PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a,b)=(z^6+z^4+z^3,z)$

GUO-NIU HAN* AND YINING HU*

Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a,b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a,b) in $\mathbb{F}_2[x]\setminus\mathbb{F}_2$. In this paper we prove the conjecture for $(a,b)=(z^6+z^4+z^3,z)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a}=(a_0(z),a_1(z),a_2(z),\ldots)$ of elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{a_2(z)+\cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z))=\cfrac{1}{a_0(z)+\cfrac{1}{a_1(z)+\cfrac{1}{\ddots+\cfrac{1}{a_n(z)}}}}\in\mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1}+c_{-2}z^{-2}+c_{-3}z^{-3}+\cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z]\setminus\mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a,b)-Thue-Morse sequence

$$\mathbf{t}=(t_0(z),t_1(z),t_2(z),\ldots)=(a,b,b,a,b,a,a,b,\ldots)$$

that we shall denote

$$\mathrm{CF}(a,b):=\mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^6+z^4+z^3,z)$, *the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$, *with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$,

$$\begin{aligned}
p_0(z)&=z^{17}+z^{14}+z^{12}+z^{11}+z^9+z^7+z^5+z^4+z^3+z^2+1,\\
p_1(z)&=z^{23}+z^{21}+z^{18}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_2(z)&=z^{24}+z^{22}+z^{16}+z^{14}+z^{10}+z^8+z^6,\\
p_3(z)&=z^{23}+z^{21}+z^{18}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_4(z)&=z^{22}+z^{20}+z^{17}+z^{12}+z^{11}+z^{10}+z^9+z^7+z^6+z^5+z^4+z^3+z^2,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$,

$$\begin{aligned}
p_0(z)&=z^{20}+z^{17}+z^{14}+z^{11}+z^9+z^8+z^7+z^5+z^4+z^3+z^2,\\
p_1(z)&=z^{23}+z^{21}+z^{18}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_2(z)&=z^{24}+z^{22}+z^{16}+z^{14}+z^{10}+z^8+z^6,\\
p_3(z)&=z^{23}+z^{21}+z^{18}+z^{12}+z^{11}+z^{10}+z^9+z^8+z^6+z^5+z^4+z^3,\\
p_4(z)&=z^{22}+z^{17}+z^{11}+z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2+1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\geq 0}$ and $(Q_n(z))_{n\geq 0}$ by

$$\begin{pmatrix}P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix}:=\begin{pmatrix}t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x)=x^{\deg(t_{2^n-1})}&\begin{pmatrix}t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix}t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(t_0)}\begin{pmatrix}t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x)=x^{\deg(\bar t_{2^n-1})}&\begin{pmatrix}\bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix}\bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix}\bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n\geq 0$

$$\begin{aligned}
M_{n+1}(x)&=W_n(x)\cdot M_n(x),\\
W_{n+1}(x)&=M_n(x)\cdot W_n(x).
\end{aligned}$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \le i,j \le 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \le i,j \le 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \le i,j \le 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \ge 0$. If for some $n \ge 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \ge 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12}\cdot\phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. □

Define

$$R^e = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Lemma 2.2. *For all integers $k \ge 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned}
M^e[:14u] &= M^e[:7u] + x^{3u}M^e[:4u] + x^{5u}M^e[:2u] + x^{3u}M^e[:7u] + x^{6u}M^e[:4u]\\
&\quad + x^{8u}M^e[:2u] + x^{5u}M^e[:7u] + x^{8u}M^e[:4u] + x^{10u}M^e[:2u]\\
&\quad + x^{6u}M^e[:7u] + x^{9u}M^e[:4u] + x^{11u}M^e[:2u] + x^{7u}M^e[:7u]\\
&\quad + x^{12u}M^e[:2u] + x^{8u}M^e[:6u] + (x^{7u} + x^{10u} + x^{12u} + x^{13u})R^e,\\
W^e[:14u] &= W^e[:7u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^{3u}W^e[:7u] + x^{6u}W^e[:4u]\\
&\quad + x^{8u}W^e[:2u] + x^{5u}W^e[:7u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u]
\end{aligned}$$

$$
\begin{aligned}
&+ x^{6u}W^e[:7u] + x^{9u}W^e[:4u] + x^{11u}W^e[:2u] + x^{7u}W^e[:7u]\\
&+ x^{12u}W^e[:2u] + x^{8u}W^e[:6u] + (x^{7u} + x^{10u} + x^{12u} + x^{13u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] = {} & M^o[:7u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^{3u}M^o[:7u] + x^{6u}M^o[:4u]\\
&+ x^{8u}M^o[:2u] + x^{5u}M^o[:7u] + x^{8u}M^o[:4u] + x^{10u}M^o[:2u]\\
&+ x^{6u}M^o[:7u] + x^{9u}M^o[:4u] + x^{11u}M^o[:2u] + x^{7u}M^o[:7u]\\
&+ x^{12u}M^o[:2u] + x^{8u}M^o[:6u] + (x^{7u} + x^{10u} + x^{12u} + x^{13u})R^o,\\
W^o[:14u] = {} & W^o[:7u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^{3u}W^o[:7u] + x^{6u}W^o[:4u]\\
&+ x^{8u}W^o[:2u] + x^{5u}W^o[:7u] + x^{8u}W^o[:4u] + x^{10u}W^o[:2u]\\
&+ x^{6u}W^o[:7u] + x^{9u}W^o[:4u] + x^{11u}W^o[:2u] + x^{7u}W^o[:7u]\\
&+ x^{12u}W^o[:2u] + x^{8u}W^o[:6u] + (x^{7u} + x^{10u} + x^{12u} + x^{13u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] = {} & T[:7u] + x^{3u}T[:4u] + x^{5u}T[:2u] + x^{3u}T[:7u] + x^{6u}T[:4u] + x^{8u}T[:2u]\\
&+ x^{5u}T[:7u] + x^{8u}T[:4u] + x^{10u}T[:2u] + x^{6u}T[:7u] + x^{9u}T[:4u]\\
&+ x^{11u}T[:2u] + x^{7u}T[:7u] + x^{12u}T[:2u] + x^{8u}T[:6u].
\end{aligned}
$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$
\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + T[7u:8u],\\
0 &= x^{8u}T[:u] + x^{7u}T[u:2u] + x^{5u}T[3u:4u] + x^{3u}T[5u:6u] + T[8u:9u],\\
0 &= x^{9u}T[:u] + x^{8u}T[u:2u] + x^{7u}T[2u:3u] + x^{5u}T[4u:5u]\\
&\quad + x^{3u}T[6u:7u] + T[9u:10u],\\
0 &= x^{10u}T[:u] + x^{9u}T[u:2u] + x^{7u}T[3u:4u] + x^{6u}T[4u:5u]\\
&\quad + x^{5u}T[5u:6u] + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{10u}T[u:2u] + x^{9u}T[2u:3u] + x^{7u}T[4u:5u]\\
&\quad + x^{6u}T[5u:6u] + x^{5u}T[6u:7u] + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{9u}T[3u:4u] + x^{8u}T[4u:5u]\\
&\quad + x^{7u}T[5u:6u] + x^{6u}T[6u:7u] + T[12u:13u],\\
0 &= x^{12u}T[u:2u] + x^{8u}T[5u:6u] + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}
$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.8}$$

$$
\begin{aligned}
0 = {} & T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2]\\
&+ T[[1001w]_2],
\end{aligned} \tag{2.9}
$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1010w]_2],\end{aligned} \tag{2.10}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1011w]_2],\end{aligned} \tag{2.11}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1100w]_2],\end{aligned} \tag{2.12}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[101w]_2] + T[[1000w]_2], \tag{2.15}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2]\\ &\quad + T[[1001w]_2],\end{aligned} \tag{2.16}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[1010w]_2],\end{aligned} \tag{2.17}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1011w]_2],\end{aligned} \tag{2.18}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2]\\ &\quad + T[[110w]_2] + T[[1100w]_2],\end{aligned} \tag{2.19}$$

$$0 = T[[1w]_2] + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 1188 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,111)), \tag{2.21}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101))\\ &\quad + \tau(A(s,1000)),\end{aligned} \tag{2.22}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100))\\ &\quad + \tau(A(s,110)) + \tau(A(s,1001)),\end{aligned} \tag{2.23}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,1010)),\end{aligned} \tag{2.24}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100))\\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1100)),\end{aligned} \tag{2.26}$$

$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.27}$$

for all $s \in E_{2k}$, and

$$0 = \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,111)), \tag{2.28}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,101)) \\ & + \tau(A(s,1000)), \end{aligned} \tag{2.29}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ & + \tau(A(s,110)) + \tau(A(s,1001)), \end{aligned} \tag{2.30}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,1010)), \end{aligned} \tag{2.31}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1011)), \end{aligned} \tag{2.32}$$

$$\begin{aligned} 0 = {} & \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,11)) + \tau(A(s,100)) \\ & + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1100)), \end{aligned} \tag{2.33}$$

$$0 = \tau(A(s,1)) + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)), \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{28}), E_{28}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 14$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$\begin{aligned} M_{2k} &= M^e[:7u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{7u} R^e, \\ W_{2k} &= W^e[:7u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{7u} R^e. \end{aligned}$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$\begin{aligned} M_{2k+1} &= M^o[:7u] + x^{3u} M^o[:4u] + x^{5u} M^o[:2u] + x^{7u} R^o, \\ W_{2k+1} &= W^o[:7u] + x^{3u} W^o[:4u] + x^{5u} W^o[:2u] + x^{7u} R^o. \end{aligned}$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$\begin{aligned} M_{2k} \wedge W_{2k} &\Rightarrow M_{2k+1} \wedge W_{2k+1}, \\ M_{2k+1} \wedge W_{2k+1} &\Rightarrow M_{2k+2} \wedge W_{2k+2}. \end{aligned}$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$\begin{aligned} W_{2k} M_{2k} = {} & \left(W^e[:7v] + x^{3v} W^e[:4v] + x^{5v} W^e[:2v] + x^{7u} R^e\right) \times \left(M^e[:7v]\right. \\ & \left. + x^{3v} M^e[:4v] + x^{5v} M^e[:2v] + x^{7u} R^e\right); \end{aligned} \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is

$O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{6v}W^e[:8v]M^e[:8v] + x^{10v}W^e[:4v]M^e[:4v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$\begin{aligned}
X &:= x^v,\\
a_n &:= W^e[n \cdot v : (n+1) \cdot v]/X^n,\\
b_n &:= M^e[n \cdot v : (n+1) \cdot v]/X^n,\\
c &:= R^e.
\end{aligned}$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. $\square$

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a, b)$; for $\mathrm{CF}(b, a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j, k \le 1$,

$$\begin{aligned}
\lim_{n\to\infty} M_{2n,j,k} &= M^e_{j,k},\\
\lim_{n\to\infty} M_{2n+1,j,k} &= M^o_{j,k},\\
\lim_{n\to\infty} W_{2n,j,k} &= W^e_{j,k},\\
\lim_{n\to\infty} W_{2n+1,j,k} &= W^o_{j,k}.
\end{aligned}$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a, b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e, 0, 1)$ and $\phi(M^e, 0, 0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x, y) = \mathrm{Res}_t\left(\phi(M^e, 0, 1)(x, t), y^{12}\phi(M^e, 0, 1)(x, t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x, y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{17} + x^{15} + x^{13} + x^{12} + x^{10} + x^7,\\
q_1(x) &= x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^6 + x^3 + x,\\
q_2(x) &= x^{18} + x^{16} + x^{14} + x^{10} + x^8 + x^2 + 1,\\
q_3(x) &= x^{21} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^6 + x^3 + x,\\
q_4(x) &= x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{15} + x^{14} + x^{13} + x^{12} + x^7 + x^4\\
&\quad + x^2.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. $\square$

