PROOF OF THE CONTINUED FRACTION CONJECTURE FOR $(a, b) = (z^5 + z^4 + z^3, z^2 + 1)$

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Abstract. The continued fraction conjecture claims that the continued fraction defined by the (a, b)-Thue-Morse sequence is algebraic over $\mathbb{F}_2(x)$ for all pairs of distinct elements (a, b) in $\mathbb{F}_2[x] \setminus \mathbb{F}_2$. In this paper we prove the conjecture for $(a, b) = (z^5 + z^4 + z^3, z^2 + 1)$.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of cardinality 2. Given a sequence of polynomials $\mathbf{a} = (a_0(z), a_1(z), a_2(z), \dots)$ of elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$, we define the infinite continued fraction

$$\mathrm{CF}(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}} \tag{1.1}$$

as the limit of the finite continued fractions

$$\mathrm{CF}_n(\mathbf{a}(z)) = \cfrac{1}{a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{\ddots + \cfrac{1}{a_n(z)}}}} \in \mathbb{F}_2((1/z)). \tag{1.2}$$

Conversely, each power series in $\mathbb{F}_2[[1/z]]$ without constant term

$$c_{-1}z^{-1} + c_{-2}z^{-2} + c_{-3}z^{-3} + \cdots \tag{1.3}$$

admits a continued fraction expansion of form (1.1).

Let a and b be two distinct elements from $\mathbb{F}_2[z] \setminus \mathbb{F}_2$. We consider the continued fraction (1.1) associated with the (a, b)-Thue-Morse sequence

$$\mathbf{t} = (t_0(z), t_1(z), t_2(z), \dots) = (a, b, b, a, b, a, a, b, \dots)$$

that we shall denote

$$\mathrm{CF}(a, b) := \mathrm{CF}(\mathbf{t}). \tag{1.4}$$

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Theorem 1.1. *When* $(a,b)=(z^5+z^4+z^3,z^2+1)$*, the two power series* $\mathrm{CF}(a,b)$ *and* $\mathrm{CF}(b,a)$ *are algebraic over* $\mathbb{F}_2(z)$*, with minimal polynomials of the form*

$$p_4(z)y^4+p_3(z)y^3+p_2(z)y^2+p_1(z)y+p_0(z)=0$$

where, for $\mathrm{CF}(a,b)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{20}+z^{19}+z^{15}+z^{13}+z^{12}+z^{11}+z^{10}+z^8+z^4+z^3+1,\\
p_1(z) &= z^{26}+z^{24}+z^{23}+z^{19}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^6+z^5+z^4\\
&\quad +z^3,\\
p_2(z) &= z^{28}+z^{18}+z^{14}+z^{10}+z^8+z^6,\\
p_3(z) &= z^{26}+z^{24}+z^{23}+z^{19}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^6+z^5+z^4\\
&\quad +z^3,\\
p_4(z) &= z^{24}+z^{21}+z^{19}+z^{18}+z^{15}+z^{14}+z^{13}+z^{12}+z^{11}+z^8+z^6+z^4+z^3,
\end{aligned}$$

and for $\mathrm{CF}(b,a)$*,*

$$\begin{aligned}
p_0(z) &= z^{21}+z^{19}+z^{16}+z^{15}+z^{13}+z^{11}+z^{10}+z^4+z^3,\\
p_1(z) &= z^{26}+z^{24}+z^{23}+z^{19}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^6+z^5+z^4\\
&\quad +z^3,\\
p_2(z) &= z^{28}+z^{18}+z^{14}+z^{10}+z^8+z^6,\\
p_3(z) &= z^{26}+z^{24}+z^{23}+z^{19}+z^{18}+z^{17}+z^{16}+z^{14}+z^{11}+z^{10}+z^6+z^5+z^4\\
&\quad +z^3,\\
p_4(z) &= z^{24}+z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{15}+z^{14}+z^{13}+z^{11}+z^6+z^4+z^3\\
&\quad +1.
\end{aligned}$$

2. Proof

Define the sequences of polynomials $(P_n(z))_{n\ge 0}$ and $(Q_n(z))_{n\ge 0}$ by

$$\begin{pmatrix} P_n(z) & Q_n(z)\\ P_{n-1}(z) & Q_{n-1}(z)\end{pmatrix} := \begin{pmatrix} t_n(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_{n-1}(z) & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} t_0(z) & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By the basic properties of continued fractions, we have

$$\mathrm{CF}_n(a,b)=\frac{P_n(z)}{Q_n(z)}\in\mathbb{F}_2((1/z)). \tag{2.1}$$

As formula (2.1) is not efficient for calculating $\mathrm{CF}_n(a,b)$, a fast method is to be derived and described as follows. Define

$$\begin{aligned}
M_n(x) &= x^{\deg(t_{2^n-1})}\begin{pmatrix} t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(t_{2^n-2})}\begin{pmatrix} t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(t_0)}\begin{pmatrix} t_0(1/x) & 1\\ 1 & 0\end{pmatrix},\\
W_n(x) &= x^{\deg(\bar t_{2^n-1})}\begin{pmatrix} \bar t_{2^n-1}(1/x) & 1\\ 1 & 0\end{pmatrix}\times x^{\deg(\bar t_{2^n-2})}\begin{pmatrix} \bar t_{2^n-2}(1/x) & 1\\ 1 & 0\end{pmatrix}\times\\
&\qquad\cdots\times x^{\deg(\bar t_0)}\begin{pmatrix} \bar t_0(1/x) & 1\\ 1 & 0\end{pmatrix},
\end{aligned}$$

where $\bar{\mathbf{t}}$ is the (b,a)-Thue-Morse sequence. By the properties of the Thue-Morse sequence, we have for $n \geq 0$

$$M_{n+1}(x) = W_n(x) \cdot M_n(x),$$
$$W_{n+1}(x) = M_n(x) \cdot W_n(x).$$

Let $x := 1/z$. For each non-zero polynomial $P(z)$, define $\tilde{P}(x)$ to be $P(1/x)$. Then,

$$\mathrm{CF}_n(a,b) = \frac{P_n(z)}{Q_n(z)} = \frac{\tilde{P}_n(x)}{\tilde{Q}_n(x)} \in \mathbb{F}_2((x)) = \mathbb{F}_2((1/z)).$$

Comparing the definition of $M_n(x)$ with the definition of $P_n(x)$ and $Q_n(x)$, we see that

$$M_n(x)_{0,1} = x^{d_n}\tilde{P}_{2^n-1}(x),$$
$$M_n(x)_{0,0} = x^{d_n}\tilde{Q}_{2^n-1}(x),$$

for some positive integer d_n. Hence,

$$\mathrm{CF}_{2^{2n}-1}(a,b) = \frac{\tilde{P}_{2^{2n}-1}(x)}{\tilde{Q}_{2^{2n}-1}(x)} = \frac{M_{2n}(x)_{0,1}}{M_{2n}(x)_{0,0}}. \tag{2.2}$$

Therefore, by the convergence theorem of continued fraction, the algebraicity of $\mathrm{CF}(a,b)$ will be established if it is shown that both $M_{2n}(x)_{0,1}$ and $M_{2n}(x)_{0,0}$ converge to algebraic series in $\mathbb{F}_2[[x]]$.

Actually, we will prove that for all $0 \leq i,j \leq 1$ the four sequences $(M_{2n}(x)_{i,j})_n$, $(M_{2n+1}(x)_{i,j})_n$, $(W_{2n}(x)_{i,j})_n$, and $(W_{2n+1}(x)_{i,j})_n$ converge to algebraic series in $\mathbb{F}_2[[x]]$. For this purpose, we define four 2×2 matrices M^e, M^o, W^e, W^o as follows: For each $T \in \{M^e, M^o, W^e, W^o\}$ and $0 \leq i,j \leq 1$, $T_{i,j}$ is defined to be the unique solution in $\mathbb{F}_2[[x]]$ of the polynomial $\phi(T,i,j)$ under certain initial conditions; the polynomials $\phi(T,i,j)$ and initial conditions are given in Section 3. We will prove that these four matrices, whose components are algebraic by definition, are the limits of $(M_{2n}(x))_n$, $(M_{2n+1}(x))_n$, $(W_{2n}(x))_n$, and $(W_{2n+1}(x))_n$.

Let us explain how the polynomials $\phi(T,i,j)$ and initial conditions are found, and why the solutions exist and are unique. For $0 \leq i,j \leq 1$, the the coefficients of the polynomial $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$) are the Padé-Hermite approximants of type

$$(175, 175, 175, 175, 175)$$

of the vector

$$(1, T^3, T^6, T^9, T^{12}),$$

where $T = M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$), found by the Derksen algorithm. We take the first 8 terms of $M_{12,i,j}$ (resp. $M_{11,i,j}$, $W_{12,i,j}$, and $W_{11,i,j}$) as initial conditions for $\phi(M^e,i,j)$ (resp. $\phi(M^o,i,j)$, $\phi(W^e,i,j)$, and $\phi(W^o,i,j)$). The following fact will be used to ensure that the solution exists and is unique: let $P(x,y) \in \mathbb{F}_2[x,y]$ and for each series $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{F}_2[[x]]$ denote the partial sum $\sum_{j=0}^{n-1} a_j x^j$ by $f_n(x)$ for $n \geq 0$. If for some $n \geq 0$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_2$ $P(x, \sum_{j=0}^{n-1} a_j x^j) = O(x^n)$ and $Q(x,y) := P(x, \sum_{j=0}^{n-1} a_j x^j + x^n y)$ can be written as $x^m \sum_{j=0}^{\infty} q_j(x) y^j$ where $q_j(x)$ are polynomials for $j \geq 0$, $q_1(0) = 1$, and $q_j(0) = 0$ for $j > 1$, then there exists a unique solution $f(x) \in \mathbb{F}_2[[x]]$ of $P(x, f(x)) = 0$ that satisfies the initial condition $f_n(x) = \sum_{j=0}^{n-1} a_j x^j$.

We state two lemmas concerning the four matrices M^e, M^o, W^e, W^o. The first one is about relations between them; the second, about the structure of the each matrix.

Lemma 2.1. *We have*

$$M^e = W^o \cdot M^o, \tag{2.3}$$

$$M^o = W^e \cdot M^e, \tag{2.4}$$

$$W^e = M^o \cdot W^o, \tag{2.5}$$

$$W^o = M^e \cdot W^e. \tag{2.6}$$

Proof. We give the proof of the identity

$$M^e_{0,0} = W^o_{0,0} M^o_{0,0} + W^o_{0,1} M^o_{1,0},$$

the proofs of the others being similar.

First, we compute the minimal polynomials of $W^o_{0,0}M^o_{0,0}$ and $W^o_{0,1}M^o_{1,0}$. We know that

$$P(x,y) = \mathrm{Res}_z\left(\phi(W^o,0,0)(x,z),\ z^{12} \cdot \phi(M^o,0,0)(x,y/z)\right)$$

is an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We use Padé-Hermite approximation to find a candidate for the minimal polynomial of $W^o_{0,0}M^o_{0,0}$, that will be called $\phi_0(x,y)$. To prove that $\phi_0(x,y)$ is indeed the minimal polynomial, it suffices to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and that $Q(x,y) := P(x,y)/\phi_0(x,y)^m$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We verify the first point directly. For the second point, we truncate $W^o_{0,0}M^o_{0,0}$ to order 630 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 630, which proves that $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}$. We find the minimal polynomial $\phi_1(x,y)$ of $W^o_{0,1}M^o_{1,0}$ in a similar way.

Now we prove that $\phi(M^e,0,0)$ is the minimal polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. We know that

$$S(x,y) = \mathrm{Res}_z\left(\phi_0(x,z),\ \phi_1(x,y+z)\right)$$

is an annihilating of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$. We verify that $\phi(M^e,0,0)$ is an irreducible factor of $S(x,y)$ of multiplicity μ, and that $Q(x,y) := S(x,y)/\phi(M^e,0,0)^\mu$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0}+W^o_{0,1}M^o_{1,0}$. To see the last point, we truncate $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ to order 770 and substitute it for y in $Q(x,y)$. We get a series of valuation less than 770, and therefore $Q(x,y)$ is not an annihilating polynomial of $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$.

Finally, the first 16 terms of $M^e_{0,0}$ and $W^o_{0,0}M^o_{0,0} + W^o_{0,1}M^o_{1,0}$ coincide. As these first terms determine a unique solution of $\phi(M^e,0,0)$, we know that the two series are one and the same. ☐

Define

$$R^e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.2. *For all integers $k \geq 2$ and $u = 2^{2k-1}$, the following identities hold.*

$$\begin{aligned} M^e[:14u] = {} & M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^u M^e[:7u] \\ & + x^{2u} M^e[:6u] + x^{4u} M^e[:4u] + x^{6u} M^e[:2u] + x^{2u} M^e[:7u] \\ & + x^{3u} M^e[:6u] + x^{5u} M^e[:4u] + x^{7u} M^e[:2u] + x^{4u} M^e[:7u] \\ & + x^{5u} M^e[:6u] + x^{7u} M^e[:4u] + x^{9u} M^e[:2u] + x^{5u} M^e[:7u] \end{aligned}$$

$$
\begin{aligned}
&+ x^{6u}M^e[:6u] + x^{8u}M^e[:4u] + x^{10u}M^e[:2u] + x^{7u}M^e[:7u] \\
&+ x^{8u}M^e[:6u] + x^{9u}M^e[:5u] + x^{10u}M^e[:4u] + x^{11u}M^e[:3u] \\
&+ x^{12u}M^e[:2u] + x^{13u}M^e[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^e, \\
W^e[:14u] &= W^e[:7u] + x^uW^e[:6u] + x^{3u}W^e[:4u] + x^{5u}W^e[:2u] + x^uW^e[:7u] \\
&+ x^{2u}W^e[:6u] + x^{4u}W^e[:4u] + x^{6u}W^e[:2u] + x^{2u}W^e[:7u] \\
&+ x^{3u}W^e[:6u] + x^{5u}W^e[:4u] + x^{7u}W^e[:2u] + x^{4u}W^e[:7u] \\
&+ x^{5u}W^e[:6u] + x^{7u}W^e[:4u] + x^{9u}W^e[:2u] + x^{5u}W^e[:7u] \\
&+ x^{6u}W^e[:6u] + x^{8u}W^e[:4u] + x^{10u}W^e[:2u] + x^{7u}W^e[:7u] \\
&+ x^{8u}W^e[:6u] + x^{9u}W^e[:5u] + x^{10u}W^e[:4u] + x^{11u}W^e[:3u] \\
&+ x^{12u}W^e[:2u] + x^{13u}W^e[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^e.
\end{aligned}
$$

For all integers $k \geq 2$ and $u = 2^{2k}$, the following identities hold.

$$
\begin{aligned}
M^o[:14u] &= M^o[:7u] + x^uM^o[:6u] + x^{3u}M^o[:4u] + x^{5u}M^o[:2u] + x^uM^o[:7u] \\
&+ x^{2u}M^o[:6u] + x^{4u}M^o[:4u] + x^{6u}M^o[:2u] + x^{2u}M^o[:7u] \\
&+ x^{3u}M^o[:6u] + x^{5u}M^o[:4u] + x^{7u}M^o[:2u] + x^{4u}M^o[:7u] \\
&+ x^{5u}M^o[:6u] + x^{7u}M^o[:4u] + x^{9u}M^o[:2u] + x^{5u}M^o[:7u] \\
&+ x^{6u}M^o[:6u] + x^{8u}M^o[:4u] + x^{10u}M^o[:2u] + x^{7u}M^o[:7u] \\
&+ x^{8u}M^o[:6u] + x^{9u}M^o[:5u] + x^{10u}M^o[:4u] + x^{11u}M^o[:3u] \\
&+ x^{12u}M^o[:2u] + x^{13u}M^o[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^o, \\
W^o[:14u] &= W^o[:7u] + x^uW^o[:6u] + x^{3u}W^o[:4u] + x^{5u}W^o[:2u] + x^uW^o[:7u] \\
&+ x^{2u}W^o[:6u] + x^{4u}W^o[:4u] + x^{6u}W^o[:2u] + x^{2u}W^o[:7u] \\
&+ x^{3u}W^o[:6u] + x^{5u}W^o[:4u] + x^{7u}W^o[:2u] + x^{4u}W^o[:7u] \\
&+ x^{5u}W^o[:6u] + x^{7u}W^o[:4u] + x^{9u}W^o[:2u] + x^{5u}W^o[:7u] \\
&+ x^{6u}W^o[:6u] + x^{8u}W^o[:4u] + x^{10u}W^o[:2u] + x^{7u}W^o[:7u] \\
&+ x^{8u}W^o[:6u] + x^{9u}W^o[:5u] + x^{10u}W^o[:4u] + x^{11u}W^o[:3u] \\
&+ x^{12u}W^o[:2u] + x^{13u}W^o[:u] \\
&+ (x^{7u} + x^{8u} + x^{9u} + x^{11u} + x^{12u})R^o.
\end{aligned}
$$

Proof. To prove Lemma 2.2 we first construct an automaton for each sequence concerned, and then transform the conditions on infinitely many k's into finitely many conditions on the states of the automaton. In the following, we will prove that for $T = M^o_{0,0}$, for all integer $k \geq 2$ and $u = 2^{2k}$,

$$
\begin{aligned}
T[:14u] &= T[:7u] + x^uT[:6u] + x^{3u}T[:4u] + x^{5u}T[:2u] + x^uT[:7u] + x^{2u}T[:6u] \\
&+ x^{4u}T[:4u] + x^{6u}T[:2u] + x^{2u}T[:7u] + x^{3u}T[:6u] + x^{5u}T[:4u] \\
&+ x^{7u}T[:2u] + x^{4u}T[:7u] + x^{5u}T[:6u] + x^{7u}T[:4u] + x^{9u}T[:2u]
\end{aligned}
$$

$$+ x^{5u}T[:7u] + x^{6u}T[:6u] + x^{8u}T[:4u] + x^{10u}T[:2u] + x^{7u}T[:7u] \\ + x^{8u}T[:6u] + x^{9u}T[:5u] + x^{10u}T[:4u] + x^{11u}T[:3u] + x^{12u}T[:2u] \\ + x^{13u}T[:u].$$

The proof of the other 15 cases are similar. We break down the above identity into 7 parts:

$$\begin{aligned}
0 &= x^{7u}T[:u] + x^{5u}T[2u:3u] + x^{3u}T[4u:5u] + x^{u}T[6u:7u] + T[7u:8u],\\
0 &= x^{7u}T[u:2u] + x^{6u}T[2u:3u] + x^{5u}T[3u:4u] + x^{4u}T[4u:5u]\\
&\quad + x^{3u}T[5u:6u] + x^{2u}T[6u:7u] + T[8u:9u],\\
0 &= x^{6u}T[3u:4u] + x^{4u}T[5u:6u] + T[9u:10u],\\
0 &= x^{6u}T[4u:5u] + x^{4u}T[6u:7u] + T[10u:11u],\\
0 &= x^{11u}T[:u] + x^{9u}T[2u:3u] + x^{7u}T[4u:5u] + x^{6u}T[5u:6u]\\
&\quad + x^{5u}T[6u:7u] + T[11u:12u],\\
0 &= x^{12u}T[:u] + x^{11u}T[u:2u] + x^{10u}T[2u:3u] + x^{9u}T[3u:4u]\\
&\quad + x^{8u}T[4u:5u] + x^{7u}T[5u:6u] + T[12u:13u],\\
0 &= x^{13u}T[:u] + x^{12u}T[u:2u] + x^{11u}T[2u:3u] + x^{10u}T[3u:4u]\\
&\quad + x^{9u}T[4u:5u] + x^{8u}T[5u:6u] + x^{7u}T[6u:7u] + T[13u:14u],
\end{aligned}$$

which can be rewritten as

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.7}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\ + T[[110w]_2] + T[[1000w]_2], \tag{2.8}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.9}$$

$$0 = T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.10}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ + T[[1011w]_2], \tag{2.11}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ + T[[101w]_2] + T[[1100w]_2], \tag{2.12}$$

$$0 = T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2], \tag{2.13}$$

for all binary word w of length $2k$ and $w \neq 0^{2k}$, and

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[110w]_2] + T[[111w]_2], \tag{2.14}$$

$$0 = T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] + T[[101w]_2] \\ + T[[110w]_2] + T[[1000w]_2], \tag{2.15}$$

$$0 = T[[11w]_2] + T[[101w]_2] + T[[1001w]_2], \tag{2.16}$$

$$0 = T[[100w]_2] + T[[110w]_2] + T[[1010w]_2], \tag{2.17}$$

$$0 = T[[w]_2] + T[[10w]_2] + T[[100w]_2] + T[[101w]_2] + T[[110w]_2] \\ + T[[1011w]_2], \tag{2.18}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[101w]_2] + T[[1100w]_2],\end{aligned} \tag{2.19}$$

$$\begin{aligned}0 &= T[[w]_2] + T[[1w]_2] + T[[10w]_2] + T[[11w]_2] + T[[100w]_2] \\ &\quad + T[[101w]_2] + T[[110w]_2] + T[[1101w]_2],\end{aligned} \tag{2.20}$$

for $w = 0^{2k}$.

First we calculate an 2-automaton that generates T from its minimal polynomial and its first terms. This automaton has 956 states; its transition function and output function can be found in the annex. Let $A(s, w)$ denote the state reached after reading w from right to left starting from the state s, and τ the output function. Define

$$E_{2k} = \{A(i, w) : |w| = 2k, w \neq 0^{2k}\}.$$

Identities (2.7) through (2.20) can be written as

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)),\end{aligned} \tag{2.21}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)),\end{aligned} \tag{2.22}$$

$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.23}$$

$$0 = \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1010)), \tag{2.24}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.25}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1100)),\end{aligned} \tag{2.26}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)),\end{aligned} \tag{2.27}$$

for all $s \in E_{2k}$, and

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,110)) \\ &\quad + \tau(A(s,111)),\end{aligned} \tag{2.28}$$

$$\begin{aligned}0 &= \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1000)),\end{aligned} \tag{2.29}$$

$$0 = \tau(A(s,11)) + \tau(A(s,101)) + \tau(A(s,1001)), \tag{2.30}$$

$$0 = \tau(A(s,100)) + \tau(A(s,110)) + \tau(A(s,1010)), \tag{2.31}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,10)) + \tau(A(s,100)) + \tau(A(s,101)) \\ &\quad + \tau(A(s,110)) + \tau(A(s,1011)),\end{aligned} \tag{2.32}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,1100)),\end{aligned} \tag{2.33}$$

$$\begin{aligned}0 &= \tau(A(s,\epsilon)) + \tau(A(s,1)) + \tau(A(s,10)) + \tau(A(s,11)) + \tau(A(s,100)) \\ &\quad + \tau(A(s,101)) + \tau(A(s,110)) + \tau(A(s,1101)),\end{aligned} \tag{2.34}$$

for $s = A(i, 0^{2k})$. We find that $(A(i, 0^{18}), E_{18}) = (A(i, 0^{16}), E_{16})$, so that we only have to verify that identities (2.21) through (2.34) hold for $2 \le k \le 9$, which turns out to be true. $\square$

In the following lemma, we express M_{2k}, M_{2k+1}, W_{2k}, and W_{2k+1} in terms of M^e, M^o, W^e, and W^o.

Lemma 2.3. *For all integer $k \ge 2$, and $u = 2^{2k-1}$,*

$$M_{2k} = M^e[:7u] + x^u M^e[:6u] + x^{3u} M^e[:4u] + x^{5u} M^e[:2u] + x^{7u} R^e,$$
$$W_{2k} = W^e[:7u] + x^u W^e[:6u] + x^{3u} W^e[:4u] + x^{5u} W^e[:2u] + x^{7u} R^e.$$

For all integer $k \ge 2$, and $u = 2^{2k}$,

$$M_{2k+1} = M^o[:7u] + x^u M^o[:6u] + x^{3u} M^o[:4u] + x^{5u} M^o[:2u] + x^{7u} R^o,$$
$$W_{2k+1} = W^o[:7u] + x^u W^o[:6u] + x^{3u} W^o[:4u] + x^{5u} W^o[:2u] + x^{7u} R^o.$$

Proof. We call the four identities in Lemma 2.3 also by the name M_{2k}, W_{2k}, M_{2k+1}, and W_{2k+1}. For $n = 2$, the identities can be verified directly. For $n \ge 2$, we claim that

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1} \wedge W_{2k+1},$$
$$M_{2k+1} \wedge W_{2k+1} \Rightarrow M_{2k+2} \wedge W_{2k+2}.$$

We give the proof of

$$M_{2k} \wedge W_{2k} \Rightarrow M_{2k+1},$$

the other ones being similar. Set $u = 2^{2k}$ and $v = 2^{2k-1}$. By definition and induction hypothesis, the left side of identity M_{2k+1} is equal to

$$W_{2k}M_{2k} = \big(W^e[:7v] + x^v W^e[:6v] + x^{3v} W^e[:4v] + x^{5v} W^e[:2v] + x^{7u} R^e\big) \times \big($$
$$M^e[:7v] + x^v M^e[:6v] + x^{3v} M^e[:4v] + x^{5v} M^e[:2v] + x^{7u} R^e\big); \tag{2.35}$$

Call this expression lhs. Note that both sides of identity M_{2k+1} have the same term of highest degree $x^{14v} R^o$. Therefore we only need to prove that their difference is $O(x^{14v})$. Using Lemma 2.1 it can be seen that the right side of identity M_{2k+1} is congruent, modulo x^{14v}, to

$$W^e[:14v]M^e[:14v] + x^{2v} W^e[:12v]M^e[:12v] + x^{6v} W^e[:8v]M^e[:8v]$$
$$+ x^{10v} W^e[:4v]M^e[:4v].$$

For all $n \le 14$, replace the occurrences of $W^e[:n \cdot v]$ and $M^e[:n \cdot v]$ in the above expression by the reduction modulo $x^{n \cdot v}$ of the right side of the corresponding identity in Lemma 2.2 and get a new expression, which we call rhs. Define

$$X := x^v,$$
$$a_n := W^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$b_n := M^e[n \cdot v : (n+1) \cdot v]/X^n,$$
$$c := R^e.$$

Using the notation introduced above, we can represent the expressions lhs (2.35) and rhs as polynomials in $\mathbb{F}_2[a_1, ..., a_{14}, b_1, ..., b_{14}, c][X]$. Note that it is not a problem that a_j commutes with b_k while W^e does not commute with M^e, because in the expressions concerned, the products of W^e-terms and M^e-terms are always in the same order. We let the computer do the simplification and check that the difference between these two polynomials is indeed $O(X^{14})$, which completes the proof. □

Proof of Theorem 1.1. We prove the theorem for $\mathrm{CF}(a,b)$; for $\mathrm{CF}(b,a)$, the proof is similar. By Lemma 2.3, we have For all $0 \le j,k \le 1$,

$$\lim_{n\to\infty} M_{2n,j,k} = M^e_{j,k},$$
$$\lim_{n\to\infty} M_{2n+1,j,k} = M^o_{j,k},$$
$$\lim_{n\to\infty} W_{2n,j,k} = W^e_{j,k},$$
$$\lim_{n\to\infty} W_{2n+1,j,k} = W^o_{j,k}.$$

Let $z = 1/x$. By the convergence theorem and identity (2.2),

$$\mathrm{CF}(a,b) = \frac{M^e_{0,1}(x)}{M^e_{0,0}(x)}.$$

By definition, $\phi(M^e,0,1)$ and $\phi(M^e,0,0)$ are minimal polynomials of $M^e_{0,1}$ and $M^e_{0,0}$. Therefore

$$P(x,y) = \mathrm{Res}_t\left(\phi(M^e,0,1)(x,t), y^{12}\phi(M^e,0,1)(x,t/y)\right)$$

is an annihilating polynomial of $f(x) = M^e_{0,1}/M^e_{0,0}$.

Define

$$Q(x,y) = q_4(x)y^4 + q_3(x)y^3 + q_2(x)y^2 + q_1(x)y + q_0(x),$$

where

$$\begin{aligned}
q_0(x) &= x^{28} + x^{25} + x^{24} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^9 + x^8 + x^7,\\
q_1(x) &= x^{25} + x^{24} + x^{23} + x^{22} + x^{18} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^5\\
&\quad + x^4 + x^2,\\
q_2(x) &= x^{22} + x^{20} + x^{18} + x^{14} + x^{10} + 1,\\
q_3(x) &= x^{25} + x^{24} + x^{23} + x^{22} + x^{18} + x^{17} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^5\\
&\quad + x^4 + x^2,\\
q_4(x) &= x^{25} + x^{24} + x^{22} + x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{10} + x^9 + x^7\\
&\quad + x^4.
\end{aligned}$$

The polynomial $Q(x,y)$ is the candidate for the minimal polynomial of $f(x)$ found by Padé-Hermite approximation. To prove that it is indeed the minimal polynomial of $f(x)$, we only need to prove that it is an irreducible factor of $P(x,y)$ of multiplicity m and $R(x,y) := P(x,y)/Q(x,y)^m$ is not an annihilating polynomial of $f(x)$. We verify the first point directly. For the second point, we find that when we truncate $f(x)$ to order 216, and substitute it for y in $R(z,y)$, we get a series with valuation smaller than 216, which proves that $R(z,y)$ is not an annihilating polynomial of $f(x)$. Finally, $z^{12}Q(1/z,y)$ is the minimal polynomial of $\mathrm{CF}(a,b) = f(1/z)$. □

3. Annexe

All 16 polynomials are of the form

$$p_0(x) + p_3(x)y^3 + p_6(x)y^6 + p_9(x)y^9 + p_{12}(x)y^{12}.$$

The coefficients $p_j(x)$ and the 16 initial terms to determine the solutions uniquely are given below.

For $\phi(M^e,0,0)$:

$$\begin{aligned}
p_0(x) &= x^{138}+x^{134}+x^{124}+x^{122}+x^{120}+x^{118}+x^{112}+x^{110}+x^{106}+x^{104}\\
&\quad +x^{102}+x^{100}+x^{90}+x^{88}+x^{82}+x^{80}+x^{78}+x^{72}+x^{70}+x^{68}+x^{66}\\
&\quad +x^{62}+x^{56}+x^{46}+x^{44}+x^{40}+x^{34}+x^{30}+x^{28}+x^{26}+x^{24},\\
p_3(x) &= x^{128}+x^{126}+x^{122}+x^{120}+x^{116}+x^{110}+x^{106}+x^{102}+x^{100}+x^{96}\\
&\quad +x^{94}+x^{88}+x^{84}+x^{78}+x^{76}+x^{74}+x^{72}+x^{70}+x^{64}+x^{60}+x^{56}\\
&\quad +x^{44}+x^{42}+x^{40}+x^{36}+x^{34}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{16},\\
p_6(x) &= x^{128}+x^{122}+x^{120}+x^{118}+x^{116}+x^{112}+x^{106}+x^{102}+x^{100}+x^{96}\\
&\quad +x^{90}+x^{86}+x^{80}+x^{78}+x^{76}+x^{72}+x^{70}+x^{60}+x^{58}+x^{54}+x^{52}\\
&\quad +x^{50}+x^{42}+x^{38}+x^{32}+x^{28}+x^{26}+x^{24}+x^{20}+x^{18}+x^{12}+x^{10}\\
&\quad +x^{8}+x^{6}+x^{2}+1,\\
p_9(x) &= x^{132}+x^{130}+x^{128}+x^{126}+x^{122}+x^{118}+x^{110}+x^{108}+x^{106}+x^{98}\\
&\quad +x^{96}+x^{94}+x^{92}+x^{86}+x^{82}+x^{68}+x^{64}+x^{60}+x^{58}+x^{54}+x^{44}\\
&\quad +x^{38}+x^{32}+x^{30}+x^{24}+x^{22}+x^{20}+x^{18}+x^{10}+x^{8}+x^{6}+x^{4},\\
p_{12}(x) &= x^{132}+x^{130}+x^{128}+x^{116}+x^{114}+x^{112}+x^{100}+x^{98}+x^{96}+x^{84}\\
&\quad +x^{82}+x^{80}+x^{52}+x^{50}+x^{48}+x^{4}+x^{2}+1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^e,0,1)$:

$$\begin{aligned}
p_0(x) &= x^{147}+x^{144}+x^{141}+x^{139}+x^{138}+x^{133}+x^{131}+x^{128}+x^{125}+x^{124}\\
&\quad +x^{123}+x^{121}+x^{118}+x^{116}+x^{112}+x^{111}+x^{110}+x^{109}+x^{108}\\
&\quad +x^{107}+x^{106}+x^{103}+x^{101}+x^{99}+x^{98}+x^{97}+x^{96}+x^{95}+x^{92}\\
&\quad +x^{91}+x^{88}+x^{87}+x^{85}+x^{82}+x^{81}+x^{78}+x^{77}+x^{76}+x^{74}+x^{72}\\
&\quad +x^{71}+x^{70}+x^{69}+x^{68}+x^{66}+x^{65}+x^{64}+x^{63}+x^{62}+x^{60}+x^{59}\\
&\quad +x^{57}+x^{55}+x^{50}+x^{49}+x^{48}+x^{46}+x^{42}+x^{41}+x^{39}+x^{36}+x^{35}\\
&\quad +x^{33},\\
p_3(x) &= x^{141}+x^{138}+x^{137}+x^{132}+x^{131}+x^{130}+x^{128}+x^{127}+x^{126}+x^{125}\\
&\quad +x^{124}+x^{119}+x^{116}+x^{115}+x^{113}+x^{108}+x^{107}+x^{106}+x^{104}\\
&\quad +x^{103}+x^{102}+x^{100}+x^{98}+x^{97}+x^{95}+x^{93}+x^{88}+x^{87}+x^{86}+x^{84}\\
&\quad +x^{82}+x^{81}+x^{79}+x^{77}+x^{72}+x^{71}+x^{70}+x^{68}+x^{66}+x^{65}+x^{63}\\
&\quad +x^{58}+x^{57}+x^{56}+x^{55}+x^{54}+x^{53}+x^{51}+x^{50}+x^{48}+x^{46}+x^{45}\\
&\quad +x^{43}+x^{41}+x^{39}+x^{37}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{23}+x^{21}\\
&\quad +x^{20}+x^{19}+x^{18},\\
p_6(x) &= x^{138}+x^{137}+x^{136}+x^{126}+x^{125}+x^{123}+x^{121}+x^{119}+x^{118}+x^{112}\\
&\quad +x^{111}+x^{110}+x^{106}+x^{105}+x^{104}+x^{102}+x^{101}+x^{99}+x^{97}+x^{96}
\end{aligned}$$

$$+ x^{94} + x^{93} + x^{92} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{70} + x^{69} + x^{68} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} + x^{28} + x^{27} + x^{25} + x^{24} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12},$$

$$p_9(x) = x^{135} + x^{133} + x^{132} + x^{130} + x^{128} + x^{127} + x^{123} + x^{122} + x^{121} + x^{119} + x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} + x^{50} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} + x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{10} + x^9 + x^8 + x^7 + x^6,$$

$$p_{12}(x) = x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{115} + x^{113} + x^{112} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} + x^{15} + x^{13} + x^{12} + x^8 + x^7 + x^5 + x^3 + x + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0].

For $\phi(M^e, 1, 0)$:

$$p_0(x) = x^{147} + x^{144} + x^{141} + x^{139} + x^{138} + x^{133} + x^{131} + x^{128} + x^{125} + x^{124} + x^{123} + x^{121} + x^{118} + x^{116} + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{106} + x^{103} + x^{101} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{88} + x^{87} + x^{85} + x^{82} + x^{81} + x^{78} + x^{77} + x^{76} + x^{74} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{60} + x^{59} + x^{57} + x^{55} + x^{50} + x^{49} + x^{48} + x^{46} + x^{42} + x^{41} + x^{39} + x^{36} + x^{35} + x^{33},$$

$$p_3(x) = x^{141} + x^{138} + x^{137} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{125} + x^{124} + x^{119} + x^{116} + x^{115} + x^{113} + x^{108} + x^{107} + x^{106} + x^{104} + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{88} + x^{87} + x^{86} + x^{84} + x^{82} + x^{81} + x^{79} + x^{77} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{37} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{23} + x^{21} + x^{20} + x^{19} + x^{18},$$

$$p_6(x) = x^{138} + x^{137} + x^{136} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{118} + x^{112} + x^{111} + x^{110} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96}$$

$$
\begin{aligned}
&+ x^{94} + x^{93} + x^{92} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{70} + x^{69}\\
&+ x^{68} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47}\\
&+ x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{135} + x^{133} + x^{132} + x^{130} + x^{128} + x^{127} + x^{123} + x^{122} + x^{121} + x^{119}\\
&+ x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{103} + x^{102}\\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{115}\\
&+ x^{113} + x^{112} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85}\\
&+ x^{83} + x^{81} + x^{80} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0].

For $\phi(M^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{156} + x^{152} + x^{150} + x^{148} + x^{146} + x^{138} + x^{136} + x^{132} + x^{128} + x^{122}\\
&+ x^{118} + x^{116} + x^{114} + x^{110} + x^{106} + x^{104} + x^{102} + x^{100} + x^{92} + x^{78}\\
&+ x^{76} + x^{72} + x^{70} + x^{64} + x^{58} + x^{56} + x^{46} + x^{44} + x^{42},\\
p_3(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{124} + x^{122} + x^{114} + x^{112} + x^{110}\\
&+ x^{108} + x^{100} + x^{92} + x^{88} + x^{76} + x^{72} + x^{68} + x^{62} + x^{60} + x^{58} + x^{56}\\
&+ x^{52} + x^{50} + x^{46} + x^{40} + x^{32} + x^{28} + x^{26} + x^{24} + x^{20} + x^{16} + x^{12},\\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{126} + x^{124} + x^{118} + x^{114} + x^{112} + x^{110} + x^{108}\\
&+ x^{106} + x^{90} + x^{88} + x^{86} + x^{82} + x^{78} + x^{76} + x^{74} + x^{72} + x^{64} + x^{56}\\
&+ x^{54} + x^{50} + x^{44} + x^{42} + x^{36} + x^{32} + x^{30} + x^{28} + x^{26} + x^{16} + x^{10}\\
&+ x^{8} + 1,\\
p_9(x) &= x^{132} + x^{128} + x^{126} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96}\\
&+ x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{60} + x^{58} + x^{54}\\
&+ x^{52} + x^{50} + x^{48} + x^{38} + x^{32} + x^{30} + x^{28} + x^{18} + x^{16} + x^{14} + x^{10}\\
&+ x^{8} + x^{4},\\
p_{12}(x) &= x^{132} + x^{128} + x^{124} + x^{120} + x^{116} + x^{112} + x^{100} + x^{96} + x^{92} + x^{88}\\
&+ x^{84} + x^{80} + x^{52} + x^{48} + x^{44} + x^{40} + x^{28} + x^{24} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 0)$:

$$\begin{aligned}
p_0(x) &= x^{156} + x^{152} + x^{150} + x^{148} + x^{146} + x^{144} + x^{138} + x^{136} + x^{132} + x^{126}\\
&\quad + x^{120} + x^{116} + x^{112} + x^{106} + x^{104} + x^{98} + x^{96} + x^{92} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{80} + x^{78} + x^{70} + x^{64} + x^{62} + x^{56} + x^{54} + x^{52} + x^{48} + x^{40}\\
&\quad + x^{38} + x^{36} + x^{32} + x^{30} + x^{28} + x^{24},\\
p_3(x) &= x^{138} + x^{136} + x^{134} + x^{132} + x^{128} + x^{126} + x^{122} + x^{112} + x^{104} + x^{90}\\
&\quad + x^{86} + x^{82} + x^{80} + x^{76} + x^{74} + x^{72} + x^{70} + x^{68} + x^{66} + x^{50} + x^{48}\\
&\quad + x^{46} + x^{38} + x^{36} + x^{30} + x^{22} + x^{18} + x^{16},\\
p_6(x) &= x^{138} + x^{136} + x^{134} + x^{120} + x^{118} + x^{106} + x^{104} + x^{100} + x^{96} + x^{88}\\
&\quad + x^{80} + x^{78} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{52}\\
&\quad + x^{50} + x^{48} + x^{42} + x^{38} + x^{36} + x^{28} + x^{22} + x^{18} + x^{16} + x^{10} + 1,\\
p_9(x) &= x^{132} + x^{128} + x^{126} + x^{116} + x^{112} + x^{108} + x^{106} + x^{104} + x^{100} + x^{96}\\
&\quad + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{66} + x^{64} + x^{60} + x^{58} + x^{54}\\
&\quad + x^{52} + x^{50} + x^{48} + x^{38} + x^{32} + x^{30} + x^{28} + x^{18} + x^{16} + x^{14} + x^{10}\\
&\quad + x^{8} + x^{4},\\
p_{12}(x) &= x^{132} + x^{128} + x^{124} + x^{120} + x^{116} + x^{112} + x^{100} + x^{96} + x^{92} + x^{88}\\
&\quad + x^{84} + x^{80} + x^{52} + x^{48} + x^{44} + x^{40} + x^{28} + x^{24} + x^{12} + x^{8} + x^{4} + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0].

For $\phi(W^e, 0, 1)$:

$$\begin{aligned}
p_0(x) &= x^{147} + x^{143} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{130} + x^{129} + x^{127}\\
&\quad + x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111}\\
&\quad + x^{110} + x^{109} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{99} + x^{96} + x^{95}\\
&\quad + x^{93} + x^{91} + x^{90} + x^{88} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78}\\
&\quad + x^{77} + x^{74} + x^{73} + x^{69} + x^{67} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{54}\\
&\quad + x^{53} + x^{52} + x^{51} + x^{49} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36}\\
&\quad + x^{35} + x^{34} + x^{33},\\
p_3(x) &= x^{141} + x^{138} + x^{137} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{125}\\
&\quad + x^{124} + x^{119} + x^{116} + x^{115} + x^{113} + x^{108} + x^{107} + x^{106} + x^{104}\\
&\quad + x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{88} + x^{87} + x^{86} + x^{84}\\
&\quad + x^{82} + x^{81} + x^{79} + x^{77} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63}\\
&\quad + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45}\\
&\quad + x^{43} + x^{41} + x^{39} + x^{37} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{23} + x^{21}\\
&\quad + x^{20} + x^{19} + x^{18},\\
p_6(x) &= x^{138} + x^{137} + x^{136} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{118} + x^{112}
\end{aligned}$$

$$
\begin{aligned}
&+ x^{111} + x^{110} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96} \\
&+ x^{94} + x^{93} + x^{92} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{70} + x^{69} \\
&+ x^{68} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47} \\
&+ x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30} \\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12}, \\
p_9(x) = {}& x^{135} + x^{133} + x^{132} + x^{130} + x^{128} + x^{127} + x^{123} + x^{122} + x^{121} + x^{119} \\
&+ x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{103} + x^{102} \\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83} \\
&+ x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68} \\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51} \\
&+ x^{50} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34} \\
&+ x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{10} \\
&+ x^{9} + x^{8} + x^{7} + x^{6}, \\
p_{12}(x) = {}& x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{115} \\
&+ x^{113} + x^{112} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85} \\
&+ x^{83} + x^{81} + x^{80} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39} \\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16} \\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1].

For $\phi(W^e, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{147} + x^{143} + x^{141} + x^{139} + x^{138} + x^{136} + x^{135} + x^{130} + x^{129} + x^{127} \\
&+ x^{125} + x^{123} + x^{121} + x^{120} + x^{119} + x^{117} + x^{116} + x^{112} + x^{111} \\
&+ x^{110} + x^{109} + x^{108} + x^{107} + x^{105} + x^{104} + x^{103} + x^{99} + x^{96} + x^{95} \\
&+ x^{93} + x^{91} + x^{90} + x^{88} + x^{84} + x^{83} + x^{82} + x^{81} + x^{80} + x^{79} + x^{78} \\
&+ x^{77} + x^{74} + x^{73} + x^{69} + x^{67} + x^{65} + x^{64} + x^{61} + x^{59} + x^{58} + x^{54} \\
&+ x^{53} + x^{52} + x^{51} + x^{49} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} \\
&+ x^{35} + x^{34} + x^{33}, \\
p_3(x) = {}& x^{141} + x^{138} + x^{137} + x^{132} + x^{131} + x^{130} + x^{128} + x^{127} + x^{126} + x^{125} \\
&+ x^{124} + x^{119} + x^{116} + x^{115} + x^{113} + x^{108} + x^{107} + x^{106} + x^{104} \\
&+ x^{103} + x^{102} + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{88} + x^{87} + x^{86} + x^{84} \\
&+ x^{82} + x^{81} + x^{79} + x^{77} + x^{72} + x^{71} + x^{70} + x^{68} + x^{66} + x^{65} + x^{63} \\
&+ x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{46} + x^{45} \\
&+ x^{43} + x^{41} + x^{39} + x^{37} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{23} + x^{21} \\
&+ x^{20} + x^{19} + x^{18}, \\
p_6(x) = {}& x^{138} + x^{137} + x^{136} + x^{126} + x^{125} + x^{123} + x^{121} + x^{119} + x^{118} + x^{112}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{111} + x^{110} + x^{106} + x^{105} + x^{104} + x^{102} + x^{101} + x^{99} + x^{97} + x^{96}\\
&+ x^{94} + x^{93} + x^{92} + x^{80} + x^{79} + x^{78} + x^{76} + x^{75} + x^{74} + x^{70} + x^{69}\\
&+ x^{68} + x^{62} + x^{61} + x^{59} + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{49} + x^{47}\\
&+ x^{45} + x^{44} + x^{42} + x^{41} + x^{39} + x^{37} + x^{36} + x^{34} + x^{33} + x^{31} + x^{30}\\
&+ x^{28} + x^{27} + x^{25} + x^{24} + x^{20} + x^{19} + x^{18} + x^{16} + x^{15} + x^{13} + x^{12},\\
p_9(x) &= x^{135} + x^{133} + x^{132} + x^{130} + x^{128} + x^{127} + x^{123} + x^{122} + x^{121} + x^{119}\\
&+ x^{118} + x^{116} + x^{115} + x^{114} + x^{113} + x^{112} + x^{105} + x^{103} + x^{102}\\
&+ x^{100} + x^{99} + x^{98} + x^{97} + x^{96} + x^{95} + x^{92} + x^{91} + x^{89} + x^{84} + x^{83}\\
&+ x^{82} + x^{81} + x^{80} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{69} + x^{68}\\
&+ x^{67} + x^{66} + x^{65} + x^{62} + x^{59} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{51}\\
&+ x^{50} + x^{48} + x^{46} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{27} + x^{25} + x^{24} + x^{22} + x^{20} + x^{19} + x^{17} + x^{15} + x^{10}\\
&+ x^{9} + x^{8} + x^{7} + x^{6},\\
p_{12}(x) &= x^{132} + x^{131} + x^{129} + x^{127} + x^{125} + x^{124} + x^{120} + x^{119} + x^{117} + x^{115}\\
&+ x^{113} + x^{112} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{88} + x^{87} + x^{85}\\
&+ x^{83} + x^{81} + x^{80} + x^{52} + x^{51} + x^{49} + x^{47} + x^{45} + x^{44} + x^{40} + x^{39}\\
&+ x^{37} + x^{36} + x^{32} + x^{31} + x^{29} + x^{28} + x^{24} + x^{23} + x^{21} + x^{20} + x^{16}\\
&+ x^{15} + x^{13} + x^{12} + x^{8} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1].

For $\phi(W^e, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{138} + x^{134} + x^{132} + x^{130} + x^{128} + x^{126} + x^{114} + x^{108} + x^{106} + x^{104}\\
&+ x^{102} + x^{96} + x^{92} + x^{88} + x^{78} + x^{76} + x^{68} + x^{66} + x^{62} + x^{58} + x^{52}\\
&+ x^{50} + x^{48} + x^{44} + x^{42},\\
p_3(x) &= x^{128} + x^{114} + x^{112} + x^{110} + x^{108} + x^{104} + x^{102} + x^{94} + x^{90} + x^{84}\\
&+ x^{76} + x^{70} + x^{68} + x^{60} + x^{54} + x^{52} + x^{48} + x^{42} + x^{40} + x^{38} + x^{26}\\
&+ x^{22} + x^{20} + x^{16} + x^{14} + x^{12},\\
p_6(x) &= x^{128} + x^{126} + x^{120} + x^{110} + x^{108} + x^{106} + x^{104} + x^{98} + x^{94} + x^{92}\\
&+ x^{84} + x^{80} + x^{76} + x^{74} + x^{72} + x^{68} + x^{66} + x^{64} + x^{62} + x^{58} + x^{54}\\
&+ x^{48} + x^{46} + x^{44} + x^{40} + x^{30} + x^{28} + x^{22} + x^{16} + x^{14} + x^{12} + x^{6}\\
&+ x^{2} + 1,\\
p_9(x) &= x^{132} + x^{130} + x^{128} + x^{126} + x^{122} + x^{118} + x^{110} + x^{108} + x^{106} + x^{98}\\
&+ x^{96} + x^{94} + x^{92} + x^{86} + x^{82} + x^{68} + x^{64} + x^{60} + x^{58} + x^{54} + x^{44}\\
&+ x^{38} + x^{32} + x^{30} + x^{24} + x^{22} + x^{20} + x^{18} + x^{10} + x^{8} + x^{6} + x^{4},\\
p_{12}(x) &= x^{132} + x^{130} + x^{128} + x^{116} + x^{114} + x^{112} + x^{100} + x^{98} + x^{96} + x^{84}\\
&+ x^{82} + x^{80} + x^{52} + x^{50} + x^{48} + x^{4} + x^{2} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0].

For $\phi(M^o, 0, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{144} + x^{138} + x^{134} + x^{133} + x^{131} + x^{129} + x^{126} + x^{125} + x^{122} \\
& + x^{120} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{106} + x^{104} + x^{99} + x^{95} \\
& + x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{85} + x^{82} + x^{81} + x^{78} + x^{76} \\
& + x^{75} + x^{73} + x^{71} + x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55} \\
& + x^{53} + x^{52} + x^{50} + x^{46} + x^{45} + x^{44} + x^{42} + x^{39} + x^{38} + x^{32} + x^{30} \\
& + x^{28} + x^{27} + x^{25} + x^{24}, \\
p_3(x) = {} & x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{135} + x^{132} + x^{129} + x^{126} + x^{125} \\
& + x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{114} + x^{113} + x^{112} \\
& + x^{111} + x^{110} + x^{109} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{92} + x^{89} \\
& + x^{88} + x^{87} + x^{78} + x^{77} + x^{73} + x^{70} + x^{66} + x^{65} + x^{63} + x^{60} + x^{57} \\
& + x^{55} + x^{51} + x^{46} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34} \\
& + x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16}, \\
p_6(x) = {} & x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{135} + x^{132} + x^{127} + x^{123} + x^{121} \\
& + x^{120} + x^{119} + x^{115} + x^{114} + x^{110} + x^{106} + x^{104} + x^{101} + x^{100} \\
& + x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{82} \\
& + x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} \\
& + x^{67} + x^{60} + x^{59} + x^{55} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42} \\
& + x^{41} + x^{39} + x^{36} + x^{31} + x^{30} + x^{29} + x^{28} + x^{23} + x^{21} + x^{19} + x^{17} \\
& + x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) = {} & x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127} \\
& + x^{125} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} + x^{104} \\
& + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{64} + x^{63} \\
& + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} \\
& + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} \\
& + x^{17} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) = {} & x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{132} + x^{124} + x^{123} \\
& + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} \\
& + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} \\
& + x^{64} + x^{63} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} \\
& + x^{43} + x^{41} + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{24} \\
& + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0].

For $\phi(M^o, 0, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{160} + x^{152} + x^{151} + x^{149} + x^{147} + x^{146} + x^{145} + x^{143} + x^{140} + x^{139} \\
& + x^{138} + x^{136} + x^{135} + x^{133} + x^{131} + x^{130} + x^{128} + x^{126} + x^{124} \\
& + x^{121} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102} \\
& + x^{101} + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{85} + x^{84} + x^{80} + x^{79} + x^{72} \\
& + x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{56} + x^{55} + x^{54} \\
& + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40} \\
& + x^{38} + x^{34} + x^{33}, \\
p_3(x) = {} & x^{139} + x^{138} + x^{135} + x^{134} + x^{133} + x^{129} + x^{128} + x^{127} + x^{123} + x^{122} \\
& + x^{121} + x^{117} + x^{116} + x^{115} + x^{111} + x^{110} + x^{109} + x^{105} + x^{104} \\
& + x^{103} + x^{97} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{81} + x^{79} \\
& + x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59} \\
& + x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42} \\
& + x^{41} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{27} + x^{26} + x^{25} \\
& + x^{21} + x^{20} + x^{18}, \\
p_6(x) = {} & x^{142} + x^{140} + x^{134} + x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{112} \\
& + x^{108} + x^{106} + x^{102} + x^{98} + x^{90} + x^{88} + x^{86} + x^{74} + x^{70} + x^{68} + x^{66} \\
& + x^{60} + x^{54} + x^{52} + x^{46} + x^{44} + x^{40} + x^{26} + x^{24} + x^{18} + x^{16} + x^{12}, \\
p_9(x) = {} & x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{132} + x^{131} \\
& + x^{130} + x^{129} + x^{122} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{111} \\
& + x^{110} + x^{109} + x^{108} + x^{106} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{90} \\
& + x^{89} + x^{88} + x^{86} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56} \\
& + x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{38} + x^{36} \\
& + x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} + x^{22} + x^{20} + x^{19} \\
& + x^{18} + x^{17} + x^{10} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) = {} & x^{148} + x^{144} + x^{136} + x^{132} + x^{128} + x^{120} + x^{116} + x^{104} + x^{100} + x^{96} \\
& + x^{88} + x^{80} + x^{68} + x^{64} + x^{56} + x^{32} + x^{20} + x^{8} + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1].

For $\phi(M^o, 1, 0)$:

$$\begin{aligned}
p_0(x) = {} & x^{142} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{132} + x^{131} + x^{128} + x^{126} \\
& + x^{122} + x^{116} + x^{115} + x^{113} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102} \\
& + x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{84} + x^{83} + x^{80} + x^{79} \\
& + x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65} \\
& + x^{63} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{42} + x^{41} + x^{40} \\
& + x^{39} + x^{38} + x^{36} + x^{35} + x^{33},
\end{aligned}$$

$$
\begin{aligned}
p_3(x) &= x^{139} + x^{138} + x^{133} + x^{128} + x^{126} + x^{124} + x^{123} + x^{120} + x^{118} + x^{117} \\
&\quad + x^{116} + x^{115} + x^{109} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92} \\
&\quad + x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70} \\
&\quad + x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} \\
&\quad + x^{45} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27} + x^{24} + x^{22} \\
&\quad + x^{21} + x^{20} + x^{18}, \\
p_6(x) &= x^{142} + x^{140} + x^{138} + x^{136} + x^{130} + x^{126} + x^{122} + x^{118} + x^{116} + x^{114} \\
&\quad + x^{112} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{84} + x^{82} \\
&\quad + x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{48} \\
&\quad + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{22} + x^{18} + x^{12}, \\
p_9(x) &= x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131} \\
&\quad + x^{128} + x^{125} + x^{124} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{111} \\
&\quad + x^{110} + x^{107} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92} \\
&\quad + x^{89} + x^{88} + x^{86} + x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{56} + x^{55} + x^{53} \\
&\quad + x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38} \\
&\quad + x^{37} + x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{23} \\
&\quad + x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{13} + x^{12} + x^{9} + x^{8} + x^{6}, \\
p_{12}(x) &= x^{148} + x^{132} + x^{128} + x^{124} + x^{112} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80} \\
&\quad + x^{68} + x^{48} + x^{44} + x^{36} + x^{28} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(M^o, 1, 1)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{143} + x^{137} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127} + x^{121} + x^{120} \\
&\quad + x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} \\
&\quad + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} \\
&\quad + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} \\
&\quad + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{50} + x^{47} + x^{42}, \\
p_3(x) &= x^{144} + x^{143} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135} + x^{131} + x^{127} + x^{126} \\
&\quad + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} \\
&\quad + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{103} + x^{102} + x^{101} \\
&\quad + x^{100} + x^{98} + x^{97} + x^{96} + x^{92} + x^{89} + x^{85} + x^{83} + x^{81} + x^{80} + x^{78} \\
&\quad + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{60} \\
&\quad + x^{58} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{40} + x^{39} + x^{37} \\
&\quad + x^{36} + x^{35} + x^{34} + x^{33} + x^{26} + x^{25} + x^{22}, \\
p_6(x) &= x^{144} + x^{143} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135} + x^{131} + x^{129} + x^{126} \\
&\quad + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{113} + x^{111} + x^{105} + x^{102}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87}\\
&+ x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69}\\
&+ x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{50} + x^{48}\\
&+ x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{36} + x^{34} + x^{33} + x^{29} + x^{27} + x^{25}\\
&+ x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12}\\
&+ x^{8} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) &= x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127}\\
&+ x^{125} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} + x^{104}\\
&+ x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{64} + x^{63}\\
&+ x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41}\\
&+ x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19}\\
&+ x^{17} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) &= x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{132} + x^{124} + x^{123}\\
&+ x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104}\\
&+ x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80}\\
&+ x^{64} + x^{63} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45}\\
&+ x^{43} + x^{41} + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{24}\\
&+ x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0].

For $\phi(W^o, 0, 0)$:

$$
\begin{aligned}
p_0(x) &= x^{147} + x^{144} + x^{138} + x^{134} + x^{133} + x^{131} + x^{129} + x^{126} + x^{125} + x^{122}\\
&+ x^{120} + x^{118} + x^{115} + x^{114} + x^{111} + x^{110} + x^{106} + x^{104} + x^{99} + x^{95}\\
&+ x^{94} + x^{93} + x^{92} + x^{90} + x^{89} + x^{87} + x^{85} + x^{82} + x^{81} + x^{78} + x^{76}\\
&+ x^{75} + x^{73} + x^{71} + x^{66} + x^{65} + x^{64} + x^{62} + x^{58} + x^{57} + x^{56} + x^{55}\\
&+ x^{53} + x^{52} + x^{50} + x^{46} + x^{45} + x^{44} + x^{42} + x^{39} + x^{38} + x^{32} + x^{30}\\
&+ x^{28} + x^{27} + x^{25} + x^{24},\\
p_3(x) &= x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{135} + x^{132} + x^{129} + x^{126} + x^{125}\\
&+ x^{124} + x^{122} + x^{121} + x^{120} + x^{119} + x^{118} + x^{114} + x^{113} + x^{112}\\
&+ x^{111} + x^{110} + x^{109} + x^{105} + x^{102} + x^{100} + x^{99} + x^{98} + x^{92} + x^{89}\\
&+ x^{88} + x^{87} + x^{78} + x^{77} + x^{73} + x^{70} + x^{66} + x^{65} + x^{63} + x^{60} + x^{57}\\
&+ x^{55} + x^{51} + x^{46} + x^{45} + x^{44} + x^{40} + x^{39} + x^{37} + x^{36} + x^{35} + x^{34}\\
&+ x^{33} + x^{32} + x^{31} + x^{29} + x^{27} + x^{25} + x^{22} + x^{20} + x^{19} + x^{17} + x^{16},\\
p_6(x) &= x^{144} + x^{143} + x^{140} + x^{139} + x^{138} + x^{135} + x^{132} + x^{127} + x^{123} + x^{121}\\
&+ x^{120} + x^{119} + x^{115} + x^{114} + x^{110} + x^{106} + x^{104} + x^{101} + x^{100}\\
&+ x^{99} + x^{98} + x^{94} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{86} + x^{84} + x^{82}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{81} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69}\\
&+ x^{67} + x^{60} + x^{59} + x^{55} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{42}\\
&+ x^{41} + x^{39} + x^{36} + x^{31} + x^{30} + x^{29} + x^{28} + x^{23} + x^{21} + x^{19} + x^{17}\\
&+ x^{16} + x^{15} + x^{14} + x^{13} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,\\
p_9(x) = &\, x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127}\\
&+ x^{125} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} + x^{104}\\
&+ x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{64} + x^{63}\\
&+ x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41}\\
&+ x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19}\\
&+ x^{17} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8},\\
p_{12}(x) = &\, x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{132} + x^{124} + x^{123}\\
&+ x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104}\\
&+ x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80}\\
&+ x^{64} + x^{63} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45}\\
&+ x^{43} + x^{41} + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{24}\\
&+ x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}
$$

and the initial terms are [1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0].

For $\phi(W^o, 0, 1)$:

$$
\begin{aligned}
p_0(x) = &\, x^{142} + x^{140} + x^{139} + x^{138} + x^{137} + x^{135} + x^{132} + x^{131} + x^{128} + x^{126}\\
&+ x^{122} + x^{116} + x^{115} + x^{113} + x^{109} + x^{108} + x^{105} + x^{103} + x^{102}\\
&+ x^{101} + x^{100} + x^{99} + x^{97} + x^{95} + x^{93} + x^{92} + x^{84} + x^{83} + x^{80} + x^{79}\\
&+ x^{78} + x^{77} + x^{75} + x^{74} + x^{73} + x^{72} + x^{70} + x^{69} + x^{67} + x^{66} + x^{65}\\
&+ x^{63} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51} + x^{42} + x^{41} + x^{40}\\
&+ x^{39} + x^{38} + x^{36} + x^{35} + x^{33},\\
p_3(x) = &\, x^{139} + x^{138} + x^{133} + x^{128} + x^{126} + x^{124} + x^{123} + x^{120} + x^{118} + x^{117}\\
&+ x^{116} + x^{115} + x^{109} + x^{104} + x^{102} + x^{100} + x^{98} + x^{96} + x^{94} + x^{92}\\
&+ x^{90} + x^{88} + x^{86} + x^{84} + x^{82} + x^{80} + x^{78} + x^{76} + x^{74} + x^{72} + x^{70}\\
&+ x^{68} + x^{66} + x^{64} + x^{62} + x^{60} + x^{59} + x^{56} + x^{54} + x^{53} + x^{52} + x^{51}\\
&+ x^{45} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{27} + x^{24} + x^{22}\\
&+ x^{21} + x^{20} + x^{18},\\
p_6(x) = &\, x^{142} + x^{140} + x^{138} + x^{136} + x^{130} + x^{126} + x^{122} + x^{118} + x^{116} + x^{114}\\
&+ x^{112} + x^{108} + x^{104} + x^{102} + x^{100} + x^{98} + x^{94} + x^{92} + x^{84} + x^{82}\\
&+ x^{76} + x^{72} + x^{70} + x^{68} + x^{66} + x^{62} + x^{60} + x^{58} + x^{56} + x^{54} + x^{48}\\
&+ x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{32} + x^{30} + x^{22} + x^{18} + x^{12},\\
p_9(x) = &\, x^{145} + x^{144} + x^{143} + x^{142} + x^{139} + x^{136} + x^{135} + x^{133} + x^{132} + x^{131}
\end{aligned}
$$

$$
\begin{aligned}
&+ x^{128} + x^{125} + x^{124} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{107} + x^{104} + x^{103} + x^{101} + x^{100} + x^{99} + x^{96} + x^{93} + x^{92}\\
&+ x^{89} + x^{88} + x^{86} + x^{65} + x^{64} + x^{63} + x^{62} + x^{59} + x^{56} + x^{55} + x^{53}\\
&+ x^{52} + x^{51} + x^{49} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{41} + x^{39} + x^{38}\\
&+ x^{37} + x^{36} + x^{35} + x^{33} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{25} + x^{23}\\
&+ x^{22} + x^{21} + x^{20} + x^{19} + x^{16} + x^{13} + x^{12} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{148} + x^{132} + x^{128} + x^{124} + x^{112} + x^{100} + x^{96} + x^{92} + x^{84} + x^{80}\\
&+ x^{68} + x^{48} + x^{44} + x^{36} + x^{28} + x^{12} + x^{4} + 1,
\end{aligned}
$$

and the initial terms are [0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0].

For $\phi(W^o, 1, 0)$:

$$
\begin{aligned}
p_0(x) = {}& x^{160} + x^{152} + x^{151} + x^{149} + x^{147} + x^{146} + x^{145} + x^{143} + x^{140} + x^{139}\\
&+ x^{138} + x^{136} + x^{135} + x^{133} + x^{131} + x^{130} + x^{128} + x^{126} + x^{124}\\
&+ x^{121} + x^{112} + x^{111} + x^{110} + x^{109} + x^{107} + x^{104} + x^{103} + x^{102}\\
&+ x^{101} + x^{99} + x^{98} + x^{96} + x^{93} + x^{92} + x^{85} + x^{84} + x^{80} + x^{79} + x^{72}\\
&+ x^{71} + x^{70} + x^{68} + x^{67} + x^{65} + x^{63} + x^{62} + x^{61} + x^{56} + x^{55} + x^{54}\\
&+ x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{43} + x^{42} + x^{41} + x^{40}\\
&+ x^{38} + x^{34} + x^{33},\\
p_3(x) = {}& x^{139} + x^{138} + x^{135} + x^{134} + x^{133} + x^{129} + x^{128} + x^{127} + x^{123} + x^{122}\\
&+ x^{121} + x^{117} + x^{116} + x^{115} + x^{111} + x^{110} + x^{109} + x^{105} + x^{104}\\
&+ x^{103} + x^{97} + x^{95} + x^{94} + x^{92} + x^{90} + x^{89} + x^{88} + x^{87} + x^{81} + x^{79}\\
&+ x^{78} + x^{76} + x^{74} + x^{73} + x^{72} + x^{71} + x^{65} + x^{63} + x^{62} + x^{60} + x^{59}\\
&+ x^{57} + x^{56} + x^{54} + x^{53} + x^{51} + x^{50} + x^{48} + x^{47} + x^{45} + x^{44} + x^{42}\\
&+ x^{41} + x^{39} + x^{38} + x^{36} + x^{34} + x^{33} + x^{32} + x^{31} + x^{27} + x^{26} + x^{25}\\
&+ x^{21} + x^{20} + x^{18},\\
p_6(x) = {}& x^{142} + x^{140} + x^{134} + x^{132} + x^{128} + x^{126} + x^{124} + x^{122} + x^{120} + x^{112}\\
&+ x^{108} + x^{106} + x^{102} + x^{98} + x^{90} + x^{88} + x^{86} + x^{74} + x^{70} + x^{68} + x^{66}\\
&+ x^{60} + x^{54} + x^{52} + x^{46} + x^{44} + x^{40} + x^{26} + x^{24} + x^{18} + x^{16} + x^{12},\\
p_9(x) = {}& x^{145} + x^{144} + x^{143} + x^{142} + x^{141} + x^{140} + x^{138} + x^{136} + x^{132} + x^{131}\\
&+ x^{130} + x^{129} + x^{122} + x^{121} + x^{120} + x^{118} + x^{113} + x^{112} + x^{111}\\
&+ x^{110} + x^{109} + x^{108} + x^{106} + x^{104} + x^{100} + x^{99} + x^{98} + x^{97} + x^{90}\\
&+ x^{89} + x^{88} + x^{86} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{58} + x^{56}\\
&+ x^{52} + x^{51} + x^{50} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{41} + x^{38} + x^{36}\\
&+ x^{35} + x^{34} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{25} + x^{22} + x^{20} + x^{19}\\
&+ x^{18} + x^{17} + x^{10} + x^{9} + x^{8} + x^{6},\\
p_{12}(x) = {}& x^{148} + x^{144} + x^{136} + x^{132} + x^{128} + x^{120} + x^{116} + x^{104} + x^{100} + x^{96}
\end{aligned}
$$

$$+ x^{88} + x^{80} + x^{68} + x^{64} + x^{56} + x^{32} + x^{20} + x^{8} + 1,$$

and the initial terms are [0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1].

For $\phi(W^o, 1, 1)$:

$$\begin{aligned}
p_0(x) = {} & x^{147} + x^{143} + x^{137} + x^{133} + x^{132} + x^{131} + x^{129} + x^{127} + x^{121} + x^{120} \\
& + x^{117} + x^{115} + x^{113} + x^{112} + x^{109} + x^{102} + x^{101} + x^{100} + x^{97} + x^{96} \\
& + x^{94} + x^{93} + x^{92} + x^{90} + x^{88} + x^{86} + x^{85} + x^{84} + x^{82} + x^{79} + x^{77} \\
& + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{64} \\
& + x^{62} + x^{61} + x^{60} + x^{58} + x^{57} + x^{55} + x^{54} + x^{52} + x^{50} + x^{47} + x^{42}, \\
p_3(x) = {} & x^{144} + x^{143} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135} + x^{131} + x^{127} + x^{126} \\
& + x^{124} + x^{123} + x^{121} + x^{120} + x^{119} + x^{118} + x^{116} + x^{115} + x^{113} \\
& + x^{112} + x^{111} + x^{110} + x^{109} + x^{108} + x^{107} + x^{103} + x^{102} + x^{101} \\
& + x^{100} + x^{98} + x^{97} + x^{96} + x^{92} + x^{89} + x^{85} + x^{83} + x^{81} + x^{80} + x^{78} \\
& + x^{77} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{67} + x^{66} + x^{64} + x^{63} + x^{60} \\
& + x^{58} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} + x^{46} + x^{40} + x^{39} + x^{37} \\
& + x^{36} + x^{35} + x^{34} + x^{33} + x^{26} + x^{25} + x^{22}, \\
p_6(x) = {} & x^{144} + x^{143} + x^{140} + x^{139} + x^{137} + x^{136} + x^{135} + x^{131} + x^{129} + x^{126} \\
& + x^{124} + x^{120} + x^{118} + x^{116} + x^{114} + x^{113} + x^{111} + x^{105} + x^{102} \\
& + x^{100} + x^{98} + x^{97} + x^{95} + x^{93} + x^{92} + x^{91} + x^{90} + x^{89} + x^{88} + x^{87} \\
& + x^{86} + x^{85} + x^{84} + x^{83} + x^{82} + x^{77} + x^{76} + x^{75} + x^{74} + x^{72} + x^{69} \\
& + x^{68} + x^{65} + x^{64} + x^{62} + x^{61} + x^{59} + x^{55} + x^{54} + x^{53} + x^{50} + x^{48} \\
& + x^{46} + x^{45} + x^{43} + x^{41} + x^{39} + x^{36} + x^{34} + x^{33} + x^{29} + x^{27} + x^{25} \\
& + x^{24} + x^{23} + x^{21} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} \\
& + x^{8} + x^{7} + x^{5} + x^{3} + x + 1, \\
p_9(x) = {} & x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{131} + x^{129} + x^{127} \\
& + x^{125} + x^{123} + x^{121} + x^{120} + x^{112} + x^{111} + x^{109} + x^{108} + x^{104} \\
& + x^{103} + x^{101} + x^{99} + x^{97} + x^{95} + x^{93} + x^{91} + x^{89} + x^{88} + x^{64} + x^{63} \\
& + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{51} + x^{49} + x^{48} + x^{44} + x^{43} + x^{41} \\
& + x^{39} + x^{37} + x^{35} + x^{33} + x^{32} + x^{28} + x^{27} + x^{25} + x^{23} + x^{21} + x^{19} \\
& + x^{17} + x^{15} + x^{13} + x^{11} + x^{9} + x^{8}, \\
p_{12}(x) = {} & x^{144} + x^{143} + x^{141} + x^{140} + x^{136} + x^{135} + x^{133} + x^{132} + x^{124} + x^{123} \\
& + x^{121} + x^{119} + x^{117} + x^{115} + x^{113} + x^{111} + x^{109} + x^{108} + x^{104} \\
& + x^{103} + x^{101} + x^{100} + x^{92} + x^{91} + x^{89} + x^{87} + x^{85} + x^{83} + x^{81} + x^{80} \\
& + x^{64} + x^{63} + x^{61} + x^{60} + x^{56} + x^{55} + x^{53} + x^{52} + x^{48} + x^{47} + x^{45} \\
& + x^{43} + x^{41} + x^{40} + x^{36} + x^{35} + x^{33} + x^{31} + x^{29} + x^{27} + x^{25} + x^{24} \\
& + x^{20} + x^{19} + x^{17} + x^{16} + x^{12} + x^{11} + x^{9} + x^{7} + x^{5} + x^{3} + x + 1,
\end{aligned}$$

and the initial terms are [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0].

Below is the transition function and output function of an 2-automaton that generates $T = M^o_{0,0}$:

Transition function $(n, j) \mapsto \delta(n, j)$ $(\Lambda(n) := [\delta(n, 0), \delta(n, 1)])$:

n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$	n	$\Lambda(n)$
0	[1, 2]	192	[341, 342]	384	[408, 162]	576	[789, 521]	768	[833, 264]
1	[3, 4]	193	[343, 344]	385	[523, 170]	577	[55, 790]	769	[834, 835]
2	[5, 6]	194	[345, 346]	386	[524, 54]	578	[154, 196]	770	[836, 837]
3	[7, 8]	195	[347, 348]	387	[525, 452]	579	[772, 485]	771	[838, 839]
4	[9, 10]	196	[349, 350]	388	[526, 69]	580	[791, 48]	772	[638, 662]
5	[11, 12]	197	[220, 351]	389	[527, 528]	581	[401, 448]	773	[840, 841]
6	[13, 14]	198	[352, 353]	390	[529, 530]	582	[792, 172]	774	[842, 843]
7	[15, 16]	199	[354, 355]	391	[531, 532]	583	[487, 423]	775	[844, 845]
8	[17, 18]	200	[356, 357]	392	[533, 534]	584	[51, 793]	776	[569, 846]
9	[19, 20]	201	[358, 359]	393	[535, 536]	585	[465, 782]	777	[847, 848]
10	[21, 22]	202	[360, 361]	394	[537, 538]	586	[209, 455]	778	[849, 850]
11	[23, 24]	203	[276, 362]	395	[539, 540]	587	[160, 471]	779	[851, 852]
12	[25, 26]	204	[363, 364]	396	[541, 542]	588	[126, 196]	780	[853, 854]
13	[27, 28]	205	[359, 365]	397	[369, 543]	589	[794, 48]	781	[855, 240]
14	[29, 30]	206	[366, 367]	398	[544, 545]	590	[449, 169]	782	[528, 856]
15	[31, 32]	207	[64, 368]	399	[546, 547]	591	[212, 202]	783	[857, 858]
16	[33, 34]	208	[208, 208]	400	[548, 549]	592	[399, 426]	784	[859, 860]
17	[35, 36]	209	[244, 369]	401	[550, 551]	593	[60, 795]	785	[861, 862]
18	[37, 38]	210	[370, 371]	402	[552, 553]	594	[177, 489]	786	[863, 864]
19	[39, 40]	211	[372, 373]	403	[554, 238]	595	[424, 53]	787	[865, 866]
20	[41, 42]	212	[374, 375]	404	[239, 555]	596	[796, 782]	788	[867, 635]
21	[43, 44]	213	[376, 377]	405	[282, 556]	597	[797, 452]	789	[868, 869]
22	[45, 46]	214	[378, 379]	406	[557, 558]	598	[504, 404]	790	[870, 871]
23	[47, 48]	215	[380, 381]	407	[559, 560]	599	[438, 188]	791	[286, 872]
24	[49, 50]	216	[382, 383]	408	[561, 562]	600	[456, 773]	792	[567, 873]
25	[51, 52]	217	[384, 385]	409	[563, 564]	601	[798, 406]	793	[874, 875]
26	[53, 54]	218	[386, 387]	410	[565, 566]	602	[799, 510]	794	[718, 876]
27	[55, 56]	219	[136, 388]	411	[305, 567]	603	[483, 10]	795	[119, 877]
28	[57, 58]	220	[174, 165]	412	[568, 569]	604	[157, 790]	796	[878, 879]
29	[59, 60]	221	[389, 210]	413	[570, 571]	605	[771, 48]	797	[880, 881]
30	[61, 62]	222	[390, 200]	414	[572, 573]	606	[156, 800]	798	[301, 882]
31	[63, 64]	223	[391, 392]	415	[574, 575]	607	[408, 801]	799	[883, 884]
32	[65, 66]	224	[393, 155]	416	[576, 577]	608	[488, 53]	800	[885, 886]
33	[67, 68]	225	[394, 395]	417	[578, 579]	609	[796, 473]	801	[887, 888]
34	[69, 70]	226	[396, 397]	418	[580, 581]	610	[802, 218]	802	[18, 889]
35	[71, 72]	227	[398, 58]	419	[582, 583]	611	[412, 503]	803	[379, 25]
36	[73, 74]	228	[399, 400]	420	[584, 585]	612	[195, 50]	804	[890, 891]
37	[28, 75]	229	[401, 149]	421	[242, 586]	613	[768, 501]	805	[892, 337]
38	[76, 77]	230	[44, 402]	422	[20, 587]	614	[438, 477]	806	[893, 894]

39	[78, 79]	231	[403, 404]	423	[588, 108]	615	[418, 42]	807	[895, 896]
40	[80, 81]	232	[405, 406]	424	[589, 590]	616	[43, 503]	808	[897, 898]
41	[82, 83]	233	[407, 408]	425	[591, 592]	617	[803, 800]	809	[899, 705]
42	[84, 85]	234	[409, 410]	426	[30, 593]	618	[158, 503]	810	[900, 901]
43	[86, 87]	235	[136, 138]	427	[594, 595]	619	[61, 169]	811	[902, 903]
44	[88, 89]	236	[139, 136]	428	[596, 597]	620	[194, 793]	812	[837, 76]
45	[90, 91]	237	[135, 34]	429	[353, 598]	621	[399, 491]	813	[904, 905]
46	[92, 93]	238	[208, 134]	430	[599, 600]	622	[143, 479]	814	[906, 838]
47	[94, 95]	239	[34, 137]	431	[601, 602]	623	[804, 428]	815	[728, 574]
48	[96, 97]	240	[138, 135]	432	[603, 604]	624	[137, 139]	816	[907, 352]
49	[98, 99]	241	[135, 411]	433	[605, 606]	625	[805, 141]	817	[908, 27]
50	[100, 101]	242	[137, 404]	434	[607, 608]	626	[806, 473]	818	[909, 910]
51	[102, 103]	243	[152, 392]	435	[609, 610]	627	[807, 503]	819	[545, 911]
52	[104, 105]	244	[412, 413]	436	[611, 612]	628	[61, 808]	820	[912, 913]
53	[106, 107]	245	[414, 32]	437	[613, 343]	629	[786, 471]	821	[914, 915]
54	[108, 109]	246	[396, 415]	438	[614, 615]	630	[506, 479]	822	[916, 917]
55	[110, 111]	247	[416, 417]	439	[616, 617]	631	[451, 182]	823	[918, 527]
56	[112, 113]	248	[418, 37]	440	[618, 619]	632	[185, 218]	824	[919, 920]
57	[114, 115]	249	[391, 419]	441	[620, 621]	633	[179, 471]	825	[921, 526]
58	[116, 117]	250	[418, 420]	442	[622, 623]	634	[142, 216]	826	[922, 923]
59	[118, 119]	251	[421, 196]	443	[624, 118]	635	[189, 468]	827	[924, 925]
60	[120, 121]	252	[422, 423]	444	[600, 625]	636	[512, 497]	828	[926, 927]
61	[122, 123]	253	[420, 400]	445	[626, 627]	637	[809, 10]	829	[241, 928]
62	[124, 125]	254	[424, 408]	446	[316, 628]	638	[423, 785]	830	[929, 849]
63	[126, 127]	255	[425, 426]	447	[629, 630]	639	[810, 131]	831	[930, 931]
64	[128, 129]	256	[427, 428]	448	[631, 632]	640	[811, 428]	832	[672, 932]
65	[130, 131]	257	[172, 429]	449	[633, 634]	641	[453, 417]	833	[781, 411]
66	[132, 133]	258	[430, 431]	450	[635, 636]	642	[812, 37]	834	[826, 182]
67	[134, 134]	259	[432, 40]	451	[637, 638]	643	[806, 782]	835	[464, 818]
68	[135, 136]	260	[183, 433]	452	[639, 640]	644	[813, 48]	836	[789, 184]
69	[136, 137]	261	[434, 435]	453	[641, 274]	645	[495, 814]	837	[768, 150]
70	[138, 139]	262	[436, 37]	454	[642, 17]	646	[815, 396]	838	[34, 388]
71	[140, 141]	263	[409, 200]	455	[643, 644]	647	[809, 479]	839	[815, 191]
72	[142, 40]	264	[437, 406]	456	[645, 646]	648	[786, 161]	840	[832, 12]
73	[143, 144]	265	[411, 137]	457	[647, 572]	649	[42, 426]	841	[178, 169]
74	[145, 146]	266	[438, 439]	458	[648, 649]	650	[816, 504]	842	[61, 475]
75	[56, 147]	267	[440, 441]	459	[650, 651]	651	[209, 396]	843	[150, 933]
76	[148, 149]	268	[198, 60]	460	[652, 653]	652	[817, 216]	844	[934, 814]
77	[150, 151]	269	[148, 442]	461	[654, 655]	653	[804, 818]	845	[174, 509]
78	[152, 40]	270	[139, 443]	462	[656, 657]	654	[819, 780]	846	[443, 137]
79	[153, 53]	271	[444, 445]	463	[658, 659]	655	[799, 820]	847	[430, 516]
80	[154, 155]	272	[446, 12]	464	[660, 661]	656	[821, 48]	848	[824, 477]
81	[156, 157]	273	[447, 448]	465	[662, 663]	657	[494, 14]	849	[437, 395]
82	[158, 44]	274	[157, 56]	466	[311, 664]	658	[822, 439]	850	[814, 208]

83	[49, 159]	275	[126, 155]	467	[665, 666]	659	[153, 167]	851	[39, 419]
84	[160, 161]	276	[449, 62]	468	[667, 668]	660	[420, 38]	852	[485, 441]
85	[53, 162]	277	[138, 33]	469	[669, 670]	661	[800, 790]	853	[828, 491]
86	[134, 163]	278	[450, 445]	470	[671, 672]	662	[823, 58]	854	[813, 508]
87	[164, 165]	279	[451, 188]	471	[673, 674]	663	[785, 54]	855	[173, 135]
88	[166, 167]	280	[434, 452]	472	[675, 676]	664	[44, 776]	856	[48, 151]
89	[168, 169]	281	[453, 184]	473	[677, 678]	665	[453, 521]	857	[403, 33]
90	[157, 170]	282	[412, 454]	474	[679, 680]	666	[153, 785]	858	[815, 481]
91	[171, 161]	283	[205, 165]	475	[681, 682]	667	[812, 420]	859	[817, 392]
92	[59, 172]	284	[455, 415]	476	[683, 684]	668	[168, 62]	860	[145, 827]
93	[49, 36]	285	[60, 429]	477	[685, 686]	669	[770, 213]	861	[823, 935]
94	[173, 33]	286	[138, 404]	478	[687, 688]	670	[824, 182]	862	[441, 782]
95	[174, 175]	287	[456, 133]	479	[689, 690]	671	[391, 216]	863	[789, 433]
96	[176, 177]	288	[457, 431]	480	[691, 692]	672	[456, 60]	864	[502, 44]
97	[178, 62]	289	[458, 54]	481	[693, 694]	673	[798, 395]	865	[12, 203]
98	[179, 180]	290	[459, 48]	482	[695, 696]	674	[496, 193]	866	[936, 210]
99	[181, 182]	291	[49, 460]	483	[697, 698]	675	[786, 461]	867	[520, 433]
100	[183, 184]	292	[171, 461]	484	[699, 700]	676	[56, 131]	868	[478, 462]
101	[185, 186]	293	[399, 38]	485	[701, 702]	677	[401, 442]	869	[41, 37]
102	[187, 188]	294	[398, 431]	486	[703, 704]	678	[155, 778]	870	[57, 935]
103	[189, 155]	295	[9, 462]	487	[705, 706]	679	[825, 443]	871	[42, 491]
104	[190, 191]	296	[463, 184]	488	[707, 708]	680	[164, 175]	872	[164, 799]
105	[192, 193]	297	[464, 428]	489	[93, 709]	681	[826, 188]	873	[174, 799]
106	[194, 52]	298	[160, 461]	490	[710, 548]	682	[787, 827]	874	[394, 406]
107	[170, 147]	299	[465, 466]	491	[612, 711]	683	[517, 175]	875	[445, 482]
108	[195, 36]	300	[467, 144]	492	[712, 713]	684	[509, 820]	876	[405, 192]
109	[196, 197]	301	[393, 468]	493	[714, 715]	685	[821, 150]	877	[163, 208]
110	[198, 172]	302	[436, 420]	494	[267, 716]	686	[422, 488]	878	[505, 52]
111	[199, 200]	303	[469, 442]	495	[717, 718]	687	[478, 10]	879	[42, 400]
112	[201, 202]	304	[173, 139]	496	[719, 720]	688	[485, 210]	880	[790, 203]
113	[56, 203]	305	[139, 34]	497	[721, 722]	689	[484, 60]	881	[523, 790]
114	[187, 182]	306	[189, 127]	498	[723, 724]	690	[803, 157]	882	[788, 431]
115	[204, 60]	307	[470, 471]	499	[725, 726]	691	[393, 127]	883	[499, 491]
116	[205, 175]	308	[472, 473]	500	[727, 728]	692	[500, 141]	884	[783, 468]
117	[206, 207]	309	[474, 60]	501	[729, 730]	693	[828, 426]	885	[829, 503]
118	[208, 163]	310	[412, 44]	502	[731, 552]	694	[807, 454]	886	[148, 448]
119	[209, 191]	311	[449, 475]	503	[732, 733]	695	[500, 793]	887	[777, 769]
120	[176, 210]	312	[170, 203]	504	[555, 624]	696	[133, 429]	888	[503, 402]
121	[211, 36]	313	[140, 476]	505	[734, 691]	697	[789, 417]	889	[46, 210]
122	[212, 213]	314	[181, 477]	506	[735, 736]	698	[423, 408]	890	[12, 937]
123	[214, 184]	315	[478, 479]	507	[737, 738]	699	[137, 33]	891	[14, 37]
124	[215, 216]	316	[189, 196]	508	[739, 740]	700	[437, 192]	892	[520, 184]
125	[217, 218]	317	[480, 481]	509	[741, 742]	701	[791, 150]	893	[770, 202]
126	[219, 220]	318	[206, 482]	510	[743, 744]	702	[777, 410]	894	[465, 473]

127	[221, 222]	319	[483, 462]	511	[745, 699]	703	[34, 504]	895	[816, 388]
128	[223, 224]	320	[153, 408]	512	[746, 611]	704	[405, 395]	896	[437, 496]
129	[225, 226]	321	[484, 172]	513	[747, 748]	705	[440, 465]	897	[938, 144]
130	[227, 228]	322	[45, 485]	514	[749, 23]	706	[57, 431]	898	[507, 435]
131	[229, 230]	323	[391, 40]	515	[750, 287]	707	[829, 44]	899	[467, 462]
132	[231, 232]	324	[440, 177]	516	[751, 752]	708	[199, 410]	900	[179, 161]
133	[233, 234]	325	[486, 44]	517	[753, 754]	709	[172, 795]	901	[441, 473]
134	[235, 236]	326	[487, 488]	518	[556, 755]	710	[830, 196]	902	[167, 162]
135	[237, 219]	327	[210, 489]	519	[756, 90]	711	[503, 776]	903	[936, 441]
136	[238, 239]	328	[490, 42]	520	[757, 758]	712	[167, 801]	904	[934, 163]
137	[240, 241]	329	[425, 491]	521	[759, 760]	713	[14, 42]	905	[444, 206]
138	[68, 86]	330	[492, 146]	522	[381, 761]	714	[493, 419]	906	[137, 135]
139	[70, 242]	331	[493, 40]	523	[762, 763]	715	[423, 167]	907	[495, 208]
140	[243, 244]	332	[55, 12]	524	[764, 765]	716	[194, 141]	908	[809, 462]
141	[245, 246]	333	[421, 155]	525	[766, 767]	717	[511, 137]	909	[810, 147]
142	[247, 248]	334	[494, 418]	526	[34, 138]	718	[135, 443]	910	[939, 146]
143	[249, 250]	335	[495, 163]	527	[768, 48]	719	[831, 147]	911	[470, 161]
144	[251, 252]	336	[405, 496]	528	[199, 769]	720	[459, 508]	912	[788, 935]
145	[253, 254]	337	[204, 133]	529	[770, 497]	721	[414, 780]	913	[773, 429]
146	[255, 256]	338	[128, 497]	530	[432, 216]	722	[445, 207]	914	[134, 814]
147	[111, 257]	339	[130, 147]	531	[187, 477]	723	[825, 411]	915	[450, 32]
148	[258, 259]	340	[498, 468]	532	[485, 177]	724	[450, 780]	916	[187, 439]
149	[260, 261]	341	[499, 426]	533	[771, 150]	725	[140, 52]	917	[41, 420]
150	[262, 263]	342	[474, 133]	534	[772, 46]	726	[441, 489]	918	[493, 392]
151	[264, 265]	343	[500, 52]	535	[511, 504]	727	[809, 144]	919	[416, 521]
152	[266, 267]	344	[501, 151]	536	[164, 509]	728	[768, 508]	920	[440, 210]
153	[268, 269]	345	[493, 216]	537	[204, 773]	729	[832, 790]	921	[781, 136]
154	[270, 271]	346	[502, 503]	538	[514, 471]	730	[211, 159]	922	[520, 417]
155	[272, 273]	347	[212, 497]	539	[458, 774]	731	[173, 404]	923	[485, 465]
156	[274, 112]	348	[9, 479]	540	[775, 503]	732	[446, 790]	924	[524, 774]
157	[275, 276]	349	[166, 408]	541	[513, 774]	733	[390, 410]	925	[940, 218]
158	[277, 278]	350	[35, 159]	542	[132, 60]	734	[438, 182]	926	[398, 935]
159	[279, 280]	351	[388, 404]	543	[413, 776]	735	[519, 439]	927	[170, 937]
160	[281, 282]	352	[136, 504]	544	[152, 216]	736	[55, 170]	928	[815, 455]
161	[283, 284]	353	[444, 32]	545	[47, 501]	737	[210, 466]	929	[208, 814]
162	[269, 285]	354	[505, 141]	546	[201, 129]	738	[46, 441]	930	[140, 793]
163	[236, 286]	355	[506, 462]	547	[42, 38]	739	[407, 167]	931	[785, 801]
164	[287, 288]	356	[215, 392]	548	[777, 200]	740	[469, 448]	932	[777, 941]
165	[289, 290]	357	[507, 452]	549	[196, 778]	741	[831, 131]	933	[95, 942]
166	[103, 291]	358	[453, 433]	550	[505, 476]	742	[775, 454]	934	[943, 914]
167	[292, 293]	359	[47, 508]	551	[779, 417]	743	[514, 461]	935	[944, 945]
168	[294, 295]	360	[190, 481]	552	[444, 780]	744	[468, 197]	936	[946, 677]
169	[296, 297]	361	[509, 510]	553	[404, 411]	745	[511, 138]	937	[24, 947]
170	[298, 299]	362	[155, 197]	554	[781, 34]	746	[822, 182]	938	[948, 949]

171	[300, 301]	363	[511, 388]	555	[134, 208]	747	[770, 129]	939	[950, 951]
172	[302, 303]	364	[450, 206]	556	[128, 202]	748	[53, 774]	940	[952, 953]
173	[304, 305]	365	[512, 202]	557	[472, 782]	749	[822, 188]	941	[954, 955]
174	[306, 307]	366	[513, 54]	558	[783, 155]	750	[467, 479]	942	[33, 411]
175	[308, 309]	367	[498, 155]	559	[456, 172]	751	[480, 191]	943	[495, 134]
176	[310, 311]	368	[127, 197]	560	[401, 784]	752	[455, 397]	944	[819, 32]
177	[91, 312]	369	[514, 161]	561	[515, 202]	753	[502, 413]	945	[496, 518]
178	[313, 314]	370	[515, 129]	562	[785, 162]	754	[805, 52]	946	[830, 155]
179	[315, 316]	371	[441, 466]	563	[786, 180]	755	[454, 776]	947	[48, 933]
180	[317, 318]	372	[398, 516]	564	[779, 184]	756	[152, 419]	948	[822, 477]
181	[319, 320]	373	[214, 417]	565	[463, 433]	757	[467, 10]	949	[157, 12]
182	[321, 322]	374	[478, 144]	566	[787, 146]	758	[418, 399]	950	[177, 466]
183	[323, 324]	375	[204, 172]	567	[139, 411]	759	[792, 60]	951	[800, 170]
184	[325, 326]	376	[517, 165]	568	[781, 443]	760	[522, 418]	952	[790, 937]
185	[327, 328]	377	[192, 518]	569	[209, 481]	761	[515, 497]	953	[488, 408]
186	[329, 330]	378	[519, 477]	570	[389, 177]	762	[794, 150]	954	[938, 479]
187	[331, 332]	379	[423, 53]	571	[447, 442]	763	[195, 159]	955	[217, 186]
188	[333, 334]	380	[520, 521]	572	[502, 454]	764	[430, 58]		
189	[335, 336]	381	[41, 399]	573	[457, 58]	765	[465, 489]		
190	[337, 338]	382	[486, 503]	574	[788, 58]	766	[37, 400]		
191	[339, 340]	383	[522, 14]	575	[508, 151]	767	[490, 37]		

Output function $n \mapsto \tau(n)$:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
0	0	137	1	274	0	411	1	548	1	685	0	822	0
1	0	138	1	275	0	412	0	549	1	686	1	823	0
2	0	139	1	276	0	413	0	550	1	687	0	824	0
3	0	140	1	277	1	414	0	551	1	688	1	825	0
4	1	141	0	278	1	415	1	552	1	689	1	826	1
5	0	142	0	279	1	416	0	553	0	690	0	827	1
6	0	143	0	280	1	417	1	554	0	691	1	828	1
7	0	144	1	281	1	418	1	555	0	692	1	829	1
8	1	145	0	282	0	419	1	556	0	693	1	830	0
9	1	146	0	283	0	420	0	557	0	694	1	831	1
10	0	147	1	284	0	421	1	558	0	695	1	832	1
11	0	148	1	285	0	422	1	559	1	696	1	833	0
12	0	149	0	286	1	423	0	560	1	697	0	834	1
13	0	150	0	287	1	424	1	561	1	698	0	835	0
14	0	151	0	288	1	425	0	562	0	699	1	836	0
15	0	152	1	289	0	426	0	563	0	700	0	837	0
16	0	153	0	290	1	427	1	564	1	701	1	838	0
17	1	154	1	291	0	428	1	565	0	702	1	839	0
18	0	155	1	292	1	429	1	566	0	703	0	840	1
19	1	156	0	293	1	430	1	567	1	704	0	841	1

20	1	157	0	294	1	431	1	568	0	705	1	842	0
21	0	158	1	295	1	432	0	569	0	706	0	843	0
22	0	159	1	296	0	433	0	570	0	707	1	844	1
23	0	160	1	297	0	434	1	571	0	708	1	845	1
24	0	161	0	298	1	435	1	572	0	709	0	846	1
25	0	162	1	299	0	436	0	573	1	710	0	847	1
26	0	163	1	300	1	437	0	574	1	711	1	848	0
27	0	164	1	301	1	438	1	575	1	712	1	849	0
28	0	165	0	302	0	439	0	576	0	713	0	850	1
29	0	166	1	303	0	440	1	577	0	714	0	851	1
30	0	167	1	304	0	441	0	578	1	715	0	852	1
31	0	168	1	305	1	442	0	579	0	716	0	853	1
32	1	169	0	306	1	443	1	580	1	717	1	854	1
33	0	170	1	307	0	444	1	581	1	718	1	855	0
34	0	171	1	308	0	445	0	582	1	719	1	856	0
35	1	172	0	309	0	446	1	583	1	720	1	857	0
36	0	173	0	310	0	447	0	584	0	721	0	858	0
37	0	174	1	311	0	448	1	585	0	722	0	859	1
38	1	175	0	312	1	449	0	586	0	723	0	860	0
39	1	176	0	313	1	450	1	587	1	724	1	861	0
40	1	177	1	314	0	451	1	588	0	725	1	862	0
41	1	178	1	315	0	452	0	589	1	726	0	863	0
42	1	179	0	316	1	453	1	590	0	727	1	864	0
43	0	180	1	317	1	454	0	591	0	728	0	865	0
44	1	181	0	318	0	455	0	592	1	729	1	866	0
45	0	182	1	319	0	456	1	593	0	730	1	867	1
46	0	183	0	320	0	457	1	594	1	731	0	868	0
47	0	184	0	321	1	458	0	595	1	732	1	869	1
48	0	185	1	322	0	459	1	596	1	733	0	870	0
49	0	186	0	323	0	460	1	597	0	734	1	871	1
50	0	187	0	324	1	461	0	598	0	735	1	872	1
51	0	188	1	325	0	462	0	599	1	736	0	873	1
52	1	189	1	326	1	463	0	600	1	737	1	874	1
53	0	190	1	327	1	464	0	601	1	738	0	875	0
54	0	191	1	328	0	465	0	602	1	739	1	876	0
55	0	192	1	329	0	466	0	603	0	740	0	877	1
56	1	193	1	330	1	467	1	604	0	741	1	878	1
57	0	194	0	331	0	468	0	605	0	742	1	879	1
58	0	195	0	332	0	469	0	606	0	743	0	880	0
59	0	196	1	333	1	470	0	607	1	744	0	881	1
60	0	197	1	334	1	471	1	608	1	745	1	882	1
61	0	198	0	335	1	472	0	609	1	746	0	883	1
62	1	199	1	336	0	473	1	610	0	747	0	884	0
63	0	200	1	337	1	474	0	611	0	748	0	885	1

64	0	201	1	338	0	475	1	612	0	749	0	886	1
65	1	202	1	339	1	476	0	613	0	750	1	887	1
66	0	203	0	340	0	477	0	614	1	751	1	888	1
67	0	204	1	341	1	478	0	615	1	752	0	889	0
68	1	205	0	342	0	479	1	616	0	753	0	890	0
69	0	206	0	343	1	480	1	617	0	754	1	891	0
70	1	207	0	344	1	481	1	618	1	755	0	892	1
71	1	208	0	345	0	482	1	619	0	756	1	893	0
72	0	209	0	346	0	483	0	620	0	757	1	894	0
73	0	210	1	347	0	484	1	621	1	758	1	895	1
74	0	211	1	348	1	485	1	622	0	759	1	896	0
75	1	212	0	349	1	486	0	623	0	760	1	897	0
76	1	213	0	350	1	487	1	624	1	761	1	898	1
77	0	214	1	351	0	488	1	625	1	762	1	899	1
78	1	215	1	352	0	489	0	626	0	763	0	900	0
79	0	216	0	353	1	490	0	627	1	764	1	901	0
80	1	217	1	354	1	491	0	628	0	765	0	902	1
81	0	218	1	355	1	492	1	629	0	766	0	903	0
82	1	219	0	356	1	493	0	630	1	767	0	904	1
83	0	220	1	357	1	494	1	631	1	768	0	905	1
84	1	221	0	358	1	495	1	632	1	769	1	906	1
85	0	222	0	359	0	496	1	633	0	770	0	907	1
86	0	223	0	360	1	497	0	634	0	771	0	908	1
87	1	224	1	361	1	498	0	635	1	772	0	909	0
88	1	225	1	362	1	499	1	636	0	773	1	910	1
89	1	226	0	363	1	500	1	637	1	774	0	911	0
90	0	227	1	364	1	501	1	638	0	775	1	912	1
91	1	228	1	365	0	502	0	639	0	776	0	913	1
92	0	229	1	366	0	503	1	640	1	777	1	914	0
93	0	230	1	367	0	504	0	641	1	778	0	915	1
94	0	231	0	368	0	505	1	642	0	779	1	916	0
95	1	232	0	369	0	506	1	643	0	780	1	917	1
96	0	233	1	370	1	507	1	644	1	781	0	918	0
97	1	234	0	371	0	508	1	645	1	782	1	919	0
98	0	235	0	372	1	509	1	646	0	783	0	920	1
99	0	236	1	373	1	510	0	647	1	784	1	921	0
100	0	237	1	374	0	511	1	648	0	785	0	922	1
101	1	238	0	375	1	512	0	649	1	786	0	923	1
102	0	239	0	376	0	513	0	650	1	787	0	924	1
103	1	240	1	377	1	514	0	651	0	788	1	925	0
104	1	241	1	378	1	515	1	652	1	789	0	926	1
105	1	242	1	379	0	516	1	653	0	790	0	927	1
106	0	243	1	380	1	517	0	654	0	791	1	928	0
107	1	244	0	381	1	518	0	655	1	792	1	929	0

108	0	245	0	382	0	519	1	656	0	793	1	930	1
109	1	246	0	383	1	520	1	657	1	794	1	931	0
110	0	247	0	384	1	521	1	658	0	795	0	932	1
111	1	248	1	385	1	522	1	659	0	796	1	933	1
112	1	249	0	386	1	523	1	660	0	797	0	934	1
113	1	250	1	387	0	524	1	661	1	798	1	935	0
114	0	251	1	388	0	525	0	662	0	799	1	936	0
115	1	252	1	389	0	526	0	663	0	800	1	937	0
116	0	253	0	390	0	527	0	664	1	801	1	938	0
117	0	254	1	391	0	528	1	665	1	802	0	939	1
118	0	255	0	392	0	529	0	666	0	803	0	940	0
119	0	256	1	393	1	530	0	667	0	804	0	941	0
120	0	257	0	394	1	531	0	668	1	805	1	942	0
121	1	258	1	395	0	532	1	669	0	806	0	943	1
122	0	259	0	396	0	533	0	670	0	807	1	944	0
123	1	260	0	397	0	534	0	671	0	808	0	945	1
124	1	261	1	398	1	535	1	672	1	809	1	946	0
125	1	262	0	399	1	536	1	673	1	810	0	947	0
126	0	263	0	400	1	537	1	674	1	811	1	948	0
127	0	264	0	401	1	538	0	675	0	812	0	949	0
128	0	265	1	402	1	539	0	676	1	813	1	950	1
129	1	266	1	403	0	540	1	677	1	814	1	951	1
130	1	267	1	404	0	541	0	678	1	815	0	952	0
131	1	268	0	405	0	542	0	679	0	816	1	953	1
132	0	269	1	406	0	543	0	680	1	817	1	954	0
133	1	270	1	407	1	544	1	681	1	818	0	955	1
134	0	271	1	408	1	545	0	682	0	819	0		
135	1	272	1	409	0	546	1	683	0	820	1		
136	0	273	0	410	0	547	1	684	1	821	0		

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