	194	1	372	1	550	0	728	1	906	1
17	0	195	0	373	1	551	1	729	0	907	1
18	0	196	0	374	0	552	0	730	0	908	0
19	0	197	0	375	0	553	1	731	1	909	1
20	0	198	1	376	1	554	1	732	1	910	1
21	0	199	0	377	0	555	1	733	0	911	1
22	1	200	0	378	1	556	0	734	0	912	1
23	1	201	0	379	0	557	1	735	1	913	1
24	0	202	0	380	1	558	0	736	1	914	1
25	1	203	1	381	0	559	1	737	1	915	1
26	1	204	1	382	0	560	0	738	1	916	1
27	1	205	1	383	1	561	0	739	1	917	0
28	0	206	0	384	0	562	1	740	0	918	1
29	0	207	1	385	1	563	0	741	0	919	0
30	0	208	1	386	0	564	0	742	1	920	0
31	0	209	1	387	0	565	0	743	0	921	0
32	0	210	1	388	1	566	0	744	0	922	1
33	1	211	1	389	1	567	1	745	1	923	0
34	0	212	0	390	1	568	1	746	0	924	1
35	1	213	0	391	0	569	1	747	1	925	0
36	0	214	1	392	1	570	1	748	1	926	0
37	0	215	1	393	0	571	1	749	1	927	1
38	0	216	0	394	1	572	1	750	1	928	0
39	0	217	1	395	1	573	0	751	1	929	1
40	0	218	0	396	0	574	1	752	0	930	0
41	0	219	0	397	0	575	1	753	0	931	1
42	0	220	1	398	0	576	0	754	1	932	0
43	1	221	1	399	0	577	1	755	1	933	1
44	0	222	0	400	1	578	1	756	0	934	1
45	1	223	0	401	1	579	0	757	1	935	1
46	1	224	1	402	0	580	1	758	1	936	0
47	1	225	0	403	1	581	1	759	0	937	0
48	1	226	1	404	0	582	0	760	1	938	1
49	1	227	1	405	1	583	0	761	1	939	0
50	1	228	1	406	1	584	0	762	1	940	0
51	1	229	0	407	1	585	0	763	0	941	1
52	1	230	0	408	0	586	1	764	0	942	1
53	1	231	0	409	0	587	1	765	1	943	1
54	0	232	0	410	0	588	0	766	0	944	0
55	0	233	1	411	1	589	1	767	1	945	0
56	1	234	0	412	0	590	0	768	0	946	1
57	0	235	1	413	0	591	1	769	0	947	1
58	1	236	1	414	1	592	1	770	0	948	1
59	0	237	0	415	1	593	1	771	0	949	0