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned} p_0(x) &= x^{114} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{94} + x^{92} + x^{90} + x^{88} \\ &\quad + x^{86} + x^{84} + x^{82} + x^{76} + x^{74} + x^{72} + x^{66} + x^{54} + x^{50} + x^{44} + x^{42} \\ &\quad + x^{34} + x^{28} + x^{18} + x^{12}, \\ p_3(x) &= x^{104} + x^{100} + x^{98} + x^{96} + x^{92} + x^{90} + x^{82} + x^{80} + x^{78} + x^{74} + x^{72} \\ &\quad + x^{66} + x^{62} + x^{58} + x^{56} + x^{52} + x^{46} + x^{40} + x^{38} + x^{32} + x^{30} + x^{26} \\ &\quad + x^{20} + x^{18} + x^{16} + x^{12} + x^{10} + x^{8}, \\ p_6(x) &= x^{104} + x^{100} + x^{96} + x^{88} + x^{84} + x^{82} + x^{76} + x^{72} + x^{68} + x^{66} + x^{60} \\ &\quad + x^{52} + x^{42} + x^{40} + x^{36} + x^{32} + x^{24} + x^{20} + x^{18} + x^{12} + x^{10} + x^{8} \\ &\quad + x^{2} + 1, \\ p_9(x) &= x^{102} + x^{100} + x^{88} + x^{86} + x^{84} + x^{82} + x^{78} + x^{76} + x^{72} + x^{70} + x^{64} \\ &\quad + x^{62} + x^{60} + x^{54} + x^{52} + x^{50} + x^{42} + x^{38} + x^{36} + x^{34} + x^{32} + x^{28} \\ &\quad + x^{26} + x^{22} + x^{20} + x^{14} + x^{10} + x^{8} + x^{4} + x^{2}, \\ p_{12}(x) &= x^{102} + x^{100} + x^{96} + x^{70} + x^{68} + x^{64} + x^{38} + x^{36} + x^{32} + x^{22} + x^{20} \\ &\quad + x^{16} + x^{6} + x^{4} + 1, \end{aligned}$$

and the initial terms are [1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned} p_0(x) &= x^{129} + x^{124} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{115} + x^{114} + x^{108} \\ &\quad + x^{105} + x^{104} + x^{99} + x^{96} + x^{90} + x^{89} + x^{88} + x^{86} + x^{81} + x^{80} + x^{79} \\ &\quad + x^{76} + x^{73} + x^{72} + x^{70} + x^{69} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58} \\ &\quad + x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42} \\ &\quad + x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{27}, \\ p_3(x) &= x^{120} + x^{117} + x^{114} + x^{112} + x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{101} \\ &\quad + x^{100} + x^{97} + x^{96} + x^{93} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{81} + x^{76} \end{aligned}$$

$$
\begin{aligned}
&+ x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{65} + x^{60} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{53} + x^{52} + x^{49} + x^{48} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{34}\\
&+ x^{31} + x^{28} + x^{26} + x^{25} + x^{23} + x^{16} + x^{13} + x^{12} + x^{9},\\
p_6(x) &= x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103}\\
&+ x^{101} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{81}\\
&+ x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{43} + x^{42} + x^{41} + x^{39}\\
&+ x^{36} + x^{35} + x^{34} + x^{32} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16}\\
&+ x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{101} + x^{100} + x^{98}\\
&+ x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{84} + x^{82} + x^{81}\\
&+ x^{79} + x^{76} + x^{73} + x^{68} + x^{66} + x^{65} + x^{63} + x^{58} + x^{55} + x^{46} + x^{44}\\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{26}\\
&+ x^{24} + x^{23} + x^{21} + x^{16} + x^{14} + x^{13} + x^{11} + x^{6} + x^{3},\\
p_{12}(x) &= x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{79}\\
&+ x^{78} + x^{76} + x^{75} + x^{74} + x^{72} + x^{67} + x^{66} + x^{64} + x^{47} + x^{46} + x^{44}\\
&+ x^{43} + x^{42} + x^{40} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26}\\
&+ x^{24} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2}\\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129} + x^{124} + x^{123} + x^{122} + x^{121} + x^{117} + x^{116} + x^{115} + x^{114} + x^{108}\\
&+ x^{105} + x^{104} + x^{99} + x^{96} + x^{90} + x^{89} + x^{88} + x^{86} + x^{81} + x^{80} + x^{79}\\
&+ x^{76} + x^{73} + x^{72} + x^{70} + x^{69} + x^{65} + x^{64} + x^{62} + x^{60} + x^{59} + x^{58}\\
&+ x^{57} + x^{53} + x^{52} + x^{51} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{42}\\
&+ x^{40} + x^{36} + x^{35} + x^{33} + x^{32} + x^{27},\\
p_3(x) &= x^{120} + x^{117} + x^{114} + x^{112} + x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{101}\\
&+ x^{100} + x^{97} + x^{96} + x^{93} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{81} + x^{76}\\
&+ x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{65} + x^{60} + x^{58} + x^{57} + x^{56}\\
&+ x^{55} + x^{53} + x^{52} + x^{49} + x^{48} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{34}\\
&+ x^{31} + x^{28} + x^{26} + x^{25} + x^{23} + x^{16} + x^{13} + x^{12} + x^{9},\\
p_6(x) &= x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103}\\
&+ x^{101} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{81}\\
&+ x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63}\\
&+ x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{43} + x^{42} + x^{41} + x^{39}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{36} + x^{35} + x^{34} + x^{32} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} \\
&+ x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{101} + x^{100} + x^{98} \\
&+ x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{84} + x^{82} + x^{81} \\
&+ x^{79} + x^{76} + x^{73} + x^{68} + x^{66} + x^{65} + x^{63} + x^{58} + x^{55} + x^{46} + x^{44} \\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{26} \\
&+ x^{24} + x^{23} + x^{21} + x^{16} + x^{14} + x^{13} + x^{11} + x^{6} + x^{3}, \\
p_{12}(x) &= x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{79} \\
&+ x^{78} + x^{76} + x^{75} + x^{74} + x^{72} + x^{67} + x^{66} + x^{64} + x^{47} + x^{46} + x^{44} \\
&+ x^{43} + x^{42} + x^{40} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} \\
&+ x^{24} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2} \\
&+ 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{138} + x^{134} + x^{128} + x^{126} + x^{124} + x^{120} + x^{116} + x^{110} \\
&+ x^{106} + x^{104} + x^{98} + x^{96} + x^{92} + x^{88} + x^{86} + x^{78} + x^{74} + x^{72} + x^{66} \\
&+ x^{64} + x^{60} + x^{58} + x^{52} + x^{48} + x^{42}, \\
p_3(x) &= x^{126} + x^{124} + x^{122} + x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{96} + x^{90} \\
&+ x^{86} + x^{70} + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52} \\
&+ x^{50} + x^{48} + x^{40} + x^{38} + x^{36} + x^{32} + x^{26} + x^{22} + x^{20} + x^{16} + x^{14} \\
&+ x^{8} + x^{6}, \\
p_6(x) &= x^{126} + x^{124} + x^{122} + x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{104} + x^{102} \\
&+ x^{98} + x^{94} + x^{92} + x^{90} + x^{88} + x^{84} + x^{74} + x^{70} + x^{68} + x^{62} + x^{58} \\
&+ x^{56} + x^{52} + x^{48} + x^{42} + x^{40} + x^{34} + x^{28} + x^{16} + x^{12} + x^{10} + x^{6} \\
&+ x^{2} + 1, \\
p_9(x) &= x^{120} + x^{114} + x^{110} + x^{106} + x^{104} + x^{96} + x^{92} + x^{90} + x^{88} + x^{86} + x^{76} \\
&+ x^{72} + x^{70} + x^{64} + x^{50} + x^{48} + x^{40} + x^{38} + x^{36} + x^{30} + x^{28} + x^{26} \\
&+ x^{24} + x^{16} + x^{14} + x^{10} + x^{8} + x^{6} + x^{4} + x^{2}, \\
p_{12}(x) &= x^{120} + x^{112} + x^{96} + x^{88} + x^{80} + x^{64} + x^{56} + x^{48} + x^{40} + x^{24} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{138} + x^{134} + x^{132} + x^{126} + x^{124} + x^{120} + x^{114} + x^{106} \\
&+ x^{100} + x^{96} + x^{92} + x^{90} + x^{84} + x^{82} + x^{74} + x^{62} + x^{58} + x^{56} + x^{54} \\
&+ x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{22} \\
&+ x^{12},
\end{aligned}
$$

$$\begin{aligned}
p_3(x) &= x^{126}+x^{124}+x^{122}+x^{120}+x^{116}+x^{114}+x^{110}+x^{108}+x^{104}+x^{102}\\
&\quad+x^{100}+x^{94}+x^{88}+x^{86}+x^{84}+x^{80}+x^{74}+x^{72}+x^{70}+x^{64}+x^{56}\\
&\quad+x^{54}+x^{48}+x^{44}+x^{42}+x^{40}+x^{38}+x^{34}+x^{32}+x^{28}+x^{22}+x^{8},\\
p_6(x) &= x^{126}+x^{124}+x^{122}+x^{108}+x^{100}+x^{98}+x^{96}+x^{76}+x^{70}+x^{68}+x^{66}\\
&\quad+x^{52}+x^{50}+x^{48}+x^{44}+x^{26}+x^{24}+x^{14}+x^{10}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{120}+x^{114}+x^{110}+x^{106}+x^{104}+x^{96}+x^{92}+x^{90}+x^{88}+x^{86}+x^{76}\\
&\quad+x^{72}+x^{70}+x^{64}+x^{50}+x^{48}+x^{40}+x^{38}+x^{36}+x^{30}+x^{28}+x^{26}\\
&\quad+x^{24}+x^{16}+x^{14}+x^{10}+x^{8}+x^{6}+x^{4}+x^{2},\\
p_{12}(x) &= x^{120}+x^{112}+x^{96}+x^{88}+x^{80}+x^{64}+x^{56}+x^{48}+x^{40}+x^{24}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0].
For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129}+x^{125}+x^{123}+x^{121}+x^{120}+x^{117}+x^{116}+x^{113}+x^{111}+x^{110}\\
&\quad+x^{108}+x^{107}+x^{106}+x^{105}+x^{103}+x^{101}+x^{100}+x^{98}+x^{96}+x^{95}\\
&\quad+x^{93}+x^{92}+x^{88}+x^{85}+x^{83}+x^{82}+x^{81}+x^{80}+x^{79}+x^{78}+x^{75}\\
&\quad+x^{74}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}+x^{66}+x^{64}+x^{63}+x^{62}+x^{60}\\
&\quad+x^{58}+x^{56}+x^{53}+x^{52}+x^{50}+x^{46}+x^{44}+x^{40}+x^{38}+x^{35}+x^{29}\\
&\quad+x^{28}+x^{26}+x^{25}+x^{24}+x^{20}+x^{18},\\
p_3(x) &= x^{120}+x^{117}+x^{114}+x^{112}+x^{111}+x^{109}+x^{108}+x^{105}+x^{104}+x^{101}\\
&\quad+x^{100}+x^{97}+x^{96}+x^{93}+x^{90}+x^{88}+x^{87}+x^{85}+x^{84}+x^{81}+x^{76}\\
&\quad+x^{74}+x^{73}+x^{72}+x^{71}+x^{69}+x^{68}+x^{65}+x^{60}+x^{58}+x^{57}+x^{56}\\
&\quad+x^{55}+x^{53}+x^{52}+x^{49}+x^{48}+x^{45}+x^{44}+x^{41}+x^{40}+x^{37}+x^{34}\\
&\quad+x^{31}+x^{28}+x^{26}+x^{25}+x^{23}+x^{16}+x^{13}+x^{12}+x^{9},\\
p_6(x) &= x^{117}+x^{116}+x^{115}+x^{113}+x^{111}+x^{109}+x^{106}+x^{105}+x^{104}+x^{103}\\
&\quad+x^{101}+x^{98}+x^{97}+x^{96}+x^{94}+x^{91}+x^{90}+x^{88}+x^{83}+x^{82}+x^{81}\\
&\quad+x^{78}+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{69}+x^{66}+x^{65}+x^{64}+x^{63}\\
&\quad+x^{60}+x^{59}+x^{58}+x^{56}+x^{55}+x^{54}+x^{52}+x^{43}+x^{42}+x^{41}+x^{39}\\
&\quad+x^{36}+x^{35}+x^{34}+x^{32}+x^{25}+x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{16}\\
&\quad+x^{13}+x^{12}+x^{10}+x^{9}+x^{8}+x^{6},\\
p_9(x) &= x^{114}+x^{112}+x^{111}+x^{110}+x^{109}+x^{107}+x^{104}+x^{101}+x^{100}+x^{98}\\
&\quad+x^{97}+x^{95}+x^{94}+x^{92}+x^{91}+x^{89}+x^{88}+x^{85}+x^{84}+x^{82}+x^{81}\\
&\quad+x^{79}+x^{76}+x^{73}+x^{68}+x^{66}+x^{65}+x^{63}+x^{58}+x^{55}+x^{46}+x^{44}\\
&\quad+x^{43}+x^{42}+x^{41}+x^{39}+x^{38}+x^{35}+x^{34}+x^{32}+x^{31}+x^{29}+x^{26}\\
&\quad+x^{24}+x^{23}+x^{21}+x^{16}+x^{14}+x^{13}+x^{11}+x^{6}+x^{3},\\
p_{12}(x) &= x^{111}+x^{110}+x^{108}+x^{107}+x^{106}+x^{104}+x^{99}+x^{98}+x^{96}+x^{79}\\
&\quad+x^{78}+x^{76}+x^{75}+x^{74}+x^{72}+x^{67}+x^{66}+x^{64}+x^{47}+x^{46}+x^{44}
\end{aligned}$$

$$+ x^{43} + x^{42} + x^{40} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2} + 1,$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{123} + x^{121} + x^{120} + x^{117} + x^{116} + x^{113} + x^{111} + x^{110} \\
&\quad + x^{108} + x^{107} + x^{106} + x^{105} + x^{103} + x^{101} + x^{100} + x^{98} + x^{96} + x^{95} \\
&\quad + x^{93} + x^{92} + x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{75} \\
&\quad + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{64} + x^{63} + x^{62} + x^{60} \\
&\quad + x^{58} + x^{56} + x^{53} + x^{52} + x^{50} + x^{46} + x^{44} + x^{40} + x^{38} + x^{35} + x^{29} \\
&\quad + x^{28} + x^{26} + x^{25} + x^{24} + x^{20} + x^{18}, \\
p_3(x) &= x^{120} + x^{117} + x^{114} + x^{112} + x^{111} + x^{109} + x^{108} + x^{105} + x^{104} + x^{101} \\
&\quad + x^{100} + x^{97} + x^{96} + x^{93} + x^{90} + x^{88} + x^{87} + x^{85} + x^{84} + x^{81} + x^{76} \\
&\quad + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{65} + x^{60} + x^{58} + x^{57} + x^{56} \\
&\quad + x^{55} + x^{53} + x^{52} + x^{49} + x^{48} + x^{45} + x^{44} + x^{41} + x^{40} + x^{37} + x^{34} \\
&\quad + x^{31} + x^{28} + x^{26} + x^{25} + x^{23} + x^{16} + x^{13} + x^{12} + x^{9}, \\
p_6(x) &= x^{117} + x^{116} + x^{115} + x^{113} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} + x^{103} \\
&\quad + x^{101} + x^{98} + x^{97} + x^{96} + x^{94} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{81} \\
&\quad + x^{78} + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{66} + x^{65} + x^{64} + x^{63} \\
&\quad + x^{60} + x^{59} + x^{58} + x^{56} + x^{55} + x^{54} + x^{52} + x^{43} + x^{42} + x^{41} + x^{39} \\
&\quad + x^{36} + x^{35} + x^{34} + x^{32} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} \\
&\quad + x^{13} + x^{12} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_9(x) &= x^{114} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{101} + x^{100} + x^{98} \\
&\quad + x^{97} + x^{95} + x^{94} + x^{92} + x^{91} + x^{89} + x^{88} + x^{85} + x^{84} + x^{82} + x^{81} \\
&\quad + x^{79} + x^{76} + x^{73} + x^{68} + x^{66} + x^{65} + x^{63} + x^{58} + x^{55} + x^{46} + x^{44} \\
&\quad + x^{43} + x^{42} + x^{41} + x^{39} + x^{38} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} + x^{26} \\
&\quad + x^{24} + x^{23} + x^{21} + x^{16} + x^{14} + x^{13} + x^{11} + x^{6} + x^{3}, \\
p_{12}(x) &= x^{111} + x^{110} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{98} + x^{96} + x^{79} \\
&\quad + x^{78} + x^{76} + x^{75} + x^{74} + x^{72} + x^{67} + x^{66} + x^{64} + x^{47} + x^{46} + x^{44} \\
&\quad + x^{43} + x^{42} + x^{40} + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} \\
&\quad + x^{24} + x^{19} + x^{18} + x^{16} + x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3} + x^{2} \\
&\quad + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0].

For $\phi(W^e, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{114} + x^{110} + x^{108} + x^{104} + x^{102} + x^{98} + x^{94} + x^{86} + x^{84} + x^{80} + x^{78} \\
&\quad + x^{76} + x^{74} + x^{70} + x^{66} + x^{64} + x^{62} + x^{56} + x^{54} + x^{50} + x^{46} + x^{44}
\end{aligned}$$

$$
\begin{aligned}
&+x^{40}+x^{36}+x^{34}+x^{28}+x^{24},\\
p_3(x) &= x^{104}+x^{100}+x^{98}+x^{92}+x^{82}+x^{78}+x^{76}+x^{72}+x^{70}+x^{66}+x^{62}\\
&\quad +x^{58}+x^{52}+x^{50}+x^{46}+x^{40}+x^{38}+x^{36}+x^{24}+x^{14}+x^{8}+x^{6},\\
p_6(x) &= x^{104}+x^{100}+x^{86}+x^{84}+x^{80}+x^{78}+x^{74}+x^{62}+x^{60}+x^{58}+x^{56}\\
&\quad +x^{54}+x^{52}+x^{50}+x^{48}+x^{42}+x^{40}+x^{28}+x^{26}+x^{24}+x^{18}+x^{14}\\
&\quad +x^{12}+x^{10}+x^{6}+x^{4}+x^{2}+1,\\
p_9(x) &= x^{102}+x^{100}+x^{88}+x^{86}+x^{84}+x^{82}+x^{78}+x^{76}+x^{72}+x^{70}+x^{64}\\
&\quad +x^{62}+x^{60}+x^{54}+x^{52}+x^{50}+x^{42}+x^{38}+x^{36}+x^{34}+x^{32}+x^{28}\\
&\quad +x^{26}+x^{22}+x^{20}+x^{14}+x^{10}+x^{8}+x^{4}+x^{2},\\
p_{12}(x) &= x^{102}+x^{100}+x^{96}+x^{70}+x^{68}+x^{64}+x^{38}+x^{36}+x^{32}+x^{22}+x^{20}\\
&\quad +x^{16}+x^{6}+x^{4}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0].
For $\phi(M^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{129}+x^{124}+x^{122}+x^{121}+x^{120}+x^{118}+x^{117}+x^{115}+x^{114}+x^{109}\\
&\quad +x^{107}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{92}+x^{85}+x^{84}+x^{82}+x^{81}\\
&\quad +x^{79}+x^{77}+x^{75}+x^{73}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{62}+x^{61}\\
&\quad +x^{58}+x^{57}+x^{54}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{44}+x^{43}+x^{42}\\
&\quad +x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{35}+x^{25}+x^{24}+x^{22}+x^{17}+x^{15}\\
&\quad +x^{12},\\
p_3(x) &= x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{116}+x^{114}+x^{112}+x^{109}+x^{106}\\
&\quad +x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{94}+x^{90}\\
&\quad +x^{89}+x^{87}+x^{83}+x^{82}+x^{80}+x^{77}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}\\
&\quad +x^{66}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{59}+x^{58}+x^{53}+x^{51}+x^{49}\\
&\quad +x^{48}+x^{47}+x^{40}+x^{39}+x^{38}+x^{36}+x^{34}+x^{31}+x^{30}+x^{26}+x^{21}\\
&\quad +x^{20}+x^{18}+x^{17}+x^{14}+x^{13}+x^{11}+x^{8},\\
p_6(x) &= x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{116}+x^{114}+x^{112}+x^{107}+x^{106}\\
&\quad +x^{105}+x^{103}+x^{102}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}\\
&\quad +x^{85}+x^{83}+x^{82}+x^{80}+x^{79}+x^{77}+x^{76}+x^{75}+x^{71}+x^{67}+x^{64}\\
&\quad +x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}\\
&\quad +x^{47}+x^{43}+x^{40}+x^{35}+x^{33}+x^{32}+x^{29}+x^{28}+x^{27}+x^{26}+x^{22}\\
&\quad +x^{21}+x^{20}+x^{19}+x^{15}+x^{14}+x^{12}+x^{9}+x^{7}+x^{4}+x^{3}+x^{2}+1,\\
p_9(x) &= x^{123}+x^{122}+x^{120}+x^{115}+x^{114}+x^{112}+x^{111}+x^{110}+x^{108}+x^{103}\\
&\quad +x^{102}+x^{100}+x^{91}+x^{90}+x^{88}+x^{83}+x^{82}+x^{80}+x^{79}+x^{78}+x^{76}\\
&\quad +x^{71}+x^{70}+x^{68}+x^{59}+x^{58}+x^{56}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}\\
&\quad +x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{38}+x^{36}+x^{35}+x^{34}+x^{32}+x^{31}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{7} + x^{6} + x^{4},\\
p_{12}(x) &= x^{123} + x^{122} + x^{120} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{104} + x^{103}\\
&+ x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80}\\
&+ x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{59} + x^{58}\\
&+ x^{56} + x^{51} + x^{50} + x^{48} + x^{39} + x^{38} + x^{36} + x^{23} + x^{22} + x^{20} + x^{11}\\
&+ x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1].
For $\phi(M^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{135} + x^{131} + x^{122} + x^{121} + x^{119} + x^{117} + x^{112} + x^{104}\\
&+ x^{101} + x^{98} + x^{96} + x^{91} + x^{90} + x^{88} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74}\\
&+ x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{64} + x^{59} + x^{56} + x^{55} + x^{53} + x^{52}\\
&+ x^{51} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{30} + x^{27},\\
p_3(x) &= x^{123} + x^{122} + x^{119} + x^{116} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104}\\
&+ x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64}\\
&+ x^{63} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{46}\\
&+ x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} + x^{29} + x^{28} + x^{26}\\
&+ x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{16} + x^{14} + x^{9},\\
p_6(x) &= x^{126} + x^{124} + x^{116} + x^{114} + x^{110} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{86} + x^{84} + x^{80} + x^{76} + x^{70} + x^{68} + x^{66} + x^{62} + x^{56} + x^{50}\\
&+ x^{46} + x^{38} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{18} + x^{16} + x^{14}\\
&+ x^{10} + x^{6},\\
p_9(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{116}\\
&+ x^{115} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{97} + x^{96}\\
&+ x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{79} + x^{77}\\
&+ x^{76} + x^{75} + x^{74} + x^{72} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59}\\
&+ x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{44} + x^{41} + x^{40}\\
&+ x^{38} + x^{36} + x^{33} + x^{32} + x^{30} + x^{28} + x^{25} + x^{24} + x^{22} + x^{20} + x^{15}\\
&+ x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3},\\
p_{12}(x) &= x^{132} + x^{128} + x^{124} + x^{116} + x^{112} + x^{92} + x^{84} + x^{80} + x^{60} + x^{44} + x^{36}\\
&+ x^{32} + x^{28} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1].
For $\phi(M^o, 1, 0)$:

$$
p_0(x) = x^{120} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{103}
$$

$$
\begin{aligned}
&+ x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{93} + x^{85} + x^{84} + x^{82} + x^{81}\\
&+ x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} + x^{65} + x^{64} + x^{62}\\
&+ x^{61} + x^{60} + x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44}\\
&+ x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29}\\
&+ x^{28} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18},\\
p_3(x) = {}& x^{111} + x^{110} + x^{106} + x^{104} + x^{102} + x^{97} + x^{87} + x^{86} + x^{82} + x^{80} + x^{78}\\
&+ x^{73} + x^{71} + x^{70} + x^{66} + x^{64} + x^{62} + x^{57} + x^{55} + x^{54} + x^{50} + x^{48}\\
&+ x^{46} + x^{41} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{18} + x^{17} + x^{16} + x^{14}\\
&+ x^{9},\\
p_6(x) = {}& x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{94} + x^{88}\\
&+ x^{80} + x^{78} + x^{72} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56} + x^{52} + x^{40} + x^{38}\\
&+ x^{36} + x^{32} + x^{22} + x^{18} + x^{16} + x^{10} + x^{6},\\
p_9(x) = {}& x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{109} + x^{108} + x^{106} + x^{104} + x^{99}\\
&+ x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{67} + x^{53}\\
&+ x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{42} + x^{40} + x^{37} + x^{36} + x^{34}\\
&+ x^{32} + x^{29} + x^{28} + x^{26} + x^{24} + x^{21} + x^{20} + x^{18} + x^{16} + x^{13} + x^{12}\\
&+ x^{10} + x^{8} + x^{3},\\
p_{12}(x) = {}& x^{120} + x^{112} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56}\\
&+ x^{48} + x^{36} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) = {}& x^{129} + x^{125} + x^{121} + x^{119} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109}\\
&+ x^{108} + x^{106} + x^{104} + x^{103} + x^{97} + x^{95} + x^{91} + x^{89} + x^{87} + x^{85}\\
&+ x^{84} + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{66}\\
&+ x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{52} + x^{46} + x^{45} + x^{44} + x^{43}\\
&+ x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33},\\
p_3(x) = {}& x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{110} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{97} + x^{94} + x^{93} + x^{92}\\
&+ x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{77} + x^{73} + x^{71} + x^{67}\\
&+ x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52}\\
&+ x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{31} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{12},\\
p_6(x) = {}& x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{110} + x^{109} + x^{108} + x^{106} + x^{102} + x^{101} + x^{95} + x^{93} + x^{92} + x^{91}\\
&+ x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{71}+x^{69}+x^{68}+x^{67}+x^{66}+x^{65}+x^{62}+x^{61}+x^{60}+x^{59}+x^{58}\\
&+x^{57}+x^{56}+x^{55}+x^{53}+x^{51}+x^{49}+x^{47}+x^{43}+x^{41}+x^{38}+x^{35}\\
&+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{25}+x^{19}+x^{18}+x^{15}+x^{14}+x^{13}\\
&+x^{11}+x^{3}+x^{2}+1,\\
p_9(x)=&\,x^{123}+x^{122}+x^{120}+x^{115}+x^{114}+x^{112}+x^{111}+x^{110}+x^{108}+x^{103}\\
&+x^{102}+x^{100}+x^{91}+x^{90}+x^{88}+x^{83}+x^{82}+x^{80}+x^{79}+x^{78}+x^{76}\\
&+x^{71}+x^{70}+x^{68}+x^{59}+x^{58}+x^{56}+x^{51}+x^{50}+x^{48}+x^{47}+x^{46}\\
&+x^{44}+x^{43}+x^{42}+x^{40}+x^{39}+x^{38}+x^{36}+x^{35}+x^{34}+x^{32}+x^{31}\\
&+x^{30}+x^{28}+x^{27}+x^{26}+x^{24}+x^{23}+x^{22}+x^{20}+x^{19}+x^{18}+x^{16}\\
&+x^{15}+x^{14}+x^{12}+x^{7}+x^{6}+x^{4},\\
p_{12}(x)=&\,x^{123}+x^{122}+x^{120}+x^{115}+x^{114}+x^{112}+x^{107}+x^{106}+x^{104}+x^{103}\\
&+x^{102}+x^{100}+x^{99}+x^{98}+x^{96}+x^{91}+x^{90}+x^{88}+x^{83}+x^{82}+x^{80}\\
&+x^{75}+x^{74}+x^{72}+x^{71}+x^{70}+x^{68}+x^{67}+x^{66}+x^{64}+x^{59}+x^{58}\\
&+x^{56}+x^{51}+x^{50}+x^{48}+x^{39}+x^{38}+x^{36}+x^{23}+x^{22}+x^{20}+x^{11}\\
&+x^{10}+x^{8}+x^{7}+x^{6}+x^{4}+x^{3}+x^{2}+1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x)=&\,x^{129}+x^{124}+x^{122}+x^{121}+x^{120}+x^{118}+x^{117}+x^{115}+x^{114}+x^{109}\\
&+x^{107}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{92}+x^{85}+x^{84}+x^{82}+x^{81}\\
&+x^{79}+x^{77}+x^{75}+x^{73}+x^{70}+x^{69}+x^{68}+x^{67}+x^{65}+x^{62}+x^{61}\\
&+x^{58}+x^{57}+x^{54}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{44}+x^{43}+x^{42}\\
&+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{35}+x^{25}+x^{24}+x^{22}+x^{17}+x^{15}\\
&+x^{12},\\
p_3(x)=&\,x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{116}+x^{114}+x^{112}+x^{109}+x^{106}\\
&+x^{104}+x^{103}+x^{102}+x^{101}+x^{100}+x^{99}+x^{97}+x^{96}+x^{94}+x^{90}\\
&+x^{89}+x^{87}+x^{83}+x^{82}+x^{80}+x^{77}+x^{72}+x^{71}+x^{70}+x^{69}+x^{68}\\
&+x^{66}+x^{65}+x^{64}+x^{63}+x^{61}+x^{60}+x^{59}+x^{58}+x^{53}+x^{51}+x^{49}\\
&+x^{48}+x^{47}+x^{40}+x^{39}+x^{38}+x^{36}+x^{34}+x^{31}+x^{30}+x^{26}+x^{21}\\
&+x^{20}+x^{18}+x^{17}+x^{14}+x^{13}+x^{11}+x^{8},\\
p_6(x)=&\,x^{123}+x^{122}+x^{121}+x^{120}+x^{119}+x^{116}+x^{114}+x^{112}+x^{107}+x^{106}\\
&+x^{105}+x^{103}+x^{102}+x^{98}+x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}+x^{90}\\
&+x^{85}+x^{83}+x^{82}+x^{80}+x^{79}+x^{77}+x^{76}+x^{75}+x^{71}+x^{67}+x^{64}\\
&+x^{61}+x^{60}+x^{59}+x^{58}+x^{57}+x^{56}+x^{54}+x^{53}+x^{52}+x^{51}+x^{49}\\
&+x^{47}+x^{43}+x^{40}+x^{35}+x^{33}+x^{32}+x^{29}+x^{28}+x^{27}+x^{26}+x^{22}\\
&+x^{21}+x^{20}+x^{19}+x^{15}+x^{14}+x^{12}+x^{9}+x^{7}+x^{4}+x^{3}+x^{2}+1,
\end{aligned}
$$

$$\begin{aligned}
p_9(x) &= x^{123} + x^{122} + x^{120} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{103} \\
&\quad + x^{102} + x^{100} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76} \\
&\quad + x^{71} + x^{70} + x^{68} + x^{59} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} \\
&\quad + x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} \\
&\quad + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16} \\
&\quad + x^{15} + x^{14} + x^{12} + x^{7} + x^{6} + x^{4}, \\
p_{12}(x) &= x^{123} + x^{122} + x^{120} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{104} + x^{103} \\
&\quad + x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} \\
&\quad + x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{59} + x^{58} \\
&\quad + x^{56} + x^{51} + x^{50} + x^{48} + x^{39} + x^{38} + x^{36} + x^{23} + x^{22} + x^{20} + x^{11} \\
&\quad + x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1].

For $\phi(W^o, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{120} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{108} + x^{107} + x^{105} + x^{103} \\
&\quad + x^{102} + x^{101} + x^{100} + x^{99} + x^{96} + x^{94} + x^{93} + x^{85} + x^{84} + x^{82} + x^{81} \\
&\quad + x^{80} + x^{79} + x^{78} + x^{75} + x^{74} + x^{72} + x^{70} + x^{68} + x^{65} + x^{64} + x^{62} \\
&\quad + x^{61} + x^{60} + x^{59} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} \\
&\quad + x^{43} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31} + x^{29} \\
&\quad + x^{28} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18}, \\
p_3(x) &= x^{111} + x^{110} + x^{106} + x^{104} + x^{102} + x^{97} + x^{87} + x^{86} + x^{82} + x^{80} + x^{78} \\
&\quad + x^{73} + x^{71} + x^{70} + x^{66} + x^{64} + x^{62} + x^{57} + x^{55} + x^{54} + x^{50} + x^{48} \\
&\quad + x^{46} + x^{41} + x^{31} + x^{30} + x^{26} + x^{24} + x^{23} + x^{18} + x^{17} + x^{16} + x^{14} \\
&\quad + x^{9}, \\
p_6(x) &= x^{114} + x^{112} + x^{110} + x^{108} + x^{106} + x^{102} + x^{100} + x^{98} + x^{94} + x^{88} \\
&\quad + x^{80} + x^{78} + x^{72} + x^{68} + x^{66} + x^{62} + x^{60} + x^{56} + x^{52} + x^{40} + x^{38} \\
&\quad + x^{36} + x^{32} + x^{22} + x^{18} + x^{16} + x^{10} + x^{6}, \\
p_9(x) &= x^{117} + x^{116} + x^{115} + x^{114} + x^{112} + x^{109} + x^{108} + x^{106} + x^{104} + x^{99} \\
&\quad + x^{85} + x^{84} + x^{83} + x^{82} + x^{80} + x^{77} + x^{76} + x^{74} + x^{72} + x^{67} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{50} + x^{48} + x^{45} + x^{44} + x^{42} + x^{40} + x^{37} + x^{36} + x^{34} \\
&\quad + x^{32} + x^{29} + x^{28} + x^{26} + x^{24} + x^{21} + x^{20} + x^{18} + x^{16} + x^{13} + x^{12} \\
&\quad + x^{10} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{120} + x^{112} + x^{104} + x^{100} + x^{96} + x^{88} + x^{80} + x^{72} + x^{68} + x^{64} + x^{56} \\
&\quad + x^{48} + x^{36} + x^{20} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1].

For $\phi(W^o, 1, 0)$:

$$\begin{aligned}
p_0(x) &= x^{144} + x^{140} + x^{135} + x^{131} + x^{122} + x^{121} + x^{119} + x^{117} + x^{112} + x^{104} \\
&\quad + x^{101} + x^{98} + x^{96} + x^{91} + x^{90} + x^{88} + x^{82} + x^{79} + x^{78} + x^{75} + x^{74} \\
&\quad + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{64} + x^{59} + x^{56} + x^{55} + x^{53} + x^{52} \\
&\quad + x^{51} + x^{48} + x^{47} + x^{46} + x^{43} + x^{42} + x^{41} + x^{40} + x^{36} + x^{30} + x^{27}, \\
p_3(x) &= x^{123} + x^{122} + x^{119} + x^{116} + x^{112} + x^{111} + x^{109} + x^{106} + x^{105} + x^{104} \\
&\quad + x^{102} + x^{99} + x^{98} + x^{97} + x^{95} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} \\
&\quad + x^{80} + x^{79} + x^{78} + x^{76} + x^{73} + x^{72} + x^{71} + x^{69} + x^{67} + x^{65} + x^{64} \\
&\quad + x^{63} + x^{62} + x^{60} + x^{57} + x^{56} + x^{55} + x^{53} + x^{50} + x^{49} + x^{48} + x^{46} \\
&\quad + x^{43} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{34} + x^{32} + x^{29} + x^{28} + x^{26} \\
&\quad + x^{25} + x^{23} + x^{22} + x^{21} + x^{18} + x^{16} + x^{14} + x^{9}, \\
p_6(x) &= x^{126} + x^{124} + x^{116} + x^{114} + x^{110} + x^{106} + x^{98} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{86} + x^{84} + x^{80} + x^{76} + x^{70} + x^{68} + x^{66} + x^{62} + x^{56} + x^{50} \\
&\quad + x^{46} + x^{38} + x^{36} + x^{34} + x^{32} + x^{28} + x^{26} + x^{24} + x^{18} + x^{16} + x^{14} \\
&\quad + x^{10} + x^{6}, \\
p_9(x) &= x^{129} + x^{128} + x^{127} + x^{126} + x^{125} + x^{123} + x^{122} + x^{121} + x^{118} + x^{116} \\
&\quad + x^{115} + x^{111} + x^{109} + x^{108} + x^{107} + x^{106} + x^{104} + x^{99} + x^{97} + x^{96} \\
&\quad + x^{95} + x^{94} + x^{93} + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{83} + x^{79} + x^{77} \\
&\quad + x^{76} + x^{75} + x^{74} + x^{72} + x^{67} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{59} \\
&\quad + x^{58} + x^{57} + x^{54} + x^{52} + x^{51} + x^{49} + x^{48} + x^{46} + x^{44} + x^{41} + x^{40} \\
&\quad + x^{38} + x^{36} + x^{33} + x^{32} + x^{30} + x^{28} + x^{25} + x^{24} + x^{22} + x^{20} + x^{15} \\
&\quad + x^{13} + x^{12} + x^{11} + x^{10} + x^{8} + x^{3}, \\
p_{12}(x) &= x^{132} + x^{128} + x^{124} + x^{116} + x^{112} + x^{92} + x^{84} + x^{80} + x^{60} + x^{44} + x^{36} \\
&\quad + x^{32} + x^{28} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) &= x^{129} + x^{125} + x^{121} + x^{119} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{109} \\
&\quad + x^{108} + x^{106} + x^{104} + x^{103} + x^{97} + x^{95} + x^{91} + x^{89} + x^{87} + x^{85} \\
&\quad + x^{84} + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{75} + x^{72} + x^{70} + x^{69} + x^{66} \\
&\quad + x^{63} + x^{62} + x^{61} + x^{59} + x^{58} + x^{56} + x^{52} + x^{46} + x^{45} + x^{44} + x^{43} \\
&\quad + x^{41} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{33}, \\
p_3(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112} \\
&\quad + x^{110} + x^{108} + x^{107} + x^{106} + x^{103} + x^{102} + x^{97} + x^{94} + x^{93} + x^{92} \\
&\quad + x^{91} + x^{90} + x^{87} + x^{86} + x^{85} + x^{84} + x^{82} + x^{77} + x^{73} + x^{71} + x^{67} \\
&\quad + x^{66} + x^{65} + x^{63} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{54} + x^{52}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{49} + x^{47} + x^{45} + x^{44} + x^{43} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33}\\
&+ x^{31} + x^{26} + x^{25} + x^{24} + x^{22} + x^{21} + x^{20} + x^{18} + x^{17} + x^{14} + x^{12},\\
p_6(x) &= x^{123} + x^{122} + x^{121} + x^{120} + x^{119} + x^{117} + x^{115} + x^{114} + x^{113} + x^{112}\\
&+ x^{110} + x^{109} + x^{108} + x^{106} + x^{102} + x^{101} + x^{95} + x^{93} + x^{92} + x^{91}\\
&+ x^{88} + x^{85} + x^{83} + x^{82} + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72}\\
&+ x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58}\\
&+ x^{57} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{47} + x^{43} + x^{41} + x^{38} + x^{35}\\
&+ x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{25} + x^{19} + x^{18} + x^{15} + x^{14} + x^{13}\\
&+ x^{11} + x^{3} + x^{2} + 1,\\
p_9(x) &= x^{123} + x^{122} + x^{120} + x^{115} + x^{114} + x^{112} + x^{111} + x^{110} + x^{108} + x^{103}\\
&+ x^{102} + x^{100} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80} + x^{79} + x^{78} + x^{76}\\
&+ x^{71} + x^{70} + x^{68} + x^{59} + x^{58} + x^{56} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46}\\
&+ x^{44} + x^{43} + x^{42} + x^{40} + x^{39} + x^{38} + x^{36} + x^{35} + x^{34} + x^{32} + x^{31}\\
&+ x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{23} + x^{22} + x^{20} + x^{19} + x^{18} + x^{16}\\
&+ x^{15} + x^{14} + x^{12} + x^{7} + x^{6} + x^{4},\\
p_{12}(x) &= x^{123} + x^{122} + x^{120} + x^{115} + x^{114} + x^{112} + x^{107} + x^{106} + x^{104} + x^{103}\\
&+ x^{102} + x^{100} + x^{99} + x^{98} + x^{96} + x^{91} + x^{90} + x^{88} + x^{83} + x^{82} + x^{80}\\
&+ x^{75} + x^{74} + x^{72} + x^{71} + x^{70} + x^{68} + x^{67} + x^{66} + x^{64} + x^{59} + x^{58}\\
&+ x^{56} + x^{51} + x^{50} + x^{48} + x^{39} + x^{38} + x^{36} + x^{23} + x^{22} + x^{20} + x^{11}\\
&+ x^{10} + x^{8} + x^{7} + x^{6} + x^{4} + x^{3} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	298	[497, 472]	596	[524, 56]	894	[1035, 359]
1	[3, 4]	299	[483, 498]	597	[502, 391]	895	[1036, 98]
2	[5, 6]	300	[499, 500]	598	[199, 508]	896	[643, 634]
3	[7, 8]	301	[501, 53]	599	[425, 536]	897	[1037, 1038]
4	[9, 10]	302	[502, 490]	600	[397, 532]	898	[651, 1039]
5	[11, 12]	303	[503, 504]	601	[532, 507]	899	[1040, 351]
6	[13, 14]	304	[505, 506]	602	[843, 452]	900	[1041, 1042]
7	[15, 16]	305	[474, 133]	603	[844, 845]	901	[1043, 970]
8	[17, 18]	306	[507, 508]	604	[467, 554]	902	[1044, 264]
9	[19, 20]	307	[199, 165]	605	[466, 458]	903	[624, 250]
10	[21, 16]	308	[507, 165]	606	[846, 162]	904	[1045, 89]
11	[22, 23]	309	[509, 181]	607	[473, 30]	905	[1046, 765]
12	[24, 25]	310	[510, 511]	608	[847, 483]	906	[1047, 1048]

13	[26, 27]	311	[512, 513]	609	[206, 848]	907	[1049, 314]
14	[28, 29]	312	[514, 183]	610	[420, 378]	908	[1050, 1051]
15	[30, 31]	313	[439, 515]	611	[849, 492]	909	[808, 1052]
16	[32, 33]	314	[52, 516]	612	[828, 539]	910	[1053, 1054]
17	[34, 35]	315	[378, 414]	613	[850, 851]	911	[1055, 802]
18	[31, 36]	316	[517, 191]	614	[45, 826]	912	[1056, 669]
19	[37, 38]	317	[518, 519]	615	[852, 2]	913	[1057, 726]
20	[39, 40]	318	[520, 521]	616	[487, 34]	914	[772, 1058]
21	[41, 42]	319	[446, 385]	617	[835, 853]	915	[1059, 796]
22	[43, 44]	320	[522, 513]	618	[854, 137]	916	[681, 1060]
23	[45, 46]	321	[131, 523]	619	[855, 556]	917	[1061, 567]
24	[47, 48]	322	[143, 379]	620	[451, 33]	918	[1062, 1063]
25	[49, 50]	323	[54, 521]	621	[856, 538]	919	[16, 209]
26	[51, 52]	324	[524, 441]	622	[142, 527]	920	[1064, 240]
27	[53, 54]	325	[406, 44]	623	[821, 857]	921	[1065, 623]
28	[55, 56]	326	[52, 194]	624	[142, 384]	922	[963, 353]
29	[57, 58]	327	[421, 168]	625	[541, 42]	923	[775, 1034]
30	[59, 60]	328	[50, 525]	626	[858, 121]	924	[349, 74]
31	[61, 62]	329	[181, 526]	627	[859, 414]	925	[1066, 1051]
32	[63, 64]	330	[32, 452]	628	[514, 860]	926	[1067, 292]
33	[65, 66]	331	[458, 382]	629	[51, 193]	927	[1068, 270]
34	[67, 68]	332	[383, 527]	630	[429, 131]	928	[1069, 649]
35	[69, 23]	333	[519, 158]	631	[861, 527]	929	[1070, 1071]
36	[70, 71]	334	[522, 528]	632	[10, 434]	930	[64, 1072]
37	[72, 73]	335	[529, 530]	633	[862, 156]	931	[1073, 1053]
38	[74, 75]	336	[531, 519]	634	[117, 545]	932	[1074, 667]
39	[76, 77]	337	[11, 496]	635	[461, 510]	933	[1075, 90]
40	[78, 79]	338	[154, 144]	636	[863, 845]	934	[365, 811]
41	[80, 81]	339	[121, 172]	637	[864, 14]	935	[321, 1076]
42	[82, 83]	340	[186, 397]	638	[508, 398]	936	[1077, 109]
43	[84, 85]	341	[187, 532]	639	[865, 120]	937	[1031, 1078]
44	[86, 87]	342	[54, 433]	640	[163, 401]	938	[1079, 661]
45	[88, 89]	343	[533, 534]	641	[866, 177]	939	[1001, 4]
46	[90, 91]	344	[535, 424]	642	[463, 848]	940	[970, 579]
47	[92, 93]	345	[536, 187]	643	[145, 433]	941	[1080, 655]
48	[94, 95]	346	[537, 427]	644	[178, 558]	942	[1081, 310]
49	[96, 97]	347	[538, 539]	645	[122, 445]	943	[1082, 261]
50	[29, 98]	348	[126, 540]	646	[867, 868]	944	[1083, 94]
51	[99, 100]	349	[541, 542]	647	[869, 458]	945	[774, 269]
52	[101, 102]	350	[381, 459]	648	[870, 563]	946	[1084, 1029]
53	[103, 104]	351	[543, 120]	649	[47, 523]	947	[1085, 976]
54	[105, 106]	352	[508, 166]	650	[871, 872]	948	[1086, 769]
55	[107, 108]	353	[544, 545]	651	[486, 873]	949	[948, 919]
56	[109, 110]	354	[527, 394]	652	[866, 431]	950	[544, 118]

57	[111, 112]	355	[546, 156]	653	[150, 148]	951	[535, 872]
58	[113, 114]	356	[547, 504]	654	[432, 521]	952	[1087, 197]
59	[115, 116]	357	[548, 394]	655	[867, 560]	953	[1088, 496]
60	[117, 118]	358	[549, 431]	656	[512, 528]	954	[1089, 453]
61	[119, 120]	359	[187, 198]	657	[493, 389]	955	[457, 175]
62	[121, 121]	360	[536, 397]	658	[537, 874]	956	[900, 193]
63	[122, 123]	361	[399, 536]	659	[510, 875]	957	[443, 12]
64	[124, 125]	362	[550, 401]	660	[850, 448]	958	[924, 207]
65	[126, 127]	363	[551, 552]	661	[119, 823]	959	[1090, 33]
66	[128, 129]	364	[553, 554]	662	[380, 876]	960	[505, 525]
67	[130, 131]	365	[555, 556]	663	[871, 424]	961	[847, 396]
68	[132, 58]	366	[147, 151]	664	[877, 878]	962	[846, 488]
69	[133, 134]	367	[456, 515]	665	[879, 34]	963	[134, 527]
70	[135, 136]	368	[557, 10]	666	[837, 556]	964	[1087, 185]
71	[137, 12]	369	[192, 558]	667	[190, 880]	965	[1091, 431]
72	[138, 45]	370	[559, 201]	668	[881, 882]	966	[449, 404]
73	[139, 140]	371	[504, 560]	669	[883, 884]	967	[471, 562]
74	[141, 142]	372	[384, 561]	670	[392, 495]	968	[152, 900]
75	[143, 36]	373	[497, 562]	671	[558, 885]	969	[13, 464]
76	[144, 145]	374	[468, 563]	672	[886, 404]	970	[37, 378]
77	[146, 40]	375	[564, 565]	673	[453, 457]	971	[200, 882]
78	[147, 148]	376	[395, 483]	674	[465, 500]	972	[519, 923]
79	[132, 149]	377	[566, 367]	675	[887, 34]	973	[447, 851]
80	[150, 151]	378	[567, 568]	676	[849, 410]	974	[1092, 472]
81	[14, 8]	379	[60, 569]	677	[462, 875]	975	[1093, 918]
82	[152, 153]	380	[570, 571]	678	[888, 205]	976	[399, 186]
83	[154, 53]	381	[572, 573]	679	[889, 472]	977	[1094, 528]
84	[155, 156]	382	[574, 575]	680	[188, 45]	978	[1095, 407]
85	[157, 158]	383	[576, 577]	681	[838, 530]	979	[460, 169]
86	[159, 160]	384	[260, 578]	682	[433, 824]	980	[1096, 897]
87	[161, 162]	385	[579, 580]	683	[890, 882]	981	[899, 153]
88	[163, 164]	386	[581, 582]	684	[10, 125]	982	[1089, 34]
89	[165, 166]	387	[583, 584]	685	[821, 891]	983	[829, 823]
90	[167, 168]	388	[585, 586]	686	[184, 197]	984	[1097, 874]
91	[169, 170]	389	[587, 108]	687	[414, 845]	985	[919, 506]
92	[171, 164]	390	[588, 231]	688	[842, 891]	986	[936, 946]
93	[172, 121]	391	[589, 590]	689	[144, 54]	987	[1098, 420]
94	[173, 174]	392	[591, 592]	690	[856, 828]	988	[905, 145]
95	[175, 176]	393	[593, 97]	691	[41, 542]	989	[547, 867]
96	[177, 178]	394	[246, 99]	692	[892, 836]	990	[919, 525]
97	[56, 179]	395	[594, 595]	693	[893, 381]	991	[1099, 398]
98	[56, 180]	396	[596, 597]	694	[894, 187]	992	[926, 145]
99	[178, 181]	397	[598, 599]	695	[895, 116]	993	[546, 1100]
100	[182, 183]	398	[600, 601]	696	[533, 884]	994	[377, 38]

101	[184, 185]	399	[102, 340]	697	[896, 897]	995	[1101, 424]
102	[186, 187]	400	[602, 20]	698	[133, 142]	996	[1102, 530]
103	[188, 189]	401	[603, 604]	699	[55, 441]	997	[1103, 31]
104	[190, 191]	402	[605, 374]	700	[898, 123]	998	[1104, 58]
105	[177, 192]	403	[606, 76]	701	[892, 853]	999	[500, 381]
106	[193, 194]	404	[607, 608]	702	[128, 556]	1000	[934, 542]
107	[192, 181]	405	[609, 610]	703	[899, 900]	1001	[529, 839]
108	[173, 195]	406	[611, 612]	704	[901, 468]	1002	[35, 479]
109	[196, 197]	407	[613, 271]	705	[902, 532]	1003	[30, 143]
110	[198, 199]	408	[73, 614]	706	[903, 821]	1004	[1105, 876]
111	[200, 201]	409	[615, 88]	707	[904, 425]	1005	[1106, 506]
112	[202, 203]	410	[616, 617]	708	[827, 538]	1006	[419, 37]
113	[204, 205]	411	[618, 296]	709	[863, 415]	1007	[1106, 525]
114	[206, 207]	412	[619, 620]	710	[501, 144]	1008	[913, 431]
115	[208, 209]	413	[621, 622]	711	[905, 54]	1009	[1107, 554]
116	[210, 211]	414	[367, 623]	712	[454, 906]	1010	[161, 488]
117	[212, 213]	415	[75, 624]	713	[907, 443]	1011	[8, 919]
118	[214, 215]	416	[625, 17]	714	[462, 511]	1012	[933, 930]
119	[216, 217]	417	[626, 61]	715	[833, 387]	1013	[831, 542]
120	[218, 219]	418	[627, 628]	716	[480, 136]	1014	[940, 202]
121	[220, 221]	419	[629, 610]	717	[908, 438]	1015	[548, 561]
122	[222, 223]	420	[630, 216]	718	[869, 381]	1016	[1098, 37]
123	[224, 225]	421	[631, 632]	719	[909, 202]	1017	[1108, 45]
124	[226, 227]	422	[633, 634]	720	[844, 415]	1018	[888, 914]
125	[20, 228]	423	[592, 635]	721	[887, 453]	1019	[403, 450]
126	[229, 230]	424	[636, 637]	722	[910, 911]	1020	[877, 2]
127	[231, 232]	425	[307, 638]	723	[912, 131]	1021	[1091, 177]
128	[233, 234]	426	[639, 640]	724	[417, 30]	1022	[1092, 562]
129	[235, 236]	427	[641, 642]	725	[913, 177]	1023	[854, 443]
130	[237, 238]	428	[77, 643]	726	[468, 442]	1024	[412, 929]
131	[239, 240]	429	[644, 215]	727	[204, 914]	1025	[938, 918]
132	[241, 242]	430	[645, 646]	728	[915, 151]	1026	[498, 169]
133	[243, 244]	431	[647, 330]	729	[916, 552]	1027	[841, 821]
134	[245, 246]	432	[648, 321]	730	[917, 37]	1028	[1109, 427]
135	[247, 248]	433	[266, 649]	731	[915, 148]	1029	[388, 494]
136	[249, 250]	434	[317, 644]	732	[861, 384]	1030	[945, 450]
137	[251, 252]	435	[650, 651]	733	[435, 918]	1031	[396, 498]
138	[253, 254]	436	[652, 653]	734	[883, 534]	1032	[1101, 872]
139	[255, 256]	437	[654, 655]	735	[49, 919]	1033	[558, 526]
140	[257, 258]	438	[656, 657]	736	[145, 521]	1034	[563, 137]
141	[259, 260]	439	[658, 659]	737	[189, 46]	1035	[1110, 536]
142	[261, 262]	440	[660, 80]	738	[920, 165]	1036	[825, 563]
143	[263, 264]	441	[661, 598]	739	[921, 414]	1037	[423, 872]
144	[265, 266]	442	[234, 24]	740	[922, 193]	1038	[532, 199]

145	[267, 268]	443	[662, 322]	741	[444, 123]	1039	[551, 1111]
146	[269, 270]	444	[663, 664]	742	[157, 923]	1040	[1112, 508]
147	[271, 272]	445	[665, 625]	743	[557, 124]	1041	[1113, 50]
148	[273, 274]	446	[666, 667]	744	[924, 848]	1042	[437, 1114]
149	[275, 276]	447	[668, 669]	745	[57, 149]	1043	[903, 842]
150	[277, 278]	448	[670, 671]	746	[925, 391]	1044	[1115, 186]
151	[279, 280]	449	[672, 673]	747	[35, 175]	1045	[1116, 507]
152	[281, 282]	450	[674, 675]	748	[198, 507]	1046	[1117, 897]
153	[283, 284]	451	[676, 677]	749	[926, 54]	1047	[135, 481]
154	[285, 286]	452	[678, 679]	750	[897, 906]	1048	[408, 845]
155	[287, 288]	453	[680, 681]	751	[405, 848]	1049	[1118, 460]
156	[289, 290]	454	[223, 306]	752	[927, 197]	1050	[907, 137]
157	[291, 292]	455	[634, 682]	753	[476, 873]	1051	[828, 832]
158	[217, 293]	456	[683, 684]	754	[402, 420]	1052	[910, 412]
159	[294, 295]	457	[81, 685]	755	[889, 562]	1053	[53, 145]
160	[296, 297]	458	[686, 359]	756	[928, 181]	1054	[909, 446]
161	[298, 299]	459	[230, 687]	757	[911, 929]	1055	[489, 151]
162	[300, 301]	460	[595, 688]	758	[159, 930]	1056	[1113, 919]
163	[302, 303]	461	[689, 690]	759	[931, 836]	1057	[482, 31]
164	[304, 305]	462	[691, 662]	760	[479, 176]	1058	[175, 1119]
165	[306, 307]	463	[692, 567]	761	[890, 201]	1059	[1120, 853]
166	[93, 308]	464	[693, 335]	762	[504, 868]	1060	[1121, 911]
167	[309, 310]	465	[694, 686]	763	[503, 867]	1061	[565, 441]
168	[311, 312]	466	[695, 696]	764	[855, 129]	1062	[1107, 468]
169	[313, 314]	467	[697, 698]	765	[139, 876]	1063	[931, 853]
170	[209, 315]	468	[699, 251]	766	[932, 378]	1064	[1122, 166]
171	[316, 317]	469	[700, 696]	767	[865, 823]	1065	[1123, 460]
172	[172, 172]	470	[701, 96]	768	[384, 394]	1066	[1124, 528]
173	[318, 319]	471	[702, 261]	769	[842, 857]	1067	[1125, 842]
174	[320, 254]	472	[703, 704]	770	[879, 453]	1068	[1126, 515]
175	[68, 321]	473	[705, 288]	771	[426, 874]	1069	[1123, 498]
176	[322, 323]	474	[706, 689]	772	[911, 413]	1070	[9, 124]
177	[324, 325]	475	[707, 573]	773	[933, 160]	1071	[460, 116]
178	[272, 326]	476	[292, 708]	774	[555, 129]	1072	[116, 1127]
179	[327, 328]	477	[709, 710]	775	[518, 157]	1073	[1120, 836]
180	[108, 329]	478	[711, 322]	776	[165, 398]	1074	[1128, 454]
181	[330, 331]	479	[620, 712]	777	[520, 433]	1075	[1129, 445]
182	[332, 333]	480	[713, 714]	778	[386, 834]	1076	[563, 443]
183	[334, 70]	481	[715, 716]	779	[146, 392]	1077	[1130, 198]
184	[335, 336]	482	[323, 717]	780	[166, 399]	1078	[1121, 412]
185	[337, 338]	483	[718, 210]	781	[927, 185]	1079	[400, 401]
186	[308, 339]	484	[325, 719]	782	[852, 878]	1080	[1131, 1100]
187	[340, 341]	485	[720, 721]	783	[934, 42]	1081	[1132, 834]
188	[342, 343]	486	[104, 722]	784	[517, 880]	1082	[896, 454]

189	[344, 345]	487	[723, 19]	785	[531, 157]	1083	[478, 175]
190	[346, 347]	488	[724, 725]	786	[498, 116]	1084	[1133, 479]
191	[348, 349]	489	[584, 726]	787	[521, 935]	1085	[1116, 199]
192	[278, 350]	490	[727, 728]	788	[886, 450]	1086	[870, 442]
193	[351, 352]	491	[729, 615]	789	[936, 565]	1087	[1134, 77]
194	[353, 354]	492	[730, 731]	790	[549, 177]	1088	[1135, 685]
195	[355, 343]	493	[732, 371]	791	[937, 510]	1089	[1136, 811]
196	[227, 356]	494	[733, 734]	792	[928, 558]	1090	[1137, 319]
197	[357, 358]	495	[336, 735]	793	[938, 436]	1091	[1138, 796]
198	[359, 360]	496	[211, 736]	794	[550, 164]	1092	[1139, 245]
199	[62, 361]	497	[728, 689]	795	[939, 830]	1093	[288, 818]
200	[362, 363]	498	[673, 737]	796	[420, 38]	1094	[363, 686]
201	[364, 365]	499	[738, 287]	797	[940, 446]	1095	[1140, 1024]
202	[366, 367]	500	[739, 230]	798	[941, 490]	1096	[1076, 312]
203	[368, 369]	501	[740, 105]	799	[922, 52]	1097	[1039, 951]
204	[370, 327]	502	[741, 742]	800	[942, 149]	1098	[1141, 74]
205	[371, 372]	503	[726, 743]	801	[937, 462]	1099	[1142, 102]
206	[373, 374]	504	[744, 745]	802	[430, 407]	1100	[1143, 609]
207	[375, 376]	505	[746, 737]	803	[393, 561]	1101	[1144, 785]
208	[377, 378]	506	[18, 747]	804	[943, 133]	1102	[1145, 586]
209	[31, 379]	507	[341, 600]	805	[944, 47]	1103	[812, 70]
210	[380, 140]	508	[360, 748]	806	[893, 458]	1104	[1146, 632]
211	[381, 382]	509	[749, 750]	807	[945, 404]	1105	[1147, 714]
212	[383, 384]	510	[751, 302]	808	[554, 563]	1106	[1148, 688]
213	[202, 385]	511	[690, 342]	809	[499, 466]	1107	[1149, 808]
214	[386, 387]	512	[752, 753]	810	[864, 464]	1108	[1150, 90]
215	[388, 389]	513	[754, 755]	811	[554, 442]	1109	[664, 1151]
216	[390, 391]	514	[756, 757]	812	[442, 137]	1110	[1152, 598]
217	[39, 392]	515	[758, 759]	813	[38, 414]	1111	[1153, 1154]
218	[393, 394]	516	[284, 760]	814	[564, 946]	1112	[1155, 93]
219	[395, 396]	517	[761, 762]	815	[943, 141]	1113	[1156, 578]
220	[397, 198]	518	[610, 763]	816	[897, 455]	1114	[1157, 574]
221	[398, 399]	519	[764, 765]	817	[947, 172]	1115	[1158, 110]
222	[400, 164]	520	[766, 712]	818	[939, 485]	1116	[1159, 1038]
223	[171, 401]	521	[765, 750]	819	[948, 50]	1117	[1160, 1047]
224	[402, 37]	522	[767, 572]	820	[949, 950]	1118	[1161, 784]
225	[403, 404]	523	[238, 768]	821	[951, 220]	1119	[685, 1162]
226	[405, 207]	524	[622, 734]	822	[952, 344]	1120	[1013, 286]
227	[406, 407]	525	[769, 253]	823	[953, 954]	1121	[286, 1070]
228	[38, 408]	526	[350, 237]	824	[268, 955]	1122	[1163, 352]
229	[409, 410]	527	[79, 769]	825	[956, 302]	1123	[1164, 313]
230	[411, 174]	528	[770, 702]	826	[254, 957]	1124	[1020, 287]
231	[412, 413]	529	[771, 772]	827	[299, 785]	1125	[1165, 1166]
232	[414, 415]	530	[773, 774]	828	[958, 313]	1126	[1167, 213]

233	[416, 162]	531	[374, 775]	829	[959, 763]	1127	[688, 107]
234	[14, 49]	532	[339, 776]	830	[960, 961]	1128	[1168, 1169]
235	[417, 418]	533	[777, 85]	831	[962, 19]	1129	[818, 1037]
236	[419, 420]	534	[778, 100]	832	[779, 963]	1130	[1170, 62]
237	[134, 384]	535	[244, 779]	833	[964, 101]	1131	[753, 61]
238	[421, 422]	536	[345, 780]	834	[965, 966]	1132	[1151, 282]
239	[423, 424]	537	[781, 782]	835	[967, 267]	1133	[1171, 252]
240	[398, 425]	538	[783, 784]	836	[968, 969]	1134	[1172, 140]
241	[426, 427]	539	[785, 786]	837	[731, 970]	1135	[1173, 839]
242	[392, 428]	540	[655, 787]	838	[971, 972]	1136	[1096, 454]
243	[429, 47]	541	[788, 69]	839	[973, 974]	1137	[155, 1100]
244	[430, 44]	542	[789, 790]	840	[975, 333]	1138	[946, 56]
245	[431, 178]	543	[667, 791]	841	[369, 716]	1139	[416, 488]
246	[131, 48]	544	[792, 670]	842	[640, 976]	1140	[1094, 513]
247	[432, 433]	545	[793, 238]	843	[977, 296]	1141	[1103, 143]
248	[124, 434]	546	[794, 795]	844	[978, 75]	1142	[1110, 186]
249	[435, 436]	547	[796, 797]	845	[623, 979]	1143	[917, 420]
250	[437, 438]	548	[798, 18]	846	[114, 27]	1144	[1174, 440]
251	[439, 440]	549	[799, 272]	847	[980, 269]	1145	[1132, 387]
252	[441, 180]	550	[800, 801]	848	[981, 982]	1146	[1175, 918]
253	[442, 443]	551	[802, 621]	849	[983, 729]	1147	[1124, 513]
254	[411, 195]	552	[803, 804]	850	[984, 94]	1148	[1176, 880]
255	[444, 445]	553	[805, 234]	851	[684, 985]	1149	[1177, 189]
256	[446, 203]	554	[806, 316]	852	[784, 368]	1150	[115, 169]
257	[447, 448]	555	[807, 808]	853	[986, 987]	1151	[916, 1111]
258	[449, 450]	556	[809, 810]	854	[988, 211]	1152	[1178, 425]
259	[138, 189]	557	[811, 801]	855	[66, 698]	1153	[1088, 12]
260	[451, 452]	558	[745, 577]	856	[27, 989]	1154	[901, 554]
261	[453, 35]	559	[282, 92]	857	[950, 990]	1155	[1179, 507]
262	[454, 455]	560	[743, 812]	858	[991, 220]	1156	[932, 38]
263	[196, 185]	561	[331, 813]	859	[992, 614]	1157	[862, 1100]
264	[166, 425]	562	[814, 815]	860	[993, 717]	1158	[1115, 536]
265	[130, 47]	563	[698, 816]	861	[994, 331]	1159	[1122, 398]
266	[456, 440]	564	[817, 818]	862	[995, 263]	1160	[921, 408]
267	[34, 457]	565	[819, 327]	863	[996, 568]	1161	[418, 31]
268	[458, 459]	566	[820, 821]	864	[997, 17]	1162	[8, 50]
269	[396, 460]	567	[464, 49]	865	[998, 775]	1163	[904, 399]
270	[461, 462]	568	[441, 179]	866	[999, 278]	1164	[153, 52]
271	[463, 207]	569	[116, 170]	867	[1000, 1001]	1165	[1133, 175]
272	[464, 8]	570	[822, 492]	868	[708, 1002]	1166	[182, 860]
273	[465, 466]	571	[40, 428]	869	[736, 574]	1167	[1093, 436]
274	[467, 468]	572	[543, 823]	870	[1003, 662]	1168	[895, 169]
275	[469, 127]	573	[425, 186]	871	[1004, 989]	1169	[908, 1114]
276	[470, 162]	574	[167, 422]	872	[1005, 1006]	1170	[1178, 399]

277	[471, 472]	575	[521, 824]	873	[1007, 1008]	1171	[1118, 498]
278	[141, 134]	576	[825, 442]	874	[1009, 1010]	1172	[1180, 248]
279	[473, 418]	577	[193, 516]	875	[4, 1011]	1173	[1181, 371]
280	[474, 141]	578	[189, 826]	876	[1012, 1013]	1174	[1182, 670]
281	[475, 397]	579	[827, 828]	877	[597, 1014]	1175	[795, 109]
282	[476, 477]	580	[378, 408]	878	[1015, 1016]	1176	[1183, 757]
283	[478, 479]	581	[829, 120]	879	[1017, 69]	1177	[1184, 295]
284	[480, 481]	582	[484, 830]	880	[1018, 1019]	1178	[1185, 345]
285	[482, 143]	583	[553, 468]	881	[582, 1020]	1179	[1186, 306]
286	[483, 460]	584	[831, 42]	882	[1021, 1022]	1180	[1131, 156]
287	[484, 485]	585	[509, 558]	883	[1023, 1024]	1181	[1175, 436]
288	[486, 477]	586	[538, 832]	884	[1025, 353]	1182	[1187, 387]
289	[487, 453]	587	[833, 834]	885	[649, 1026]	1183	[1187, 834]
290	[470, 488]	588	[822, 410]	886	[653, 595]	1184	[859, 408]
291	[489, 148]	589	[469, 540]	887	[1027, 81]	1185	[1099, 166]
292	[390, 490]	590	[835, 836]	888	[1028, 1029]	1186	[1179, 199]
293	[457, 479]	591	[837, 129]	889	[1030, 1031]	1187	[782, 351]
294	[491, 492]	592	[838, 839]	890	[1032, 239]		
295	[493, 494]	593	[840, 191]	891	[716, 1033]		
296	[40, 495]	594	[841, 842]	892	[642, 105]		
297	[137, 496]	595	[431, 192]	893	[1034, 657]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	199	1	398	1	597	1	796	1	995	0
1	0	200	1	399	0	598	1	797	0	996	1
2	1	201	0	400	0	599	1	798	0	997	0
3	0	202	0	401	0	600	1	799	0	998	0
4	0	203	1	402	1	601	1	800	1	999	0
5	1	204	0	403	1	602	0	801	1	1000	1
6	1	205	0	404	1	603	0	802	0	1001	0
7	0	206	1	405	1	604	0	803	1	1002	1
8	0	207	1	406	1	605	1	804	0	1003	0
9	0	208	0	407	0	606	1	805	0	1004	1
10	0	209	0	408	1	607	1	806	1	1005	0
11	1	210	0	409	0	608	1	807	1	1006	1
12	0	211	1	410	1	609	1	808	1	1007	0
13	1	212	1	411	0	610	1	809	1	1008	1
14	0	213	0	412	0	611	1	810	0	1009	1
15	0	214	1	413	1	612	0	811	1	1010	1
16	0	215	1	414	0	613	0	812	0	1011	0
17	0	216	0	415	1	614	1	813	0	1012	0
18	0	217	0	416	0	615	0	814	1	1013	1
19	0	218	1	417	0	616	1	815	0	1014	0

20	0	219	0	418	1	617	0	816	1	1015	0
21	0	220	1	419	1	618	0	817	1	1016	0
22	1	221	1	420	1	619	0	818	0	1017	0
23	1	222	0	421	0	620	1	819	0	1018	1
24	0	223	0	422	0	621	1	820	0	1019	1
25	1	224	1	423	1	622	1	821	0	1020	1
26	1	225	1	424	1	623	0	822	0	1021	1
27	1	226	1	425	1	624	1	823	0	1022	0
28	0	227	1	426	0	625	0	824	0	1023	0
29	1	228	0	427	0	626	0	825	1	1024	0
30	0	229	0	428	1	627	1	826	0	1025	0
31	0	230	0	429	1	628	0	827	0	1026	1
32	0	231	0	430	0	629	1	828	0	1027	0
33	0	232	0	431	0	630	1	829	1	1028	1
34	0	233	0	432	0	631	0	830	1	1029	1
35	1	234	0	433	0	632	0	831	1	1030	1
36	0	235	0	434	1	633	0	832	1	1031	1
37	0	236	1	435	1	634	1	833	0	1032	0
38	0	237	0	436	0	635	0	834	1	1033	1
39	0	238	0	437	0	636	1	835	0	1034	1
40	0	239	1	438	1	637	0	836	1	1035	0
41	0	240	1	439	1	638	1	837	1	1036	1
42	1	241	0	440	0	639	0	838	1	1037	1
43	1	242	1	441	0	640	1	839	1	1038	1
44	1	243	1	442	0	641	0	840	1	1039	0
45	1	244	0	443	0	642	0	841	0	1040	0
46	1	245	0	444	1	643	0	842	1	1041	1
47	0	246	1	445	0	644	1	843	0	1042	0
48	1	247	0	446	1	645	0	844	0	1043	1
49	1	248	1	447	1	646	1	845	0	1044	1
50	1	249	1	448	1	647	0	846	1	1045	1
51	1	250	0	449	0	648	0	847	1	1046	0
52	0	251	1	450	0	649	0	848	0	1047	0
53	1	252	0	451	1	650	1	849	1	1048	1
54	1	253	0	452	1	651	1	850	0	1049	1
55	0	254	0	453	1	652	0	851	0	1050	1
56	1	255	1	454	0	653	0	852	0	1051	0
57	1	256	1	455	1	654	0	853	0	1052	1
58	0	257	1	456	0	655	1	854	0	1053	1
59	0	258	0	457	0	656	1	855	0	1054	1
60	1	259	0	458	0	657	0	856	1	1055	1
61	0	260	1	459	0	658	1	857	0	1056	1
62	1	261	1	460	0	659	1	858	0	1057	1
63	0	262	0	461	0	660	0	859	1	1058	0

64	1	263	1	462	0	661	0	860	1	1059	1
65	0	264	0	463	0	662	0	861	0	1060	0
66	0	265	0	464	1	663	1	862	0	1061	0
67	0	266	0	465	0	664	1	863	1	1062	1
68	0	267	0	466	1	665	0	864	0	1063	1
69	1	268	0	467	0	666	1	865	0	1064	1
70	0	269	1	468	0	667	1	866	0	1065	0
71	1	270	0	469	1	668	1	867	1	1066	1
72	0	271	0	470	0	669	0	868	0	1067	1
73	1	272	1	471	0	670	1	869	0	1068	1
74	0	273	0	472	0	671	1	870	0	1069	0
75	1	274	0	473	1	672	0	871	1	1070	0
76	0	275	1	474	1	673	1	872	0	1071	0
77	1	276	0	475	1	674	0	873	0	1072	0
78	0	277	0	476	0	675	0	874	1	1073	1
79	0	278	0	477	1	676	1	875	0	1074	1
80	0	279	1	478	0	677	0	876	0	1075	0
81	0	280	1	479	1	678	1	877	1	1076	1
82	1	281	1	480	1	679	1	878	0	1077	0
83	1	282	0	481	0	680	1	879	0	1078	0
84	1	283	0	482	1	681	1	880	1	1079	0
85	1	284	1	483	0	682	0	881	1	1080	0
86	1	285	1	484	1	683	0	882	1	1081	1
87	1	286	0	485	0	684	0	883	0	1082	0
88	1	287	1	486	1	685	0	884	0	1083	0
89	0	288	1	487	1	686	0	885	0	1084	1
90	1	289	1	488	0	687	0	886	0	1085	1
91	1	290	0	489	1	688	1	887	0	1086	0
92	0	291	1	490	0	689	0	888	1	1087	0
93	0	292	0	491	1	690	1	889	1	1088	0
94	1	293	0	492	0	691	0	890	0	1089	1
95	0	294	1	493	0	692	0	891	1	1090	1
96	1	295	0	494	1	693	1	892	0	1091	1
97	1	296	0	495	0	694	0	893	1	1092	0
98	1	297	1	496	1	695	1	894	0	1093	1
99	1	298	1	497	1	696	1	895	1	1094	0
100	1	299	0	498	1	697	0	896	0	1095	0
101	0	300	1	499	1	698	1	897	1	1096	1
102	0	301	0	500	0	699	0	898	1	1097	0
103	1	302	1	501	0	700	1	899	0	1098	0
104	1	303	0	502	1	701	0	900	1	1099	0
105	1	304	1	503	0	702	0	901	1	1100	0
106	1	305	1	504	0	703	0	902	1	1101	0
107	0	306	0	505	1	704	1	903	1	1102	1

108	1	307	1	506	0	705	1	904	1	1103	0
109	1	308	0	507	0	706	1	905	0	1104	0
110	0	309	1	508	1	707	1	906	0	1105	1
111	1	310	1	509	1	708	0	907	1	1106	0
112	0	311	1	510	1	709	1	908	1	1107	1
113	0	312	0	511	1	710	0	909	1	1108	0
114	1	313	1	512	1	711	0	910	1	1109	1
115	0	314	0	513	1	712	0	911	1	1110	0
116	0	315	1	514	0	713	1	912	1	1111	0
117	1	316	0	515	1	714	0	913	1	1112	0
118	1	317	1	516	1	715	0	914	1	1113	1
119	0	318	1	517	0	716	1	915	1	1114	0
120	1	319	1	518	1	717	1	916	1	1115	1
121	1	320	0	519	0	718	0	917	0	1116	1
122	0	321	1	520	1	719	1	918	1	1117	0
123	1	322	1	521	1	720	0	919	0	1118	1
124	1	323	1	522	0	721	0	920	1	1119	0
125	0	324	1	523	0	722	1	921	0	1120	1
126	0	325	1	524	1	723	1	922	0	1121	0
127	0	326	0	525	1	724	0	923	1	1122	1
128	0	327	0	526	1	725	1	924	0	1123	0
129	0	328	1	527	0	726	0	925	1	1124	1
130	0	329	0	528	0	727	0	926	1	1125	1
131	1	330	0	529	0	728	1	927	1	1126	1
132	0	331	0	530	0	729	1	928	0	1127	1
133	1	332	1	531	0	730	0	929	0	1128	1
134	0	333	0	532	1	731	1	930	1	1129	0
135	0	334	0	533	1	732	0	931	1	1130	0
136	1	335	0	534	1	733	1	932	1	1131	0
137	1	336	0	535	0	734	0	933	0	1132	1
138	0	337	1	536	1	735	1	934	1	1133	1
139	1	338	1	537	1	736	0	935	1	1134	0
140	1	339	1	538	1	737	0	936	0	1135	0
141	0	340	0	539	0	738	1	937	1	1136	1
142	1	341	0	540	1	739	0	938	0	1137	1
143	1	342	1	541	0	740	0	939	0	1138	1
144	0	343	1	542	0	741	1	940	0	1139	0
145	0	344	0	543	1	742	1	941	0	1140	0
146	1	345	1	544	0	743	1	942	1	1141	0
147	0	346	1	545	0	744	0	943	0	1142	0
148	0	347	1	546	1	745	1	944	0	1143	0
149	1	348	0	547	1	746	1	945	1	1144	0
150	0	349	0	548	0	747	1	946	1	1145	1
151	1	350	1	549	0	748	0	947	1	1146	0

152	1	351	1	550	1	749	1	948	0	1147	1
153	0	352	1	551	0	750	1	949	0	1148	0
154	1	353	0	552	1	751	1	950	0	1149	1
155	1	354	0	553	0	752	1	951	0	1150	0
156	1	355	1	554	1	753	0	952	0	1151	1
157	1	356	1	555	1	754	1	953	0	1152	0
158	0	357	0	556	1	755	1	954	1	1153	0
159	1	358	0	557	1	756	0	955	0	1154	1
160	0	359	0	558	1	757	1	956	1	1155	0
161	1	360	1	559	0	758	1	957	0	1156	1
162	1	361	0	560	1	759	1	958	0	1157	0
163	1	362	1	561	0	760	1	959	1	1158	1
164	1	363	0	562	1	761	0	960	1	1159	1
165	0	364	0	563	1	762	0	961	1	1160	0
166	0	365	1	564	1	763	0	962	1	1161	1
167	1	366	0	565	0	764	0	963	0	1162	0
168	1	367	0	566	0	765	1	964	0	1163	1
169	1	368	1	567	1	766	1	965	1	1164	0
170	0	369	0	568	0	767	0	966	0	1165	1
171	0	370	0	569	0	768	1	967	0	1166	1
172	0	371	0	570	0	769	1	968	1	1167	1
173	1	372	1	571	0	770	0	969	1	1168	1
174	0	373	1	572	1	771	0	970	0	1169	1
175	0	374	0	573	1	772	1	971	1	1170	0
176	1	375	1	574	1	773	0	972	0	1171	1
177	1	376	0	575	1	774	1	973	1	1172	0
178	1	377	0	576	1	775	1	974	0	1173	0
179	0	378	1	577	1	776	0	975	1	1174	0
180	1	379	1	578	0	777	1	976	0	1175	0
181	0	380	0	579	0	778	1	977	0	1176	0
182	1	381	1	580	1	779	1	978	0	1177	1
183	0	382	1	581	1	780	0	979	0	1178	0
184	0	383	1	582	1	781	1	980	1	1179	0
185	1	384	1	583	0	782	0	981	0	1180	0
186	0	385	0	584	1	783	1	982	1	1181	0
187	0	386	1	585	1	784	0	983	1	1182	0
188	1	387	0	586	1	785	0	984	0	1183	0
189	0	388	1	587	0	786	1	985	0	1184	1
190	1	389	0	588	0	787	1	986	0	1185	0
191	0	390	0	589	1	788	0	987	0	1186	0
192	0	391	1	590	0	789	0	988	0	1187	0
193	1	392	1	591	1	790	0	989	1		
194	0	393	1	592	1	791	1	990	0		
195	1	394	1	593	1	792	0	991	0		

196	1	395	0	594	0	793	0	992	1	
197	0	396	1	595	0	794	1	993	1	
198	0	397	1	596	1	795	0	994	0	

Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: guoniu.han@unistra.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, PR China
Email address: huyining@hust.edu.cn