60	1	238	0	416	0	594	1	772	1	950	1
61	0	239	1	417	1	595	1	773	1	951	0
62	0	240	1	418	0	596	0	774	1	952	0
63	1	241	0	419	1	597	0	775	0	953	0
64	1	242	0	420	0	598	1	776	1	954	1
65	0	243	0	421	1	599	1	777	0	955	0
66	0	244	1	422	0	600	1	778	1	956	1
67	1	245	1	423	0	601	1	779	1	957	0
68	0	246	1	424	0	602	0	780	1	958	1
69	0	247	0	425	1	603	1	781	0	959	0
70	1	248	1	426	1	604	0	782	1	960	0
71	0	249	1	427	1	605	0	783	0	961	0
72	1	250	1	428	1	606	1	784	0	962	0
73	0	251	0	429	1	607	0	785	1	963	1
74	0	252	0	430	0	608	0	786	0	964	1
75	0	253	0	431	1	609	1	787	1	965	1
76	0	254	1	432	0	610	0	788	0	966	0
77	0	255	0	433	1	611	0	789	0	967	1
78	0	256	0	434	0	612	1	790	0	968	1
79	0	257	1	435	0	613	0	791	1	969	0
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81	1	259	0	437	0	615	0	793	0	971	0
82	1	260	1	438	0	616	0	794	1	972	1
83	1	261	0	439	0	617	0	795	1	973	0
84	0	262	0	440	1	618	1	796	0	974	0
85	0	263	1	441	1	619	1	797	1	975	1
86	1	264	0	442	0	620	0	798	0	976	1
87	1	265	0	443	1	621	1	799	1	977	0
88	1	266	1	444	0	622	0	800	0	978	1
89	0	267	1	445	0	623	1	801	0	979	1
90	1	268	1	446	1	624	1	802	1	980	1
91	0	269	1	447	0	625	0	803	1	981	0
92	1	270	1	448	1	626	0	804	0	982	1
93	1	271	1	449	0	627	1	805	1	983	1
94	1	272	1	450	0	628	0	806	0	984	0
95	1	273	0	451	0	629	0	807	1	985	1
96	1	274	1	452	1	630	1	808	1	986	0
97	0	275	0	453	0	631	0	809	1	987	0
98	1	276	0	454	1	632	0	810	1	988	0
99	0	277	0	455	1	633	1	811	0	989	0
100	1	278	0	456	1	634	0	812	0	990	0
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102	1	280	1	458	1	636	0	814	0	992	0
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104	0	282	0	460	0	638	0	816	0	994	0
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107	1	285	1	463	0	641	0	819	0	997	1
108	1	286	1	464	1	642	0	820	1	998	0
109	1	287	0	465	0	643	1	821	0	999	1
110	0	288	0	466	0	644	1	822	0	1000	1
111	0	289	1	467	0	645	1	823	0	1001	1
112	1	290	1	468	1	646	0	824	0	1002	0
113	1	291	1	469	0	647	1	825	1	1003	1
114	0	292	1	470	1	648	0	826	1	1004	1
115	0	293	1	471	1	649	1	827	0	1005	1
116	1	294	1	472	0	650	1	828	1	1006	1
117	0	295	1	473	1	651	0	829	1	1007	0
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122	0	300	1	478	1	656	1	834	0	1012	0
123	0	301	1	479	0	657	0	835	1	1013	0
124	0	302	0	480	0	658	1	836	0	1014	1
125	1	303	0	481	1	659	1	837	0	1015	1
126	0	304	0	482	0	660	0	838	0	1016	0
127	0	305	0	483	1	661	0	839	1	1017	1
128	1	306	1	484	1	662	0	840	1	1018	0
129	0	307	0	485	1	663	1	841	0	1019	0
130	0	308	1	486	0	664	1	842	0	1020	1
131	1	309	1	487	0	665	0	843	0	1021	0
132	1	310	0	488	0	666	0	844	1	1022	0
133	0	311	1	489	0	667	0	845	0	1023	1
134	0	312	0	490	1	668	0	846	1	1024	0
135	1	313	0	491	1	669	0	847	1	1025	1
136	0	314	1	492	1	670	1	848	0	1026	0
137	0	315	1	493	1	671	1	849	0	1027	0
138	1	316	1	494	0	672	0	850	0	1028	0
139	0	317	0	495	1	673	1	851	1	1029	0
140	1	318	1	496	0	674	1	852	1	1030	0
141	0	319	0	497	1	675	1	853	0	1031	0
142	0	320	1	498	0	676	0	854	0	1032	1
143	0	321	0	499	1	677	0	855	1	1033	0
144	1	322	0	500	0	678	1	856	1	1034	0
145	0	323	0	501	1	679	1	857	1	1035	1
146	0	324	0	502	0	680	0	858	1	1036	1
147	1	325	1	503	0	681	1	859	1	1037	1

148	0	326	1	504	1	682	0	860	0	1038	1
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150	1	328	1	506	0	684	0	862	0	1040	1
151	1	329	1	507	0	685	0	863	0	1041	0
152	1	330	0	508	0	686	0	864	0	1042	1
153	1	331	0	509	0	687	0	865	0	1043	1
154	1	332	0	510	1	688	1	866	1	1044	1
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156	0	334	1	512	1	690	0	868	1	1046	1
157	0	335	0	513	1	691	0	869	1	1047	0
158	1	336	0	514	0	692	1	870	1	1048	1
159	1	337	1	515	0	693	1	871	1	1049	0
160	0	338	1	516	0	694	0	872	1	1050	1
161	0	339	1	517	1	695	0	873	1	1051	0
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165	1	343	0	521	1	699	0	877	0	1055	0
166	0	344	0	522	0	700	1	878	0	1056	0
167	1	345	0	523	0	701	0	879	1	1057	0
168	1	346	1	524	0	702	1	880	1	1058	1
169	1	347	1	525	1	703	0	881	1	1059	1
170	1	348	0	526	0	704	1	882	0	1060	1
171	0	349	0	527	1	705	1	883	1	1061	1
172	0	350	1	528	0	706	0	884	1	1062	1
173	1	351	1	529	0	707	1	885	0	1063	1
174	0	352	1	530	1	708	0	886	1	1064	1
175	0	353	1	531	1	709	0	887	1	1065	1
176	0	354	0	532	1	710	0	888	0		
177	1	355	1	533	0	711	1	889	1		

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn